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Analysis of a hidden memory variably distributed-order space-fractional diffusion equation

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ABSTRACT

We prove the wellposedness of a space-fractional diffusion equation governed by a hidden memory variably distributed-order fractional derivative, which provides a competitive means to describe the history memory property of anomalous diffusive transport of solutes in heterogeneous porous media.

1. Introduction

Space-fractional diffusion equations (sFDEs) could describe the superdiffusive transport of solutes characterized by highly skewed power-law decays [1–6]. As the order of sFDEs is determined by the fractal dimension of the surrounding porous medium via the Hurst index, a constant-order sFDE has difficulties to model the superdiffusive transport of solutes in highly heterogeneous porous media [7–13], the strong heterogeneity of the porous media may lead to spatially-dependent distributed-order fractional differential operators. A hidden memory variably distributed-order space-fractional derivative is defined by

$$ {}_0^C\mathbb{D}_x^{\omega} u(x) := \int_0^x u''(s) \int_0^1 \frac{\omega(\gamma, s)d\gamma}{\Gamma(1-\gamma)(x-s)^{\gamma}} ds := {}_0\mathbb{I}_x^{\omega} u'', \tag{1} $$

where $x \in [0,1]$, $\Gamma(\cdot)$ is the Gamma function, $\omega(\gamma, x)$ is the probability density function (pdf) satisfies $\displaystyle\int_0^1 \omega(\gamma, x) d\gamma = 1$ on $[0,1]$.

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In this paper we consider the following hidden memory variably distributed-order space-fractional diffusion equation

$$\begin{aligned} -u''(x) - d(x)\,{}_0^C\mathbb{D}_x^{\omega} u = f(x), \quad 0 < x < 1, \\ u(0) = u(1) = 0, \end{aligned} \tag{2}$$

where $d(x) \geq 0$ with $d(x)/(1+d(x))$ be the proportion of the superdiffusive phase versus the total solute mass at x, $f(x)$ is the source or sink term.

If the pdf ω is a delta function with $\omega(\gamma, x) = 0$ for $\gamma \neq \gamma_0$ $(0 < \gamma_0 < 1)$ and $\int_0^1 \omega(\gamma, x) d\gamma = 1$ for $x \in [0,1]$, (1) reduces to the standard Caputo fractional derivative. Wellposedness and smoothness of Caputo fractional differential equations have been extensively investigated in [14–17]. It is worth to mention that ω in (2) contains history memory itself, which stems from $\omega(\gamma, s)$ and its support $[0, \overline{\gamma}(s)]$ in (1), results in the integration by parts formula which is a key to prove the high order regularity in [18–20] inapplicable here. To analyze the wellposedness and smoothness properties of solutions to (2), we make the following assumptions on ω:

Assumptions on ω:

(a) supp $\omega(\cdot, x) \in [0, \overline{\gamma}(x)]$ for $0 \leq \overline{\gamma}(x) \leq \gamma^*$ with $0 < \gamma^* < 1, x \in [0,1]$.

(b) $\overline{\gamma} \in C^1[0, \gamma^*]$ and $\omega, \omega_x \in C([0,\gamma^*];[0,1])$.

Throughout this paper we use Q_1 to denote a fixed positive constant and Q to denote a generic positive constant that may assume different values at different occurrences.

2. Wellposedness of (2)

We let $v = u''(x)$ and move the fractional derivative to the right hand side to get the following Volterra integral equation

$$v(x) := -d(x)\,{}_0\mathbb{I}_x^{\omega} v(x) - f(x). \tag{3}$$

The solution u to (2) can be recovered from v by

$$u(x) = \int_0^x v(s)(x-s)ds - x\int_0^1 v(s)(1-s)ds.$$

Theorem 2.1. *Suppose that Assumption (a) holds, $d, f \in C[0,1]$, then (2) has a unique solution $u \in C^2[0,1]$ with*

$$\|u\|_{C^2[0,1]} \leq Q\|f\|_{C[0,1]}, \quad Q = Q(\|d\|_{C[0,1]}, \gamma^*). \tag{4}$$

Proof. We define a functional sequence $\{v_n(x)\}_{n=0}^{\infty}$ on $[0,1]$ by

$$v_0(x) := -f(x), \quad v_{n+1}(x) := v_0(x) - d(x)\,{}_0\mathbb{I}_x^{\omega} v_n(x), \quad n \geq 0.$$

We subtract $v_n(x)$ from $v_{n+1}(x)$ for $n = 1, 2, \ldots$ to get

$$v_{n+1}(x) - v_n(x) = -d(x)\,{}_0\mathbb{I}_x^{\omega}\big(v_n - v_{n-1}\big), \tag{5}$$

and use the fact that $\|v_0\|_{C[0,1]} = \|f\|_{C[0,1]}$ to obtain

$$\begin{aligned} \|v_1 - v_0\|_{C[0,1]} &= \Big\| d(x)\,{}_0\mathbb{I}_x^{\omega} v_0(x)\Big\|_{C[0,1]} \\ &= \Big\| d(x)\int_0^x v_0(s)\int_0^{\overline{\gamma}(s)} \frac{\omega(\gamma,s)d\gamma}{\Gamma(1-\gamma)(x-s)^{\gamma}} ds\Big\|_{C[0,1]} \\ &\leq \|d\|_{C[0,1]}\|v_0\|_{C[0,1]}\Big\| \int_0^x \frac{1}{(x-s)^{\gamma^*}}\int_0^{\overline{\gamma}(s)} \frac{\omega(\gamma,s)d\gamma}{\Gamma(1-\gamma)} ds\Big\|_{C[0,1]} \\ &\leq \|f\|_{C[0,1]}\|d\|_{C[0,1]}\int_0^x \frac{ds}{(x-s)^{\gamma^*}} := \frac{Q_1\|f\|_{C[0,1]}}{\Gamma(2-\gamma^*)}x^{1-\gamma^*}. \end{aligned} \tag{6}$$

We assume that

$$\|v_m - v_{m-1}\|_{C[0,1]} \le \frac{Q_1^m \|f\|_{C[0,1]} x^{m(1-\gamma^*)}}{\Gamma(1+m(1-\gamma^*))}, \quad m = 1,2,\dots,n. \tag{7}$$

Substitute (7) into (5) to get

$$\begin{aligned}
\|v_{n+1} - v_n\|_{C[0,1]} &\le \|d\|_{C[0,1]} \big\| {}_0\mathbb{I}_x^{\omega}(v_n - v_{n-1}) \big\|_{C[0,1]} \\
&\le \frac{\|d\|_{C[0,1]}\|f\|_{C[0,1]} Q_1^n}{\Gamma(1+n(1-\gamma^*))} \Big\| \int_0^x s^{n(1-\gamma^*)} \int_0^{\overline{\gamma}(s)} \frac{\omega(\gamma,s)d\gamma}{\Gamma(1-\gamma)(x-s)^{\gamma}} ds \Big\|_{C[0,1]} \\
&\le \frac{\|f\|_{C[0,1]} Q_1^{n+1}}{\Gamma(1+n(1-\gamma^*))\Gamma(1-\gamma^*)} \int_0^x s^{n(1-\gamma^*)}(x-s)^{-\gamma^*} ds \\
&\le \frac{\|f\|_{C[0,1]} Q_1^{n+1} B(1+n(1-\gamma^*), 1-\gamma^*) x^{(n+1)(1-\gamma^*)}}{\Gamma(1+n(1-\gamma^*))\Gamma(1-\gamma^*)} \\
&= \frac{\|f\|_{C[0,1]} Q_1^{n+1} x^{(n+1)(1-\gamma^*)}}{\Gamma(1+(n+1)(1-\gamma^*))},
\end{aligned}$$

where $B(p,q)$ is the Beta function. Thus, assumption (7) holds for $n \in \mathbb{N}^+$ by mathematical induction.

As the accumulation of the right hand side terms of (7) converges as

$$\sum_{n=1}^{\infty} \frac{\|f\|_{C[0,1]} Q_1^n x^{n(1-\gamma^*)}}{\Gamma(1+n(1-\gamma^*))} = \|f\|_{C[0,1]} E_{1-\gamma^*,1}(Q_1 x^{1-\gamma^*}) < \infty, \quad x \in [0,1],$$

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function, we conclude that $\{v_n(x)\}_{n=0}^{\infty}$ converges uniformly on $[0,1]$. We use

$$v(x) := \lim_{n\to\infty} v_n(t) = \sum_{n=1}^{\infty}(v_n - v_{n-1}) + v_0(x),$$

to conclude estimate (4).

The uniqueness can be proved follows from the stability estimate and we skip it here.

Theorem 2.2. *Suppose that Assumptions (a) and (b) hold, if $d, f \in C^1[0,1]$, then $u \in C^3[x,1]$ for $0 < x \ll 1$ with*

$$\|u\|_{C^3[x,1]} \le Q\|f\|_{C^1[0,1]} x^{-\overline{\gamma}(0)}, \tag{8}$$

where $Q = Q(\|d\|_{C^1[0,1]}, \|\overline{\gamma}\|_{C^1[0,1]}, \|\omega\|_{C([0,\gamma^*];[0,1])}, \|\omega_x\|_{C([0,\gamma^*];[0,1])})$.

Proof. We differentiate (3) to get

$$v'(x) = -d(x)\big({}_0\mathbb{I}_x^{\omega} v\big)' - d'(x)\,{}_0\mathbb{I}_x^{\omega} v(x) - f'(x). \tag{9}$$

We apply Theorem 2.1 to estimate the second and third terms on the right hand side of (9) by

$$\begin{aligned}
&\big\| d'(x)\,{}_0\mathbb{I}_x^{\omega} v(x) + f'(x) \big\|_{C[0,1]} \\
&\le \|d\|_{C^1[0,1]} \int_0^x \|v\|_{C[0,1]} \int_0^{\overline{\gamma}(s)} \frac{\omega(\gamma,s)d\gamma}{\Gamma(1-\gamma)(x-s)^{\gamma}} ds + \|f'\|_{C[0,1]} \\
&\le Q\|f\|_{C^1[0,1]}.
\end{aligned}$$

Affected by the history memory of the variably distributed-order fractional derivative, the integration by parts formula cannot be applied directly. In order to analyze the first term on the right hand side of (9), we use

$$\int_0^{\overline{\gamma}(s)} \frac{\omega(\gamma,s)d\gamma}{\Gamma(1-\gamma)(x-s)^{\gamma}} = \int_0^{\overline{\gamma}(x)} \frac{\omega(\gamma,s)d\gamma}{\Gamma(1-\gamma)(x-s)^{\gamma}} + \int_{\overline{\gamma}(x)}^{\overline{\gamma}(s)} \frac{\omega(\gamma,s)d\gamma}{\Gamma(1-\gamma)(x-s)^{\gamma}}$$

to divide ${}_0\mathbb{I}_x^{\omega} v$ into two parts,

$$\begin{aligned}{}_0\mathbb{I}_x^{\omega} v &= \int_0^x v(s)\int_0^{\overline{\gamma}(x)} \frac{\omega(\gamma,s)d\gamma}{\Gamma(1-\gamma)(x-s)^{\gamma}}ds + \int_0^x v(s)\int_{\overline{\gamma}(x)}^{\overline{\gamma}(s)} \frac{\omega(\gamma,s)d\gamma}{\Gamma(1-\gamma)(x-s)^{\gamma}}ds \\ &:= I_1 + I_2.\end{aligned}$$

Now we analyze I_1 and I_2, respectively.

We integrate the singular kernal $(x-s)^{-\gamma}$ by parts to rewrite I_1 as

$$\begin{aligned}I_1 &= \int_0^{\overline{\gamma}(x)} \frac{1}{\Gamma(1-\gamma)}\int_0^x \frac{v(s)\omega(\gamma,s)ds}{(x-s)^{\gamma}}d\gamma \\ &= -\int_0^{\overline{\gamma}(x)} \frac{1}{\Gamma(2-\gamma)}\Big(\int_0^x v(s)\omega(\gamma,s)d(x-s)^{1-\gamma}\Big)d\gamma \\ &= \int_0^{\overline{\gamma}(x)} \frac{1}{\Gamma(2-\gamma)}\int_0^x (x-s)^{1-\gamma}\Big[v'(s)\omega(\gamma,s)+v(s)\omega_s(\gamma,s)\Big]dsd\gamma \\ &\quad + \int_0^{\overline{\gamma}(x)} \frac{\omega(\gamma,0)x^{1-\gamma}d\gamma}{\Gamma(2-\gamma)}v_0.\end{aligned}$$

We differentiate I_1 to get

$$\begin{aligned}I_1' &= \Big(\frac{\overline{\gamma}'(x)\omega(\overline{\gamma}(x),0)x^{1-\overline{\gamma}(x)}}{\Gamma(2-\overline{\gamma}(x))} + \int_0^{\overline{\gamma}(x)} \frac{\omega(\gamma,0)d\gamma}{\Gamma(1-\gamma)x^{\gamma}}\Big)v_0 \\ &\quad + \frac{\overline{\gamma}'(x)}{\Gamma(2-\overline{\gamma}(x))}\int_0^x (x-s)^{1-\overline{\gamma}(x)}\Big[v'(s)\omega(\overline{\gamma}(x),s)+v(s)\omega_s(\overline{\gamma}(x),s)\Big]ds \\ &\quad + \int_0^{\overline{\gamma}(x)} \frac{1}{\Gamma(1-\gamma)}\int_0^x \frac{\big[v'(s)\omega(\gamma,s)+v(s)\omega_s(\gamma,s)\big]ds}{(x-s)^{\gamma}}d\gamma.\end{aligned} \tag{10}$$

We use Assumptions (a)–(b) and the fact that for $0 \le \gamma \le \overline{\gamma}(x)$,

$$x^{-\gamma} \le x^{-\overline{\gamma}(x)} = x^{-\overline{\gamma}(0)}x^{\overline{\gamma}(0)-\overline{\gamma}(x)} = x^{-\overline{\gamma}(0)}e^{(\overline{\gamma}(0)-\overline{\gamma}(x))\ln x} \le Qx^{-\overline{\gamma}(0)}$$

to bound the second term on the right hand side of (10) by

$$\Big|\int_0^{\overline{\gamma}(x)} \frac{\omega(\gamma,0)d\gamma}{\Gamma(1-\gamma)x^{\gamma}}v_0\Big| \le Q|v(0)|x^{-\overline{\gamma}(0)}.$$

We use Theorem 2.1 to estimate the rest terms on the right hand side of (10) by

$$Q\|v\|_{C[0,1]} + Q\int_0^x \frac{|v'(s)|}{(x-s)^{\gamma^*}}ds.$$

Thus I_1 can be bounded by

$$|I_1| \le Q\|v\|_{C[0,1]}x^{-\overline{\gamma}(0)} + Q\int_0^x \frac{|v'(s)|ds}{(x-s)^{\gamma^*}}. \tag{11}$$

For $0 \le s < x$, if $\overline{\gamma}(x) > \overline{\gamma}(s)$, then $\lim_{s\to x}(x-s)^{\overline{\gamma}(x)-\gamma} = 0$, else if $\overline{\gamma}(x) \le \overline{\gamma}(s)$, we have

$$(x-s)^{\overline{\gamma}(x)-\gamma} = (x-s)^{\overline{\gamma}(x)-\overline{\gamma}(s)}(x-s)^{\overline{\gamma}(s)-\gamma} \le e^{(\overline{\gamma}(x)-\overline{\gamma}(s))\ln(x-s)} \to 1,\ s \to x.$$

So we conclude that $\lim_{s\to x}(x-s)^{\overline{\gamma}(x)-\gamma}$ is bounded for γ between $\overline{\gamma}(s)$ and $\overline{\gamma}(x)$. Apply the integration by parts formula, we rewrite I_2 as

$$\begin{aligned}
I_2 &= -\frac{1}{1-\overline{\gamma}(x)}\int_0^x v(s)\int_{\overline{\gamma}(x)}^{\overline{\gamma}(s)}\frac{\omega(\gamma,s)(x-s)^{\overline{\gamma}(x)-\gamma}d\gamma}{\Gamma(1-\gamma)}d(x-s)^{1-\overline{\gamma}(x)}\\
&= \frac{v(0)}{1-\overline{\gamma}(x)}\int_{\overline{\gamma}(x)}^{\overline{\gamma}(0)}\frac{\omega(\gamma,0)x^{1-\gamma}d\gamma}{\Gamma(1-\gamma)}\\
&\quad +\frac{1}{1-\overline{\gamma}(x)}\int_0^x v'(s)\int_{\overline{\gamma}(x)}^{\overline{\gamma}(s)}\frac{\omega(\gamma,s)(x-s)^{1-\gamma}d\gamma}{\Gamma(1-\gamma)}ds\\
&\quad +\frac{1}{1-\overline{\gamma}(x)}\int_0^x v(s)\int_{\overline{\gamma}(x)}^{\overline{\gamma}(s)}\frac{\omega_s(\gamma,s)(x-s)^{1-\gamma}d\gamma}{\Gamma(1-\gamma)}ds\\
&\quad -\frac{1}{1-\overline{\gamma}(x)}\int_0^x v(s)\int_{\overline{\gamma}(x)}^{\overline{\gamma}(s)}\frac{(\overline{\gamma}(x)-\gamma)\omega(\gamma,s)d\gamma}{\Gamma(1-\gamma)(x-s)^{\gamma}}ds.
\end{aligned}$$

We observe that the first three terms on the right hand side of previous formula have continuous kernels. By the Assumption (b), the kernel of the last term could be bounded by

$$\Big|\int_{\overline{\gamma}(x)}^{\overline{\gamma}(s)}\frac{(\overline{\gamma}(x)-\gamma)\omega(\gamma,s)}{\Gamma(1-\gamma)(x-s)^{\gamma}}d\gamma\Big| \le \frac{Q|\overline{\gamma}(s)-\overline{\gamma}(x)|}{(x-s)^{\max\{\overline{\gamma}(x),\overline{\gamma}(s)\}}} \le Q(x-s)^{1-\gamma^*}.$$

Consequently, we bound I_2 by

$$|I_2| \le Q\|v\|_{C[0,1]} + Q\int_0^x |v'(s)|ds.$$

We then differentiate I_2 to get

$$\begin{aligned}
I_2' &= \frac{\overline{\gamma}'(x)}{1-\overline{\gamma}(x)}I_2 - \frac{v(0)\overline{\gamma}'(x)\omega(\overline{\gamma}(x),0)x^{1-\overline{\gamma}(x)}}{\Gamma(2-\overline{\gamma}(x))}\\
&\quad +\frac{v(0)}{1-\overline{\gamma}(x)}\int_{\overline{\gamma}(x)}^{\overline{\gamma}(0)}\frac{\omega(\gamma,0)(1-\gamma)d\gamma}{\Gamma(1-\gamma)x^{\gamma}}\\
&\quad -\frac{\overline{\gamma}'(x)}{\Gamma(2-\overline{\gamma}(x))}\int_0^x v'(s)\omega(\overline{\gamma}(x),s)(x-s)^{1-\overline{\gamma}(x)}ds\\
&\quad +\frac{1}{1-\overline{\gamma}(x)}\int_0^x v'(s)\int_{\overline{\gamma}(x)}^{\overline{\gamma}(s)}\frac{\omega(\gamma,s)(1-\gamma)d\gamma}{\Gamma(1-\gamma)(x-s)^{\gamma}}ds\\
&\quad -\frac{\overline{\gamma}'(x)}{\Gamma(2-\overline{\gamma}(x))}\int_0^x v(s)\omega_s(\overline{\gamma}(x),s)(x-s)^{1-\overline{\gamma}(x)}ds\\
&\quad +\frac{1}{1-\overline{\gamma}(x)}\int_0^x v(s)\int_{\overline{\gamma}(x)}^{\overline{\gamma}(s)}\frac{\omega_s(\gamma,s)(1-\gamma)d\gamma}{\Gamma(1-\gamma)(x-s)^{\gamma}}ds\\
&\quad -\frac{\overline{\gamma}'(x)}{1-\overline{\gamma}(x)}\int_0^x v(s)\int_{\overline{\gamma}(x)}^{\overline{\gamma}(s)}\frac{\omega(\gamma,s)d\gamma}{\Gamma(1-\gamma)(x-s)^{\gamma}}ds\\
&\quad +\frac{1}{1-\overline{\gamma}(x)}\int_0^x v(s)\int_{\overline{\gamma}(x)}^{\overline{\gamma}(s)}\frac{(\overline{\gamma}(x)-\gamma)\omega(\gamma,s)\gamma d\gamma}{\Gamma(1-\gamma)(x-s)^{1+\gamma}}ds.
\end{aligned}\tag{12}$$

We estimate the fifth term on the right hand side of (12) by

$$\begin{aligned}
&\Big|\int_0^x v'(s)\int_{\overline{\gamma}(x)}^{\overline{\gamma}(s)}\frac{\omega(\gamma,s)(1-\gamma)d\gamma}{\Gamma(1-\gamma)(x-s)^{\gamma}}ds\Big|\\
&\quad \le Q\Big|\int_0^x \frac{v'(s)|\overline{\gamma}(s)-\overline{\gamma}(x)|}{(x-s)^{\max\{\overline{\gamma}(x),\overline{\gamma}(s)\}}}ds\Big| \le Q\int_0^x |v'(s)|ds.
\end{aligned}$$

The third and the last terms on the right hand side of (12) could be bounded by

$$\begin{aligned}&\Big|\frac{v(0)}{1-\overline{\gamma}(x)}\int_{\overline{\gamma}(x)}^{\overline{\gamma}(0)}\frac{\omega(\gamma,0)(1-\gamma)d\gamma}{\Gamma(1-\gamma)x^{\gamma}}\Big|\leq\frac{Q|v(0)||\overline{\gamma}(0)-\overline{\gamma}(x)|}{x^{\max\{\overline{\gamma}(0),\overline{\gamma}(x)\}}}\\&\leq Q|v_0|\|\overline{\gamma}\|_{C^1[0,1]}x^{1-\max\{\overline{\gamma}(x),\overline{\gamma}(0)\}}\leq Q\|v\|_{C[0,T]},\end{aligned}$$

and

$$\begin{aligned}&\Big|\int_0^x v(s)\int_{\overline{\gamma}(x)}^{\overline{\gamma}(s)}\frac{(\overline{\gamma}(x)-\gamma)\omega(\gamma,s)\gamma d\gamma}{\Gamma(1-\gamma)(x-s)^{1+\gamma}}ds\Big|\\&\leq\|\overline{\gamma}\|_{C^1[0,1]}\Big|\int_0^x\frac{v(s)(s-x)ds}{(x-s)^{1+\max\{\overline{\gamma}(x),\overline{\gamma}(s)\}}}\Big|\leq Q\|v\|_{C[0,T]}.\end{aligned}$$

Similarly, the rest terms on the right hand side of (12) could be bounded by $Q\|v\|_{C[0,1]}$. Finally, we obtain

$$|I_2'(x)|\leq Q\|v\|_{C[0,1]}+Q\int_0^x|v'(s)|ds.$$

Incorporate with (11), we conclude that

$$\big|\big({}_0\mathbb{I}_x^{\omega}v\big)'\big|\leq Q\|v\|_{C[0,1]}(x^{-\overline{\gamma}(0)}+1)+Q\int_0^x\frac{|v'(s)|ds}{(x-s)^{\gamma^*}}.$$

We substitute this result into (9) to get

$$|v'(x)|\leq Q\|f\|_{C^1[0,1]}(x^{-\overline{\gamma}(0)}+1)+Q\int_0^x\frac{|v'(s)|ds}{(x-s)^{\gamma^*}},$$

by applying the generalized Gronwall inequality [21], we obtain

$$\begin{aligned}|v'(x)|&\leq Q\|f\|_{C^1[0,1]}\Big(x^{-\overline{\gamma}(0)}+\sum_{n=1}^{\infty}\frac{(Q\Gamma(1-\gamma^*))^n}{\Gamma(n(1-\gamma^*))}\int_0^x(x-s)^{n(1-\gamma^*)-1}s^{-\overline{\gamma}(0)}ds\Big)\\&\leq Q\|f\|_{C^1[0,1]}x^{-\overline{\gamma}(0)}.\end{aligned}$$

Thus we finish the proof of (8).

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