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Toroidal prefactorization algebras associated to holomorphic fibrations and a relationship to vertex algebras

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A B S T R A C T

Let X be a complex manifold, $\pi : E \to X$ a locally trivial holomorphic fibration with fiber F, and $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ a Lie algebra with an invariant symmetric form. We associate to this data a holomorphic prefactorization algebra $\mathcal{F}_{\mathfrak{g},\pi}$ on X in the formalism of Costello-Gwilliam. When $X = \mathbb{C}$, $\mathfrak{g}$ is simple, and F is a smooth affine variety, we extract from $\mathcal{F}_{\mathfrak{g},\pi}$ a vertex algebra which is a vacuum module for the universal central extension of the Lie algebra $\mathfrak{g} \otimes H^0(F, \mathcal{O})[z, z^{-1}]$. As a special case, when F is an algebraic torus $(\mathbb{C}^*)^n$, we obtain a vertex algebra naturally associated to an $(n+1)$–toroidal algebra, generalizing the affine vacuum module.

Contents

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1. Introduction

Affine Kac-Moody Lie algebras [10] are a class of infinite-dimensional Lie algebras which play a central role in representation theory and conformal field theory. Given a finite-dimensional complex Lie algebra with invariant form $(\mathfrak{g}, \langle, \rangle)$, the corresponding affine Lie algebra $\hat{\mathfrak{g}}$ is a central extension of the loop algebra $\mathfrak{g}[z, z^{-1}] = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ by a one-dimensional center $\mathbb{C}\mathbf{k}$. It is well-known (see e.g. [7]) that for each $K \in \mathbb{C}$, the *vacuum module*

$$V_K(\hat{\mathfrak{g}}) = \mathrm{Ind}_{\hat{\mathfrak{g}}^+}^{\hat{\mathfrak{g}}} \mathbb{C}_K$$

(where $\hat{\mathfrak{g}}^+ = \mathfrak{g}[z] \oplus \mathbb{C}\mathbf{k} \subset \hat{\mathfrak{g}}$, and $\mathbb{C}_K$ denotes its one-dimensional representation on which the first summand acts trivially and $\mathbf{k}$ acts by K) has the structure of a vertex algebra. The representation theory of $\hat{\mathfrak{g}}$ and $V_K(\hat{\mathfrak{g}})$ are inextricably linked - $V_K(\hat{\mathfrak{g}})$ picks out interesting categories of representations of $\hat{\mathfrak{g}}$, and provides tools for studying these. $V_K(\hat{\mathfrak{g}})$ can also be realized geometrically on a smooth complex curve X (see [2,4,7]), and tied closely to the geometry of the moduli space $Bun_G(X)$ of principal G-bundles on X.

More generally, for a commutative $\mathbb{C}$-algebra R, one can consider the Lie algebra $\mathfrak{g}_R = \mathfrak{g} \otimes_{\mathbb{C}} R$, and its universal central extension $\hat{\mathfrak{g}}_R$. The case $R = \mathbb{C}[z, z^{-1}]$ corresponds to the affine Kac-Moody algebra, and when $R = \mathbb{C}[z^{\pm 1}, z_1^{\pm}, \cdots z_n^{\pm}]$, $\hat{\mathfrak{g}}_R$ is known as the $(n+1)$-toroidal algebra. It is a natural question whether one can associate to $\hat{\mathfrak{g}}_R$ a vertex algebra analogous to $V_K(\hat{\mathfrak{g}})$, and if so, whether it has a "geometric" realization. Our goal in this paper is to show that the answer is affirmative in the case when $R = A[z, z^{-1}]$ for a commutative $\mathbb{C}$-algebra A. Connections between toroidal algebras and vertex algebras have also been explored in [1,6,14,16].

When $R = A[z, z^{-1}]$, $\hat{\mathfrak{g}}_R$ contains a subalgebra $\hat{\mathfrak{g}}_R^+$ corresponding to non-negative powers of z. By analogy with the affine case, we may form the induced module

$$V(\hat{\mathfrak{g}}_R) := \mathrm{Ind}_{\hat{\mathfrak{g}}_R^+}^{\hat{\mathfrak{g}}_R} \mathbb{C}$$

where $\mathbb{C}$ denotes the trivial representation of $\hat{\mathfrak{g}}_R^+$. We prove the following:

Theorem 1.1 *(Theorem 2.9 and Proposition 2.10). When $R = A[z, z^{-1}]$, $V(\hat{\mathfrak{g}}_R)$ has the structure of a vertex algebra. Moreover, this structure is functorial in A.*

The bulk of the paper is devoted to giving a geometric realization of $V(\hat{\mathfrak{g}}_R)$ in the language of factorization algebras, which we briefly recall.

1.1. (Pre)factorization algebras

The formalism of (pre)factorization algebras was developed by Kevin Costello and Owen Gwilliam in [4] to describe the algebraic structure of observables in quantum field theory, as well as their symmetries. Roughly speaking, a prefactorization algebra $\mathcal{F}$ on a manifold X assigns to each open subset $U \subset X$ a cochain complex $\mathcal{F}(U)$, and to each inclusion

$$U_1 \sqcup U_2 \sqcup \cdots \sqcup U_n \subset V$$

of disjoint open subsets U_i of V, a map

$$m_V^{U_1, \cdots, U_n} : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \mapsto \mathcal{F}(V) \tag{1}$$

subject to some natural compatibility conditions. If $\mathcal{F}$ is the prefactorization algebra of observables in a quantum field theory, the cohomology groups $H^i(\mathcal{F}(U))$ can be interpreted as the observables of the theory on U as well as their (higher) symmetries. This structure is reminiscent of a multiplicative cosheaf, and just as in the theory of sheaves/cosheaves a gluing axiom distinguishes factorization algebras from mere prefactorization algebras.

An important source of prefactorization algebras are *factorization enveloping algebras*. Let $\mathcal{L}$ be a fine sheaf of dg (or L_∞ algebras) on X. Denoting by $\mathcal{L}_c$ the cosheaf of compactly supported sections of $\mathcal{L}$, we have maps

$$\oplus_{i=1}^n \mathcal{L}_c(U_i) \cong \mathcal{L}_c(U_1 \cup \cdots \cup U_n) \mapsto \mathcal{L}_c(V) \tag{2}$$

for disjoint opens $U_i \subset V$, where the map on the right is extension by 0. Applying the functor C_*^{Lie} of Chevalley chains to (2) yields maps

$$\otimes_{i=1}^n C_*^{Lie}(\mathcal{L}_c(U_i)) \mapsto C_*^{Lie}(\mathcal{L}_c(V))$$

The argument just sketched shows that the assignment

$$U \mapsto C_*^{Lie}(\mathcal{L}_c(U)) \tag{3}$$

defines a prefactorization algebra. It is called the *factorization enveloping algebra of* $\mathcal{L}$ and denoted $\mathbb{U}\mathcal{L}$.

Costello-Gwilliam showed that there is a close relationship between a certain class of prefactorization algebras on $X = \mathbb{C}$ and vertex algebras. The following result from [4] (paraphrased for the sake of brevity) is central to our construction:

Theorem 1.2 *([4], Theorem 5.3.3). Let* $\mathcal{F}$ *be a unital,* S^1*-equivariant, holomorphically translation invariant prefactorization algebra on* $\mathbb{C}$ *satisfying certain natural conditions. Then the vector space*

$$V(\mathcal{F}) := \bigoplus_{l\in\mathbb{Z}} H^*(\mathcal{F}^{(l)}(\mathbb{C})) \tag{4}$$

has the structure of a vertex algebra, where $\mathcal{F}^{(l)}(\mathbb{C})$ *denotes the* l*-th eigenspace of* S^1 *in* $\mathcal{F}(\mathbb{C})$.

1.2. Prefactorization algebras from holomorphic fibrations

In this paper we construct prefactorization algebras starting with two pieces of data:

- A locally trivial holomorphic fibration $\pi : E \to X$ of complex manifolds with fiber F.
- A Lie algebra $(\mathfrak{g}, \langle\,,\,\rangle)$ with invariant bilinear form.

We begin with a sheaf of dg Lie algebras (DGLA's) on X

$$\mathfrak{g}_\pi = (\mathfrak{g} \otimes \pi_*\Omega_E^{0,*}, \bar{\partial})$$

with bracket

$$[J \otimes \alpha, J' \otimes \beta] = [J, J'] \otimes \alpha \wedge \beta, \quad J, J' \in \mathfrak{g}, \ \alpha, \beta \in \pi_*\Omega_E^{0,*}.$$

$\mathfrak{g}_\pi$ has an L_∞ central extension $\widehat{\mathfrak{g}}_\pi$ whose underlying complex of sheaves is of the form

$$\widehat{\mathfrak{g}}_\pi = \mathfrak{g}_\pi \oplus \mathcal{K}_\pi,$$

with $\mathcal{K}_\pi$ a certain three-term complex. Our prefactorization algebra is

$$\mathcal{F}_{\pi,\mathfrak{g}} := C_*^{Lie}(\widehat{\mathfrak{g}}_{\pi,c}),$$

where $\widehat{\mathfrak{g}}_{\pi,c}$ denotes the cosheaf of sections with compact support. This is an instance of the factorization enveloping algebra as in Equation (3) described above.

When F is a smooth affine complex variety, and $E = X \times F$ is a trivial fibration, we obtain a chain of inclusions of factorization enveloping algebras

$$\mathcal{G}^{alg}_{\pi,\mathfrak{g}} \subset \mathcal{G}_{\pi,\mathfrak{g}} \subset \mathcal{F}_{\pi,\mathfrak{g}}$$

corresponding to the inclusion of sheaves of DGLA's

$$(\mathfrak{g} \otimes H^0(F, \mathcal{O}_F^{alg}) \otimes \Omega_X^{0,*}, \bar{\partial}) \subset (\mathfrak{g} \otimes \Gamma(F, \Omega_F^{0,*}) \otimes \Omega_X^{0,*}, \bar{\partial}) \subset \mathfrak{g}_\pi$$

which extends to the central extensions. Here, $\mathcal{O}_F^{alg}$ denotes the sheaf of algebraic functions on F.

When $X = \mathbb{C}$ (and E is necessarily trivial), we may attempt to extract from $\mathcal{G}^{alg}_{\mathfrak{g},\pi}$ a vertex algebra using Theorem 1.2. Our main result is the following:

Theorem 1.3 *(Theorem 5.2). Let F be a smooth complex affine variety, and $\pi : \mathbb{C} \times F \to \mathbb{C}$ the trivial fibration with fiber F. Then*

(1) *The toroidal prefactorization algebra $\mathcal{G}^{alg}_{\mathfrak{g},\pi}$ satisfies the hypotheses of Theorem 1.2.*
(2) *The vertex algebra $\mathbb{V}(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$ is isomorphic to the toroidal vertex algebra $\mathbb{V}(\mathfrak{g}_R)$, with $R = H^0(F, \mathcal{O}^{alg}_F)[t, t^{-1}]$.*

We may view these results as follows. When X is a (arbitrary) Riemann surface, and $p \in X$ a point, we may choose a coordinate z centered at p, and a local trivialization of E near p. The cohomology prefactorization algebra $H^*(\mathcal{F}_{\pi,\mathfrak{g}})$ is then locally modeled by the vertex algebra $\mathbb{V}(\mathfrak{g}_F)$ via the dense inclusion $\mathcal{G}^{alg}_{\pi,\mathfrak{g}} \subset \mathcal{F}_{\pi,\mathfrak{g}}$.

Most of the work in this paper goes into proving Theorem 1.3, which involves two main steps. First, we verify that the various technical hypotheses of Theorem 1.2 are satisfied. The second step is a somewhat lengthy direct calculation following the approach taken in [19] for the Virasoro factorization algebra and in [4] (Section 5.5.5) for the affine factorization algebra. This involves constructing explicit representatives in $H^*(\mathcal{G}^{alg}_{\pi,\mathfrak{g}})$, and verifying that operator product expansions match those of $\mathbb{V}(\mathfrak{g}_R)$, where $R = H^0(F, \mathcal{O}^{alg}_F)[t, t^{-1}]$.

1.3. Outline of paper

In section 2 we recall universal central extensions, the construction of $\mathfrak{g}_R$, and vertex algebras. We construct an L_∞ model of $\mathfrak{g}_R$ which is later used to build our prefactorization algebra $\mathcal{F}_{\pi,\mathfrak{g}}$. We also show how to associate to the algebra $R = A[z, z^{-1}]$, where A is a commutative $\mathbb{C}$–algebra, a vertex algebra generalizing the affine vacuum module. Our later geometric construction will be a special case of this. In section 3 we recall some basic facts about prefactorization algebras. The construction of $\mathcal{F}_{\mathfrak{g},\pi}$ and the related prefactorization algebra $\mathcal{G}^{alg}_{\mathfrak{g},\pi}$ happens in section 4. Finally, in section (5) we consider the special case when $X = \mathbb{C}$, and prove Theorem 1.3 using the approach outlined above.

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2. Lie algebras and vertex algebras

We begin by recalling some aspects of toroidal Lie algebras, then move towards a slight variant that will be useful for our purposes.

2.1. Central extensions

Let $\mathfrak{g}$ be a complex Lie algebra equipped with an invariant bilinear form $\langle \bullet, \bullet \rangle$. Also, fix a commutative $\mathbb{C}$-algebra R. Then, $\mathfrak{g}_R := \mathfrak{g} \otimes_{\mathbb{C}} R$ carries a natural complex Lie algebra structure with bracket

$$[J \otimes r, J' \otimes s] = [J, J'] \otimes rs$$

where $J, J' \in \mathfrak{g}$ and $r, s \in R$. It is shown by Kassel [11] that there exists a universal central extension of the form

$$0 \to H_2^{\mathrm{Lie}}(\mathfrak{g}_R) \to \hat{\mathfrak{g}}_R \to \mathfrak{g}_R \to 0.$$

Furthermore, when $\mathfrak{g}$ is simple and $\langle \bullet, \bullet \rangle$ the Killing form, there is an isomorphism of the Lie algebra homology $H_2(\mathfrak{g}_R) \cong \Omega^1_R/dR$ where Ω^1_R is the R-module of Kähler differentials of $R/\mathbb{C}$ and $d : R \to \Omega^1_R$ is the universal derivation. The bracket on

$$\hat{\mathfrak{g}}_R \cong \mathfrak{g} \otimes R \oplus \Omega^1_R/dR, \tag{5}$$

is given by

$$\begin{aligned}
[J \otimes r, J' \otimes s] &= [J, J'] \otimes rs + \langle J, J' \rangle \overline{rds} \\
&= [J, J'] \otimes rs + \frac{1}{2} \langle J, J' \rangle \overline{(rds - sdr)}
\end{aligned}$$

where $\overline{\omega}$ denotes the class of $\omega \in \Omega^1_R$ in Ω^1_R/dR. We will find the second form of the central cocycle more convenient to use.

Example 2.1. Let $n \geq 0$ be an integer. An important class of examples is obtained by taking

$$R := \mathbb{C}[t_0^{\pm 1}, \cdots, t_n^{\pm 1}].$$

This is the algebra of functions on the $(n + 1)$-dimensional algebraic torus.

When $n = 0$, the vector space Ω^1_R/dR is one-dimensional with an explicit isomorphism given by the residue

$$\mathrm{Res} : \Omega^1_R/dR \xrightarrow{\cong} \mathbb{C}.$$

The resulting Lie algebra $\hat{\mathfrak{g}}_R$ is the ordinary affine Kac-Moody algebra usually denoted by $\hat{\mathfrak{g}}$. For $n \geq 1$, the vector space Ω^1_R/dR is infinite dimensional. Indeed, let us denote $k_i = t_i^{-1} dt_i$. The space Ω^1_R/dR is generated over the ring $\mathbb{C}[t_0^{\pm 1}, \ldots, t_n^{\pm 1}]$ by the symbols $k_0, \ldots, k_n$ subject to the relation

$$\sum_{i=0}^{n} m_i t_0^{m_0} \cdots t_n^{m_n} k_i = 0$$

where $(m_0, \ldots, m_n)$ is any n-tuple of integers. The Lie algebra $\mathfrak{g}_R$ is called the $(n+1)$-*toroidal* Lie algebra associated to $\mathfrak{g}$.

It will be useful for us to have a model for the Lie algebra $\mathfrak{g}_R$ as an L_∞ algebra. We recall the definition (see Section 3.2.3 of [12], or Section 10.1.6 of [15] for instance).

Definition 2.2. An L_∞ structure on a graded vector space $\mathfrak{h}$ is the data of cohomological degree 1 coderivation D of the cofree cocommutative coalgebra

$$\mathrm{Sym}(\mathfrak{h}[1])$$

satisfying $D^2 = 0$. We may write $D = \sum_{m=1}^{\infty} l_m$, where

$$l_m : \mathfrak{h}^{\otimes m} \to \mathfrak{h}[2-m]$$

and the l_m's are extended to the symmetric coalgebra as coderivations.

Given such a square zero coderivation D the cochain complex

$$C_*^{Lie}(\mathfrak{g}) = (\mathrm{Sym}(\mathfrak{h}[1]), D)$$

is called the *Chevalley-Eilenberg chain complex*. In what follows, we usually denote the differential by $D = d_{CE}$.

An ordinary Lie algebra corresponds to the case where $l_m = 0$ for $m \neq 2$, and l_2 is the Lie bracket. For an ordinary Lie algebra, the differential d_{CE} is given on generators by $d_{CE}(xy) = [x, y]$. In this case, the complex computes Lie algebra homology with trivial coefficients.

The linear dual of $C_*^{Lie}(\mathfrak{h})$ is

$$C^*_{Lie}(\mathfrak{h}) = \left(\mathrm{Sym}(\mathfrak{h}^*[-1]), d^*_{CE}\right).$$

Here, d^*_{CE} is the linear dual of the map d_{CE}. This cochain complex computes Lie algebra *co*homology of $\mathfrak{h}$ with trivial coefficients.

Suppose (M, d_M) is a cochain complex with the structure of an $\mathfrak{h}$-module. This means that underlying graded vector space M has an $\mathfrak{h}$-module structure and this $\mathfrak{h}$-action commutes with the differential d_M. Then, the complex

$$C^*_{Lie}(\mathfrak{h}; M) = \left(\mathrm{Sym}(\mathfrak{h}^*[-1]) \otimes M, d^*_{CE} + d_M + d_{\mathfrak{g},M}\right)$$

is defined. Here, the additional differential $d_{\mathfrak{g},M}$ encodes the module structure, for a precise formula we refer to [15]. This complex computes the Lie algebra cohomology $H^*(\mathfrak{h}; M)$ of $\mathfrak{h}$ with coefficients in M.

If M is a trivial module, concentrated in cohomological degree zero, then it is a standard fact that cohomology classes in $H^2(\mathfrak{h}; M)$ correspond to central extension of $\mathfrak{h}$ by M. Similarly, if M is a trivial dg $\mathfrak{h}$-module, then degree two cocycles in $C^*_{Lie}(\mathfrak{h}; M)$ give rise to extensions of $\mathfrak{h}$ by M as an L_∞ algebra.

We return to the Lie algebra $\hat{\mathfrak{g}}_R$. The L_∞ model for the central extension amounts to replacing the vector space Ω^1_R/dR appearing as the central term by the cochain complex

$$\mathrm{K}_R = \mathrm{Ker}(d)[2] \to R[1] \xrightarrow{d} \Omega^1_R.$$

Just as the Lie algebra $\hat{\mathfrak{g}}_R$ is a central extension of $\mathfrak{g}_R = \mathfrak{g} \otimes R$ by the trivial module Ω^1/dR, the L_∞ model we to construct is a central extension of $\mathfrak{g}_R$ by the cochain complex K_R, thought of as a trivial dg module for $\mathfrak{g}_R$.

The central extension is determined by a cocycle in the Chevalley–Eilenberg cochain complex

$$C^*_{Lie}(\mathfrak{g}_R, \mathrm{K}_R) = \left(\mathrm{Sym}(\mathfrak{g}^*_R[-1]) \otimes \mathrm{K}_R\, ,\ d_{CE} + d_{\mathrm{K}}\right)$$

of total degree two. Here d_{CE} is the Chevalley–Eilenberg differential for $\mathfrak{g}_R$ and d_{K} is the differential induced from the differential on the complex K_R.

The cocycle is of the form $\varphi = \varphi^{(0)} + \varphi^{(1)}$ where

$$\begin{aligned} \varphi^{(1)} : \quad (\mathfrak{g}_R)^{\otimes 2} &\to \Omega^1_R \\ (J \otimes r) \otimes (J' \otimes s) &\mapsto \tfrac{1}{2}\langle J, J'\rangle (r\,ds - s\,dr) \end{aligned}$$

and

$$\begin{aligned} \varphi^{(0)} : \quad (\mathfrak{g}_R)^{\otimes 3} &\to R \\ (J \otimes r) \otimes (J' \otimes s) \otimes (J'' \otimes t) &\mapsto \tfrac{1}{2}\langle [J, J'], J''\rangle rst \end{aligned}$$

Lemma 2.3. *The functional φ defines a cocycle in $C^*(\mathfrak{g}_R, \mathrm{K}_R)$ of total degree two.*

Proof. The differential in the cochain complex $C^*(\mathfrak{g}_R, \mathrm{K}_R)$ is of the form $d_{CE} + \mathrm{d}_{\mathrm{K}}$ where d is the de Rham differential defining the complex K_R, and d_{CE} is the Chevalley-Eilenberg differential encoding the Lie bracket of $\mathfrak{g}_R$. It is immediate that $d_{\mathrm{K}}\varphi^{(1)} = 0$, $d_{CE}\varphi^{(0)} = 0$ by the Jacobi identity for $\mathfrak{g}$ and invariance of the pairing $\langle \bullet, \bullet \rangle$, and $d_{\mathrm{K}}\varphi^{(0)} + d_{CE}\varphi^{(1)} = 0$ by direct calculation. Thus $(d_{CE} + d_{\mathrm{K}})\varphi = 0$ as desired. $\square$

The cocycle φ defines an L_∞ central extension

$$\mathrm{K}_R \to \hat{\mathfrak{g}}_R \to \mathfrak{g}_R.$$

As a vector space, $\tilde{\mathfrak{g}}_R = \mathfrak{g}_R \oplus \mathcal{K}_R$, and the L_∞ operations are defined by $\ell_1 = d$, $\ell_2 = [\cdot,\cdot]_{\mathfrak{g}_R} + \varphi^{(1)}$, and $\ell_3 = \varphi^{(0)}$. The following is immediate from our definitions:

Lemma 2.4. *There is an isomorphism of Lie algebras* $H^*(\tilde{\mathfrak{g}}_R, \ell_1) \cong \hat{\mathfrak{g}}_R$.

Proof. The cohomology of $\tilde{\mathfrak{g}}_R$ is concentrated in degree zero, and isomorphic to

$$\mathfrak{g}_R \oplus H^0(\mathcal{K}_R) = \mathfrak{g}_R \oplus \Omega^1_R/dR$$

as a vector space. Our definition of φ implies that

$$\varphi^{(0)}((J \otimes r) \otimes (J' \otimes s)) = \frac{1}{2}\langle J, J'\rangle(r\,ds - s\,dr) = r\,ds \mod dR$$

so that the resulting Lie bracket is the same as that of $\hat{\mathfrak{g}}_R$. ∎

2.2. Vertex algebras

We proceed to briefly recall the basics of vertex algebras and discuss an important class of examples, which will later be constructed geometrically via factorization algebras. We refer the reader to [7,9] for details.

Definition 2.5. A vertex algebra $(V, |0\rangle, T, Y)$ is a complex vector space V along with the following data:

- A vacuum vector $|0\rangle \in V$.
- A linear map $T : V \to V$ (the translation operator).
- A linear map $Y(-, z) : V \to \mathrm{End}(V)[[z^{\pm 1}]]$ (the vertex operator). We write $Y(v, z) = \sum_{n\in\mathbb{Z}} A^v_n z^{-n}$ where $A^v_n \in \mathrm{End}(V)$.

satisfying the following axioms:

- For all $v, v' \in V$ there exists an $N \geq 0$ such that $A^v_n v' = 0$ for all $n > N$. (This says that $Y(v, z)$ is a *field* for all v).
- (vacuum axiom) $Y(|0\rangle, z) = \mathrm{id}_V$ and $Y(v, z)|0\rangle \in v + zV[[z]]$ for all $v \in V$
- (translation) $[T, Y(v, z)] = \partial_z Y(v, z)$ for all $v \in V$. Moreover $T|0\rangle = 0$.
- (locality) For all $v, v' \in V$, there exists $N \geq 0$ such that

$$(z - w)^N [Y(v, z), Y(v', w)] = 0$$

in $\mathrm{End}(V)[[z^{\pm 1}, w^{\pm 1}]]$.

In order to prove that a given $(V, |0\rangle, T, Y)$ forms a vertex algebra, the following "reconstruction" or "extension" theorem is very useful. It shows that any collection of local fields generates a vertex algebra in a suitable sense.

Theorem 2.6 *([7], [5]). Let V be a complex vector space equipped with: an element $|0\rangle \in V$, a linear map $T : V \to V$, a set of vectors $\{a^s\}_{s\in S} \subset V$ indexed by a set S, and fields $A^s(z) = \sum_{n\in \mathbb{Z}} A^s_n z^{-n-1}$ for each $s \in S$ such that:*

- *For all $s \in S$ $A^s(z)|0\rangle \in a^s + zV[[z]]$;*
- *$T|0\rangle = 0$ and $[T, A^s(z)] = \partial_z A^s(z)$;*
- *$A^s(z)$ are mutually local;*
- *and V is spanned by $\{A^{s_1}_{j_1} \cdots A^{s_m}_{j_m}|0\rangle\}$ as the j_i's range over negative integers.*

Then, the data $(V, |0\rangle, T, Y)$ defines a unique vertex algebra satisfying

$$Y(a^s, z) = A^s(z).$$

Remark 2.7. The version stated above appears in [5], and is slightly more general than the version stated in [7].

2.3. The vertex algebras $V(\mathfrak{g})$ and $V(\mathfrak{g}_R)$

A number of vertex algebras are constructed from vacuum representations of affine Lie algebras and their generalizations. We proceed to review the vertex algebra structure on the affine Kac-Moody vacuum module $V(\hat{\mathfrak{g}})$ and extend the construction to vacuum representations of $\hat{\mathfrak{g}}_R$, where $R = A[t, t^{-1}]$ for some $\mathbb{C}$-algebra A.

2.3.1. $V(\hat{\mathfrak{g}})$

Let $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$ be the affine Kac-Moody algebra, $\hat{\mathfrak{g}}^+ = \mathfrak{g}[t]$ denote the positive sub-algebra, and $\mathbb{C}$ denote the trivial representation of $\hat{\mathfrak{g}}^+$. For $J \in \mathfrak{g}$, denote $J \otimes t^n$ by J_n, and $1 \in \mathbb{C}$ by $|0\rangle$.

It is well-known (see for instance [7]) that the induced vacuum representation

$$V(\hat{\mathfrak{g}}) := \mathrm{Ind}^{\hat{\mathfrak{g}}}_{\hat{\mathfrak{g}}^+} \mathbb{C} := U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t])} \mathbb{C}$$

has a $\mathbb{C}[\mathbf{k}]$-linear vertex algebra structure, which is generated, in the sense of the above reconstruction theorem, by the fields

$$J^i(z) := Y(J^i_{-1}|0\rangle, z) = \sum_{n\in\mathbb{Z}} J^i_n z^{-n-1},$$

where $\{J^i\}$ is a basis for $\mathfrak{g}$. These satisfy the commutation relations

$$[J^i(z), J^j(w)] = [J^i, J^j](w)\delta(z-w) + \langle J^i, J^j\rangle\mathbf{k}\partial_w\delta(z-w)$$

where

$$\delta(z-w) = \sum_m z^m w^{-m-1}$$

The translation operator T is determined by the properties

$$T|0\rangle = 0, [T, J^i_n] = -nJ^i_{n-1}.$$

Remark 2.8. The construction above produces a generic version of the affine Kac-Moody vacuum module, in the sense that $\mathbf{k}$ is not specialized to be a complex number. In the vertex algebra literature one typically specifies a level $K \in \mathbb{C}$, and defines

$$V_K(\hat{\mathfrak{g}}) := \mathrm{Ind}^{\hat{\mathfrak{g}}}_{\mathfrak{g}[t]\oplus\mathbb{C}\mathbf{k}} \mathbb{C},$$

where $\mathbb{C}$ denotes the one-dimensional representation of $\mathfrak{g}[t]\oplus\mathbb{C}\mathbf{k}$ on which the first factor acts by 0 and $\mathbf{k}$ acts by K. We have an isomorphism

$$V(\hat{\mathfrak{g}})/I \cong V_K(\mathfrak{g})$$

where I is the vertex ideal generated by $K|0\rangle - \mathbf{k}|0\rangle$. $V(\mathfrak{g})$ can therefore be viewed as a family of vertex algebras over $\mathrm{spec}(\mathbb{C}[\mathbf{k}])$, with fiber $V_K(\mathfrak{g})$ at $\mathbf{k} = K$.

2.3.2. The generalized toroidal vertex algebra

In this section we generalize the construction of the affine Kac-Moody vacuum module above to the Lie algebra $\hat{\mathfrak{g}}_R$, for $R = A[t, t^{-1}]$, where A is a commutative $\mathbb{C}$–algebra. The construction specializes to $V(\hat{\mathfrak{g}})$ for $A = \mathbb{C}$.

Let A be a commutative $\mathbb{C}$-algebra, $R = A[t, t^{-1}] := A \otimes \mathbb{C}[t, t^{-1}]$, and $\hat{\mathfrak{g}}_R$ the Lie algebra (5). We have a Lie subalgebra

$$\hat{\mathfrak{g}}^+_R := \mathfrak{g} \otimes A[t] \oplus \Omega^1_{A[t]}/dA[t] \hookrightarrow \hat{\mathfrak{g}}_R.$$

Let

$$V(\hat{\mathfrak{g}}_R) := \mathrm{Ind}^{\hat{\mathfrak{g}}_R}_{\hat{\mathfrak{g}}^+_R} \mathbb{C} \tag{6}$$

where $\mathbb{C}$ denotes the trivial representation of $\hat{\mathfrak{g}}^+_R$. Our goal is to define the structure of a vertex algebra on $V(\hat{\mathfrak{g}}_R)$.

The vacuum vector is simply $|0\rangle := 1 \in \mathbb{C}$. The fields of the vertex algebra split into three classes and are defined as follows.

$$J_u(z) := Y(J\otimes ut^{-1}|0\rangle, z) := \sum_{n\in\mathbb{Z}} (J\otimes ut^n) z^{-n-1}, \tag{7}$$

$$K_{u\frac{dt}{t}}(z) := Y(t^{-1}udt|0\rangle, z) := \sum_{n\in\mathbb{Z}} (ut^{n-1}dt) z^{-n}, \tag{8}$$

$$K_{t^{-1}\omega}(z) := Y(t^{-1}\omega|0\rangle, z) := \sum_{n\in\mathbb{Z}} (t^n\omega) z^{-n-1} \tag{9}$$

where $J\in\mathfrak{g}, u\in A, \omega\in\Omega^1_A$.

The commutation relations between these fields are easily checked to be

$$\begin{aligned}[J^1_u(z), J^2_v(w)] = &\left([J^1, J^2]_{uv}(w) + \langle J^1, J^2\rangle K_{t^{-1}udv}(w)\right)\delta(z-w) \\ &+ \langle J^1, J^2\rangle K_{uv\frac{dt}{t}}(w)\partial_w\delta(z-w)\end{aligned} \tag{10}$$

with all other commutators 0.

The operator T, corresponding to the Lie derivative $L_{-\partial_t}$, is defined by

$$T|0\rangle = 0,\ [T, J^i\otimes ut^n] = -nJ^i\otimes ut^{n-1},\ [T, ut^n dt] = -nft^{n-1}dt,\ [T, t^n\omega] = -nt^{n-1}\omega.$$

Theorem 2.9. *The above field assignments, together with T equip $V(\mathfrak{g}_R)$ with the structure of a vertex algebra.*

Proof. We begin by checking that the field assignment above is well-defined. This amounts to verifying that $Y(d(t^{-1}u)|0\rangle, z) = 0$. We have

$$\begin{aligned}Y(d(t^{-1}u)|0\rangle, z) &= Y(t^{-1}du|0\rangle, z) - Y(t^{-2}udt|0\rangle, z) \\ &= Y(t^{-1}du|0\rangle, z) - Y([T, t^{-1}udt]|0\rangle, z) \\ &= Y(t^{-1}du|0\rangle, z) - \partial_z Y(t^{-1}udt|0\rangle, z) \\ &= \sum_n (t^n du + nt^{n-1}u) z^{-n-1} = \sum_n d(t^n u) z^{-n-1} = 0\end{aligned}$$

To obtain the structure of a vertex algebra, we apply the reconstruction Theorem 2.6 to $V(\mathfrak{g}_R)$ and the fields $\{J_u(z), K_{u\frac{dt}{t}}(z), K_{t^{-1}\omega}(z)\}$ for $J\in\mathfrak{g}, f\in A, \omega\in\Omega^1_A$. The only nontrivial axiom to check is mutual locality for the $J(z)$-fields, which follows from the explicit commutator (10). $\square$

2.4. Some properties of $V(\mathfrak{g}_R)$

The map sending a $\mathbb{C}$-algebra A to $V(\mathfrak{g}_R)$, with $R = A[t, t^{-1}]$ has a number of pleasing properties. While these are not used in the remainder of this paper, they are simple to establish and useful for the study of the representation theory of $V(\mathfrak{g}_R)$ and its conformal blocks, which we plan to pursue in future work.

Denote by C − Alg the category of commutative C-algebras and Vert the category of vertex algebras. We have the following result:

Proposition 2.10. *The map*

$$\begin{aligned} \text{C-Alg} &\to \text{Vert} \\ A &\mapsto V(\mathfrak{g}_{A[t,t^{-1}]}) \end{aligned}$$

defines a functor.

Proof. We must check that a homomorphism of C-algebras $\psi : A \to B$ induces a vertex algebra homomorphism $\hat{\psi} : V(\mathfrak{g}_{A[t,t^{-1}]}) \to V(\mathfrak{g}_{B[t,t^{-1}]})$. We begin by constructing a Lie algebra homomorphism

$$\overline{\psi} : \mathfrak{g}_{A[t,t^{-1}]} \to \mathfrak{g}_{B[t,t^{-1}]}.$$

Recall that a C-algebra homomorphism $\sigma : R \to S$ induces a map on Kahler differentials (as vector spaces) $\sigma_* : \Omega_R \to \Omega_S$ given by $\sigma_*(rdr') = \sigma(r)d\sigma(r')$, which sends exact elements to exact elements, inducing a map

$$\sigma_* : \Omega_R/dR \to \Omega_S/dS$$

Extending $\psi : A \to B$ to a homomorphism (abusively also denoted ψ) $\psi : A[t, t^{-1}] \to B[t, t^{-1}]$, and taking $\sigma = \psi$ yields a map

$$\psi_* : \Omega_{A[t,t^{-1}]}/dA[t, t^{-1}] \to \Omega_{B[t,t^{-1}]}/dB[t, t^{-1}]$$

Now, $\overline{\psi} : \mathfrak{g}_{A[t,t^{-1}]} \to \mathfrak{g}_{B[t,t^{-1}]}$ is defined by

$$\overline{\psi}(Jut^n + \omega) = J\psi(u)t^n + \psi_*\omega \quad \text{where } J \in \mathfrak{g},\ u \in A, \omega \in \Omega_{A[t,t^{-1}]}/dA[t, t^{-1}]$$

and easily checked to be a Lie algebra homomorphism. Finally,

$$\hat{\psi} : V(\mathfrak{g}_{A[t,t^{-1}]}) \to V(\mathfrak{g}_{B[t,t^{-1}]})$$

may be defined on the generating fields $J_u(z), K_{u\frac{dt}{t}}, K_{t^{-1}\omega}$ in the obvious way by:

$$\begin{aligned} \hat{\psi}(J_u(z)) &:= \sum_{n\in\mathbb{Z}} (J \otimes \psi(u)t^n)z^{-n-1}, \\ \hat{\psi}(K_{u\frac{dt}{t}}(z)) &:= K_{\psi_*(u\frac{dt}{t})}(z) = \sum_{n\in\mathbb{Z}} (\psi(u)t^{n-1}dt)z^{-n}, \\ \hat{\psi}(K_{t^{-1}\omega}(z)) &:= K_{\psi_*(t^{-1}\omega)}(z) = \sum_{n\in\mathbb{Z}} (t^n\psi_*\omega)z^{-n-1} \end{aligned}$$

where $u \in A, \omega \in \Omega_A$. One easily checks that

$$\hat{\psi}([J_u(z), J_\omega^{\%}(w)]) = [\hat{\psi}(J_u(z)), \hat{\psi}(J_\omega^{\%}(w))],$$

which shows that $\hat{\psi}$ respects the only non-trivial OPE among the generating fields. It follows that $\hat{\psi}$ is a vertex algebra homomorphism. !

If A is any $\mathbb{C}$-algebra, we may apply this result to the structure map $\mathbb{C} \to A$, to obtain:

Corollary 2.11. *The structure map $\mathbb{C} \to A$ induces an embedding of vertex algebras*

$$V(\hat{\mathfrak{g}}) \to V(\hat{\mathfrak{g}}_{A[t,t^{-1}]}).$$

Remark 2.12. As explained in Remark 2.8, we have

$$V_K(\mathfrak{g}) \cong V(\hat{\mathfrak{g}}_{\mathbb{C}[t,t^{-1}]})/I,$$

where $V_K(\mathfrak{g})$ denotes the "usual" affine vacuum module at level $K \in \mathbb{C}$, and I is the vertex ideal generated by $\mathbf{k}|0\rangle - K|0\rangle$. When $K \neq -h^\vee$ (where $h^\vee$ denotes the dual Coxeter number of $\mathfrak{g}$), $V_K(\mathfrak{g})$ is a conformal vertex algebra with conformal Segal-Sugawara vector

$$S = \frac{1}{2(K + h^\vee)} \sum_{i=1}^{d} (J_i \otimes t^{-1})(J_i \otimes t^{-1})|0\rangle,$$

where $\{J_i\}_{i=1}^d$ an orthonormal basis of $\mathfrak{g}$ with respect to the invariant pairing $\langle \cdot, \cdot \rangle$. It follows that when $K \neq -h^\vee$, S defines a conformal vector in $V(\hat{\mathfrak{g}}_{A[t,t^{-1}]})/I^{\%}$, where $I^{\%}$ is the vertex ideal generated by $\frac{dt}{t}|0\rangle - K|0\rangle$.

As another consequence of Proposition 2.10, we note that any algebra automorphism $\psi : A \to A$ induces a vertex algebra automorphism of $V(\hat{\mathfrak{g}}_{A[t,t^{-1}]})$:

Corollary 2.13. *There is a natural group homomorphism*

$$\mathrm{Aut}(A) \to \mathrm{Aut}(V(\hat{\mathfrak{g}}_{A[t,t^{-1}]}))$$
$$\psi \to \hat{\psi}.$$

3. (Pre)factorization algebras and examples

In this section we recall basic notions pertaining to pre-factorization algebras. We refer the reader to [4] for details.

Let X be a smooth manifold, and $\mathbf{C}^{\otimes}$ a symmetric monoidal category.

Definition 3.1. A *prefactorization algebra* $\mathcal{F}$ on X with values in $\mathbf{C}^{\otimes}$ consists of the following data:

- for each open $U \subset M$, an object $\mathcal{F}(U) \in \mathbf{C}^{\otimes}$,
- for each finite collection of pairwise disjoint opens $U_1, \ldots, U_n$ and an open V containing every U_i, a morphism

$$m_V^{U_1, \cdots, U_n} : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V), \tag{11}$$

and satisfying the following conditions:

- composition is associative, so that the triangle

$$\begin{array}{ccc} \ast_i \ast_j \mathcal{F}(T_{ij}) & \longrightarrow & \ast_i \mathcal{F}(U_i) \\ & \searrow \quad \swarrow & \\ & \mathcal{F}(V) & \end{array}$$

  commutes for any disjoint collection $\{U_i\}$ contained in V, and disjoint collections $\{T_{ij}\}_j \subset U_i$

- the morphisms $m_V^{U_1, \cdots, U_n}$ are equivariant under permutation of labels, so that the triangle

$$\begin{array}{ccc} \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) & \xrightarrow{\ast} & \mathcal{F}(U_{\sigma(1)}) \otimes \cdots \otimes \mathcal{F}(U_{\sigma(n)}) \\ & \searrow \quad \swarrow & \\ & \mathcal{F}(V) & \end{array}$$

  commutes for any $\sigma \in S_n$.

In this paper, we will take the target category $\mathbf{C}^{\otimes}$ to be Vect, dg-Vect, or their smooth enhancements DVS described below.

Remark 3.2. A *factorization algebra* is a prefactorization algebra satisfying a descent (or gluing) axiom with respect to a class of special covers called *Weiss covers*. As Theorem 1.2 connecting the Costello-Gwilliam formalism to vertex algebras does not require this axiom, it will not play a role in this paper. We refer the interested reader to [4] for more details on descent..

Example 3.3 *([4], Section 3.2).* Given an associative algebra over $\mathbb{C}$, one can construct a prefactorization algebra $\mathcal{F}_A$ in Vect on $\mathbb{R}$ by declaring $\mathcal{F}_A(I) = A$ for a connected open interval $I \subset \mathbb{R}$, and defining the structure maps in terms of the multiplication on A. For instance, if $I = (a, b)$, $J = (c, d)$, $K = (e, f)$, with $e < a < b < c < d < f$, the structure map is

$$\mathcal{F}_A(I) \otimes \mathcal{F}_A(J) \longrightarrow \mathcal{F}_A(K)$$
$$a \otimes b \longmapsto ab$$

$\mathcal{F}_A$ has the property that it is *locally constant*, in the sense that if $I \subset I'$ are connected intervals, then $\mathcal{F}_A(I) \longrightarrow \mathcal{F}_A(I')$ is an isomorphism. It is shown in Section 3.2 of [4] that locally constant prefactorization algebras on $\mathbb{R}$ in Vect correspond precisely to associative algebras.

Prefactorization algebras can be pushed forward under smooth maps as follows. Suppose $f : X \longrightarrow Y$ is a smooth map of smooth manifolds, and $\mathcal{F}$ a prefactorization algebra on X. One then defines the prefactorization algebra $f_*\mathcal{F}$ on Y by

$$f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U))$$

The structure maps of $f_*\mathcal{F}$ are defined in the obvious way.

If $\mathcal{F}, \mathcal{G}$ are prefactorization algebras on X with values in $\mathbf{C}^{\otimes}$, then a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is the data of maps

$$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U) \in \mathrm{Hom}_{\mathbf{C}^{\otimes}}(\mathcal{F}(U), \mathcal{G}(U))$$

for each open $U \subset X$, compatible with all structure maps (11).

3.1. Prefactorization enveloping algebras

We will define a prefactorization algebra associated to the data of a holomorphic fibration. Such a factorization algebra is an instance of a *prefactorization enveloping algebra*, which we proceed to briefly review following Section 3.6 of [4].[1]

Let $\mathcal{L}$ be a fine sheaf of L_∞ algebras, and $\mathcal{L}_c$ its associated cosheaf of sections with compact support. The *prefactorization enveloping algebra of* $\mathcal{L}$, $\mathbf{U}\mathcal{L}$ is the complex of Chevalley chains of $\mathcal{L}_c$. In other words, for each open $U \subset X$

$$\mathbf{U}\mathcal{L}(U) := C_*^{Lie}(\mathcal{L}_c(U)) \tag{12}$$

The structure maps are given explicitly as follows. Let $U_1, \cdots U_k$ be disjoint open subsets of an open $V \subset X$. The cosheaf $\mathcal{L}_c$ induces a map of L_∞–algebras

$$\oplus_{i=1}^k \mathcal{L}_c(U_i) \longrightarrow \mathcal{L}_c(V)$$

[1] In [4] this is called the "factorization enveloping algebra" but as we mentioned above, we will not use the gluing axiom.

Applying the Chevalley chains functor (which sends sums to tensor products) to this sequence yields structure maps

$$\otimes_{i=1}^{k} C_*^{Lie}(\mathrm{L}_c(U_i)) \to C_*^{Lie}(\mathrm{L}_c(V)).$$

See Section 6.6 of [4] for more details.

Theorem 3.4. *If* L *is a fine cosheaf of* L_∞ *algebras, then* $\mathbf{U}$L *is a prefactorization algebra in* dg-Vect*. The cohomology* $H^*(\mathbf{U}\mathrm{L})$ *is a prefactorization algebra in* Vect*.*

Example 3.5 *(Section 3.4 [4]).* Let $X = \mathbb{R}$, and $\mathfrak{g}_{dR} := (\mathfrak{g} \otimes \Omega^*_{\mathbb{R}}, d_{dR})$ the sheaf of DGLA's on $\mathbb{R}$ obtained by tensoring $\mathfrak{g}$ with the de Rham complex. The prefactorization enveloping algebra $\mathbf{U}(\mathfrak{g}_{dR})$ is locally constant, and the cohomology factorization algebra $H^*(\mathbf{U}(\mathfrak{g} \otimes \Omega^*_{\mathbb{R}}), d_{dR})$ is a locally constant prefactorization algebra in Vect, corresponding to the enveloping algebra $U(\mathfrak{g})$ as in Example (3.3).

Example 3.6 *(Section 5.4 [8]).* Let $X = \mathbb{C}^n$, and $(\mathfrak{g} \otimes \Omega^{0,*}_{\mathbb{C}^n}, \overline{\partial})$ the sheaf of DGLA's on $\mathbb{C}^n$ obtained by tensoring $\mathfrak{g}$ with the Dolbeault complex of forms of type $(0,q)$, $q \geq 0$. As explained below, when $n = 1$, the factorization algebra $\mathbf{U}(\mathfrak{g} \otimes \Omega^{0,*}_{\mathbb{C}}, \overline{\partial})$ allows one to recover the affine vertex algebra $V(\hat{\mathfrak{g}})$ (at level 0).

3.2. Differentiable vector spaces

The prefactorization algebras considered in this paper typically assign to each open subset $U \subset X$ a cochain complex of infinite-dimensional vector spaces. This is apparent already in the Example 3.5 above, where the graded components of $\mathbf{U}(\mathfrak{g}_{dR}) = \mathbf{U}(\mathfrak{g} \otimes \Omega^*_{\mathbb{R}})(U)$ for $U \subset \mathbb{R}$ are tensors in $\mathfrak{g} \otimes \Omega^*_{\mathbb{R}}(U)_c$. The structure maps (11) are thus multilinear maps between such complexes. In order to formulate the notion of translation-invariance for prefactorization algebras in the next section, we will have to discuss what it means for these to depend smoothly on the positions of the open sets $U_i \subset X$. This raises some functional-analytic issues, which in turn complicate homological algebra involving these objects.

In [4] these technical issues are resolved by introducing the category DVS of *Differentiable Vector Spaces* together with certain sub-categories. DVS provides a flexible framework within which one can discuss smooth families of smooth maps between infinite-dimensional cochain complexes parametrized by auxiliary manifolds, and carry out homological constructions. We briefly sketch this category below, and refer to appendices B and C in [4] for all details.

Definition 3.7. Let C^∞ denote the sheaf of rings on the site of smooth manifolds sending each manifold M to the ring of smooth functions $C^\infty(M)$, and assigning to each smooth map $f : M \to N$ the pullback $f^* : C^\infty(N) \to C^\infty(M)$. A C^∞*-module* $\mathcal{F}$ is a sheaf of modules over C^∞. In other words, $\mathcal{F}$ assigns to each M a $C^\infty(M)$-module $\mathcal{F}(M)$, and to

$f : M \to N$ a pullback map $F(f) : F(N) \to F(M)$ of $C^\infty(N)$-modules. A *differentiable vector space* is a C^∞-module equipped with a flat connection. Explicitly, this amounts to assigning a flat connection

$$\nabla : F(M) \to F(M) \otimes_{C^\infty(M)} \Omega^1(M)$$

for each manifold M, compatible with pullbacks. The objects of the category DVS are differentiable vector spaces, and the morphisms $Hom_{DVS}(F, G)$ maps of C^∞-modules intertwining the connections.

Any locally convex topological vector space V gives rise to a differentiable vector space as follows. There is a good notion of a smooth map from any manifold M to V introduced by Kriegl and Michor (see [13]), and we denote by $C^\infty(M, V)$ the space of such. The space $C^\infty(M, V)$ is naturally a $C^\infty(M)$-module, and carries a natural flat connection whose horizontal sections are constant maps $M \to V$. The assignment $M \to C^\infty(M, V)$ thus produces an object of DVS. Multi-linear maps

$$F_1 \times F_2 \times \cdots \times F_r \mapsto G \quad F_i, G \in \text{DVS} \tag{13}$$

equip DVS with the structure of a multi-category (or equivalently, a colored operad) by inserting the output of a multilinear map into another. We denote the space of such maps by $\text{DVS}(F_1, \cdots, F_r | G)$.

The multicategory DVS allows us to formulate the notion of a smooth family of multilinear operations parametrized by an auxiliary manifold M. For $F \in \text{DVS}$, one first defines the mapping space $\mathbf{C}^\infty(M, F) \in \text{DVS}$ as the differentiable vector space given by the assignment $N \mapsto F(N \times M)$. As explained in [4], it has a natural flat connection along N.

Definition 3.8. Let $F_1, \cdots, F_r, G \in \text{DVS}$. A smooth family of multilinear operations $F_1 \times \cdots \times F_r \mapsto G$ parametrized by a manifold M is by definition an element of

$$\text{DVS}(F_1, \cdots, F_r | \mathbf{C}^\infty(M, G))$$

where $\mathbf{C}^\infty(M, G)$ is as explained in the preceding paragraph.

DVS has several good properties. Among these are:

- DVS is complete and co-complete.
- DVS is a Grothendieck Abelian Category.

The second property ensures that all standard constructions in homological algebra behave well in DVS. This is in contrast to the category of topological vector spaces, which is not even Abelian. As the authors explain in [4], this is because DVS has essentially been defined as the category of sheaves on a site.

Finally, we review some examples of differentiable vector spaces which will be useful to us.

Example 3.9. The following is an important example from [4]. Suppose $p : W \to X$ is a vector bundle over the manifold X. Then $V = \Gamma(X, W)$ is naturally a Frechet space, and so locally convex. $C^\infty(M, V)$ is then identified with $\Gamma(M \times X, \pi_X^* E)$, where $\pi_X : M \times X \to X$ is the projection on X. In particular, taking W to be the trivial bundle, we have $C^\infty(M, C^\infty(X)) = C^\infty(M \times X)$. The same line of reasoning shows that the space of compactly supported sections of W, $V^{\%} = \Gamma_c(X, W)$ has a DVS structure.

Example 3.10. The following generalization of the previous example will be useful in Sections 4, 5. Let $\pi : E \to X$ be a smooth map, and $p : W \to E$ a vector bundle on E. Denote by $\mathcal{W}$ the sheaf of smooth sections of W on E. Then $V = \Gamma(X, \pi_* \mathcal{W})$ yields a differentiable vector space, with $C^\infty(M, V) = \Gamma(M \times X, \tilde{\pi}_* \widetilde{\mathcal{W}})$, where $\tilde{\pi} : E \times M \to X \times M$ is defined by $\tilde{\pi}(e, m) = (\pi(e), m)$, and $\widetilde{\mathcal{W}}$ denotes the sheaf of sections of $\pi_E^* W$ on $E \times M$, with $\pi_E : E \times M \to E$ the projection on the first factor. The connection is determined by the condition that the horizontal sections are those constant in the M direction. When $E = X$ and $\pi = id_X$, this example reduces to the previous one. We may similarly equip $V^{\%} = \Gamma_c(X, \pi_* \mathcal{W})$ with a DVS structure.

Example 3.11. Suppose that $\mathcal{F} \in$ DVS, and V is any vector space (note that we don't specify a topology on V). Then the assignment $M \mapsto \mathcal{F}(M) \otimes_{\mathbb{R}} V$, with the connection acting trivially on the V factor, yields an object of DVS which we denote $\mathcal{F}_V$. When V is finite-dimensional, this amounts to a finite direct sum of $\mathcal{F}$.

3.2.1. Monoidal structures on DVS

To discuss prefactorization algebras with values in DVS, we must specify a symmetric monoidal structure, which is used in defining the structure maps (11). Certain subtleties arise on this point, typical of the issues one encounters when trying to define tensor products of infinite-dimensional topological vector spaces. We restrict ourselves to a few brief remarks, and refer the interested reader to appendices B and C of [4] for details.

- Given $\mathcal{F}, \mathcal{G} \in$ DVS, we can define $\mathcal{F} \otimes \mathcal{G}$ as the sheafification of the presheaf $X \mapsto \mathcal{F}(X) \otimes_{C^\infty(X)} \mathcal{G}(X)$, equipped with the flat connection $\nabla^{\mathcal{F}} \otimes Id + Id \otimes \nabla^{\mathcal{G}}$. When $\mathcal{F} = C^\infty(M)$, $\mathcal{G} = C^\infty(N)$, and $X = pt$ is a point, this yields $\mathcal{F} \otimes \mathcal{G}(pt) = C^\infty(M) \otimes_{\mathbb{R}} C^\infty(N)$. We call this symmetric monoidal structure the *naive tensor product* in DVS.
- The naive tensor product has certain shortcomings. Most importantly, it does not represent the space of multilinear maps (13). In order to remedy this situation, a certain completed tensor product $\hat{\otimes}_\beta$ has to be introduced. This operation is only defined on a certain sub-category of DVS however. In the last example, we would

obtain $\mathcal{F}\hat{\otimes}_\beta\mathcal{G}(pt) = C^\infty(M\times N)$. We will refer to this operation as the *completed tensor product*.

Using $\hat{\otimes}_\beta$ rather than $\otimes$ is important if one wishes to obtain a factorization, rather than merely a prefactorization algebra. As we work with prefactorization algebras in this paper, the naive tensor product is adequate, and will be the symmetric monoidal structure on DVS throughout.

3.3. Translation-invariant prefactorization algebras

Our construction in Section 4, when applied to the trivial fibration $E = F\times\mathbb{C}^n \to \mathbb{C}^n$, produces a prefactorization algebra which is *holomorphically translation-invariant*. This property will be used when extracting a vertex algebra in Section 5 in the case $n = 1$. We proceed to briefly review this notion and refer the interested reader to Sections 4.8 and 5.2 of [4] for details.

3.3.1. Discrete translation-invariance

Suppose now that $\mathcal{F}$ is prefactorization algebra on $\mathbb{C}^n$ in the category of complex vector spaces. $\mathbb{C}^n$ acts on itself by translations. For an open subset $U\subset\mathbb{C}^n$ and $x\in\mathbb{C}^n$, let

$$\tau_x U := \{y\in\mathbb{C}^n \,|\, y - x\in U\}$$

Clearly, $\tau_x(\tau_y U) = \tau_{x+y}U$. We say that $\mathcal{F}$ is *discretely translation-invariant* if we are given isomorphisms

$$\varphi_x : \mathcal{F}(U)\to\mathcal{F}(\tau_x U) \tag{14}$$

for each $x\in\mathbb{C}^n$ compatible with composition and the structure maps of $\mathcal{F}$. We refer to section 4.8 of [4] for details.

Example 3.12. For any Lie algebra $\mathfrak{g}$, $\mathbf{U}(\mathfrak{g}\otimes\Omega^{0,*}_{\mathbb{C}^n})$ in Example 3.6 is discretely translation-invariant.

3.3.2. Smooth and holomorphic translation-invariance

A refined version of translation-invariance expresses the fact that the maps φ_x, and hence the structure maps $m_V^{U_1,\cdots,U_n}$ depend smoothly/holomorphically on the positions of the open sets U_i. This notion is operadic in flavor.

For $z\in\mathbb{C}^n$ and $r > 0$ let $\mathrm{PD}(z, r)$ denote the polydisk

$$\mathrm{PD}_r(z) = \{w\in\mathbb{C}^n \,||\, w_i - z_i| < r, 1\le i\le n\}$$

and let

$$\mathrm{PD}(r_1, \cdots, r_k|s) \subset (\mathbb{C}^n)^k$$

denote the open subset $(z_1, \cdots, z_k) \in (\mathbb{C}^n)^k$ such that the polydisks $\mathrm{PD}_{r_i}(z_i)$ have disjoint closures and are all contained in $\mathrm{PD}_s(0)$. The collections $\mathrm{PD}(r_1, \cdots, r_k|s)$ form an $\mathbb{R}_{>0}$–colored operad in the category of complex manifolds under insertions of polydisks.

Suppose that $\mathcal{F}$ is a discretely translation-invariant prefactorization algebra on $\mathbb{C}^n$ with values in DVS. We may then identify $\mathcal{F}(\mathrm{PD}_r(z)) \cong \mathcal{F}(\mathrm{PD}_r(z'))$ for any two $z, z' \in \mathbb{C}^n$ using the isomorphisms (14), and denote the corresponding complex simply by $\mathcal{F}_r$. For each $p \in \mathrm{PD}(r_1, \cdots, r_k|s)$, we have a multilinear map

$$m[p] : \mathcal{F}_{r_1} \times \cdots \times \mathcal{F}_{r_k} \to \mathcal{F}_s \tag{15}$$

As explained in Section 3.2, we say that $m[p]$ *depends smoothly on* p if

$$m \in \mathrm{DVS}(\mathcal{F}_{r_1}, \cdots, \mathcal{F}_{r_k}, \mathbf{C}^{\infty}(\mathrm{PD}(r_1, \cdots, r_k|s), \mathcal{F}_s)).$$

To formulate the definition of smooth translation-invariance, we will need the notion of a derivation of a prefactorization algebra.

Definition 3.13 *([4], Definition 4.8.2).* A *degree k derivation* of a prefactorization algebra $\mathcal{F}$ is a collection of maps $D_U : \mathcal{F}(U) \to \mathcal{F}(U)$ of cohomological degree k for each open subset $U \subset M$, with the property that for any finite collection $U_1, \cdots, U_n \subset V$ of disjoint opens and elements $\alpha_i \in \mathcal{F}(U_i)$, the following version of the Leibniz rule holds

$$\begin{aligned} &D_V m_V^{U_1,\cdots,U_n}(\alpha_1, \cdots, \alpha_n) \\ &= \sum_{i=1}^{n} (-1)^{k(|\alpha_1|+\cdots+|\alpha_{i-1}|)} m_V^{U_1,\cdots,U_n}(\alpha_1, \cdots, \alpha_{i-1}, D_{U_i}\alpha_i, \cdots, \alpha_n) \end{aligned}$$

The derivations of $\mathcal{F}$ form a DGLA, with bracket $[D, D']_U = [D_U, D'_U]$ and differential d given by $dD_U = [d_U, D_U]$, where d_U is the differential on $\mathcal{F}(U)$.

The notion of smoothly translation-invariant prefactorization algebra $\mathcal{F}$ on $\mathbb{C}^n$ can now be formulated as follows:

Definition 3.14 *([4], Definition 4.8.3).* A prefactorization algebra $\mathcal{F}$ on $\mathbb{C}^n$ with values in DVS is *(smoothly) translation-invariant* if:

(1) $\mathcal{F}$ is discretely translation-invariant.
(2) The maps (15) are smooth as functions of $p \in \mathrm{PD}(r_1, \cdots, r_k|s)$
(3) $\mathcal{F}$ carries an action of the complex Abelian Lie algebra $\mathbb{C}^n$ by derivations compatible with differentiating $m[p]$.

We can further refine the notion of translation-invariance to consider the holomorphic structure. We say that $\mathcal{F}$ is *holomorphically translation invariant* if

- $\mathcal{F}$ is smoothly translation invariant.
- There exist degree-1 derivations $\eta_i : \mathcal{F} \to \mathcal{F}$ such that
  - $[d, \eta_i] = \frac{\partial}{\partial \bar{z}_i}$ (as derivations of $\mathcal{F}$)
  - $[\eta_i, \eta_j] = [\eta_i, \frac{\partial}{\partial z_j}] = 0$

  for $i = 1, \cdots, n$, and where d is the differential on $\mathcal{F}$.

This condition means that anti-holomorphic vector fields act homotopically trivially on $\mathcal{F}$.

As explained in Section 5.2 of [4], if $\mathcal{F}$ is a holomorphically translation-invariant prefactorization algebra, then upon passing to cohomology, the induced structure maps

$$m[p] : H^*(\mathcal{F}_{r_1}) \times \cdots \times H^*(\mathcal{F}_{r_k}) \to H^*(\mathcal{F}_s) \tag{16}$$

are holomorphic as functions of $p \in \mathrm{PD}(r_1, \cdots r_k | s)$. In other words, m can be viewed as a map

$$m : H^*(\mathcal{F}_{r_1}) \times \cdots \times H^*(\mathcal{F}_{r_k}) \to \mathrm{Hol}(\mathrm{PD}(r_1, \cdots r_k | s), H^*(\mathcal{F}_s)) \tag{17}$$

where Hol denotes the space of holomorphic maps in DVS.

Example 3.15. For any Lie algebra $\mathfrak{g}$, the prefactorization enveloping algebra $\mathbf{U}(\mathfrak{g} \otimes \Omega^{0,*}_{\mathbb{C}^n})$ of Example 3.6 is holomorphically translation-invariant. The action by translation invariant vector fields is simply by Lie derivative. The homotopy for anti-holomorphic translations is $\eta_i = \iota_{\partial/\partial \bar{z}_i}$ the contraction by the anti-holomorphic vector field $\partial/\partial \bar{z}_i$. The fact that this is a homotopy as above follows from Cartan's formula for operators on the Dolbeault complex

$$[\bar{\partial}, \iota_{\partial/\partial \bar{z}_i}] = L_{\partial/\partial \bar{z}_i}.$$

4. Prefactorization algebras from holomorphic fibrations

In this section, we describe our main construction of prefactorization algebras from locally trivial holomorphic fibrations.

Our starting point is the following data:

- Complex manifolds F, X.
- $(\mathfrak{g}, \langle , \rangle)$ a Lie algebra with an invariant bilinear form.
- A locally trivial holomorphic fibration $\pi : E \to X$ with fiber F.

From this data we will construct a sheaf of L_∞ algebras on the total space E of the fibration. In turn, we obtain a prefactorization algebra on E upon taking its prefactorization enveloping algebra.

4.1. A sheaf of Lie algebras

Let $\Omega_E^{0,*}$ be the sheaf of dg vector spaces given by the Dolbeault complex of the complex manifold E, equipped with the $\overline{\partial}$ operator. For any Lie algebra $\mathfrak{g}$, we can define the sheaf of dg Lie algebras

$$\mathfrak{g}_E = \mathfrak{g} \otimes \Omega_E^{0,*}$$

which on an open set $U \subset E$ assigns $\mathfrak{g} \otimes \Omega_E^{0,*}(U)$. The differential is again given by the $\overline{\partial}$ operator, and the Lie bracket is defined by

$$[X \otimes \alpha, Y \otimes \beta] = [X, Y] \otimes (\alpha \wedge \beta)$$

where $X, Y \in \mathfrak{g}$, $\alpha, \beta \in \Omega^{0,*}(U)$.

Let K_E be the bigraded complex of sheaves

$$\mathbb{C}[2] \to \Omega_E^{0,*}[1] \xrightarrow{\partial} \Omega_E^{1,*}$$

where the first arrow is an inclusion, and the second is given by the holomorphic de Rham operator ∂. We view K_E as a bigraded complex with $\overline{\partial}$ acting as vertical differential on $\Omega_E^{p,*}$, $p = 0, 1$. Let $\mathrm{K}_E = \mathrm{Tot}(K_E)$ be the totalization of this bigraded complex.

We consider the complex of sheaves K_E as a trivial dg module for the sheaf of dg Lie algebras $\mathfrak{g}_E$. We will construct a cocycle on $\mathfrak{g}_E$ with values in this trivial dg module. To this end, define the following maps of sheaves

$$\varphi^{(1)} : (\mathfrak{g}_E)^{\otimes 2} \to \Omega_E^{1,*}$$

$$\varphi^{(1)}((X \otimes \alpha) \otimes (Y \otimes \beta)) = \frac{1}{2}\langle X, Y\rangle(\alpha \wedge \partial\beta - (-1)^{|\alpha|}\partial\alpha \wedge \beta),$$

and

$$\varphi^{(0)} : (\mathfrak{g}_E)^{\otimes 3} \to \Omega_E^{0,*}[1]$$

$$\varphi^{(0)}((X \otimes \alpha) \otimes (Y \otimes \beta) \otimes (Z \otimes \gamma)) = \frac{1}{2}\langle [X, Y], Z\rangle(\alpha \wedge \beta \wedge \gamma)$$

The sum $\varphi = \varphi^{(0)} + \varphi^{(1)}$ is a cochain in the Chevalley-Eilenberg complex

$$\varphi \in C^{\bullet}(\mathfrak{g}_E, \mathrm{K}_E)$$

of total degree 2.

By an identical calculation as in Lemma 2.3 we obtain the following.

Lemma 4.1. *φ defines a cocycle in $C^*_{Lie}(\mathfrak{g}_\pi, \mathrm{K}_E)$ of total degree 2.*

The cocycle φ defines a central extension $\tilde{\mathfrak{g}}_E$ of the sheaf $\mathfrak{g}_E$ as a sheaf of L_∞ algebras which has non-vanishing ℓ'_1, ℓ'_2 and ℓ'_3. More directly, we can define the cochain complex computing the Lie algebra homology of this sheaf of L_∞ algebras.

Definition 4.2. Define the sheaf of dg vector spaces

$$C_*^{Lie}(\tilde{\mathfrak{g}}_E) := (\mathrm{Sym}(\mathfrak{g}_E[1] \oplus K_E[1]), d + d_{CE} + \varphi)$$

where

- $d = \bar{\partial} + d_{K_E}$ is the sum of the differentials on $\mathfrak{g}_E$ and K_E.
- d_{CE} is the Chevalley-Eilenberg differential of the original Lie algebra $\mathfrak{g}_E$.
- the linear map φ is extended to $\mathrm{Sym}(\mathfrak{g}_E[1] \oplus K_E[1])$ as a co-derivation.

The fact that $(d + d_{CE} + \varphi)^2 = 0$ follows from the fact that φ is a cocycle.

4.1.1. Relation to local cocycles

There is a relationship between our construction and the theory "twisted" factorization enveloping algebras given in Section 4.4 of [3] when the complex dimension $\dim_{\mathbb{C}}(E) = 1$. There, the data one uses to twist is that of a local cocycle which lives in the local cohomology of a sheaf of Lie algebras. We don't recall the precise definition, but if $\mathcal{L}$ is a sheaf of Lie algebras obtained from a bundle L, the local cohomology $C^*_{loc}(\mathcal{L})$ is a subcomplex

$$C^*_{loc}(\mathcal{L}) \subset C^*_{Lie}(\mathcal{L}_c)$$

where $\mathcal{L}_c$ denotes the cosheaf of compactly supported sections. The condition for a cochain in $C^*_{Lie}(\mathcal{L}_c)$ to be local is that it is given by integrating a "Lagrangian density". Such a Lagrangian density is a differential form valued cochain which only depends on the ∞-jet of the sections of L, that is, it is given by a product of polydifferential operators.

We have defined the cocycle φ as an element in $C^*_{Lie}(\mathfrak{g}_E, K_E)$. The complex $C^*_{Lie}(\mathfrak{g}_E, K_E)$ is neither a sheaf or a cosheaf, however, the object

$$C^*_{Lie}(\mathfrak{g}_{E,c}, K_E)$$

is a sheaf. Here, we restrict to cochains defined on compactly supported sections of $\mathfrak{g}_E$. The cocycle φ is a section of this sheaf, meaning it is compatible with the natural restriction maps.

The cocycle φ is not just any section of this sheaf. For any open $U \subset E$, it actually lies in the subcomplex

$$\varphi(U) \in C^*_{Lie}(\mathfrak{g}_{E,c}(U), K_{E,c}(U)) \subset C^*_{Lie}(\mathfrak{g}_{E,c}, K_E)(U).$$

In other words, φ preserves the condition of being compactly supported.

Now, the cosheaf $\mathcal{K}_{E,c}$ admits a natural integration map

$$\int : \mathcal{K}_{E,c} \to \underline{\mathbb{C}}[-1]$$

where $\underline{\mathbb{C}}[-1]$ is the constant cosheaf concentrated in degree +1. (Integration is only nonzero on the $\Omega_c^{1,1}$, which accounts for the shift above) Thus, for every $U \subset E$, we obtain a cocycle

$$\varphi(U) \in C^*_{Lie}(\mathfrak{g}_{E,c}(U)).$$

The cocycle φ is clearly built from differential operators, which implies that $\varphi(U)$ is actually a element of the local cochain complex

$$\varphi(U) \in C^*_{loc}(\mathfrak{g}_E)(U).$$

In conclusion, upon integration, we see that φ determines a degree one cocycle in the local cohomology of the sheaf of Lie algebras $\mathfrak{g}_E$.

In higher dimensions, there is a similar relationship to local functionals which hence determine one dimensional central extensions of Kac-Moody type algebras in any dimension. This class of cocycles is studied in detail in the context of "higher dimensional" Kac-Moody algebras in [8].

4.2. The prefactorization algebra $\mathcal{F}_{\mathfrak{g},\pi}$

We proceed to construct a prefactorization algebra on X — the base of the holomorphic fibration $\pi : E \to X$.

Let $\mathfrak{G}_\pi := \pi_*(\mathfrak{G}_E)$ - a sheaf of L_∞ algebras on X, and let $\mathfrak{G}^c_\pi$ denote the cosheaf of sections of $\mathfrak{G}_\pi$ with compact support. Explicitly, for an open subset $U \subset X$ we have

$$\begin{aligned} \mathfrak{G}^c_\pi(U) &= \Gamma_c(U, \mathfrak{G}_\pi) \\ &= \Gamma_c(U, \pi_* \mathfrak{g}_E) \oplus \Gamma_c(U, \pi_* \mathcal{K}_E). \end{aligned} \tag{18}$$

Remark 4.3. Though the assignment $U \mapsto \mathfrak{g}^c_\pi(U)$ is a cosheaf of dg vector spaces, it is just a *pre*cosheaf of L_∞ algebras on X (with L_∞ structure defined by the cocycle φ in the previous subsection). This subtle issue arises since direct sum is not the categorical coproduct in the category of Lie algebras, but it will play no essential role for us.

Definition 4.4. Define the cosheaf

$$\mathcal{F}_{\mathfrak{g},\pi} := C^{Lie}_*(\mathfrak{G}^c_\pi) \tag{19}$$

as the Chevalley-Eilenberg complex for Lie algebra homology for the precosheaf of L_∞ algebras $\mathfrak{g}^c_\pi$. For each open $U \subset X$ this cosheaf assigns the complex

$$\mathcal{F}_{\mathfrak{g},\pi}(U) = C^{Lie}_*(\mathfrak{g}^c_\pi(U)) = (\mathrm{Sym}(\Gamma_c(U, \pi_*\mathfrak{g}_E)[1] \oplus \Gamma_c(U, \pi_*K_E)[1]), d + d_{CE} + \varphi) \quad (20)$$

Remark 4.5. The cosheaf $\mathfrak{g}^c_\pi$ is equipped with a DVS structure as in Example 3.10, and therefore so is $\mathcal{F}_{\mathfrak{g},\pi}$, being constructed from algebraic tensor product.

Proposition 4.6. *$\mathcal{F}_{\mathfrak{g},\pi}$ has the structure of a prefactorization algebra on X valued in dg −DVS. When $X = \mathbb{C}^n$, $\mathcal{F}_{\mathfrak{g},\pi}$ is holomorphically translation-invariant.*

Proof. Note that if $\pi : E \to X$ is a locally trivial fibration and W is a smooth vector bundle on E, then π_*W is a fine sheaf. The smooth translation-invariance of $\mathcal{F}_{\mathfrak{g},\pi}$ is established just as in the example of the free scalar field in Section 4.8 of [4]. To see that the prefactorization algebra is holomorphically translation-invariant we choose holomorphic coordinates $\{z_i\}$ for $\mathbb{C}^n$. Then, the operators $\eta_i = \iota_{\partial/\partial \bar{z}_i}$, $i = 1, \ldots, n$, given by contraction with the vector fields $\partial/\partial \bar{z}_i$, $i = 1, \ldots, n$ are degree (-1) derivations satisfying the conditions in Definition 3.14, see Example 3.15. □

4.3. The prefactorization algebra $\mathcal{G}_{\mathfrak{g},\pi}$

In this section we discuss some prefactorization algebras closely related to $\mathcal{F}_{\mathfrak{g},\pi}$, which are both more convenient from a computational standpoint and more closely related to the class of toroidal algebras we have introduced previously. Concretely we will construct a sequence of prefactorization algebras on X

$$\mathcal{G}^{alg}_{\mathfrak{g},\pi} \to \mathcal{G}_{\mathfrak{g},\pi} \to \mathcal{F}_{\mathfrak{g},\pi} \quad (21)$$

with the property that each map above is an inclusion at the level of graded vector spaces. (Strictly speaking, the final map requires a choice of trivialization of the fibration - see Lemma 4.11.) While the definition of $\mathcal{F}_{\mathfrak{g},\pi}$ is reasonably simple, explicit calculations of $H^*(\mathcal{F}_{\mathfrak{g},\pi}(U))$ for an open subset $U \subset X$ require the $\bar{\partial}$-cohomology of the complex $\Gamma_c(U, \pi_*(\mathfrak{g}_E))$ in Equation (18). This complex involves forms with compact support along the base X and arbitrary support along the fiber F, and its $\bar{\partial}$-cohomology even when E is a trivial fibration is a certain completion of $H^{0,*}_c(U) \otimes H^{0,*}(F)$ whose explicit description involves non-trivial analytic issues, due to the failure of naive Kunneth-type theorems for Dolbeault cohomology. As a hint of the types of issues involved, we note that the space of holomorphic functions on $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$ is not simply the algebraic tensor product of holomorphic functions on the factors $\mathbb{C}$, though the latter forms a dense subspace.

The advantage of the prefactorization algebra $\mathcal{G}_{\mathfrak{g},\pi}$ is that its cohomology is simple to describe, as the Kunneth formula holds at this level. The use of the even smaller prefactorization $\mathcal{G}^{alg}_{\mathfrak{g},\pi}$ is that it is closely related to vertex algebras and the toroidal algebras we have met earlier in the paper.

Let us first suppose that $E = X \times F$ is a trivial fibration. For each p, q, p', q' we have a map of cosheaves

$$\Omega^{p,q}_{X,c} \otimes \Gamma(F, \Omega^{p',q'}_F) \to (\pi_* \Omega^{p+p',q+q'}_E)_c$$

where the subscript c denotes sections with compact support and we use the ordinary (algebraic) tensor product. Explicitly, for an open subset $U \subset X$, this map is just the wedge product

$$\begin{gathered}\Omega^{p,q}_{X,c}(U) \otimes \Gamma(F, \Omega^{p',q'}_F) \to \Gamma_c(U, \pi_* \Omega^{p+p',q+q'}_E) \\ \alpha \otimes \beta \to \alpha \wedge \beta\end{gathered}$$

It is injective provided all three factors are non-zero.

Our first definition uses this injection to split off the dependence of the fiber F in the definition of the cosheaf $\mathfrak{g}^c_\pi$.

Definition 4.7. Define a sub cosheaf $\mathfrak{g}^{\#\,c}_\pi$ of $\mathfrak{g}^c_\pi$ by

$$\mathfrak{g}^{\#\,c}_\pi := \mathfrak{g} \otimes \Omega^{0,*}_{X,c} \otimes \Gamma(F, \Omega^{0,*}_F) \oplus \mathcal{K}^\#_\pi$$

where

$$\mathcal{K}^\#_\pi := \mathrm{Tot}(\Omega^{0,*}_{X,c} \otimes \Gamma(F, \Omega^{0,*}_F)[1] \xrightarrow{\partial} \Omega^{1,*}_{X,c} \otimes \Gamma(F, \Omega^{0,*}_F) \oplus \Omega^{0,*}_{X,c} \otimes \Gamma(F, \Omega^{1,0}_F)).$$

Here, $\overline{\partial}$ acts "vertically" within each term.

Remark 4.8. The cosheaf $\mathfrak{g}^{\#\,c}_\pi$ may be equipped with a dg –DVS structure as in Example 3.11.

The L_∞ structure on $\mathfrak{g}^c_\pi$ induces one on the sub-complex $\mathfrak{g}^{\#\,c}_\pi$ (this follows from the fact that the cocycle φ restricts). The advantage of $\mathfrak{g}^{\#\,c}_\pi$ lies in the fact that it's constructed from ordinary (algebraic) tensor products of complexes whose cohomology is easy to describe.

Definition 4.9. Define the precosheaf $\mathcal{G}_{\mathfrak{g},\pi}$ on X by

$$\mathcal{G}_{\mathfrak{g},\pi} := C^{Lie}_*(\mathfrak{g}^{\#\,c}_\pi)$$

The same argument for $\mathcal{F}_{\mathfrak{g},\pi}$ in Proposition 4.6 implies that $\mathcal{G}_{\mathfrak{g},\pi}$ has the structure of a prefactorization algebra on X.

The arguments of the previous section show:

Proposition 4.10. *$\mathcal{G}_{\mathfrak{g},\pi}$ has the structure of a prefactorization algebra on X. When $X = \mathbb{C}^n$, $n \geq 1$, $\mathcal{G}_{\mathfrak{g},\pi}$ is holomorphically translation-invariant.*

When $E = X \times F$, we have the following relationship between $\mathcal{G}_{\mathfrak{g},\pi}$ and $\mathcal{F}_{\mathfrak{g},\pi}$:

Lemma 4.11. *Suppose $\pi : E = X \times F \to X$ is trivial. Then, there exists a map of prefactorization algebras on X:*

$$\mathcal{G}_{\mathfrak{g},\pi} \to \mathcal{F}_{\mathfrak{g},\pi} \tag{22}$$

Proof. We have just constructed a map of DGLA's $\mathfrak{g}^{\#\,c}_{\pi} \to \mathfrak{g}^{c}_{\pi}$, which induces a map $\mathcal{G}_{\mathfrak{g},\pi} \to \mathcal{F}_{\mathfrak{g},\pi}$ upon taking the factorization enveloping algebra. !

For a general locally trivial holomorphic fibration $\pi : E \to X$, we can construct a map of prefactorization algebras $\mathcal{G}_{\mathfrak{g},\pi} \to \mathcal{F}_{\mathfrak{g},\pi}$ locally on X. Indeed, for any $x \in X$ there exists a local trivialization of E on an open neighborhood V of x. On V, we obtain the map (22). It depends on the choice of trivialization however.

4.3.1. An algebraic variant

When the fiber F is a smooth complex affine variety and $\pi : E = X \times F \to X$ is trivial, we may further refine $\mathcal{G}_{\mathfrak{g},\pi}$ to obtain a prefactorization algebra $\mathcal{G}^{alg}_{\mathfrak{g},\pi}$ with stronger finiteness properties, by considering the algebraic rather than analytic cohomology of $\mathcal{O}_F$. This variation will be important in the next section, when we make contact with vertex algebras. Let $\mathcal{O}^{alg}_F$ denote the sheaf of algebraic regular functions on F, and $\Omega^{1,alg}$ the sheaf of Kahler differentials. We have

$$H^0(F, \mathcal{O}^{alg}_F) \subset H^0(F, \mathcal{O}_F) \hookrightarrow (\Omega^{0,*}_F, \bar{\partial})$$
$$H^0(F, \Omega^{1,alg}_F) \subset H^0(F, \Omega^1_F) \hookrightarrow (\Omega^{1,*}_F, \bar{\partial})$$

Definition 4.12. We define

$$\mathfrak{g}^{\#\,c,alg}_{\pi} := \mathfrak{g} \otimes \Omega^{0,*}_{X,c} \otimes H^0(F, \mathcal{O}^{alg}_F) \oplus \mathcal{K}^{\#\,,alg}_{\pi}$$

where

$$\mathcal{K}^{\#\,,alg}_{\pi} := \mathrm{Tot}(\Omega^{0,*}_{X,c} \otimes H^0(F, \mathcal{O}^{alg}_F)[1] \xrightarrow{\partial} \Omega^{1,*}_{X,c} \otimes H^0(F, \mathcal{O}^{alg}_F) \oplus \Omega^{0,*}_{X,c} \otimes H^0(F, \Omega^{1,alg}_F)).$$

The totalization is with respect to the horizontal ∂-operator and the vertical $\bar{\partial}$-operator acting on $\Omega^{p,*}_{X,c}$.

Remark 4.13. Again, $\mathfrak{g}^{\#\,c,alg}_{\pi}$ may be equipped with the DVS structure of Example 3.11, yielding a prefactorization algebra in dg –DVS.

We can now define the main object of study for us.

Definition 4.14. The n-dimensional *toroidal prefactorization algebra* associated with the trivial holomorphic fibration $F \to X \times F \xrightarrow{\pi} X$ is the prefactorization algebra

$$\mathcal{G}^{alg}_{\mathfrak{g},\pi} := C^{Lie}_{*}\left(\mathfrak{g}^{\#\, c,alg}_{\pi}\right)$$

with the structure maps induced from those of $\mathcal{G}_{\mathfrak{g},\pi}$.

Proposition 4.15. *Suppose that F is a smooth complex affine variety and $X = \mathbb{C}^n$. Then $\mathcal{G}^{alg}_{\mathfrak{g},\pi}$ has the structure of a holomorphically translation-invariant pre-factorization algebra valued in dg –DVS.*

The reasoning at the end of the previous section shows that for a general locally trivial fibration $\pi : E \to X$, a point $x \in X$, and a choice of trivialization of $E|_U$ on a neighborhood U of x, one obtains prefactorization algebra maps

$$\mathcal{G}^{alg}_{\mathfrak{g},\pi} \to \mathcal{G}_{\mathfrak{g},\pi} \to \mathcal{F}_{\mathfrak{g},\pi}|_U$$

Remark 4.16. We remark that in this paper we are primarily concerned with the case in which the fiber F is a smooth complex affine variety. There is another interesting case in which we take the fibers to be compact. One can still study the pushforward $\pi_*(\mathfrak{g}_E)$ as a sheaf of L_∞ algebras, and its factorization enveloping algebra. Unlike the case of affine fibers, this pushforward may have interesting higher cohomology in the fiber direction, and moreover the factorization enveloping algebra is equipped with the analytic Gauss-Manin connection. The case of a trivial fibration has been studied in Section 4.3 of [8].

5. $X = \mathbb{C}$ and vertex algebras

In [4], it is shown that prefactorization algebras on $X = \mathbb{C}$ which are holomorphically translation-invariant and S^1-equivariant for the natural action by rotations are closely related to vertex algebras. More precisely, given such a prefactorization algebra $\mathcal{F}$, the vector space

$$V(\mathcal{F}) = \bigoplus_{l} H^*(\mathcal{F}^{(l)}(\mathbb{C}))$$

equal to the direct sum of S^1-eigenspaces in the cohomology $H^*(\mathcal{F}(\mathbb{C}))$ has a vertex algebra structure. We begin by reviewing this correspondence following [4], and then apply it to the case of the one-dimensional toroidal prefactorization algebra $\mathcal{G}^{alg}_{\mathfrak{g},\pi}$, where $\pi : \mathbb{C} \times F \to \mathbb{C}$ is the trivial fibration on $\mathbb{C}$ with fiber a smooth complex affine variety F. We show that resulting vertex algebra is isomorphic to $V(\mathfrak{g}_R)$ where

$R = H^0(F, \mathcal{O}_F^{alg})[t, t^{-1}]$ from Section 2.3.2. As a special case, when $F = (\mathbb{C}^*)^k$, we recover a toroidal vertex algebra.

5.1. Prefactorization algebras on $\mathbb{C}$ and vertex algebras

We review here the correspondence between prefactorization algebras on $\mathbb{C}$ and vertex algebras established in [4], where we refer the reader for details. Recall that S^1 acts on $\mathbb{C}$ by rotations via $z \mapsto \exp(i\theta) z$. Suppose that $\mathcal{F}$ is a prefactorization algebra on $\mathbb{C}$ that is holomorphically translation-invariant and S^1-equivariant. Let $\mathcal{F}(r) := \mathcal{F}(D(0, r))$ be the complex assigned by $\mathcal{F}$ to a disk of radius r (we allow here $r = \infty$, in which case $D(0, \infty) = \mathbb{C}$), and $\mathcal{F}^{(l)}(r) \subset \mathcal{F}(r)$ be the lth eigenspace for the S^1-action. The following theorem from [4] establishes a bridge between prefactorization and vertex algebras:

Theorem 5.1 *(Theorem 5.3.3 [4]). Let $\mathcal{F}$ be a unital S^1-equivariant holomorphically translation invariant prefactorization algebra on $\mathbb{C}$. Suppose*

- *The action of S^1 on $\mathcal{F}(r)$ extends smoothly to an action of the algebra of distributions on S^1.*
- *For $r < r'$ the map*

$$\mathcal{F}^{(l)}(r) \to \mathcal{F}^{(l)}(r')$$

  is a quasi-isomorphism.
- *The cohomology $H^*(\mathcal{F}^{(l)}(r))$ vanishes for $l \gg 0$.*
- *For each l and $r > 0$ we require that $H^*(\mathcal{F}^{(l)}(r))$ is isomorphic to a countable sequential colimit of finite dimensional vector spaces.*

Then $V(\mathcal{F}) := \bigoplus_{l \in \mathbb{Z}} H^*(\mathcal{F}^{(l)}(r))$ (which is independent of r by assumption) has the structure of a vertex algebra.

We briefly sketch how the vertex algebra structure on $V(\mathcal{F})$ can be extracted from the prefactorization structure on $\mathcal{F}$.

- Using the notation of Section 3.3.2, polydisks in one dimension are simply disks, and we denote $\mathrm{PD}(r_1, \cdots r_k | s)$ by $\mathrm{Discs}(r_1, \cdots r_k | s)$. If $r_i' < r_i$, we obtain an inclusion

$$\mathrm{Discs}(r_1, \cdots r_k | s) \subset \mathrm{Discs}(r_1', \cdots r_k' | s) \tag{23}$$

  In the limit $\lim_{r_i \to 0}$, these spaces approach Conf_k, the configuration space of k distinct points in $\mathbb{C}$.
- The structure maps (17) are compatible with the maps $\mathcal{F}(r_i') \to \mathcal{F}(r_i)$ and the inclusions (23), and one may take $\lim_{r_i \to 0}$, $s = \infty$, obtaining maps

$$m : (\lim_{r\to 0} H^*(\mathcal{F}(r)))^{\otimes k} \to \mathrm{Hol}(\mathrm{Conf}_k, H^*(\mathcal{F}(\mathbb{C}))) \tag{24}$$

- We set $k = 2$, and fix one of the points to be the origin. There is a natural map $V(\mathcal{F}) \hookrightarrow \lim_{r\to 0} H^*(\mathcal{F}(r))$, as well as projections $H^*(\mathcal{F}(\mathbb{C})) \to H^*(\mathcal{F}^{(l)}(\mathbb{C}))$. Pre and post-composing by these in (24), yields a map

$$\overline{m_{0,z}} : V(\mathcal{F}) \otimes V(\mathcal{F}) \to \prod_l^0 \mathrm{Hol}(\mathbb{C}^\times, V(\mathcal{F})_l) \tag{25}$$

where $V(\mathcal{F})_l = H^*(\mathcal{F}^{(l)}(\mathbb{C}))$. Laurent expanding $\overline{m_{0,z}}$ we obtain

$$\overline{m_{0,z}} : V(\mathcal{F}) \otimes V(\mathcal{F}) \to \prod_l^0 V(\mathcal{F})_l[[z, z^{-1}]]$$

whose image can be shown to lie in $V(\mathcal{F})((z))$. The vertex operator can now be defined by

$$Y : V(\mathcal{F}) \to \mathrm{End}(V(\mathcal{F}))[[z, z^{-1}]]$$
$$Y(v, z)v' = m_{0,z}(v', v)$$

- Holomorphic translation invariance yields an action of ∂_z

$$\partial_z : \mathcal{F}^{(l)}(r) \to \mathcal{F}^{(l-1)}(r).$$

which descends to $H^*(\mathcal{F}^l(r))$. This induces the translation operator $T : V(\mathcal{F}) \to V(\mathcal{F})$.
- The vacuum vector is obtained from the unit in $\mathcal{F}(\emptyset)$.

5.2. The main theorem

Our goal in this section is to prove the following theorem

Theorem 5.2. *Let F be a smooth complex affine variety, and $\pi : \mathbb{C} \times F \to \mathbb{C}$ the trivial fibration with fiber F. Then*

(1) The toroidal prefactorization algebra $\mathcal{G}^{alg}_{\mathfrak{g},\pi}$ satisfies the hypotheses of Theorem 5.1.
(2) The vertex algebra $V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$ is isomorphic to the toroidal vertex algebra $V(\mathfrak{g}_R)$, with $R = H^0(F, \mathcal{O}^{alg}_F)[t, t^{-1}]$ defined in Section 2.3.2.

Throughout this section, R will denote the algebra $H^0(F, \mathcal{O}^{alg}_F)[t, t^{-1}]$. We will denote $H^0(F, \mathcal{O}^{alg}_F)$ simply by $\mathbb{C}[F]$, so $R = \mathbb{C}[F][t, t^{-1}]$. Recall that

$$\mathfrak{g}_R = \mathfrak{g} \otimes \mathbb{C}[F][t, t^{-1}] \oplus \Omega^1_{\mathbb{C}[F][t,t^{-1}]}/d(\mathbb{C}[F][t, t^{-1}])$$
$$= \mathfrak{g} \otimes \mathbb{C}[F][t, t^{-1}] \oplus \frac{\mathbb{C}[t, t^{-1}] \otimes \Omega^1_{\mathbb{C}[F]} \oplus \mathbb{C}[F] \otimes \Omega^1_{\mathbb{C}[t,t^{-1}]}}{\langle t^k du + kt^{k-1} u dt \rangle}$$

5.2.1. Recollections on Dolbeault cohomology

In this section we recall some facts regarding ordinary and compactly supported Dolbeault cohomology and apply these to compute $H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(U))$ over opens $U \subset \mathbb{C}$. These results will be used in proving Theorem 5.2.

Stein manifolds are complex analytic analogues of smooth affine varieties over $\mathbb{C}$ [18]. In particular, $\mathbb{C}^n$ and smooth affine complex varieties are Stein. In addition, all open subsets $U \subset \mathbb{C}$ are Stein. We recall the following classic result pertaining to Stein manifolds:

Theorem 5.3 *(Cartan's Theorem B). Let X be a Stein manifold. Then*

$$H^k(\Omega^{p,*}(X), \bar{\partial}) = \begin{cases} 0 & k \neq 0 \\ \Omega^p_X & k = 0 \end{cases}$$

where Ω^p_X denotes the space of holomorphic p-forms on Z.

Remark 5.4. When $n > 1$ the open subset $\mathbb{C}^n \setminus 0 \subset \mathbb{C}^n$ is *not* Stein, since it has higher cohomology.

On a complex manifold X of dimension n, Serre duality implies that there is a non-degenerate pairing between ordinary and compactly supported forms

$$\Omega^{p,q}_{X,c} \otimes \Omega^{n-p,n-q}_X \to \mathbb{C}$$
$$\alpha \otimes \beta \mapsto \int_X \alpha \wedge \beta.$$

Thus, compactly supported differential forms yield continuous linear functionals on differential forms. At the level Dolbeault cohomology, one obtains the following corollary to Theorem 5.3 noted by Serre ([17]):

Corollary 5.5. *Let X be a Stein manifold. Then*

$$H^k(\Omega^{p,*}_c(X), \bar{\partial}) = \begin{cases} 0 & k \neq dim(X) \\ (\Omega^{n-p}_X(X))^\vee & k = n = dim(X) \end{cases}$$

where $(\Omega^{n-p}_X(X))^\vee$ denotes the continuous dual to the space of holomorphic $n-p$ forms with respect to the Frechet topology.

We now specialize to our situation, where $\pi : \mathbb{C} \times F \to \mathbb{C}$ is the trivial fibration with F a smooth complex affine variety. The cosheaf $\mathfrak{g}_\pi^{\#\, c,alg}$ on $\mathbb{C}$ defined in Section 4.3.1 has the form

$$\mathfrak{g}_\pi^{\#\, c,alg} := \mathfrak{g} \otimes \mathbb{C}[F] \otimes \Omega^{0,*}_{X,c} \oplus \mathcal{K}_\pi^{\#,alg}$$

where $\mathcal{K}_\pi^{\#,alg}$ is the total complex of the following double complex

$$\begin{array}{ccc} \Omega_c^{0,1} \otimes \mathbb{C}[F] & \xrightarrow{\partial + d} & \Omega_c^{1,1} \otimes \mathbb{C}[F] \oplus \Omega_c^{0,1} \otimes \Omega^1_{\mathbb{C}[F]} \\ \uparrow \bar{\partial} & & \bar{\partial} \uparrow \\ \Omega_c^{0,0} \otimes \mathbb{C}[F] & \xrightarrow{\partial + d} & \Omega_c^{1,0} \otimes \mathbb{C}[F] \oplus \Omega_c^{0,0} \otimes \Omega^1_{\mathbb{C}[F]}. \end{array}$$

Here, $\Omega_c^{p,q}$ denotes the cosheaf of compactly supported forms on $\mathbb{C}$, and $\Omega^1_{\mathbb{C}[F]}$ is the space of *algebraic* 1-forms (i.e. Kahler differentials) on F. Recall, from this cosheaf we have defined the factorization algebra

$$\mathcal{G}^{alg}_{\mathfrak{g},\pi} = C_*^{Lie}(\mathfrak{g}_\pi^{\#\, c,alg}) = \mathrm{Sym}(\mathfrak{g}_\pi^{\#\, c,alg}[1], d')$$

where the differential d' may be decomposed as $d' = d_1 + d_2 + d_3$, with

$$d_i : \mathrm{Sym}^i(\mathfrak{g}_\pi^{\#\, c,alg}[1]) \to \mathfrak{g}_\pi^{\#\, c,alg}[1]$$

of cohomological degree 1 defined by:

- $d_1 = \bar{\partial} + d_{\mathcal{K}_\pi^{\#,alg}}$, the linear differential operators defining the underlying cochain complexes of Dolbeault forms;
- $d_2 = d_{CE,\mathfrak{g}}$, the Chevalley-Eilenberg differential induced from the Lie bracket on $\mathfrak{g}$;
- $d_3 = \varphi$, where φ is the cocycle defined in Section 4.1, which extends to $C_*^{Lie}(-)$ by the rule that it is a coderivation.

The complex $C_*^{Lie}(\mathfrak{g}_\pi^{\#\, c,alg})$ has an increasing filtration by symmetric degree, leading to a spectral sequence whose E_1 page is

$$H^*(\mathrm{Sym}(\mathfrak{g}_\pi^{\#\, c,alg}[1], d_1)) = \mathrm{Sym}(H^*(\mathfrak{g}_\pi^{\#\, c,alg}[1], d_1))$$

Now $(\mathfrak{g}_\pi^{\#\, c,alg}, d_1)$ is the direct sum of the complexes $(\mathfrak{g} \otimes \mathbb{C}[F] \otimes \Omega_c^{0,*}, \bar{\partial})$ and $\mathcal{K}_\pi^{\#,alg}$. Applying Theorem 5.3 and Corollary 5.5 on an open Stein subset $U \subset \mathbb{C}$, the cohomology of the first is

$$\mathfrak{g} \otimes \mathbb{C}[F] \otimes (\Omega^1(U))^\vee[-1].$$

Similarly, by first computing the $\bar{\partial}$ cohomology in $\mathcal{K}_\pi^{\#,alg}$, we have

$$H^*(\mathcal{K}^{\#,alg}_{\pi}(U)) = \mathrm{Coker}\Big((\Omega^1_X(U))^\vee \otimes \mathbb{C}[F]) \xrightarrow{1\otimes\partial + \partial^\vee\otimes 1} (\Omega^1_X(U))^\vee \otimes \Omega^1_{\mathbb{C}[F]} \oplus (\mathcal{O}(U))^\vee \otimes \mathbb{C}[F]\Big]$$

where $\partial^\vee$ denotes the transpose of $\partial : \Omega^{n-1}_X(U) \to \Omega^n_X(U)$. We have the following:

Lemma 5.6. *Let $U \subset \mathbb{C}$ be an open subset. Then*

$$H^*(\mathfrak{y}^{\#\, c,alg}_{\pi}(U),\ d) = \mathfrak{g} \otimes (\Omega^1_X(U))^\vee \otimes \mathbb{C}[F] \Big[\mathrm{Coker}\Big((\Omega^1_X(U))^\vee \otimes \mathbb{C}[F]) \xrightarrow{1\otimes\partial + \partial^\vee\otimes 1} (\Omega^1_X(U))^\vee \otimes \Omega^1_{\mathbb{C}[F]} \oplus (\mathcal{O}(U))^\vee \otimes \mathbb{C}[F]\Big]$$

where $\partial^\vee$ denotes the transpose of $\partial : \mathcal{O}_X(U) \to \Omega^1_X(U)$.

It follows that $H^*(\mathfrak{y}^{\#\, c,alg}_{\pi}[1], d_1)$ is concentrated in cohomological degree 0. Since d' has cohomological degree 1, this means that the spectral sequence computing $H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(U))$ collapses at E_1, and we have an isomorphism of vector spaces

$$H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(U)) \cong \mathrm{Sym}(H^*(\mathfrak{y}^{\#\, c,alg}_{\pi}(U)[1],\ d)) \tag{26}$$

Next, we note that the value of $\mathcal{G}^{alg}_{\mathfrak{g},\pi}$ on a disk $U = D$ has a natural S^1-action induced by rotations. Rotations act by holomorphic diffeomorphisms and so extend to an action on Dolbeault forms, and hence to the factorization algebra $\mathcal{G}^{alg}_{\mathfrak{g},\pi}$. This action clearly preserves the first term in the differential d_1. Furthermore, S^1 acts by the identity on the Lie algebra $\mathfrak{g}$, so its action preserves the differential d_2 as well. Finally, to see that the S^1 action preserves the differential d_3 note that the cocycle φ is defined in terms of diffeomorphism invariant operators.

Applied to a disk $U = D$, the sum of the S^1-eigenvalues of the cohomology of $\mathcal{G}^{alg}_{\mathfrak{g},\pi}$ naturally embeds into the left hand side of Equation (26). We now characterize this cohomology.

Lemma 5.7. *There is an isomorphism of vector spaces*

$$V(\mathcal{G}^{alg}_{\mathfrak{g},\pi}) \cong \mathrm{Sym}(\mathfrak{y}_S/\mathfrak{y}^+_S) \cong U(\mathfrak{y}_S) \otimes_{U(\mathfrak{y}^+_S)} \mathbb{C} \tag{27}$$

where $S = \mathbb{C}[F][z, z^{-1}]$.

Proof. We introduce the vector spaces

$$\begin{aligned}
S^+ &= \mathbb{C}[F][z] \\
S^- &= \mathbb{C}[F] \otimes z^{-1}\mathbb{C}[z^{-1}] \\
\Omega^1_{S^+} &= \Omega^1_{\mathbb{C}[F]} \otimes \mathbb{C}[z] \oplus \mathbb{C}[z]dz \otimes \mathbb{C}[F] \\
\Omega^1_{S^-} &= \Omega^1_{\mathbb{C}[F]} \otimes z^{-1}\mathbb{C}[z^{-1}] \oplus z^{-1}\mathbb{C}[z^{-1}]dz \otimes \mathbb{C}[F]
\end{aligned}$$

We have $S = S^+ \oplus S^-$ and $\Omega^1_S = \Omega^1_{S^+} \oplus \Omega^1_{S^-}$ as vector spaces, and these decompositions are moreover compatible with the differential, in the sense that $d(S^\pm) \in \Omega^1_{S^\pm}$. Hence

$$\Omega^1_S/dS \cong \Omega^1_{S^+}/dS^+ \oplus \Omega^1_{S^-}/dS^-$$

which implies that as vector spaces

$$\hat{\mathfrak{g}}_S/\hat{\mathfrak{g}}^+_S \cong \mathfrak{g} \otimes S^- \oplus \Omega^1_{S^-}/dS^-$$

Now, the residue identifies $z^{-1}\mathbb{C}[z^{-1}]$ with a subspace of $(\Omega^1(D))^\vee$. With respect to this embedding, the weight l-eigenspace of the S^1 action on this space is $\mathbb{C}\{z^l\}$. Similarly, the residue identifies $z^{-1}\mathbb{C}[z^{-1}]dz$ with a subspace of $(\Omega^0(D))^\vee$. The weight l-eigenspace of the S^1 action is $\mathbb{C}\{z^{l-1}dz\}$.

Using this, we can read of the cohomology of the corresponding eigenspaces as follows. For $l < 0$ we have

$$H^*(\hat{\mathfrak{g}}^{\#\,c,alg}_{\pi}(D), d_1)^{(l)} \cong \left[\mathfrak{g} \otimes \mathbb{C}[F] \otimes \{z^l\} \oplus \Omega^1_{\mathbb{C}[F]} \otimes \{z^l\} \oplus \mathbb{C}[F] \otimes \{z^{l-1}dz\}\right] / \mathrm{im}(d').$$

For $l = 0$ we have

$$H^*(\hat{\mathfrak{g}}^{\#\,c,alg}_{\pi}(D), d_1)^{(0)} \cong \left(\mathbb{C}[F] \otimes \{z^{-1}dz\}\right) / \mathrm{im}(d').$$

When $l > 0$ the cohomology $H^*(\hat{\mathfrak{g}}^{\#\,c,alg}_{\pi}(D), d_1)^{(l)}$ vanishes since $(\Omega^p(D))^\vee$ has non-positive S^1 spectrum for $p = 0, 1$ for any disk D.

Therefore, as vector spaces

$$\begin{aligned} V(\mathcal{G}^{alg}_{\mathfrak{g},\pi}) &= \mathrm{Sym}\left[\bigoplus_{l \in \mathbb{Z}} H^*(\hat{\mathfrak{g}}^{\#\,c,alg}_{\pi}(D), d_1)^{(l)}\right] \\ &= \mathrm{Sym}(\mathfrak{g} \otimes S^- \oplus \Omega^1_{S^-}/dS^-) = \mathrm{Sym}(\hat{\mathfrak{g}}_S/\hat{\mathfrak{g}}^+_S) \qquad \blacksquare \end{aligned}$$

5.2.2. Verifying the hypotheses of Theorem 5.1

We proceed to verify the hypotheses of Theorem 5.1, establishing part (1) of Theorem 5.2 above.

- The first hypothesis is verified as in Section 5.3.1 of [4].
- The second and third hypotheses follow from Lemma 5.7, from which it follows in particular that $H^*((\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(0,r)))^{(l)})$ is non-zero only if $l \leq 0$.
- The last hypothesis requires some attention. By Lemma 5.7 $H^*((\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(0,r)))^{(l)})$ may be identified with the elements of weight l in

$$\mathrm{Sym}\left(\mathfrak{g} \otimes S^- \oplus \Omega^1_{S^-}/dS^-\right).$$

We begin by showing that $\mathbb{C}[F]$ and $\Omega^1_{\mathbb{C}[F]}$ are naturally a sequential colimit of finite-dimensional vector spaces. This can be done as follows. Embed $F \subset \mathbb{A}^N = \operatorname{Spec} \mathbb{C}[x_1, \cdots x_N]$. This induces an increasing filtration $F^k\mathbb{C}[F]$, $k \geq 0$, where $F^k\mathbb{C}[F]$ is spanned by the images of polynomials of degree $\leq k$ in $x_1, \cdots x_N$. $\mathbb{C}[F]$ and by the same reasoning $\Omega^1_{\mathbb{C}[F]}$ can therefore be expressed as a countable union of finite-dimensional vector spaces. This induces a filtration on $\mathfrak{g}^{\#\, c,alg}_{\pi}$ compatible with the DVS structure, which in turn induces one on $H^*((\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(0,r)))^{(l)})$.

5.2.3. Constructing the isomorphism

We proceed to prove part (2) of Theorem 5.2. The proof is a variation on the approach taken in [19] with respect to the Virasoro factorization algebra, and involves three main steps:

(1) Showing that $V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$ has the structure of a $\mathfrak{g}_R$-module.
(2) Showing that $V(\mathcal{G}^{alg}_{\mathfrak{g},\pi}) \cong V(\mathfrak{g}_R)$ as $\mathfrak{g}_R$-modules.
(3) Checking that the vertex algebra structures agree by using the reconstruction Theorem 2.6.

Let $\rho : \mathbb{C}^\times \to \mathbb{R}_{>0}$ be the map $\rho(z) = z\bar{z} = |z|^2$. The universal enveloping algebra $U(\mathfrak{g}_R)$ defines a locally constant prefactorization algebra on $\mathbb{R}_{>0}$ which we denote $\mathcal{A}U(\mathfrak{g}_R)$.

Lemma 5.8. *There is a homomorphism* $\Phi : \mathcal{A}U(\mathfrak{g}_R) \to \rho_* H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$ *of prefactorization algebras on* $\mathbb{R}_{>0}$.

Proof. • It is shown in Section 3.2 of [4] that a map of prefactorization algebras on $\mathbb{R}_{>0}$ is determined by the maps Φ_I on connected open intervals. For each open interval $I \subset \mathbb{R}_{>0}$, $A_I = \rho^{-1}(I)$ is an annulus. We choose for each such a bump function $f_I : A_I \to \mathbb{R}$ having the properties

– f_I is a function of $r^2 = z\bar{z}$ only.
– $f_I \geq 0$ and f is supported in A_I.
– $\int_A f_I \, dz d\bar{z} = 1$.

The map Φ_I is uniquely determined by where it sends the generators of $\mathfrak{g}_R$. We define Φ_I on these linear generators by the assignments:

$$\Phi_I(J \otimes ut^k) = -[J \otimes uz^{k+1} f_I \, d\bar{z}]$$
$$\Phi_I(t^k\omega) = [z^{k+1} f_I \, \omega \wedge d\bar{z}]$$
$$\Phi_I(t^k u dt) = [uz^{k+1} f_I \, dz d\bar{z}]$$

where $J \in \mathfrak{g}$, $u \in \mathbb{C}[F]$, $\omega \in \Omega^1_{\mathbb{C}[F]}$, and $[-] \in H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(A_I))$ denotes the $\bar{\partial}$-cohomology class of the closed differential form. The elements on the right are clearly closed for the differential d', and the corresponding d'-cohomology classes are easily seen to be independent of the choice of the function f_I.

- We first check that Φ_I is well-defined, which amounts to verifying that

$$\Phi_I(d(ut^k)) = \Phi_I(t^k du + kt^{k-1}udt) = 0 \in H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(A_I))$$

for each $u \in \mathbb{C}[F]$, $k \in \mathbb{Z}$. We have

$$\Phi_I(t^k du + kt^{k-1}udt) = -\left([z^{k+1} f_I d\bar{z}du] + [kz^k uf_I dzd\bar{z}]\right).$$

Notice that

$$\begin{aligned}
&\Phi_I(t^k du + kt^{k-1}udt) + [d_{\mathcal{K}^{\#}_{\pi},alg}(z^{k+1} uf_I d\bar{z})] \\
&= -[z^{k+1} f_I dud\bar{z}] - [kz^k uf_I dzd\bar{z}] \\
&\quad + [z^{k+1} f_I dud\bar{z} + (k+1)z^k uf_I dzd\bar{z} + uz^{k+1}\partial f_I \wedge d\bar{z}] \\
&= +[u(z^k f_I dzd\bar{z} + z^{k+1}\partial f_I \wedge d\bar{z})]
\end{aligned}$$

It therefore suffices to show that $[z^k f_I dzd\bar{z} + z^{k+1}\partial f_I \wedge d\bar{z}] = 0 \in (\mathcal{O}_X(A_I))^\vee$, or equivalently, that

$$\int_{A_I} z^m \wedge (z^k f_I dzd\bar{z} + z^{k+1}\partial f_I \wedge d\bar{z}) = 0 \quad \forall m \in \mathbb{Z}$$

This follows from the identities

$$\int_{A_I} z^a f_I dzd\bar{z} = \delta_{a,0}, \quad \int_U z^b \partial f_I \wedge d\bar{z} = -\delta_{b,1} \tag{28}$$

for $a, b \in \mathbb{Z}$ that we obtain via integration by parts.

- Consider three disjoint open intervals $I_1, I_2, I_3 \subset \mathbb{R}_{>0}$, such that I_{i+1} is located to the right of I_i, all contained in a larger interval I. Their inverse images under ρ correspond to three nested annuli A_{I_i} inside a larger annulus A_I. We have structure maps

$$\bullet_{i,i+1} : \rho_* H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi})(I_i) \otimes \rho_* H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi})(I_{i+1}) \to \rho_* H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi})(I) \quad i = 1, 2$$

To show that Φ is a prefactorization algebra homomorphism, we have to check that for $X, Y \in \mathfrak{g}_R$,

$$\Phi_{I_1}(X) \bullet_{1,2} \Phi_{I_2}(Y) - \Phi_{I_2}(Y) \bullet_{2,3} \Phi_{I_3}(X) = \Phi_I([X, Y])$$

Let

$$F_m(z,\bar z) = z^m \int_0^{z\bar z} (f_{I_1}(s) - f_{I_3}(s))\, ds$$

Then on A_{I_2}, $F_m = z^m$, and moreover,

$$\begin{aligned}
\bar\partial F_m(z,\bar z) &= z^m \bar\partial\left(\int_0^{z\bar z} (f_{I_1}(s) - f_{I_3}(s))\, ds\right) \\
&= z^m \frac{\partial(z\bar z)}{\partial \bar z} \frac{\partial}{\partial(z\bar z)}\left(\int_0^{z\bar z} (f_{I_1}(s) - f_{I_3}(s))\, ds\right) d\bar z \\
&= z^{m+1} (f_{I_1}(z\bar z) - f_{I_3}(z\bar z))\, d\bar z
\end{aligned}$$

Let $J_1, J_2 \in \mathfrak{g}$, $u, v \in \mathbb{C}[F]$. Then

$$\begin{aligned}
&\Phi_{I_1}(J_1 ut^k) \bullet_{1,2} \Phi_{I_2}(J_2 vt^l) - \Phi_{I_2}(J_2 vt^l) \bullet_{2,3} \Phi_{I_3}(J_1 ut^k) - \Phi_{I_2}([J_1 ut^k, J_2 vt^l]) \\
&\quad = \Phi_{I_1}(J_1 ut^k) \bullet_{1,2} \Phi_{I_2}(J_2 vt^l) - \Phi_{I_2}(J_2 vt^l) \bullet_{2,3} \Phi_{I_3}(J_1 ut^k) \\
&\qquad - \Phi_{I_2}\left([J_1, J_2] uvt^{k+l} + \frac{1}{2}\langle J_1, J_2\rangle (ut^k d(vt^l) - vt^l d(ut^k))\right) \\
&\quad = \left([J_1 uz^{k+1} f_{I_1} d\bar z] \cdot [J_2 vz^{l+1} f_{I_2} d\bar z] - [J_2 vz^{l+1} f_{I_2} d\bar z] \cdot [J_1 uz^{k+1} f_{I_3} d\bar z]\right) + \\
&\qquad + [[J_1, J_2] uvz^{k+l+1} f_{I_2} d\bar z] + \frac{1}{2}\langle J_1, J_2\rangle [z^{k+l+1} f_{I_2} (udv - vdu) d\bar z \\
&\qquad + (l - k) uvz^{k+l} f_{I_2} dz d\bar z]
\end{aligned}$$

We also have

$$\begin{aligned}
&d'\left([J_1 uF_k] \cdot [J_2 vz^{l+1} f_{I_2} d\bar z]\right) \\
&\quad = \left([J_1 uz^{k+1} f_{I_1} d\bar z] \cdot [J_2 vz^{l+1} f_{I_2} d\bar z] - [J_2 vz^{l+1} f_{I_2} d\bar z] \cdot [J_1 uz^{k+1} f_{I_3} d\bar z]\right) \\
&\qquad + [[J_1, J_2] uvz^{k+l+1} f_{I_2} d\bar z] + \frac{1}{2}\langle J_1, J_2\rangle [uF_k \partial(vz^{l+1} f_{I_2} d\bar z) - \partial(F_k u) \\
&\qquad \wedge (z^{l+1} vf_{I_2} d\bar z)] \\
&\quad = \left([J_1 uz^{k+1} f_{I_1} d\bar z] \cdot [J_2 vz^{l+1} f_{I_2} d\bar z] - [J_2 vz^{l+1} f_{I_2} d\bar z] \cdot [J_1 uz^{k+1} f_{I_3} d\bar z]\right) \\
&\qquad + [[J_1, J_2] uvz^{k+l+1} f_{I_2} d\bar z] + \frac{1}{2}\langle J_1, J_2\rangle [z^{k+l+1} (udv - vdu) f_{I_2} d\bar z \\
&\qquad + uv((l - k + 1) f_{I_2} z^{k+l} dz d\bar z - z^{k+l+1} \partial f_{I_2} \wedge d\bar z)]
\end{aligned}$$

where we have used the fact that over the support of f_{I_2}, $F_k = z^k$. Using the identities (28), we obtain

$$[(l - k + 1) z^{k+l} f_{I_2} dz d\bar z - z^{k+l+1} \partial f_{I_2} \wedge d\bar z] = [(l - k) z^{k+l} f_{I_2} dz d\bar z].$$

It follows that

$$\Phi_{I_1}(J_1 ut^k) \bullet_{1,2} \Phi_{I_2}(J_2 vt^l) - \Phi_{I_2}(J_2 vt^l) \bullet_{2,3} \Phi_{I_3}(J_1 ut^k) - \Phi_{I_2}([J_1 ut^k, J_2 vt^l]) = 0 \in \rho_* H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi})(I)$$

proving the lemma. □

The homomorphism Φ of Proposition (5.8) equips $V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$ with the structure of a $\mathfrak{g}_R$-module. Let us fix $0 < r < r' < R$ We have the following commutative diagram:

$$\begin{array}{ccc} H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(0,r))) \otimes H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(A(r',R))) & \xrightarrow{m} & H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(0,R))) \\ \uparrow{\scriptstyle \iota\otimes\Phi} & & \uparrow{\scriptstyle \iota} \\ V(\mathcal{G}^{alg}_{\mathfrak{g},\pi}) \otimes \mathrm{AU}(\mathfrak{g}_R) & \dashrightarrow & V(\mathcal{G}^{alg}_{\mathfrak{g},\pi}) \end{array}$$

where ι denotes the inclusion of $V(\mathcal{G}^{alg}_{\mathfrak{g},\pi}) \subset H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(0,r)))$ (for any r), and m is the prefactorization structure map. As explained in the proof of Theorem 5.3.3 in [4], the existence of the dotted arrow (i.e. the fact that the $U(\mathfrak{g}_R)$-action preserves the subspace $V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$) follows from the fact that the structure map m is S^1-equivariant.

In concrete terms, the action of $X \in \mathfrak{g}_R$ on $v \in V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$ is given as follows: we may represent v by a closed chain $\tilde{v} \in C^{Lie}_*(\mathfrak{g}^{\# c,alg}_{\pi}(D(0,r))$ - then $X \cdot v$ is represented by $\Phi_{(r',R)}(X) \cdot \tilde{v}$.

Lemma 5.9. *There is an isomorphism of $\mathfrak{g}_R$-modules*

$$\eta : V(\mathfrak{g}_R) \to V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$$

which sends $|0\rangle \in V(\mathfrak{g}_R)$ to $1 \in V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$.

Proof. Let $h(z,\bar{z}) = \int_0^{z\bar{z}} f(s)ds$. By the chain rule, we have that

$$\bar{\partial}(z^n h(z,\bar{z})) = z^{n+1} f(z\bar{z}) d\bar{z}$$

Thus in $H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(0,R))$, we have for $k \geq 0$:

$$\begin{aligned} \Phi_{(r',R)}(Jut^k) &= [Juz^{k+1} f(z\bar{z}) d\bar{z}] = d(Juz^k h(z,\bar{z})) \\ \Phi_{(r',R)}(t^k udv) &= [z^{k+1} f(z\bar{z}) ud\bar{z}dv] = d(z^k h(z,\bar{z}) udv) \\ \Phi_{(r',R)}(ut^k dt) &= [uz^{k+1} f(z\bar{z}) d\bar{z}dz] = d(uz^k h(z,\bar{z}) dz) \end{aligned}$$

In other words, if $X \in \mathfrak{g}^+_R$, then $\Phi_{(r',R)}(X) = 0 \in H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(0,R))$. This shows that the vector $1 \in V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$ is annihilated by $\mathfrak{g}^+_R$. It follows that there exists a unique map of $\mathfrak{g}_R$-modules $\eta : V(\mathfrak{g}_R) \to V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$ sending $|0\rangle \to 1$. It remains to show this is an isomorphism, which can be done as in [4,19] for the affine and Virasoro algebra, so we will be brief. Both $V(\mathfrak{g}_R)$ and $V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$ have the structure of filtered $U(\mathfrak{g}_R)$-modules, where in each case the filtration is induced by symmetric degree. It is straightforward to verify that η induces an isomorphism at the level of associated graded modules, proving the result. $\square$

To complete the proof of Theorem 5.2, we check that η induces an isomorphism of vertex algebras. Suppose that $z \in A(r', R)$. Recall that the operation

$$Y : V(\mathcal{G}^{alg}_{\mathfrak{g},\pi}) \otimes V(\mathcal{G}^{alg}_{\mathfrak{g},\pi}) \to V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})((z))$$

is induced from the diagram

$$\begin{array}{ccc} V(\mathcal{G}^{alg}_{\mathfrak{g},\pi}) \otimes V(\mathcal{G}^{alg}_{\mathfrak{g},\pi}) & & \\ \downarrow{\scriptstyle \iota\otimes\iota_z} & & \\ H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(z,2))) \otimes H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(0,r))) & \xrightarrow{m_{z,0}} & H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(0,R))) \end{array}$$

as the Laurent expansion of the map $m_{z,0} \circ \iota \otimes \iota_z$. By the Reconstruction Theorem 2.6, it suffices to show that the generating field assignments agree, that is we need to verify that for $v \in V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$,

$$m_{z,0}(\iota_z(\eta(Jut^{-1}\cdot|0\rangle)), \iota(v)) = \sum_{n\in\mathbb{Z}} (\Phi(Jut^n)\cdot v)z^{-n-1}$$

$$m_{z,0}(\iota_z(\eta(ut^{-1}dt\cdot|0\rangle)), \iota(v)) = \sum_{n\in\mathbb{Z}} (\Phi(ut^{n-1}dt)\cdot v)z^{-n}$$

$$m_{z,0}(\iota_z(\eta(t^{-1}\omega\cdot|0\rangle)), \iota(v)) = \sum_{n\in\mathbb{Z}} (\Phi(t^n\omega)\cdot v)z^{-n-1}$$

$\iota_z(\eta(Jut^{-1}\cdot|0\rangle))$ may be identified with the element $Ju\psi_z \in \mathfrak{g}\otimes\mathbb{C}[F]\otimes(\Omega^1(D(z,2)))^\vee$, where $\psi_z \in (\Omega^1(D(z,2)))^\vee$ is defined by

$$\psi_z(h(w)dw) = \frac{1}{2\pi i}\int_{C(z,\delta)} \frac{h(w)dw}{w-z}$$

By the residue theorem, for $h(w)dw \in \Omega^1(A(r,R))$, we may switch contours, to write

$$\int_{C(z,\delta)} \frac{h(w)dw}{w-z} = \int_{C(0,R-\delta)} \frac{h(w)dw}{w-z} - \int_{C(0,r'+\delta)} \frac{h(w)dw}{w-z}$$
$$= \sum_{n\geq 0}\left(\int_{C(0,R-\delta)} w^{-n-1}h(w)dw\right)z^n + \sum_{n<0}\left(\int_{C(0,r'+\delta)} w^{-n-1}h(w)dw\right)z^n$$

where in the second line we have expanded $\frac{1}{w-z}$ into a geometric series in the domains $|w| > |z|$ and $|w| < |z|$ respectively. Using the fact that

$$Res_0 h(w) w^{-n-1} dw = \int_{A(r\%,R)} h(w) w^{-n} f_{(r,R)} dw d\overline{w}$$

and $\Phi(Jut^{-n-1}) \cdot v = [Juz^{-n} f_{(r\%,R)} d\overline{z}] \cdot v$ we obtain the first identity. Similarly, we may identify $\iota_z(\eta(ut^{-1}dt \cdot \emptyset''))$ with the element $u\xi_z \in \mathbb{C}[F] \otimes (\mathcal{O}(D(z,2)))^\vee$, where

$$\xi_z(h(w)) = h(z) = \frac{1}{2\pi i} \int_{C(z,\delta)} \frac{h(w)dw}{w-z}$$

and $\iota_z(\eta(t^{-1}\omega \cdot \emptyset''))$ with $\omega\psi_z \in \Omega^1_{\mathbb{C}[F]} \otimes (\Omega^1(D(z,2)))^\vee$. Expanding these in contour integrals centered at 0, and identifying the coefficients with appropriate elements in the image of Φ as above proves the remaining two identities.

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