Notre Dame Journal of Formal Logic
Volume 61, Number 2, 2020

Distality for the Asymptotic Couple of the Field of Logarithmic Transseries

Allen Gehret and Elliot Kaplan

Abstract We show that the theory $T_{\log}$ of the asymptotic couple of the field of logarithmic transseries is distal. As distal theories are NIP (have the non-independence property), this provides a new proof that $T_{\log}$ is NIP.

1 Introduction

Distal theories and structures were introduced by Simon [13] as a way to distinguish those NIP theories which are in some sense *purely unstable*, that is, where absolutely no stable behavior of any kind occurs. We sometimes think of distality as meaning: everything in sight is completely controlled or dominated by linear orders, either overtly or covertly (see Definition 2.1). Any o-minimal theory is distal, and the p-adic fields are distal as well. In an o-minimal structure, everything is controlled by the obvious underlying linear order. In the p-adics, there is no underlying linear order; however, everything is still controlled in some sense by the totally ordered value group (up to a finite residue field). A non-example is the theory of algebraically closed valued fields (ACVF). Indeed, the interpretable residue field is an algebraically closed field, a purely stable structure which is not being controlled by any linear order.

More recently, Chernikov, Galvin, and Starchenko showed that a strong Szemerédi-type regularity lemma and other combinatorial results hold in all distal structures (see [2], [4]). Consequently, there has been increased interest in classifying which NIP structures are distal, as well as classifying which NIP structures have distal expansions. In this paper we prove that a particular structure, the *asymptotic couple* $(\Gamma_{\log}, \psi)$ *of the ordered valued differential field* $\mathbb{T}_{\log}$ *of logarithmic transseries*, is

Received May 22, 2018; accepted December 8, 2019
First published online April 7, 2020
2010 Mathematics Subject Classification: Primary 03C64; Secondary 03C45, 06F20
Keywords: asymptotic couples, asymptotic integration, distality, dp-rank, independence property, indiscernible sequences, logarithmic transseries
 10.1215/00294527-2020-0010

distal. Asymptotic couples arise as the value groups of certain types of valued differential fields: the so-called *asymptotic fields*. See Aschenbrenner, van den Dries, and van der Hoeven [1] for the full story. We now define the object $(\Gamma_{\log}, \psi)$.

Throughout, m and n range over $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $\bigoplus_n \mathbb{R}e_n$ be a vector space over $\mathbb{R}$ with basis (e_n). Then $\bigoplus_n \mathbb{R}e_n$ can be made into an ordered group using the usual lexicographical order, that is, by requiring for nonzero $\sum_i r_i e_i$ that

$$\sum r_i e_i > 0 \iff r_n > 0 \quad \text{for the least } n \text{ such that } r_n \neq 0.$$

Let $\Gamma_{\log}$ be the above ordered abelian group $\bigoplus_n \mathbb{R}e_n$. It is often convenient to think of an element $\sum r_i e_i$ as the vector $(r_0, r_1, r_2, \ldots)$. We follow Rosenlicht [12] in taking the function

$$\psi : \Gamma_{\log} \setminus \{0\} \to \Gamma_{\log}$$

defined by

$$(\underbrace{0, \ldots, 0}_{n}, \underbrace{r_n}_{\neq 0}, r_{n+1}, \ldots) \mapsto (\underbrace{1, \ldots, 1}_{n+1}, 0, 0, \ldots)$$

as a new primitive, calling the pair $(\Gamma_{\log}, \psi)$ an *asymptotic couple* (the asymptotic couple of $\mathbb{T}_{\log}$). In [6] and [7], the first author studied the model theory of $(\Gamma_{\log}, \psi)$ in detail. There, $(\Gamma_{\log}, \psi)$ is construed as an $\mathcal{L}_{\log}$-structure for a certain first-order language $\mathcal{L}_{\log}$ (see, e.g., [6, Section 5]). In this paper we continue the study of the theory $T_{\log} = \mathrm{Th}_{\mathcal{L}_{\log}}(\Gamma_{\log}, \psi)$. The main result is the following.

Theorem 1.1 *$T_{\log}$ is distal.*

We provide a proof in Proposition 2.9 of the well-known fact that all distal theories are NIP. Thus, Theorem 1.1 has the following immediate consequence.

Corollary 1.2 *$T_{\log}$ is NIP.*

This provides a new proof of the main result from [7]. The original proof that $T_{\log}$ is NIP in [7] involved a counting-types argument which invoked a consistency result of Mitchell and used the fact that the statement "$T_{\log}$ is NIP" is absolute. The appeal of the new proof of Corollary 1.2 is that it is algebraic, and avoids any set-theoretic black boxes by taking place entirely within ZFC.

Theorem 1.1, together with [4, Corollary 6.3], also has the following amusing consequence.

Corollary 1.3 *No model of $T_{\log}$ interprets an infinite field of positive characteristic.*

We believe that no model of $T_{\log}$ interprets a field of characteristic zero either, although we leave that story for another time and place.

In Section 2 we recall some definitions and basic facts around distality and NIP. We also state and prove a general criterion for showing that a theory of a certain form is distal. This criterion is based on one developed by Hieronymi and Nell [11]. There they use it to show that certain "pairs" such as $(\mathbb{R}; 0, +, \cdot, <, 2^{\mathbb{Z}})$ are distal. We are able to adapt it for use in our setting due to certain superficial syntactic similarities between our structure and their pairs.

In Section 3 we discuss the basics of H-asymptotic couples with asymptotic integration. We also define the language $\mathcal{L}_{\log}$ and discuss the theory $T_{\log}$. We restate some useful facts about models of $T_{\log}$ which were established in [6] and [7]. In

Section 4 we go on to prove some additional lemmas concerning the behavior of indiscernible sequences in models of $T_{\log}$.

In Section 5 we introduce a concept of an indiscernible sequence $(a_i)_{i\in I}$ in a model of $T_{\log}$ being *spread out* by a parameter b. Roughly speaking, this means that the sequence $(a_i - b)_{i\in I}$ is sufficiently widely distributed in the convex hull of the Ψ-set. We then proceed to show that in such a situation, there is a certain desirable monotone interaction between the translated sequence and the Ψ-set (Lemma 5.3). This is one of the key steps in the proof of distality of $T_{\log}$.

In Section 6 we prove several finiteness results concerning finite-rank extensions of the underlying groups of models of $T_{\log}$, and their relationship to the functions ψ, s, and p from $\mathcal{L}_{\log}$.

In Section 7 we bring everything together to prove Theorem 1.1.

Ordered set conventions By *ordered set* we mean *totally ordered set*.

Let S be an ordered set. Below, the ordering on S will be denoted by $\leq$, and a subset of S is viewed as ordered by the induced ordering. We put $S_\infty := S \cup \{\infty\}$, where $\infty \notin S$, with the ordering on S extended to a (total) ordering on S_∞ by $S < \infty$. Suppose that B is a subset of S. We put $S^{>B} := \{s \in S : s > b \text{ for every } b \in B\}$, and we denote $S^{>\{a\}}$ as just $S^{>a}$; similarly for $\geq$, $<$, and $\leq$ instead of $>$. For $A \subseteq S$, we let

$$\operatorname{conv}(A) := \{x \in S : a \leq x \leq b \text{ for some } a, b \in A\}$$

be the *convex hull of* A in S, that is, the smallest convex subset of S containing A. For $A \subseteq S$, we put

$$A^{\downarrow} := \{s \in S : s \leq a \text{ for some } a \in A\},$$

which is the smallest downward closed subset of S containing A.

We say that S is a *successor set* if every element $x \in S$ has an *immediate successor* $y \in S$, that is, $x < y$ and for all $z \in S$, if $x < z$, then $y \leq z$. For example, $\mathbb{N}$ and $\mathbb{Z}$ with their usual orderings are successor sets. We say that S is a *copy* of $\mathbb{Z}$ if $(S, <)$ is isomorphic to $(\mathbb{Z}, <)$.

Ordered abelian group conventions Suppose that G is an ordered abelian group. Then we set $G^{\neq} := G \setminus \{0\}$. Also $G^{<} := G^{<0}$; similarly for $\geq$, $\leq$, and $>$ instead of $<$. We define $|g| := \max(g, -g)$ for $g \in G$. For $a \in G$, the *archimedean class* of a is defined by

$$[a] := \big\{g \in G : |a| \leq n|g| \text{ and } |g| \leq n|a| \text{ for some } n \geq 1\big\}.$$

The archimedean classes partition G. Each archimedean class $[a]$ with $a \neq 0$ is the disjoint union of the two convex sets $[a] \cap G^{<}$ and $[a] \cap G^{>}$. We order the set $[G] := \{[a] : a \in G\}$ of archimedean classes by

$$[a] < [b] :\iff n|a| < |b| \quad \text{for all } n \geq 1.$$

We have $[0] < [a]$ for all $a \in G^{\neq}$, and

$$[a] \leq [b] \iff |a| \leq n|b| \quad \text{for some } n \geq 1.$$

Model theory conventions In general, we adopt the model-theoretic conventions of Appendix B of [1]. In particular, $\mathcal{L}$ can be a many-sorted language. For a complete $\mathcal{L}$-theory T, we will sometimes consider a model $\mathbb{M} \models T$ and a cardinal $\kappa(\mathbb{M}) > |\mathcal{L}|$ such that $\mathbb{M}$ is $\kappa(\mathbb{M})$-saturated and every reduct of $\mathbb{M}$ is strongly $\kappa(\mathbb{M})$-homogeneous. Such a model is called a *monster model* of T. In particular, every model of T of size at most $\kappa(\mathbb{M})$ has an elementary embedding into $\mathbb{M}$. All variables are finite multivariables. By convention, we will write *indiscernible sequence* when we mean *$\emptyset$-indiscernible sequence*.

Sequence conventions Suppose that $(a_i)_{i \in I}$ is a sequence of distinct elements from some set indexed by a linear order I. Given a subset or subsequence $A \subseteq (a_i)$, we let $I^{>A}$ denote the index set

$$I^{>A} := \bigcap_{a_{i_0} \in A} \{i \in I : i > i_0\} \subseteq I$$

(similarly for $I^{<A}$). Furthermore, given $I_0 \subseteq I$ we denote by $A \restriction I_0$ the set

$$A \restriction I_0 := \{a_i \in A : i \in I_0\} \subseteq A.$$

2 Distality and NIP

This section contains all of the general model-theoretic content we need for this paper. This includes a definition of distality, a criterion for proving that theories of a certain form are distal, and a proof that distal theories are NIP. *Throughout this section $\mathcal{L}$ is a language and T is a complete $\mathcal{L}$-theory.*

Definition of distality *In this subsection we fix a monster model $\mathbb{M}$ of T. We also let I_1, I_2 range over infinite linearly ordered index sets.* The definitions do not depend on the choice of this monster model. We define *distality* in Definition 2.1 below in terms of "upgradability" of a certain indiscernible sequence configuration. In practice, this seems to be one of the more convenient definitions to work with, and it is the only one we use in this paper. For other equivalent definitions of distality see Simon [13] or [14, Chapter 9].

Definition 2.1 Given I_1 and I_2, we say that T is *I_1, I_2-distal* if for every $A \subseteq \mathbb{M}$, for every x, and for every indiscernible sequence $(a_i)_{i \in I}$ from $\mathbb{M}_x$, if

(1) $I = I_1 + (c) + I_2$, and
(2) $(a_i)_{i \in I_1 + I_2}$ is A-indiscernible,

then $(a_i)_{i \in I}$ is A-indiscernible. We say that T is *distal* if T is I_1, I_2-distal for every I_1 and I_2. Finally, we say that an $\mathcal{L}$-structure $\mathcal{M}$ is *distal* if $\mathrm{Th}(\mathcal{M})$ is distal.

It is also convenient to define what it means for a formula $\varphi(x; y)$ to be distal.

Definition 2.2 Given I_1 and I_2, we say that a formula $\varphi(x; y)$ is *I_1, I_2-distal* if for every $b \in \mathbb{M}_y$ and every indiscernible sequence $(a_i)_{i \in I}$ from $\mathbb{M}_x$ such that

(1) $I = I_1 + (c) + I_2$, and
(2) $(a_i)_{i \in I_1 + I_2}$ is b-indiscernible,

then $\models \varphi(a_c; b) \leftrightarrow \varphi(a_i; b)$ for every $i \in I$. We say that the formula $\varphi(x; y)$ is *distal* if it is I_1, I_2-distal for every I_1 and I_2.

It is well known that when checking distality, either for an individual formula or an entire theory, one is free to use any specific I_1 and I_2 they wish. To make this sentiment precise we have introduced the provisional terminology "I_1, I_2-distal" which is not standard (see Lemmas 2.3 and 2.4). We exploit this freedom in the proof of Distal Criterion 2.7 below.

Lemma 2.3 *The following are equivalent for a formula $\varphi(x; y)$:*

(1) *$\varphi(x; y)$ is distal;*
(2) *$\varphi(x; y)$ is I_1, I_2-distal for some I_1 and I_2.*

Proof The *standard lemma* (see Tent and Ziegler [15, Lemma 5.1.3]) allows one to convert a counterexample of I_1, I_2-distality into a counterexample of J_1, J_2-distality, where J_1, J_2 are two other infinite linear orders. The details are left to the reader. □

Lemma 2.4 *The following are equivalent:*

(1) *T is distal.*
(2) *There are I_1 and I_2 such that T is I_1, I_2-distal.*
(3) *Every $\varphi(x; y) \in \mathfrak{L}$ is distal.*
(4) *There are I_1 and I_2 such that every $\varphi(x; y) \in \mathfrak{L}$ is I_1, I_2-distal.*

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Leftrightarrow$ (3) follow by definition and Lemma 2.3. For (3) $\Rightarrow$ (1), assume that T is not distal. Then there are I_1 and I_2 such that T is not I_1, I_2-distal. This failure of I_1, I_2-distality is witnessed by some x, y, some sequence $(a_i)_{i \in I_1 + (c) + I_2}$ from $\mathbb{M}_x$, some formula $\varphi(x_1, \ldots, x_n; y)$ where each x_i is of the same sort as x, and some parameter $b \in \mathbb{M}_y$. By adjusting this counterexample—through a combination of joining outer elements of the sequence with the parameter b, and/or grouping elements of the sequence together to create a new "thickened" sequence—one arrives at a formula $\varphi(x'; y')$ (same formula, possibly different presentation of free variables) which is not distal. The argument is routine and left to the reader, although a word of caution is in order: in general I_1 and I_2 are not dense linear orders, and they may or may not have endpoints, and so on. So the reduction as described above really depends on I_1 and I_2 and where the elements from (a_i) which witness the failure of distality are located on these sequences. □

Lemma 2.4 permits us to work with *any* I_1, I_2 we wish. It will be convenient for us to work with I_1, I_2 of a special form.

Definition 2.5 We say that a linear order $I = I_1 + (c) + I_2$ is in *distal configuration* at c if I_1 and I_2 are infinite, I_1 does not have a greatest element, and I_2 does not have a least element.

Working with sequences in distal configuration is primarily used in the proofs of Distal Criterion 2.7 and Proposition 5.4 below. It is not clear how to remove the assumption of distal configuration from these arguments, at least without making things more complicated.

Remark 2.6 The collection of all distal formulas in the variables $(x; y)$ is closed under arbitrary boolean combinations, including negations. In the literature, there is another local notion of distality: that of a formula $\varphi(x; y)$ having a *strong honest definition* (see Chernikov and Simon [3, Theorem 21]). The collection of formulas $\varphi(x; y)$ which have a strong honest definition is in general only closed under *positive* boolean combinations.

A criterion for distality To set the stage for Distal Criterion 2.7 below, we now consider an extension $\mathcal{L}(\mathfrak{F}) := \mathcal{L} \cup \mathfrak{F}$ of the language $\mathcal{L}$ by a set $\mathfrak{F}$ of new unary function symbols involving sorts which are already present in $\mathcal{L}$. We also consider $T(\mathfrak{F})$, a complete $\mathcal{L}(\mathfrak{F})$-theory extending T. Given a model $\mathcal{M} \models T$, we denote by $(\mathcal{M}, \mathfrak{F})$ an expansion of $\mathcal{M}$ to a model of $T(\mathfrak{F})$. For a subset X of a model $\mathcal{M}$, we let $\langle X \rangle$ denote the $\mathcal{L}$-substructure of $\mathcal{M}$ generated by X. If $\mathcal{M}$ is a submodel of $\mathcal{N}$, then we let $\mathcal{M}\langle X \rangle$ denote $\langle M \cup X \rangle \subseteq \mathbf{N}$. For this subsection we also fix a monster model $\mathbb{M}$ of $T(\mathfrak{F})$. Note that $\mathbb{M} \upharpoonright \mathcal{L}$ is then a monster model of T.

Distal Criterion 2.7 is a many-sorted, many-function generalization of [11, Theorem 2.1]. We give a proof below. *In the statement of Distal Criterion 2.7 and its proof, x, x', x_i, y, z, w, w_i, and so on are variables.*

Distal Criterion 2.7 (Hieronymi and Nell) *Suppose that T is a distal theory and the following conditions hold:*

(1) *The theory $T(\mathfrak{F})$ has quantifier elimination.*

(2) *For every $\mathfrak{f} \in \mathfrak{F}$, every model $(\mathcal{N}, \mathfrak{F}) \models T(\mathfrak{F})$, every substructure $\mathcal{M} \subseteq \mathcal{N}$ such that $\mathfrak{g}(M) \subseteq M$ for all $\mathfrak{g} \in \mathfrak{F}$, every x, and every $c \in N_x$, there is a y and $d \in \mathfrak{f}(\mathcal{M}\langle c \rangle)_y$ such that*

$$\mathfrak{f}\big(\mathcal{M}\langle c \rangle\big) \subseteq \big\langle \mathfrak{f}(M), d \big\rangle.$$

(3) *For every $\mathfrak{f} \in \mathfrak{F}$, the following holds. Suppose that x' is an initial segment of x, that g, h are $\mathcal{L}$-terms of arities xy and $x'z$, respectively, and that $b_1 \in \mathbb{M}_y$ and $b_2 \in \mathfrak{f}(\mathbb{M})_z$. If $(a_i)_{i \in I}$ is an indiscernible sequence from $\mathfrak{f}(\mathbb{M})_{x'} \times \mathbb{M}_{x \setminus x'}$ such that*

   (a) *$I = I_1 + (c) + I_2$ is in distal configuration at c, and $(a_i)_{i \in I_1 + I_2}$ is $b_1 b_2$-indiscernible, and*

   (b) *$\mathfrak{f}(g(a_i, b_1)) = h(a_i, b_2)$ for every $i \in I_1 + I_2$,*

   then $\mathfrak{f}(g(a_c, b_1)) = h(a_c, b_2)$.

Then $T(\mathfrak{F})$ is distal.

We refer the reader to Figure 1 which illustrates the bookkeeping being done in the proof of Distal Criterion 2.7.

Proof Fix an infinite linear order $I = I_1 + (c) + I_2$ which is in distal configuration at c. By (1) and Lemma 2.4, it is enough to show that every quantifier-free $\mathcal{L}(\mathfrak{F})$-formula $\varphi(x; y)$ is I_1, I_2-distal. We prove this by induction on the number of times $e(\psi)$ that any symbol from $\mathfrak{F}$ occurs in ψ. If $e(\psi) = 0$, this follows from

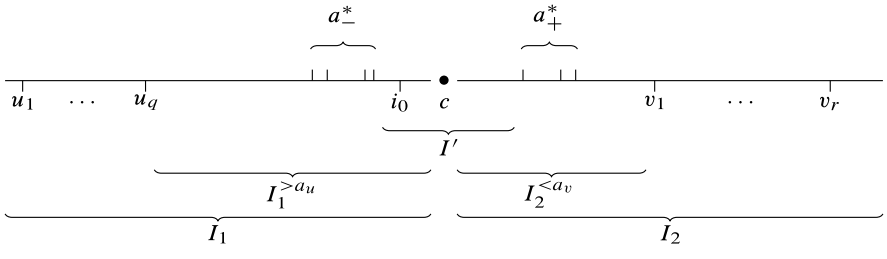

Figure 1 Bookkeeping in the proof of Distal Criterion 2.7.

the assumption that T is distal. Let $e > 0$, and suppose that for all $\mathcal{L}(\mathfrak{L})$-formulas ψ' with $e(\psi') < e$, ψ' is distal. Let $\psi(x;y)$ be a quantifier-free $\mathcal{L}(\mathfrak{F})$-formula with $e(\psi) = e$. We will show that $\psi(x;y)$ is I_1, I_2-distal. Take an indiscernible sequence $(a_i)_{i\in I}$ from $\mathbb{M}_x$ and $b \in \mathbb{M}_y$ such that $I = I_1 + (c) + I_2$ and $(a_i)_{i\in I_1+I_2}$ is b-indiscernible.

Since $e > 0$, there is an $\mathcal{L}$-term g and some $\mathfrak{f} \in \mathfrak{F}$ such that the term $\mathfrak{f}(g(x;y))$ appears in ψ. In other words, there is a quantifier-free $\mathcal{L}(\mathfrak{F})$-formula $\psi'(x;y;z)$ such that $e(\psi') < e$ and

$$\psi(x;y) \;=\; \psi'\big(x;y;\mathfrak{f}\big(g(x;y)\big)\big).$$

Let $\mathcal{M}$ be the $\mathcal{L}(\mathfrak{F})$-substructure of $\mathbb{M}$ generated by $\{a_i : i \in I_1 + I_2\}$ with reduct $\mathcal{M}_{\mathcal{L}} := \mathcal{M} \restriction \mathcal{L}$. By (2) applied to $\mathcal{M} \subseteq \mathbb{M} \restriction \mathcal{L}$, there is $d \in \mathfrak{f}(\mathcal{M}_{\mathcal{L}}\langle b\rangle)_w$ for some w such that

$$\mathfrak{f}\big(\mathcal{M}_{\mathcal{L}}\langle b\rangle\big) \;\subseteq\; \big\langle \mathfrak{f}(M), d\big\rangle. \tag{A}$$

By (A), we have:

$$\text{for every } i \in I_1 + I_2, \quad \mathfrak{f}\big(g(a_i;b)\big) \in \big\langle \mathfrak{f}(M), d\big\rangle. \tag{B}$$

Next, take $q, r \in \mathbb{N}$ and $u_1 < \cdots < u_q \in I_1$ and $v_1 < \cdots < v_r \in I_2$ such that d is in the $\mathcal{L}(\mathfrak{F})$-structure generated by $a_u a_v b$, where $a_u := (a_{u_1}, \ldots, a_{u_q})$ and $a_v := (a_{v_1}, \ldots, a_{v_r})$.

Next, take $i_0 \in I_1^{>a_u}$. Then applying (B) to this i_0, there is an $\mathcal{L}$-term h and $\mathcal{L}(\mathfrak{F})$-terms $t_1, \ldots, t_l$ (all of the form $\mathfrak{f}(s_i)$ for $\mathcal{L}(\mathfrak{F})$-terms s_i) such that

$$\mathfrak{f}\big(g(a_{i_0};b)\big) \;=\; h\big(t_1(a_u, a_{i_0}, a_v, a^*), \ldots, t_l(a_u, a_{i_0}, a_v, a^*), d\big),$$

where a^* is a tuple of new parameters from $(a_i)_{i\in I_1+I_2}$ not yet mentioned (i.e., disjoint from $a_u a_v a_{i_0}$). By $a_u a_v b d$-indiscernibility of $(a_i)_{i\in(I_1^{>a_u}+I_2^{<a_v})}$, we can arrange that i_0 is the largest index in I_1 among all indices specified so far (by sliding the elements $a^* \restriction I_1^{>a_{i_0}}$ up to I_2). Now, let $a^*_- := a^* \restriction I_1$, and let $a^*_+ := a^* \restriction I_2$.

Define $I' := I_1^{>a_u a^*_-} + (c) + I_2^{<a_v a^*_+}$, and note that $i_0 \in I'$ and that I' is also in distal configuration at c. Since $(a_i)_{i\in I'\setminus(c)}$ is $a_u a_v a^* b d$-indiscernible, it follows that

$$\mathfrak{f}\big(g(a_i;b)\big) \;=\; h\big(t_1(a_u, a_i, a_v, a^*), \ldots, t_l(a_u, a_i, a_v, a^*), d\big) \tag{C}$$

for every $i \in I' \setminus (c)$.

For each $i \in I'$, set

$$a'_i \;:=\; \big(t_1(a_u, a_i, a_v, a^*), \ldots, t_l(a_u, a_i, a_v, a^*), a_i\big).$$

Here $(a'_i)_{i\in I'}$ is an indiscernible sequence from $\mathfrak{f}(\mathbb{M})^l \times \mathbb{M}_x$ and $(a'_i)_{i\in I'\setminus(c)}$ is $a_u a_v a^* b d$-indiscernible. Then by (C) and (3), it follows that

$$\mathfrak{f}\big(g(a_c;b)\big) \;=\; h\big(t_1(a_u, a_c, a_v, a^*), \ldots, t_l(a_u, a_c, a_v, a^*), d\big). \tag{D}$$

Finally, we note that

$$\begin{aligned}
&\models \psi(a_c;b)\\
&\quad\iff \models \psi'\big(a_c;b;\mathfrak{f}\big(g(a_c;b)\big)\big) \quad \text{by definition of } \psi'\\
&\quad\iff \models \psi'\big(a_c;b;h\big(t_1(a_u, a_c, a_v, a^*), \ldots, t_l(a_u, a_c, a_v, a^*), d\big)\big) \quad \text{by (D)}\\
&\quad\iff \models \psi'\big(a_i;b;h\big(t_1(a_u, a_i, a_v, a^*), \ldots, t_l(a_u, a_i, a_v, a^*), d\big)\big)\\
&\qquad \text{(for } i \in I' \setminus (c))
\end{aligned}$$

by inductive hypothesis: $e(\psi') = e(\psi'(x; y; h(w_1, \ldots, w_l, w))) < e$
where $w_1, \ldots, w_l$ and w are variables of the appropriate sort and
the partition of this last formula is $(w_1 \cdots w_l x; yw)$

$\iff \models \psi'(a_i; b; \mathfrak{f}(g(a_i; b)))$ (for $i \in I' \setminus (c)$) by (C)

$\iff \models \psi(a_i; b)$ (for $i \in I' \setminus (c)$).

This finishes the proof. □

Connection to NIP Distality was first introduced as a property which a NIP theory may or may not have (see [13]). In this paper, we have defined what it means for an arbitrary theory to be distal. This does not actually give us any additional generality since Proposition 2.9 below shows that every distal theory is necessarily NIP. However, this does allow us to use distality as a means for establishing that a theory is NIP, although there are more direct ways to do this (see Remark 2.10). *Let* $\mathbb{M}$ *be a monster model of* T.

Definition 2.8 We say that a partitioned $\mathcal{L}$-formula $\varphi(x; y)$ has the *non-independence property* (or *is NIP*) if for every $b \in \mathbb{M}_y$ and for every indiscernible sequence $(a_i)_{i \in I}$ from $\mathbb{M}_x$, there is $\varepsilon \in \{0, 1\}$ and an index $i_0 \in I$ such that

$$\models \varphi(a_i; b)^{\varepsilon} \quad \text{for every } i \in I^{>i_0}.$$

We say that T *is NIP* if every partitioned $\mathcal{L}$-formula is NIP.

It is known that distality implies NIP (see, e.g., [4, Remark 2.6]). The proof of this fact that we include below was communicated to us by the authors of [11] and uses only the definitions of distality and NIP given in this paper.

Proposition 2.9 *If* T *is distal, then* T *is NIP.*

Proof Suppose that $\varphi(x; y)$ is not NIP. Then there is an indiscernible sequence $(c_n)_{n<\omega}$ in $\mathbb{M}_x$ and $b \in \mathbb{M}_y$ such that $\models \varphi(c_n; b)$ if and only if n is even. Now define $d_n := (c_{2n}, c_{2n+1}) \in \mathbb{M}_x \times \mathbb{M}_x$, and note that the sequence $(d_n)_{n<\omega}$ satisfies that

(1) the sequence $(d_{n,m})_{(n,m)\in\omega\times 2}$ with the lexicographical ordering on $\omega \times 2$ is indiscernible, and
(2) for every n, $\models \varphi(d_{n,0}; b) \wedge \neg\varphi(d_{n,1}; b)$.

By the *standard lemma* (see [15, Lemma 5.1.3]), there is a b-indiscernible sequence $(e_i)_{i\in\mathbb{Q}}$ in $\mathbb{M}_x \times \mathbb{M}_x$ such that $\mathrm{EM}((e_i)_{i\in\mathbb{Q}}/b) = \mathrm{EM}((d_n)_{n<\omega}/b)$. In particular,

(3) the sequence $(e_{i,m})_{(i,m)\in\mathbb{Q}\times 2}$ with the lexicographical ordering on $\mathbb{Q} \times 2$ is indiscernible, and
(4) for every i, $\models \varphi(e_{i,0}; b) \wedge \neg\varphi(e_{i,1}; b)$.

Finally, for each $i \in \mathbb{Q}$ define the following element of $\mathbb{M}_x$:

$$a_i := \begin{cases} e_{i,0} & \text{if } i \neq 0, \\ e_{i,1} & \text{if } i = 0. \end{cases}$$

We claim that the sequence $(a_i)_{i\in\mathbb{Q}}$ witnesses that $\varphi(x; y)$ is not distal. Indeed,

(5) $\mathbb{Q} = (-\infty, 0) + (0) + (0, +\infty)$ is in distal configuration at 0,
(6) $(a_i)_{i\in(-\infty,0)+(0,+\infty)}$ is b-indiscernible, but
(7) $\models \varphi(a_1; b) \wedge \neg\varphi(a_0; b)$.

□

Remark 2.10 Our route for proving Corollary 1.2 goes through Proposition 2.9 and Theorem 1.1, which uses Distal Criterion 2.7, a generalization of [11, Theorem 2.1]. If our only goal in this paper was to establish Corollary 1.2, we could have taken a slightly more direct path by generalizing in a similar way Günaydin and Hieronymi's [10, Theorem 4.1] to obtain a "NIP criterion." In which case, we could then use "NIP versions" of the results in Sections 4 and 5 to establish Corollary 1.2.

3 Asymptotic Couples and $T_{\log}$

In this section we give a summary of the basics of H-asymptotic couples with asymptotic integration, as well as describing the language $\mathcal{L}_{\log}$ and the theory $T_{\log} = \mathrm{Th}_{\mathcal{L}_{\log}}(\Gamma_{\log}, \psi)$.

Overview of asymptotic couples An *asymptotic couple* is a pair (Γ, ψ) where Γ is an ordered abelian group and $\psi : \Gamma^{\neq} \to \Gamma$ satisfies for all $\alpha, \beta \in \Gamma^{\neq}$,

(AC1) $\alpha + \beta \neq 0 \Longrightarrow \psi(\alpha + \beta) \geq \min(\psi(\alpha), \psi(\beta))$;
(AC2) $\psi(k\alpha) = \psi(\alpha)$ for all $k \in \mathbb{Z}^{\neq}$, in particular, $\psi(-\alpha) = \psi(\alpha)$;
(AC3) $\alpha > 0 \Longrightarrow \alpha + \psi(\alpha) > \psi(\beta)$.

If in addition for all $\alpha, \beta \in \Gamma$,

(HC) $0 < \alpha \leq \beta \Longrightarrow \psi(\alpha) \geq \psi(\beta)$,

then (Γ, ψ) is said to be of *H-type*, or to be an *H-asymptotic couple*.

For the rest of this subsection, (Γ, ψ) is an H-asymptotic couple. By convention, we extend ψ to all of Γ by setting $\psi(0) := \infty$. Then $\psi(\alpha + \beta) \geq \min(\psi(\alpha), \psi(\beta))$ holds for all $\alpha, \beta \in \Gamma$, and $\psi : \Gamma \to \Gamma_\infty$ is a (non-surjective) convex valuation on the ordered abelian group Γ. The following basic fact about valuations is used often.

Fact 3.1 If $\alpha, \beta \in \Gamma$ and $\psi(\alpha) < \psi(\beta)$, then $\psi(\alpha + \beta) = \psi(\alpha)$.

For $\alpha \in \Gamma^{\neq}$, we define $\alpha' := \alpha + \psi(\alpha)$. The following subsets of Γ play special roles:

$$(\Gamma^{\neq})' := \{\gamma' : \gamma \in \Gamma^{\neq}\}, \qquad (\Gamma^{>})' := \{\gamma' : \gamma \in \Gamma^{>}\},$$
$$(\Gamma^{<})' := \{\gamma' : \gamma \in \Gamma^{<}\}, \qquad \Psi := \{\psi(\gamma) : \gamma \in \Gamma^{\neq}\}.$$

We think of the map $\mathrm{id} + \psi : \Gamma^{\neq} \to \Gamma$ as the *derivative* (this is because asymptotic couples often arise in nature as the value groups of certain valued differential fields; see [1]). When *antiderivatives* exist, they are unique.

Fact 3.2 ([1, Lemma 6.5.4(iii)]) The map $\gamma \mapsto \gamma' = \gamma + \psi(\gamma) : \Gamma^{\neq} \to \Gamma$ is strictly increasing.

This allows us to talk about *asymptotic integration*.

Definition 3.3 If $\Gamma = (\Gamma^{\neq})'$, then we say that (Γ, ψ) has *asymptotic integration*. Suppose that (Γ, ψ) has asymptotic integration. Given $\alpha \in \Gamma$, we let $\int \alpha$ denote the unique $\beta \in \Gamma^{\neq}$ such that $\beta' = \alpha$ and we call $\beta = \int \alpha$ the *integral* of α. This gives us a function $\int : \Gamma \to \Gamma^{\neq}$ which is the inverse of $\gamma \mapsto \gamma' : \Gamma^{\neq} \to \Gamma$.

We now further assume that (Γ, ψ) has asymptotic integration. A closely related function to $\int$ is the *successor function* $s : \Gamma \to \Psi$ defined by $\alpha \mapsto s\alpha := \psi(\int \alpha)$. The successor function gets its name due to its behavior on the Ψ-set in the asymptotic couple $(\Gamma_{\log}, \psi)$. More generally, we have the following.

Example 3.4 The asymptotic couple $(\Gamma_{\log}, \psi)$ is of H-type and has asymptotic integration. The functions $\int$ and s behave as follows:

(1) (Integral) For $\alpha = (r_0, r_1, r_2, \ldots) \in \Gamma_{\log}$, take the unique n such that $r_n \neq 1$ and $r_m = 1$ for $m < n$. Then

$$\alpha = (\underbrace{1, \ldots, 1}_{n}, \underbrace{r_n}_{\neq 1}, r_{n+1}, r_{n+2}, \ldots) \mapsto \int \alpha = (\underbrace{0, \ldots, 0}_{n}, r_n - 1, r_{n+1}, r_{n+2}, \ldots).$$

(2) (Successor) For $\alpha = (r_0, r_1, r_2, \ldots) \in \Gamma_{\log}$, take the unique n such that $r_n \neq 1$ and $r_m = 1$ for $m < n$. Then

$$\alpha = (\underbrace{1, \ldots, 1}_{n}, \underbrace{r_n}_{\neq 1}, r_{n+1}, r_{n+2}, \ldots) \mapsto s\alpha = (\underbrace{1, \ldots, 1}_{n+1}, 0, 0, \ldots).$$

In particular, $s0 = (1, 0, 0, \ldots)$.

We conclude this subsection with some general facts about the s-function which we will need later.

Fact 3.5 Let $\alpha, \beta \in \Gamma$. Then

1. if $\alpha \in s(\Gamma)$, then $s^{-1}(\alpha) \cap (\Gamma^{>})'$ and $s^{-1}(\alpha) \cap (\Gamma^{<})'$ are convex in Γ,
2. (successor identity) if $s\alpha < s\beta$, then $\psi(\alpha - \beta) = s\alpha$,
3. if $\alpha \in (\Gamma^{<})'$ and $n \geq 1$, then $\alpha + (n+1)(s\alpha - \alpha) \in (\Gamma^{>})'$.

Proof Item (1) is [6, Corollary 3.6], (2) is [6, Lemma 3.4], and (3) is [8, Lemma 3.10]. □

The theory $T_{\log}$ Let $\mathcal{L}_{\mathrm{AC}}$ be the "natural" language of asymptotic couples:

$$\mathcal{L}_{\mathrm{AC}} = \{0, +, -, <, \psi, \infty\},$$

where $0, \infty$ are constant symbols, $+$ is a binary function symbol, $-$ and ψ are unary function symbols, and $<$ is a binary relation symbol. We consider an asymptotic couple (Γ, ψ) as an $\mathcal{L}_{\mathrm{AC}}$-structure with underlying set Γ_∞ and the obvious interpretation of the symbols of $\mathcal{L}_{\mathrm{AC}}$. We interpret the new constant symbol ∞ as a *default value*, that is, we declare that

$$-\infty = \gamma + \infty = \infty + \gamma = \infty + \infty = \psi(0) = \psi(\infty) := \infty$$

for all $\gamma \in \Gamma$.

Let T_{AC} be the $\mathcal{L}_{\mathrm{AC}}$-theory whose models are the divisible H-asymptotic couples with asymptotic integration such that

(1) Ψ has least element $s0$ as an ordered subset of Γ,
(2) $s0 > 0$,
(3) Ψ is a successor set as an ordered subset of Γ,
(4) for each $\alpha \in \Psi$, the immediate successor of α in Ψ is $s\alpha$, and
(5) $\gamma \mapsto s\gamma : \Psi \to \Psi^{>s0}$ is a bijection.

It is clear that $(\Gamma_{\log}, \psi)$ is a model of T_{AC}. For a model (Γ, ψ) of T_{AC}, we define the function $p : \Psi^{>s0} \to \Psi$ to be the inverse of the function $\gamma \mapsto s\gamma : \Psi \to \Psi^{>s0}$. We extend p to a function $\Gamma_\infty \to \Gamma_\infty$ by setting $p(\alpha) := \infty$ for $\alpha \in \Gamma_\infty \setminus \Psi^{>s0}$.

Next, let $\mathcal{L}_{\log} = \mathcal{L}_{AC} \cup \{s, p, \delta_1, \delta_2, \delta_3, \ldots\}$ where s, p, and δ_n for $n \geq 1$ are unary function symbols. All models of T_{AC} are considered as $\mathcal{L}_{\log}$-structures where

s and p are interpreted as above and where δ_n is interpreted as division by n, again with ∞ as a default value (so $\int \infty = s\infty = \delta_n(\infty) := \infty$ for all $n \geq 1$).

We let $T_{\log}$ be the $\mathcal{L}_{\log}$-theory whose models are the models of T_{AC}. Here are some of the main model-theoretic results concerning $T_{\log}$.

Theorem 3.6 *The $\mathcal{L}_{\log}$-theory $T_{\log}$*

(1) *has a universal axiomatization,*
(2) *has quantifier elimination,*
(3) *is complete,*
(4) *is decidable,*
(5) *is model complete,*
(6) *has NIP,*
(7) *does not have the Steinitz exchange property, and*
(8) *is not strongly NIP (in the sense of [5]).*

Proof Items (1)–(5) are in [6], and (6) and (7) are in [7]. Item (8) is proved by the first author in [9, Section 4.6], although we repeat the proof here.

Let $\mathcal{N}$ be a saturated model of $T_{\log}$. By Dolich and Goodrick [5, Corollary 2.14], it is enough to show that for every $\epsilon > 0$ in $\mathcal{N}$ there is an infinite definable discrete set $X \subseteq \mathcal{N}$ such that $X \subseteq (0, \epsilon)$. The infinite definable set $\Psi_{\mathcal{N}}$ is discrete and has the property that for every $\alpha \in \Psi_{\mathcal{N}}$, the set $\Psi_{\mathcal{N}}^{>\alpha}$ is also infinite and discrete. Let $\epsilon > 0$, and take $\alpha \in \Psi_{\mathcal{N}}$ such that $(\alpha + 2(s\alpha - \alpha)) - \alpha = -2\int \alpha < \epsilon$. Note that then $\alpha + 2(s\alpha - \alpha) > \Psi_{\mathcal{N}}$ by Fact 3.5(3). The definable infinite discrete set $X := \Psi_{\mathcal{N}}^{>\alpha} - \alpha$ has the desired property. □

We shall also need the following facts about models of $T_{\log}$. *For Facts 3.7 and 3.8, we let* $(\Gamma, \psi) \models T_{\log}$.

Fact 3.7 ([6, Lemma 6.8]) Ψ is a linearly independent subset of Γ as a vector space over $\mathbb{Q}$.

Fact 3.8 ([6, Corollary 6.5]) Let $n \geq 1$, let $\alpha_1 < \cdots < \alpha_n \in \Psi$, and let $\alpha = \sum_{j=1}^{n} q_j \alpha_j$ for $q_1, \ldots, q_n \in \mathbb{Q}^{\neq}$. Then

(1) $\sum_{j=1}^{n} q_j \neq 1 \Longrightarrow s\alpha = s0$,
(2) $\sum_{j=1}^{n} q_j = 1 \Longrightarrow s\alpha = s\alpha_1$.

4 Some Indiscernible Lemmas

In this section $\mathbb{M}$ is a monster model of $T_{\log}$ with underlying set Γ_∞. Furthermore, $I = I_1 + (c) + I_2$ is an ordered index set with infinite I_1 and I_2, and i, j, k range over I. In this section we will handle most of the cases that will arise when verifying hypothesis (3) from Distal Criterion 2.7 in our proof of distality for $T_{\log}$.

In terms of Distal Criterion 2.7(3), Lemma 4.1 will handle most of the cases where the sequence $(g(a_i, b_1))$ which gets plugged into † is nonconstant, but the output sequence $(h(a_i, b_2))$ is constant. The arguments involved use essentially that ψ is a convex valuation on $\Gamma^{\neq}$, and that s behaves in a similar topological manner to ψ.

Lemma 4.1 *Let $(a_i)_{i \in I}$ be a nonconstant indiscernible sequence from Γ, and suppose that $(b, \tilde{b}) \in \Gamma \times \Psi$ is such that $(a_i)_{i \in I_1 + I_2}$ is $b\tilde{b}$-indiscernible. Then*

(1) *if $\psi(a_i - b) = \tilde{b}$ for all $i \neq c$, then $\psi(a_c - b) = \tilde{b}$;*

(2) *if* $s(a_i - b) = \tilde{b}$ *for all* $i \neq c$, *then* $s(a_c - b) = \tilde{b}$;
(3) $\dagger(a_i - b) \neq \infty$ *for all* i, *where* $\dagger \in \{\psi, s\}$;
(4) $p(a_i - b) \neq \tilde{b}$ *for every* $i \in I_1 + I_2$.

Proof For (1), assume that $\psi(a_i - b) = \tilde{b}$ for all $i \neq c$. By $b\tilde{b}$-indiscernibility, the sequence $(a_i - b)_{i \in I_1 + I_2}$ is contained entirely within the convex set $\Gamma^< \cap \psi^{-1}(\tilde{b})$ or it is contained entirely within the convex set $\Gamma^> \cap \psi^{-1}(\tilde{b})$. In either case, $a_c - b$ is also contained in the same convex set by monotonicity of $(a_i - b)_{i \in I}$. Thus $\psi(a_c - b) = \tilde{b}$.

The argument for (2) is similar to the argument for (1), except that the two convex sets are $(\Gamma^<)' \cap s^{-1}(\tilde{b})$ and $(\Gamma^>)' \cap s^{-1}(b)$ (see Fact 3.5(1)).

For (3), the case $\dagger = s$ follows from $s(\Gamma) = \Psi$. Suppose that $\dagger = \psi$. If $\psi(a_j - b) = \infty$ for some j, then $a_j = b$. However, if $j = c$, then since $(a_i)_{i \in I}$ is nonconstant and monotone, we would get that $(a_i)_{i \in I_1}$ and $(a_i)_{i \in I_2}$ would lie on separate sides of b, contradicting b-indiscernibility of $(a_i)_{i \in I_1 + I_2}$. If $j \neq c$, then again by b-indiscernibility we would get that $a_i = b$ for every i, contradicting the nonconstant assumption.

For (4), suppose that $p(a_i - b) = \tilde{b}$ for some $i \in I_1 + I_2$. Then by $b\tilde{b}$-indiscernibility, we get that $p(a_j - b) = \tilde{b}$ for every $j \in I_1 + I_2$. However, by definition of p this implies that $(a_i)_{i \in I_1 + I_2}$ is a constant sequence, which is a contradiction. □

Lemma 4.2 will handle the cases where both sequences $(g(a_i, b_1))$ and $(h(a_i, b_2))$ are nonconstant.

Lemma 4.2 *Let* $(a_i \tilde{a}_i)_{i \in I}$ *be an indiscernible sequence from* $\Gamma \times \Psi$ *such that* (a_i) *and* $(\tilde{a}_i)$ *are each nonconstant, and suppose that* $b \in \Gamma$ *is such that* $(a_i \tilde{a}_i)_{i \in I_1 + I_2}$ *is* b*-indiscernible. Then*

(1) *if* $\psi(a_i - b) = \tilde{a}_i$ *for all* $i \neq c$, *then* $\psi(a_c - b) = \tilde{a}_c$;
(2) *if* $s(a_i - b) = \tilde{a}_i$ *for all* $i \neq c$, *then* $s(a_c - b) = \tilde{a}_c$;
(3) *if* $p(a_i - b) = \tilde{a}_i$ *for all* $i \neq c$, *then* $p(a_c - b) = \tilde{a}_c$.

Proof Without loss of generality, we will assume that $(\tilde{a}_i)$ is strictly increasing.

For (1), we first observe that if $i < j$, then $\psi(a_i - a_j) = \tilde{a}_i$. Indeed, by indiscernibility, it suffices to show this for $i < j$ both from $I_1 + I_2$. Given such $i < j$, we have

$$\psi(a_i - a_j) = \psi\big((a_i - b) + (b - a_j)\big) = \min(\tilde{a}_i, \tilde{a}_j) = \tilde{a}_i.$$

Next, suppose that $j \in I_2$, and note that

$$\psi(a_c - b) = \psi\big((a_c - a_j) + (a_j - b)\big) = \min(\tilde{a}_c, \tilde{a}_j) = \tilde{a}_c.$$

For (2), we first claim that if $i < j$, then $\psi(a_i - a_j) = \tilde{a}_i$. By indiscernibility, it suffices to show this for $i < j$ both from $I_1 + I_2$. Given such $i < j$, we have $\tilde{a}_i = s(a_i - b) < s(a_j - b) = \tilde{a}_j$. By the successor identity (Fact 3.5(2)), we have

$$\tilde{a}_i = s(a_i - b) = \psi\big((a_i - b) - (a_j - b)\big) = \psi(a_i - a_j).$$

Next, suppose that $j < k$ are both from I_2. Then by monotonicity of $(a_i - b)_{i \in I}$ and b-indiscernibility of $(a_i)_{i \in I_1 + I_2}$,

$$s(a_c - b) \leq s(a_j - b) < s(a_k - b).$$

By the successor identity, we have

$$\tilde{a}_c \;=\; \psi(a_c - a_k) \;=\; \psi\big((a_c - b) - (a_k - b)\big) \;=\; s(a_c - b).$$

Finally for (3), we note that if $i \in I_1 + I_2$, then $a_i - b \in \Psi^{>s0}$ and so

$$a_i - b = sp(a_i - b) = s\tilde{a}_i.$$

From this we see that $a_i - s\tilde{a}_i = b$ and, in particular, $(a_i - s\tilde{a}_i)_{i\in I_1+I_2}$ is constant. By indiscernibility of $(a_i\tilde{a}_i)_{i\in I}$, we have $a_c - s\tilde{a}_c = b$ and so $p(a_c - b) = ps\tilde{a}_c$. As $\tilde{a}_i \in \Psi$ for all $i \in I_1 + I_2$ we have $\tilde{a}_c \in \Psi$, and so $ps\tilde{a}_c = \tilde{a}_c$. □

Lemma 4.3 will handle the cases where both sequences $(g(a_i, b_1))$ and $(h(a_i, b_2))$ are constant. This case is trivial, but it does implicitly rely on the behavior $\mathcal{L}$-terms, where $\mathcal{L} := \{0, +, -, <, (\delta_n)_{n<\omega}, \infty\} \subseteq \mathcal{L}_{\log}$.

Lemma 4.3 *Given $\mathfrak{f} \in \{\psi, s, p\}$, let g, h be $\mathcal{L}$-terms of arities $n + k$ and $m + l$, respectively, with $m \leq n$. Let $b_1 \in \mathbb{M}^k$, let $b_2 \in \mathfrak{f}(\mathbb{M})^l$, and let $(a_i)_{i\in I}$ be an indiscernible sequence from $\mathfrak{f}(\mathbb{M})^m \times \mathbb{M}^{n-m}$ such that*

(1) *$(a_i)_{i\in I_1+I_2}$ is b_1b_2-indiscernible,*
(2) *$\mathfrak{f}(g(a_i, b_1)) = h(a_i, b_2)$ for every $i \neq c$,*
(3) *$(g(a_i, b_1))_{i\in I_1+I_2}$ is a constant sequence.*

Then $\mathfrak{f}(g(a_c, b_1)) = h(a_c, b_2)$.

Proof We will essentially do a case distinction on "whether ∞ plays a nontrivial role or not." This will be made more precise in the proof of the following general claim.

Claim *Suppose that $t(u, v)$ is an $\mathcal{L}$-term of arity $r + s$, that $(c_i)_{i\in I}$ is an indiscernible sequence from $\mathbb{M}^r$, and that $d \in \mathbb{M}^s$. Then if $(t(c_i, d))_{i\in I_1+I_2}$ is a constant sequence, $(t(c_i, d))_{i\in I}$ is also a constant sequence.*

Proof of claim We will consider several cases relating to the role of ∞.

Case 1: *Suppose that the constant ∞ occurs in the term t.* Then by definition of how the symbol ∞ is interpreted, we have $t(c_i, d) = \infty$ for every i, so $(t(c_i, d))_{i\in I}$ takes constant value ∞.

For the rest of this proof, we may assume that ∞ does not occur in t. We now write the variables u, v as $u = (u_1, \ldots, u_r)$ and $v = (v_1, \ldots, v_s)$. By writing $t = t(u, v)$, we mean that the free variables of the term t are among $u_1, \ldots, u_r, v_1, \ldots, v_s$. Let $u_{i_1}, \ldots, u_{i_p}, v_{j_1}, \ldots, v_{j_q}$ be those variables which actually occur in the term t. Then there are rational numbers $\rho_{i_1}, \ldots, \rho_{i_p}, \rho_{j_1}, \ldots, \rho_{j_q} \in \mathbb{Q}$ such that

$$t(u, v) \;=\; \rho_{i_1}u_{i_1} + \cdots + \rho_{i_p}u_{i_p} + \rho_{j_1}v_{j_1} + \cdots + \rho_{j_q}v_{j_q},$$

where equality here is in the sense that the interpretations of both sides as functions $\mathbb{M}^{r+s} \to \mathbb{M}$ agree. For the following two cases, it will be convenient to write out $d = (d_1, \ldots, d_s)$, and for each index $i_0 \in I$, we have $c_{i_0} = (c_{i_0,1}, c_{i_0,2}, \ldots, c_{i_0,r})$.

Case 2: *Suppose that one of $d_{j_1}, \ldots, d_{j_q}$ is equal to ∞.* In this case, it follows that $t(c_i, d) = \infty$ for all i, so again we have a constant sequence.

Case 3: *Suppose that there is some $i_0 \in I_1 + I_2$ such that one of $c_{i_0,i_1}, c_{i_0,i_2}, \ldots, c_{i_0,i_p}$ is equal to ∞.* Without loss of generality, say that $c_{i_0,i_1} = \infty$. Then by indiscernibility we have $c_{i,i_1} = \infty$ for all $i \in I$. Thus, since the variable u_{i_1} actually shows up in t, it follows that $t(c_i, d) = \infty$ for all i, and so $(t(c_i, d))_{i\in I}$ is a constant sequence.

Case 4: *Suppose that we are not in Case 1, 2, or 3.* In this case, $t(c_i, d)$ is a legitimate $\mathbb{Q}$-linear combination of elements of Γ (not ∞). In particular, we have

$$\begin{aligned}\big(t(c_i,d)\big)_{i\in I_1+I_2} \text{ is constant} &\implies (\rho_{i_1}c_{i,i_1}+\cdots+\rho_{i_p}c_{i,i_p})_{i\in I_1+I_2} \text{ is constant}\\ &\implies (\rho_{i_1}c_{i,i_1}+\cdots+\rho_{i_p}c_{i,i_p})_{i\in I} \text{ is constant}\\ &\implies \big(t(c_i,d)\big)_{i\in I} \text{ is constant,}\end{aligned}$$

where the second implication follows from indiscernibility. Note: in Case 4 it is possible that ∞ shows up somewhere in the sequence (c_i) or in the parameter d, just not in a way where it is actually being plugged in to the term t. □

The lemma now is an immediate consequence of the above claim applied separately to the sequences $(g(a_i, b_1))_{i\in I}$ and $(h(a_i, b_2))_{i\in I}$. □

The following lemma greatly simplifies the sequence $(h(a_i, b_2))$. *We no longer assume that j and k range over I.*

Lemma 4.4 *Let $h(x, y)$ be an $\mathfrak{L}$-term of arity $m + n$, let $b \in \mathbb{M}^n$, and let $(a_i)_{i\in I}$ be an indiscernible sequence from Ψ_∞^m, with $a_i = (a_{i,1}, \ldots, a_{i,m})$. Assume that $h(a_i, b) \in \Psi_\infty$ for infinitely many i. Then one of the following is true:*

(1) *$h(a_i, b) = \infty$ for every i;*
(2) *there is $\beta \in \Psi$ such that $h(a_i, b) = \beta$ for every i;*
(3) *there is $l \in \{1, \ldots, m\}$ such that $h(a_i, b) = a_{i,l}$ for every i.*

Proof If one of the components of b which corresponds to a free variable which actually occurs in h is ∞, then $h(a_i, b) = \infty$ for every i. Similarly, if the constant ∞ occurs in the term h, then $h(a_i, b) = \infty$ for every i as well. Thus we may assume for the remainder of the proof that none of the components of b are ∞ and that ∞ does not occur in the term h. We may then write $h(x, b) = (\sum_{j=1}^m q_j x_j) + c$ where $c \neq \infty$ is a $\mathbb{Q}$-linear combination of the components of b and $q_1, \ldots, q_m \in \mathbb{Q}$. We consider three disjoint cases.

Case 1: *There is $i_0 \in I$ with $h(a_{i_0}, b) = \infty$.* Then there must be $j \in \{1, \ldots, m\}$ with $a_{i_0,j} = \infty$, and so $a_{i,j} = \infty$ for every i. We conclude that $h(a_i, b) = \infty$ for every i.

Case 2: *$h(a_i, b) \neq \infty$ for every i, and there are distinct $i_0, i_1 \in I$ and $\beta \in \Gamma$ with $h(a_{i_0}, b) = h(a_{i_1}, b) = \beta$.* We see then that $\sum_{j=1}^m q_j a_{i_0,j} = \sum_{j=1}^m q_j a_{i_1,j}$ and so $\sum_{j=1}^m q_j a_{i,j} = \sum_{j=1}^m q_j a_{i',j}$ for every $i, i' \in I$ by indiscernibility. We conclude that $h(a_i, b) = \beta$ for every i and since $h(a_i, b) \in \Psi_\infty$ for infinitely many i, we see that $\beta \in \Psi$.

Case 3: *$h(a_i, b) \neq \infty$ for every i, and $h(a_i, b) \neq h(a_{i'}, b)$ for all distinct i and i'.* We will first clean up the summation by removing constant and redundant sequences. By indiscernibility, there are $m_0 \geq 1$ and natural numbers $1 \leq \eta(1) < \cdots < \eta(m_0) \leq m$ such that

(1) for every $j \in \{\eta(1), \ldots, \eta(m_0)\}$, the sequence $(a_{i,j})_{i\in I}$ is nonconstant,
(2) for every $j, j' \in \{\eta(1), \ldots, \eta(m_0)\}$ such that $j \neq j'$, $a_{i,j} \neq a_{i,j'}$ for every i, and
(3) given $j \in \{1, \ldots, m\} \setminus \{\eta(1), \ldots, \eta(m_0)\}$, either
    (a) the sequence $(a_{i,j})_{i\in I}$ is constant, or
    (b) there is $j' \in \{\eta(1), \ldots, \eta(m_0)\}$ such that $a_{i,j} = a_{i,j'}$ for every i.

By rearranging the components of (a_i) and the q_j's, we may assume for the rest of the proof that $\eta(j) = j$ for $j = 1, \dots, m_0$.

Next, for $j \in \{1, \dots, m_0\}$, define

$$\tilde{q}_j := \sum_{j' \in A} q_{j'},$$

where $A := \{j' : a_{i,j} = a_{i,j'} \text{ for every } i\}$ and

$$\tilde{c} := c + \sum_{j \in B} a_{i_0,j},$$

where $B := \{j : (a_{i,j})_{i \in I} \text{ is a constant sequence}\}$ and $i_0 \in I$ is some fixed index. We now have that for every i,

$$h(a_i, b) = \Big(\sum_{j=1}^{m_0} \tilde{q}_j a_{i,j}\Big) + \tilde{c},$$

and for every i, i' and $j, j' \in \{1, \dots, m_0\}$, $a_{i,j} \neq a_{i',j'}$ whenever $(i, j) \neq (i', j')$.

Choose distinct $i_0, i_1, \dots, i_{m_0+1} \in I$ such that $h(a_{i_k}, b) \in \Psi$ for $k \in \{0, \dots, m_0 + 1\}$. We have for each k that

$$\Big(\sum_{j=1}^{m_0} \tilde{q}_j a_{i_0,j}\Big) - h(a_{i_0}, b) \;=\; -\tilde{c} \;=\; \Big(\sum_{j=1}^{m_0} \tilde{q}_j a_{i_k,j}\Big) - h(a_{i_k}, b)$$

and so

$$h(a_{i_k}, b) \;=\; \Big(\sum_{j=1}^{m_0} \tilde{q}_j a_{i_k,j}\Big) - \Big(\sum_{j=1}^{m_0} \tilde{q}_j a_{i_0,j}\Big) + h(a_{i_0}, b). \tag{$*$}$$

Since $a_{i_0,1}, \dots, a_{i_0,m_0}, a_{i_k,1}, \dots, a_{i_k,m_0}$ are distinct elements of Ψ, and $h(a_{i_k}, b)$ and $h(a_{i_0}, b)$ are also distinct elements from Ψ, we deduce from the $\mathbb{Q}$-linear independence of Ψ (Fact 3.7) that

$$h(a_{i_k}, b) \;\in\; \{a_{i_0,1}, \dots, a_{i_0,m_0}, a_{i_k,1}, \dots, a_{i_k,m_0}\}.$$

We claim that $h(a_{i_k}, b) \in \{a_{i_k,1}, \dots, a_{i_k,m_0}\}$ for at least one $k \in \{1, \dots, m_0 + 1\}$. Suppose not. Then there is a function $\sigma : \{1, \dots, m_0 + 1\} \to \{1, \dots, m_0\}$ such that $h(a_{i_k}, b) = a_{i_0,\sigma(k)}$. As $h(a_{i_k}, b) \neq h(a_{i_{k'}}, b)$ for all $k, k' \in \{1, \dots, m_0 + 1\}$ such that $k \neq k'$, we must have that σ is injective, which is a contradiction. Therefore, we can take $k \in \{1, \dots, m_0 + 1\}$ and $l \in \{1, \dots, m_0\}$ such that $h(a_{i_k}, b) = a_{i_k,l}$. In particular, $a_{i_k,l} \neq 0$ and so $a_{i_0,l} \neq 0$. Again from $(*)$, the $\mathbb{Q}$-linear independence of Ψ, and the fact that $a_{i_0,1}, \dots, a_{i_0,m_0}, a_{i_k,1}, \dots, a_{i_k,m_0}$ are all distinct, we deduce that $h(a_{i_0}, b) = a_{i_0,l}$, that $\tilde{q}_l = 1$, and that $\tilde{q}_j = 0$ for $j \neq l$. From this, we deduce that $\tilde{c} = 0$, and so $h(a_i, b) = a_{i,l}$ for every i. □

5 Spread Out Sequences

In this section $\mathbb{M}$ is a monster model of $T_{\log}$ with underlying set Γ_∞. Furthermore, I is an infinite ordered index set, i, j range over I, and $(a_i)_{i \in I}$ is a strictly increasing indiscernible sequence from Γ.

Definition 5.1 Given $a, b \in \operatorname{conv}(\Psi)$, we write $a \ll b$ if $s^n a < b$ for every n. If there is $b \in \Gamma$ such that $(a_i - b)_{i \in I} \subseteq \operatorname{conv}(\Psi)$, and for every $i < j$, $s0 \ll a_i - b \ll a_j - b$, then we say that $(a_i)_{i \in I}$ is *spread out by b*.

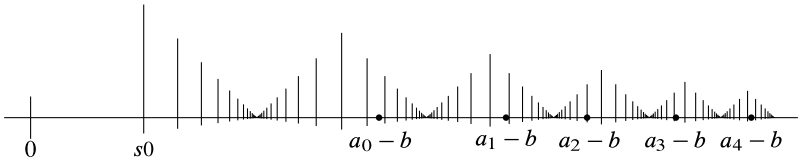

Figure 2 Five elements of a sequence being spread out by b.

Intuitively, the idea behind $(a_i)_{i\in I}$ being spread out by b is that upon translating $(a_i)_{i\in I}$ by b, all elements of the sequence $(a_i - b)_{i\in I}$ are now in the convex hull of the Ψ-set, and each element of the sequence "lives on" its own copy of $\mathbb{Z}$ (or really, the convex hull of a copy of $\mathbb{Z}$). In Figure 2 we suppose that $(a_i)_{i\in I} = (a_n)_{n<\omega}$ is an indiscernible sequence spread out by b, and we illustrate the positions of $(a_n - b)_{n<5}$. Furthermore, each element $a_i - b$ will have a "nearest" element of the Ψ-set, namely, $ps(a_i - b)$. The next lemma shows that these nearest elements do not depend on b.

Lemma 5.2 *Suppose that $(a_i)_{i\in I}$ is spread out by b. Then for every $i < j$,*

$$s(a_i - b) \;=\; \psi(a_i - a_j).$$

Proof Let $i < j$ be arbitrary. Then by definition, $a_i - b \ll a_j - b$, and by the successor identity,

$$\psi(a_i - a_j) \;=\; \psi\big((a_j - b) - (a_i - b)\big) \;=\; s(a_i - b). \qquad \square$$

If (a_i) is spread out by b, then the sign of the difference $(a_i - b) - ps(a_i - b)$ can depend on i and b, although in a very *dependent* way (see Figure 3).

Lemma 5.3 *Suppose that $(a_i)_{i\in I}$ is spread out by b. Let*

$$I^* := \begin{cases} I & \text{if } I \text{ does not have a greatest index,} \\ I^{<d} & \text{if } I \text{ has a greatest index } d. \end{cases}$$

Then the function $i \mapsto a_i - ps(a_i - b) : I^* \to \Gamma$ is either constant, strictly increasing, or strictly decreasing.

Proof Fix indices $i_0 < j_0$ from I^*, and fix $\star \in \{=, <, >\}$ such that

$$\big(a_{i_0} - ps(a_{i_0} - b)\big) \;\star\; \big(a_{j_0} - ps(a_{j_0} - b)\big).$$

Next, let $j < k$ be arbitrary indices from I^*, and fix an index $d \in I$ which is greater than j_0 and k. Note that the sequence $(a_i)_{i\in I^{<d}}$ is a_d-indiscernible. Thus

$$\begin{aligned} &\big(a_{i_0} - ps(a_{i_0} - b)\big) \;\star\; \big(a_{j_0} - ps(a_{j_0} - b)\big) \\ &\quad\iff \big(a_{i_0} - p\psi(a_{i_0} - a_d)\big) \;\star\; \big(a_{j_0} - p\psi(a_{j_0} - a_d)\big) \\ &\quad\iff \big(a_j - p\psi(a_j - a_d)\big) \;\star\; \big(a_k - p\psi(a_k - a_d)\big) \\ &\quad\iff \big(a_j - ps(a_j - b)\big) \;\star\; \big(a_k - ps(a_k - b)\big). \end{aligned}$$

Thus the function $i \mapsto a_i - ps(a_i - b) : I^* \to \Gamma$ is either constant, strictly increasing, or strictly decreasing, depending on $\star$. $\square$

The following is the key proposition.

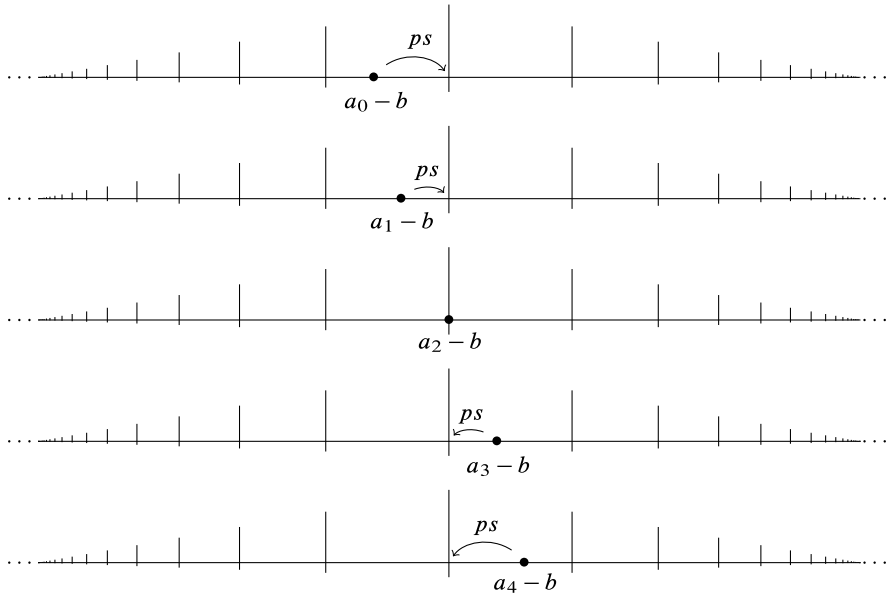

Figure 3 To illustrate Lemma 5.3 we apply ps to the sequence $(a_n - b)_{n<5}$ from Figure 2.

Proposition 5.4 *Suppose that $I = I_1 + (c) + I_2$ is in distal configuration at c. Further suppose that $b \in \Gamma$ is such that $(a_i)_{i \in I_1 + I_2}$ is b-indiscernible. If $p(a_i - b) = \infty$ for every $i \in I_1 + I_2$, then $p(a_c - b) = \infty$.*

Proof We have several cases to consider.

Case 1: *There is $i_0 \neq c$ such that $a_{i_0} - b > \Psi$.* Then by b-indiscernibility and monotonicity, $a_i - b > \Psi$ for every $i \in I$, and so $a_c - b \notin \Psi$, so $p(a_c - b) = \infty$.

Case 2: *There is n and $i_0 \neq c$ such that $a_{i_0} - b \leq s^n 0$.* Again by b-indiscernibility and monotonicity, we have that $a_i - b < s^n 0$ for every i. Assume toward a contradiction that $p(a_c - b) \neq \infty$. Then $a_c - b = s^m 0$ for some $1 < m < n$, and so for $i \neq c$, $a_i - b < s^m 0$ if and only if $i < c$, contradicting the indiscernibility of $(a_i - b)_{i \in I_1 + I_2}$.

We may now assume by monotonicity that $(a_i - b)_{i \in I} \subseteq \Gamma^{>(s^n 0)_n} \cap \Psi^{\downarrow}$. In particular, $s0 \ll a_i - b$ for every i.

Case 3: *The sequence $(ps(a_i - b))_{i \in I_1 + I_2}$ takes the constant value β.* Then by b-indiscernibility and monotonicity, either for all i, $a_i - b < \beta$ or for all i, $a_i - b > \beta$. In either case, $a_c - b \neq \beta$, however $ps(a_c - b) = \beta$. Thus $a_c - b \neq \Psi$.

Case 4: *The sequence $(ps(a_i - b))_{i \in I_1 + I_2}$ is strictly increasing.* Then the sequence $(ps(a_i - b))_{i \in I}$ is strictly increasing, and it follows from b-indiscernibility of $(a_i)_{i \in I_1 + I_2}$ and I being in distal configuration at c that $(a_i)_{i \in I}$ is spread out by b. By Lemma 5.3 and b-indiscernibility of $(a_i)_{i \in I_1 + I_2}$, the function $i \mapsto a_i - ps(a_i - b) : I \to \Gamma$ is either constant, strictly increasing, or strictly decreasing. By b-indiscernibility and the assumption that $p(a_i - b) = \infty$ for every

$i \in I_1 + I_2$, it follows that this function takes values entirely below b, or entirely above b. Thus $a_c - ps(a_c - b) \neq b$, and so $a_c - b \notin \Psi$. □

6 Extensions

In this section (Γ, ψ) *is a model of* $T_{\log}$. Here we will prove the relevant facts that will allow us to later verify condition (2) in Distal Criterion 2.7, with (Γ, ψ) playing the role of "$(\mathcal{N}, \mathfrak{F})$." In Distal Criterion 2.7(2), we are allowed to assume that the substructure $\mathcal{M}$ of $\mathcal{N}$ is closed under every function from $\mathfrak{F}$.

For the case $\mathfrak{f} = \psi$, we do not need the subgroup Γ_0 of Γ to be closed under any of the functions ψ, s, or p.

Proposition 6.1 *Suppose that* Γ_0 *is a divisible ordered subgroup of* Γ. *Given* $c_1, \ldots, c_m \in \Gamma \setminus \Gamma_0$, *we have*

$$\#\Big(\psi\Big(\Big(\Gamma_0 + \sum_{i=1}^{m} \mathbb{Q}c_i\Big)^{\neq}\Big) \setminus \psi(\Gamma_0^{\neq})\Big) \leq m.$$

In particular, there are $n \leq m$ *and distinct*

$$d_1, \ldots, d_n \in \psi\Big(\Big(\Gamma_0 + \sum_{i=1}^{m} \mathbb{Q}c_i\Big)^{\neq}\Big) \setminus \psi(\Gamma_0^{\neq})$$

such that

$$\psi\Big(\Big(\Gamma_0 + \sum_{i=1}^{m} \mathbb{Q}c_i\Big)^{\neq}\Big) \subseteq \Big(\bigoplus_{\alpha \in \psi(\Gamma_0^{\neq})} \mathbb{Q}\alpha\Big) \oplus \Big(\bigoplus_{j=1}^{n} \mathbb{Q}d_j\Big).$$

Proof This follows by induction on m. To simplify notation, we will show the inductive step only for $m = 1$. Let $c \in \Gamma \setminus \Gamma_0$. If $\psi(\Gamma_0 + \mathbb{Q}c) = \psi(\Gamma_0)$, then we are done. Otherwise, suppose that $\psi(\Gamma_0 + \mathbb{Q}c) \neq \psi(\Gamma_0)$. As ψ is constant on archimedean classes, we must have that $[\Gamma_0 + \mathbb{Q}c] \neq [\Gamma_0]$. By [1, Lemma 2.4.4], there is $c^* \in \Gamma_0 + \mathbb{Q}c$ with $[\Gamma_0 + \mathbb{Q}c] = [\Gamma_0] \cup \{[c^*]\}$ and so $\psi(\Gamma_0 + \mathbb{Q}c) = \psi(\Gamma_0) \cup \{\psi(c^*)\}$. □

For the case $\mathfrak{f} = s$, we will need the subgroup to be closed under the functions ψ and s, as the following example illustrates.

Example 6.2 Suppose that (Γ, ψ) has an element $\alpha \in \Psi$ such that $\alpha > s^n 0$ for every n. Fix such an element α. Let Γ_0 be the divisible ordered subgroup of Γ generated by

$$\{s^n 0 : n \geq 1\} \cup \{\alpha_1 - \alpha_0 : \alpha_1, \alpha_0 \in \Psi \ \& \ s0 \ll \alpha_0 < \alpha_1\}.$$

By Fact 3.8, we have $s(\Gamma_0) = \{s^n 0 : n \geq 1\} \subseteq \Gamma_0$. However, $\Psi \subseteq \Gamma_0 + \mathbb{Q}\alpha$ and thus

$$\#\big(s(\Gamma_0 + \mathbb{Q}\alpha) \setminus s(\Gamma_0)\big) = \#\big(\Psi \setminus \{s^n 0 : n \geq 1\}\big) = \infty.$$

Proposition 6.3 *Suppose that* Γ_0 *is a divisible ordered subgroup of* Γ *such that* $s(\Gamma_0) \subseteq \Gamma_0$ *and* $\psi(\Gamma_0^{\neq}) \subseteq \Gamma_0$. *Given* $c_1, \ldots, c_m \in \Gamma \setminus \Gamma_0$, *we have*

$$\#\Big(s\Big(\Gamma_0 + \sum_{i=1}^{m} \mathbb{Q}c_i\Big) \setminus s(\Gamma_0)\Big) \leq m + 1.$$

In particular, there are $n \leq m + 1$ *and distinct*

$$d_1, \dots, d_n \in s\Big(\Gamma_0 + \sum_{i=1}^{m} \mathbb{Q}c_i\Big) \Big\setminus s(\Gamma_0)$$

such that

$$s\Big(\Gamma_0 + \sum_{i=1}^{m} \mathbb{Q}c_i\Big) \subseteq \Big(\bigoplus_{\alpha \in s(\Gamma_0)} \mathbb{Q}\alpha\Big) \oplus \Big(\bigoplus_{j=1}^{n} \mathbb{Q}d_j\Big).$$

Proof Suppose that $c_1, \dots, c_m \in \Gamma \setminus \Gamma_0$, and set $\Gamma_1 := \Gamma_0 + \sum_{i=1}^{m} \mathbb{Q}c_i$. Assume toward a contradiction that there are $m+2$ distinct elements in $s(\Gamma_1) \setminus s(\Gamma_0)$. Choose $e_1, \dots, e_{m+2} \in \Gamma_1$ such that $se_1 < \cdots < se_{m+2}$ and such that $se_i \notin s(\Gamma_0)$ for each $i = 1, \dots, m+2$. By the successor identity, we have that $\psi(e_{i+1} - e_i) = se_i$ for each $i = 1, \dots, m+1$. The closure assumptions on Γ_0 imply that (Γ_0, ψ) is an asymptotic couple with asymptotic integration, and so $s(\Gamma_0) = \psi(\Gamma_0)$. Thus there are $m+1$ distinct elements in $\psi(\Gamma_1) \setminus \psi(\Gamma_0)$, contradicting Proposition 6.1 above. □

Proposition 6.4 *Suppose that* $(\Gamma_0, \psi) \preccurlyeq (\Gamma, \psi)$. *Given* $c_1, \dots, c_m \in \Gamma \setminus \Gamma_0$, *we have*

$$\#\Big(p\Big(\Gamma_0 + \sum_{i=1}^{m} \mathbb{Q}c_i\Big) \Big\setminus p(\Gamma_0)\Big) \leq m+1.$$

In particular, there are $n \leq m+1$ *and distinct*

$$d_1, \dots, d_n \in p\Big(\Gamma_0 + \sum_{i=1}^{m} \mathbb{Q}c_i\Big) \Big\setminus p(\Gamma_0)$$

such that

$$p\Big(\Gamma_0 + \sum_{i=1}^{m} \mathbb{Q}c_i\Big) \subseteq \Big(\bigoplus_{\alpha \in p(\Gamma_0), \alpha \neq \infty} \mathbb{Q}\alpha\Big) \oplus \Big(\bigoplus_{j=1}^{n} \mathbb{Q}d_j\Big) \cup \{\infty\}.$$

Proof Suppose that $c_1, \dots, c_m \in \Gamma \setminus \Gamma_0$, and set $\Gamma_1 := \Gamma_0 + \sum_{i=1}^{m} \mathbb{Q}c_i$. Assume toward a contradiction that there are $m+2$ elements $e_1, \dots, e_{m+2} \in \Gamma_1$ with $p(e_i) \in p(\Gamma_1) \setminus p(\Gamma_0)$ for each i and with $p(e_i) \neq p(e_j)$ for all $i \neq j$. Then e_i is in Ψ for each i and, as s is injective on Ψ, we have that $se_i \neq se_j$ for all $i \neq j$. As

$$p(\Gamma_0) = s(\Gamma_0) \cup \{\infty\} = (\Psi \cap \Gamma_0) \cup \{\infty\},$$

we have that $se_i \notin s(\Gamma_0)$ for each i, contradicting Proposition 6.3. □

7 Proof of Theorem 1.1

In this section we prove Theorem 1.1 by applying Distal Criterion 2.7. In the language of Distal Criterion 2.7, the role of T will be played by the reduct $T := T_{\log} \restriction \mathcal{L}$, where $\mathcal{L} = \{0, +, -, <, (\delta_n)_{n<\omega}, \infty\}$. The $\mathcal{L}$-theory T is essentially the same as the theory of ordered divisible abelian groups, except that it contains the element ∞ which serves as a default value "at infinity." It follows that T is o-minimal and therefore it is distal by [13, Lemma 2.10], since o-minimal theories are dp-minimal.

We now construe $T_{\log}$ as $T_{\log} = T(\mathfrak{F})$, with $\mathfrak{F} = \{\psi, s, p\}$. In particular, $\mathcal{L}_{\log} = \mathcal{L}(\mathfrak{F})$. By [6, Theorem 5.2], $T(\mathfrak{F})$ has quantifier elimination, which is condition (1) in Distal Criterion 2.7. Verifying Distal Criterion 2.7(2) involves three cases: $\mathfrak{f} = \psi$, s, and p. These cases are handled, respectively, by Propositions 6.1, 6.3, and 6.4.

Finally, we will show how to verify condition (3) in Distal Criterion 2.7. Fix a monster model $\mathbb{M}$ of T with underlying set Γ_∞. Let $\mathfrak{f} \in \mathfrak{F}$, and let g, h be $\mathcal{L}$-terms of arities $n+k$ and $m+l$, respectively, with $m \leq n$. Let $b_1 \in \mathbb{M}^k$, let $b_2 \in \mathfrak{f}(\mathbb{M})^l$, and let $(a_i)_{i\in I}$ be an indiscernible sequence from $\mathfrak{f}(\mathbb{M})^m \times \mathbb{M}^{n-m}$ such that

(a) $I = I_1 + (c) + I_2$ is in distal configuration at c, and $(a_i)_{i\in I_1+I_2}$ is b_1b_2-indiscernible, and
(b) $\mathfrak{f}(g(a_i, b_1)) = h(a_i, b_2)$ for every $i \in I_1 + I_2$.

Our job is to show that $\mathfrak{f}(g(a_c, b_1)) = h(a_c, b_2)$. We have several cases to consider.

Case 1: $(g(a_i, b_1))_{i\in I_1+I_2}$ *is a constant sequence.* In this case, it follows from Lemma 4.3 that $\mathfrak{f}(g(a_c, b_1)) = h(a_c, b_2)$.

For the remainder of the proof, we assume that $(g(a_i, b_1))_{i\in I_1+I_2}$ *is not a constant sequence.* In particular, the symbol ∞ does not appear in g, in b_1, or in the last n coordinates of a_i, so the $\mathcal{L}$-term $g(x, y)$ is equal to a $\mathbb{Q}$-linear combination of its arguments. By grouping these $\mathbb{Q}$-linear combinations, we get $b \in \Gamma$, and a nonconstant indiscernible sequence $(a'_i)_{i\in I}$ from $\mathbb{M}$ such that

(c) $g(a_i, b_1) = a'_i - b$ for every $i \in I$,
(d) $(a_i a'_i)_{i\in I}$ is an indiscernible sequence from $\mathfrak{f}(\mathbb{M})^m \times \mathbb{M}^{n-m+1}$,
(e) $(a_i a'_i)_{i\in I_1+I_2}$ is b_1b_2b-indiscernible, and
(f) $\mathfrak{f}(a'_i - b) = h(a_i, b_2)$ for every $i \in I_1 + I_2$.

Our job now is to show that $\mathfrak{f}(\tilde{a}_c - b) = h(a_c, b_2)$. Since $\mathfrak{f}(\mathbb{M}) \subseteq \Psi_\infty$ for each $\mathfrak{f}$, by Lemma 4.4 we get three more cases.

Case 2: $h(a_i, b_2) = \infty$ *for every* $i \in I$. By Lemma 4.1(3), this case cannot happen for $\mathfrak{f} \in \{\psi, s\}$. If $\mathfrak{f} = p$, then this case is handled by Proposition 5.4.

Case 3: *There is* $\beta \in \Psi$ *such that* $h(a_i, b_2) = \beta$ *for every* $i \in I$. If $\mathfrak{f} = \psi$ or $\mathfrak{f} = s$, then this case is handled by Lemma 4.1(1) or (2). By Lemma 4.1(4), this case cannot happen for $\mathfrak{f} = p$.

Case 4: *There is* $l \in \{1, \ldots, m\}$ *such that* $h(a_i, b_2) = a_{i,l}$ *for every* $i \in I$. This case is handled by Lemma 4.2.

This completes the verification of condition (3) in Distal Criterion 2.7 and so we are done with our proof of Theorem 1.1.

References

[1] Aschenbrenner, M., L. van den Dries, and J. van der Hoeven, *Asymptotic Differential Algebra and Model Theory of Transseries*, vol. 195 of *Annals of Mathematics Studies*, Princeton University Press, Princeton, 2017. Zbl 06684722. MR 3585498. DOI 10.1515/9781400885411. 342, 344, 349, 358

[2] Chernikov, A., D. Galvin, and S. Starchenko, "Cutting lemma and Zarankiewicz's problem in distal structures," preprint, arXiv:1612.00908v1 [math.LO]. 341

[3] Chernikov, A., and P. Simon, "Externally definable sets and dependent pairs, II," *Transactions of the American Mathematical Society*, vol. 367 (2015), pp. 5217–35. Zbl 1388.03035. MR 3335415. DOI 10.1090/S0002-9947-2015-06210-2. 345

[4] Chernikov, A., and S. Starchenko, "Regularity lemma for distal structures," *Journal of the European Mathematical Society (JEMS)*, vol. 20 (2018), pp. 2437–66. Zbl 06966829. MR 3852184. DOI 10.4171/JEMS/816. 341, 342, 348

[5] Dolich, A., and J. Goodrick, "Strong theories of ordered Abelian groups," *Fundamenta Mathematicae*, vol. 236 (2017), pp. 269–96. Zbl 1420.03065. MR 3600762. DOI 10.4064/fm256-5-2016. 351

[6] Gehret, A., "The asymptotic couple of the field of logarithmic transseries," *Journal of Algebra*, vol. 470 (2017), pp. 1–36. Zbl 1423.03133. MR 3565423. DOI 10.1016/j.jalgebra.2016.08.016. 342, 350, 351, 359

[7] Gehret, A., "NIP for the asymptotic couple of the field of logarithmic transseries," *Journal of Symbolic Logic*, vol. 82 (2017), pp. 35–61. Zbl 1419.03035. MR 3631276. DOI 10.1017/jsl.2016.59. 342, 351

[8] Gehret, A., "A tale of two Liouville closures," *Pacific Journal of Mathematics*, vol. 290 (2017), pp. 41–76. Zbl 1387.12004. MR 3673079. DOI 10.2140/pjm.2017.290.41. 350

[9] Gehret, A., "Towards a model theory of logarithmic transseries," Ph.D. dissertation, University of Illinois at Urbana-Champaign, Illinois, 2017. 351

[10] Günaydin, A., and P. Hieronymi, "Dependent pairs," *Journal of Symbolic Logic*, vol. 76 (2011), pp. 377–90. Zbl 1220.03031. MR 2830406. DOI 10.2178/jsl/1305810753. 349

[11] Hieronymi, P., and T. Nell, "Distal and non-distal pairs," *Journal of Symbolic Logic*, vol. 82 (2017), pp. 375–83. Zbl 1419.03036. MR 3631293. DOI 10.1017/jsl.2016.28. 342, 346, 348, 349

[12] Rosenlicht, M., "On the value group of a differential valuation, II," *American Journal of Mathematics*, vol. 103 (1981), pp. 977–96. Zbl 0474.12020. MR 0630775. DOI 10.2307/2374255. 342

[13] Simon, P., "Distal and non-distal NIP theories," *Annals of Pure and Applied Logic*, vol. 164 (2013), pp. 294–318. Zbl 1269.03037. MR 3001548. DOI 10.1016/j.apal.2012.10.015. 341, 344, 348, 359

[14] Simon, P., *A Guide to NIP Theories*, vol. 44 of *Lecture Notes in Logic*, Cambridge University Press, Cambridge, 2015. Zbl 1332.03001. MR 3560428. DOI 10.1017/CBO9781107415133. 344

[15] Tent, K., and M. Ziegler, *A Course in Model Theory*, vol. 40 of *Lecture Notes in Logic*, Cambridge University Press, Cambridge, 2012. Zbl 1245.03002. MR 2908005. DOI 10.1017/CBO9781139015417. 345, 348

Acknowledgments

The authors thank Matthias Aschenbrenner, Artem Chernikov, Lou van den Dries, John Goodrick, Philipp Hieronymi, and Travis Nell for various conversations and correspondences around the topics in this paper, and the anonymous referee for suggesting improvements and spotting some errors.

Gehret's work was partially supported by National Science Foundation grant 1703709.

Gehret
Department of Mathematics
University of California, Los Angeles
Los Angeles, California
USA
allen@math.ucla.edu

Kaplan
Department of Mathematics
University of Illinois at Urbana-Champaign
Urbana, Illinois
USA
eakapla2@illinois.edu