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RESEARCH/REVIEW ARTICLE

Witten's conjecture and recursions for κ classes

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Abstract

We construct a countable number of differential operators $\mathcal{L}_n$ that annihilate a generating function for intersection numbers of κ classes on $\overline{\mathrm{M}}_g$ (the κ-potential). This produces recursions among intersection numbers of κ classes which determine all such numbers from a single initial condition. The starting point of the work is a combinatorial formula relating intersection numbers of ψ and κ classes. Such a formula produces an exponential differential operator acting on the Gromov–Witten potential to produce the κ-potential; after restricting to a hyperplane, we have an explicit change of variables relating the two generating functions, and we conjugate the "classical" Virasoro operators to obtain the operators $\mathcal{L}_n$.

Keywords Moduli space of curves · κ classes · ψ classes · Witten conjecture · Enumerative geometry · Intersection theory

Mathematics Subject Classification 14N10 · 14N35

1 Introduction

Main result

We state immediately the main result of this manuscript, and refer the reader to the next section in the introduction for a more leisurely discussion leading to it.

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Definition 1.1 For $n = 0, 1, \widehat{\mathcal{L}}_n$ denotes the differential operator

$$\widehat{\mathcal{L}}_0 = -\frac{3}{2}\partial_{p_0} + \sum_{m=0}^{\infty} m p_m \partial_{p_m} + \frac{1}{16},$$

$$\widehat{\mathcal{L}}_1 = -\frac{15}{4}\partial_{p_1} + \sum_{m=0}^{\infty} m(m+4)\, p_m \partial_{p_{m+1}} - \sum_{l,m=0}^{\infty} l m p_l p_m \partial_{p_{m+l+1}}$$
$$+ \frac{(ue^{p_0})^2}{8}\left[\sum_{m=0}^{\infty} (S_{m+2}(\mathbf{p}) - S_{m+2}(2\mathbf{p}))\partial_{p_m} + \sum_{l,m=0}^{\infty} S_{l+1}(\mathbf{p}) S_{m+1}(\mathbf{p}) \partial_{p_l}\partial_{p_m}\right];$$

for all $n \geq 2$,

$$\widehat{\mathcal{L}}_n = \sum_{d=0}^{n+1} \alpha_{n,d}\left[-\sum_{m=0}^{\infty} [B_d(q_1, \ldots, q_d)]_{z^m}\partial_{p_{m+n}}\right.$$
$$+ \frac{(ue^{p_0})^2}{2}\left(\sum_{i=0}^{n-1} \frac{(2i+1)!!\,(2n-2i-1)!!}{2^{n+1}} \sum_{m=0}^{\infty} S_m(2\mathbf{p})\partial_{p_{m+n-3}}\right.$$
$$\left.\left. + \sum_{i=0}^{n-1} \frac{(2i+1)!!\,(2n-2i-1)!!}{2^{n+1}} \sum_{m,l=0}^{\infty} S_m(\mathbf{p}) S_l(\mathbf{p})\partial_{p_{m+n-2-i}}\partial_{p_{l+i-1}}\right)\right],$$

where $\mathbf{p} = (p_1, p_2, p_3, \ldots)$ denotes a countable list of variables, S_i is the i-th elementary Schur polynomial (Definition 4.3), B_d denotes the d-th Bell polynomial (Definition 5.3); the symbols $q_i = -\sum_k k^i p_k z^k$ and $\alpha_{n,d} = \left[\prod_{i=0}^{n}(x + i + 3/2)\right]_{x^d}$.

Let $K(p_0, \mathbf{p})$ denote a generating function for top intersection numbers of κ classes (Definition 3.5).

Theorem 1.2 *For all $n \geq 0$, we have*

$$\widehat{\mathcal{L}}_n(e^{K}) = 0.$$

The vanishing of the coefficients of monomials in $\widehat{\mathcal{L}}_n(e^{K})$ gives recursive relations among intersection numbers of κ classes; the collection of all such recursions uniquely determines K from the initial condition $\partial_{p_0}K|_{(p_0,\mathbf{p})=(0,\mathbf{0})} = 1/24$.

Context and motivation

The study of the tautological intersection theory of the moduli space of curves was initiated in the seminal [18]. In Mumford's words, this entails

> [...] *setting up a Chow ring for the moduli space M_g of curves of genus g and its compactification $\overline{M}_g$, defining what seem to be the most important classes in this ring* [...]

Besides boundary strata, two families of classes played a prominent role ψ classes (Definition 2.1) and κ classes (Definition 2.3). Geometrically, ψ classes arise when taking non-transversal intersections of boundary strata. The most remarkable feature of intersection numbers of ψ classes is Witten's conjecture / Kontsevich's theorem [15, 22]: a generating function F for intersection numbers of ψ classes (the Gromov–Witten potential) is a τ function for the KdV hierarchy. One formulation of this statement says there exist a countable number of differential operators $L_n, n \geq -1$, that annihilate e^F. The vanishing of the coefficient of each monomial in e^F gives a recursion on the intersection numbers of ψ classes, and such recursions allow computations of all top intersection numbers of ψ classes from the initial condition $\int_{\overline{M}_{0,3}} 1 = 1$.

Mumford's conjecture [18], now a theorem by Madsen and Weiss [16], brought κ classes to the foreground: the stable cohomology of M_g is a polynomial ring freely generated by the κ classes. More recently, Pandharipande [19] studied the restriction of the κ-ring, the part of the tautological ring of the moduli space of curves generated by κ classes, to the locus of curves of compact type; he unveiled the following interesting structure: the higher genus κ-rings are quotients of the genus zero ones in a canonical way. A survey on recent developments in the study of the κ-ring appears in [20].

The intersection theories of ψ and κ classes both exhibit rich combinatorial structure and are born out of operations involving Chern classes of the relative dualizing sheaf and tautological morphisms. It should not be completely surprising that these two theories are closely related.

It is mentioned in [2], and there credited to Carel Faber, that the degree of a top intersection monomial in ψ classes on $\overline{M}_{g,n}$ may be expressed as a polynomial in κ classes on $\overline{M}_g$, explicitly described as a sum of monomials indexed by elements in the symmetric group S_n. In [22], Witten remarks that knowing the intersection numbers of κ classes on $\overline{M}_g$ is equivalent to knowing intersection numbers of monomials in ψ classes on all the $\overline{M}_{g,n}$. An explicit formula to express a monomial in κ classes as a polynomial in ψ classes is derived independently in [1,5,11] (see Corollary 2.15).

Manin and Zograf [17, Theorem 4.1] show the formulae to express a single monomial in κ classes on $\overline{M}_g$ as a polynomial in ψ classes on different $\overline{M}_{g,n}$ directly imply that generating functions encoding intersection numbers of κ (the κ-potential) and ψ classes are related by an explicitly described change of variables.

Our first result is a new proof of Manin and Zograf's result, introducing a few intermediate steps which make the proof more conceptual and easy to understand. First, formulae (8) allow us to construct an operator L (see (13)), such that e^L acts on (the $g \geq 1$ part of) the Gromov–Witten potential to produce a potential for ω classes (Definition 2.2).

Theorem 1.3 *Denote by* $F(\mathbf{t})$ *the positive genus part of the Gromov–Witten potential, and by* $S(\mathbf{s})$ *the* ω*-potential. We exhibit an operator* L *such that*

$$(e^L F)_{|\mathbf{t}=0} = S.$$

We have an explicit expression for the operator $L = \sum f_n(\mathbf{s})\partial_{t_n}$ as a vector field. It follows that the two potentials are related by a change of variables given by the coefficients f_n (Lemma 4.1).

It is a simple consequence of projection formula that the ω-potential $S(\mathbf{s})$ and κ-potential $K(\mathbf{p})$ are related by specializing the variable $s_0 = 0$ and shifting down the other variables, so that $p_{n-1} = s_n$ (Lemma 3.6).

Combining these results, Theorem 4.5 is a reformulation of Manin and Zograf's [17, Theorem 4.1], restricted to intersections of κ classes on moduli spaces of unpointed curves $\overline{M}_g$.

In the process of proving Theorem 4.5, we note that we are restricting the domain of the Gromov–Witten potential to the hyperplane $t_0 = 0$. After dilaton shift and some cosmetic change of variables mostly involving shifts and signs, this simplifies the change of variables relating $F_{|t_0=0}$ and K to the following expression (where the coefficient of each power of z identifies a variable $\hat{t}_i$ as a function of the variables $\hat{p}_k$):

$$\sum_{i=0}^{\infty} \hat{t}_i z^i = e^{\sum_{k=0}^{\infty} \hat{p}_k z^k}. \tag{1}$$

We then turn our attention to the Virasoro operators. The idea is simple: rewriting the differential operators L_n in the variables for the κ-potential should produce a hierarchy of operators annihilating e^{K}, thereby producing relations among intersection numbers of κ classes. In order to carry this out we use the string equation L_{-1} to remove any occurrence of ∂_{t_0} from the L_ns. We then exploit the structure of the Jacobian of the change of variables (1)—which can be thought of as an infinite upper triangular matrix constant along translates of the diagonal—to reduce performing the chain rule to manipulation operations on generating series.

While Theorem 1.2 is shown as a consequence of Witten's conjecture, in fact Witten's conjecture is equivalent to Theorem 1.2 plus the string equation L_{-1}. The string equation has an elementary geometric proof, following from the behavior of ψ classes when pulled-back via forgetful morphisms (Lemma 2.5). Therefore an independent proof of the relations among κ classes arising from Theorem 1.2 would in particular give another proof of Witten's conjecture.

This work fits in the broader context of wall-crossings among Gromov–Witten potentials of Hassett spaces of weighted stable curves: the intersection theory of κ classes arises as a limit when the weights of some of the points are approaching 0. By using this language, and considering descendant invariants on heavy-light Hassett spaces one could recover the full statement of Manin–Zograf's theorem, which compares the Gromov–Witten potential to the κ-potential for all $\overline{M}_{g,n}$; in this work we chose to restrict our attention to the special case where all points are light, both because it highlights the most essential combinatorial relation between ψ and κ classes and because it allows us to reduce the bookkeeping and tell a compelling story within the classical context of intersection theory on $\overline{M}_{g,n}$. The more general picture for Hassett spaces will be treated in [6].

Organization of the paper and communication

While the ultimate motivation of this work is to further our understanding of the geometry of moduli spaces of curves $\overline{M}_{g,n}$, the techniques used are combinatorial and analytic in nature. We are making a conscious effort to communicate to the union, rather than the intersection, of these three mathematical communities. Section 2 gives a review of all relevant definitions of tautological classes on $\overline{M}_{g,n}$, as well as their behavior under pull-back/push-forward via tautological morphisms. Since some of these statements are well-known to the experts but somewhat hard to track down in the literature, we include several sketches of proofs. Section 3 introduces the generating functions for ψ, ω, and κ classes and translates the combinatorial formula (2.15) relating intersection numbers of ψ and ω classes to a differential operator acting on the Gromov–Witten potential. In Sect. 4 we use this interpretation to deduce that F and K are related by a specialization/change of variables and write explicitly such a transformation. Section 5 derives the operators $\tilde{L}_n$ in Definition 1.1. Since such a derivation is rather technical and bookkeeping intensive, we break it down into several subsections, with the intention that it should be possible for a reader interested in only a part of the computation to easily isolate it. The proof of Theorem 1.2 is then an immediate consequence of the construction of the operators $\tilde{L}_n$. Finally, in Sect. 6 we write down some of the relations among κ classes that are produced with this method.

2 Background and preliminaries

This section contains basic information about tautological classes on moduli spaces of curves and their behavior with respect to pull-backs and push-forward via tautological morphisms. While we expect most readers to have some familiarity with these topics (otherwise excellent places to start are [9,21]), we intend this section to establish notation and to highlight concepts that are relevant to this work.

Given two non-negative integers $g\,n$ satisfying $2g-2+n>0$, we denote by $\overline{M}_{g,n}$ the fine moduli space for families of Deligne–Mumford stable curves of genus g with n marked points. The space $\overline{M}_{g,n}$ is a smooth and projective DM stack of dimension $3g-3+n$ [12,13]. We recall the tautological morphisms that provide connections among different moduli spaces.

Let g_1, g_2 be two non-negative integers adding to g, and (P_1, P_2) a partition[1] of the set $[n] = \{1, \ldots, n\}$. The *gluing morphism*

$$\mathrm{gl}_{(g_1,P_1)|(g_2,P_2)} : \overline{M}_{g_1,P_1\cup\{\bullet\}} \times \overline{M}_{g_2,P_2\cup\{\star\}} \to \overline{M}_{g,n}$$

assigns to a pair $((C_1; \{p_i\}_{i\in P_1}, \bullet), (C_2; \{p_j\}_{j\in P_2}, \star))$ the pointed nodal curve $(C_1 \cup_{\bullet=\star} C_2; p_1, \ldots, p_n)$ obtained by identifying the marks denoted by $\star$ and $\bullet$. The image of the gluing morphism is an irreducible, closed subvariety of $\overline{M}_{g,n}$ called a *boundary divisor* and denoted $D((g_1, P_1)|(g_2, P_2))$.

More generally, given a nodal, pointed, stable curve $(C; p_1, \ldots, p_n)$, we can identify its topological type by its dual graph Γ. We define a more general gluing morphism

$$\mathrm{gl}_\Gamma : \prod_{v\in V(\Gamma)} \overline{M}_{g(v),\mathrm{val}(v)} \to \overline{M}_{g,n}$$

[1] If $g_i = 0$, we require $|P_i| \geq 2$.

to be the morphism that glues marked points corresponding to pairs of flags that form an edge of the dual graph. The morphism gl_Γ is finite of degree $|\mathrm{Aut}(\Gamma)|$ onto its image. We call the image of gl_Γ a *boundary stratum* and denote it by Δ_Γ. As a cycle class:

$$[\Delta_\Gamma] = \frac{1}{|\mathrm{Aut}(\Gamma)|} gl_{\Gamma *}(1). \tag{2}$$

For any mark $i \in [n+1]$, there is a *forgetful morphism*

$$\pi_i : \overline{\mathrm{M}}_{g,n+1} \to \overline{\mathrm{M}}_{g,[n+1]\setminus\{i\}} \cong \overline{\mathrm{M}}_{g,n},$$

which assigns to an $(n+1)$-pointed curve $(C; p_1, \ldots, p_{n+1})$ the n-pointed curve obtained by forgetting the i-th marked point and contracting any rational component of C which has less than three special points (marks or nodes). The morphism π_i is a universal family for $\overline{\mathrm{M}}_{g,n}$, and so in particular the universal curve $\overline{\mathrm{U}}_{g,n} \to \overline{\mathrm{M}}_{g,n}$ may be identified with $\overline{\mathrm{M}}_{g,n+1}$.

The *i-th tautological section*

$$\sigma_i : \overline{\mathrm{M}}_{g,n} \to \overline{\mathrm{U}}_{g,n} \cong \overline{\mathrm{M}}_{g,n+1}$$

assigns to an n-pointed curve $(C; p_1, \ldots, p_n)$ the point p_i in the fiber over $(C; p_1, \ldots, p_n)$ in the universal curve. Such a point corresponds to the $(n+1)$-pointed curve obtained by attaching a rational component to the point $p_i \in C$ and placing the marks p_i and p_{n+1} arbitrarily on the new rational component. Via the identification of the universal map with a forgetful morphism, the section σ_i may be viewed as a gluing morphism and its image as a boundary stratum, denoted $\Delta_{i,n+1}$. The following diagram illustrates this concept:

$$\begin{array}{ccc}
\overline{\mathrm{U}}_{g,n} & \xrightarrow{\cong} & \overline{\mathrm{M}}_{g,n+1} \\
\cup\!\mid & & \cup\!\mid \\
\mathrm{Im}(\sigma_i) & & \Delta_{i,n+1} \\
\uparrow \sigma_i & & \uparrow gl_{((g,[n]\setminus\{i\}\cup\{\bullet\})|(0,\{\bullet,i,n+1\}))} \\
\overline{\mathrm{M}}_{g,n} & \xrightarrow{\cong} & \overline{\mathrm{M}}_{g,[n]\setminus\{i\}\cup\{\bullet\}} \times \overline{\mathrm{M}}_{0,\{\bullet,i,n+1\}}
\end{array}$$

We consider all $\overline{\mathrm{M}}_{g,n}$ (for all values of g n) as a system of moduli spaces connected by the tautological morphisms and define the *tautological ring* $\mathcal{R} = \{R^*(\overline{\mathrm{M}}_{g,n})\}_{g,n}$ of this system to be the smallest system of subrings of the Chow ring of each $\overline{\mathrm{M}}_{g,n}$ containing all fundamental classes $[\overline{\mathrm{M}}_{g,n}]$ and closed under push-forwards and pull-backs via the tautological (gluing and forgetful) morphisms. By (2), classes of boundary strata are elements of the tautological ring.

We now introduce some other families of tautological classes which are studied in this work.

Definition 2.1 For any choice of mark $i \in [n]$, the class $\psi_i \in R^1(\overline{M}_{g,n})$ is defined to be

$$\psi_i := c_1(\sigma_i^*(\omega_\pi)),$$

where ω_π denotes the relative dualizing sheaf of the universal family $\pi : \overline{U}_{g,n} \to \overline{M}_{g,n}$.

While it is not obvious from Definition 2.1 that ψ classes are tautological, this follows from the fact that $\psi_i = \pi_{n+1*}(-\ {}^{2}_{i,n+1})$.

Definition 2.2 Let $g, n \geq 1$, $i \in [n]$, and let $\rho_i : \overline{M}_{g,n} \to \overline{M}_{g,\{i\}}$ be the composition of forgetful morphisms for all but the i-th mark. Then we define

$$\omega_i := \rho_i^* \psi_i$$

in $R^1(\overline{M}_{g,n})$.

Definition 2.3 For a non-negative integer i, the class $\kappa_i \in R^i(\overline{M}_{g,n})$ is

$$\kappa_i := \pi_{n+1*}(\psi_{n+1}^{i+1}).$$

Remark 2.4 In [18], Mumford introduces first κ classes on $\overline{M}_g$ as

$$\kappa_i := \pi_*(c_1(\omega_\pi)^{i+1}),$$

where $\pi : \overline{M}_{g,1} \cong \overline{U}_g \to \overline{M}_g$ denotes the universal family. One may verify that on $\overline{M}_{g,1}$,

$$\psi_1 = c_1(\omega_\pi),$$

which makes the (unique) ψ class on $\overline{M}_{g,1}$ an uncontroversially natural and canonically constructed class. There are two different ways to generalize this class to spaces with more than one mark: simply pulling-back the class ψ gives rise to the class ω, whereas focusing on the fact that ψ is the Euler class of a line bundle whose fiber over the moduli point $(C; p)$ is $T_p^*(C)$ generalizes to the definition of ψ class given above.

The following lemma shows how ψ classes behave when pulled-back via forgetful morphisms.

Lemma 2.5 ([14]) *Consider the forgetful morphism $\pi_{n+1} : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$, and let the context determine whether ψ_i denotes the class on $\overline{M}_{g,n}$ or $\overline{M}_{g,n+1}$. For $i \in [n]$, we have*

$$\psi_i = \pi_{n+1}^*(\psi_i) + D_{i,n+1},$$

where $D_{i,n+1}$ denotes the class of the image of the section σ_i, or equivalently the class of the boundary divisor $\partial_{i,n+1}$, generically parameterizing nodal curves where one component is rational and hosts the i-th and $(n+1)$-th marks.

Iterated applications of Lemma 2.5 show the relation between the classes ψ_i and ω_i on $\overline{\mathrm{M}}_{g,n}$. Denote by $D(A|B)$ the divisor $D((g, A)|(0, B))$. We call any boundary stratum where all the genus is concentrated at one vertex of the dual graph a stratum of *rational tails* type.

Lemma 2.6 ([5]) *Let $g, n \geq 1$ and $i \in [n]$. Then*

$$\psi_i = \omega_i + \sum_{B \ni i} D(A|B).$$

In words, this means that ψ_i is obtained from ω_i by adding all divisors of rational tails type where the i-th mark is contained in the rational component.

The behavior of κ classes under pull-back via forgetful morphisms was studied in [2].

Lemma 2.7 ([2]) *Consider the forgetful morphism $\pi_{n+1} : \overline{\mathrm{M}}_{g,n+1} \to \overline{\mathrm{M}}_{g,n}$, and let the context determine whether κ_i denotes the class on $\overline{\mathrm{M}}_{g,n}$ or $\overline{\mathrm{M}}_{g,n+1}$. We have*

$$\kappa_i = \pi^*_{n+1}(\kappa_i) + \psi^i_{n+1}.$$

The next group of lemmas gives information about the behavior of tautological classes under push-forward via forgetful morphisms. These are familiar facts for people in the field, but it is non-trivial to track down appropriate references; for this reason we add brief sketches of proofs that could be completed by the interested reader. Pushing-forward a monomial in ψ classes along a morphism that forgets a mark that does not support a ψ class one obtains the so-called *string recursion*.

Lemma 2.8 (String) *Consider the forgetful morphism $\pi_{n+1} : \overline{\mathrm{M}}_{g,n+1} \to \overline{\mathrm{M}}_{g,n}$. Let $K \in \mathbb{Z}^n$ denote the vector $(k_1, \ldots k_n)$ and e_i the i-th standard basis vector. By ψ^K we mean $\prod \psi_i^{k_i}$ and we adopt the convention that $\psi_i^m = 0$ whenever $m < 0$. Then*

$$\pi_{n+1*}(\psi^K) = \sum_{i=1}^{n} \psi^{K-e_i}. \tag{3}$$

Proof Equation (3) is proved by using Lemma 2.5 to replace each $\psi_i^{k_i}$ with $\pi^*_{n+1}(\psi_i)^{k_i} + D_{i,n+1}\pi^*_{n+1}(\psi_i)^{k_i-1}$, and then applying projection formula to obtain the right- hand side of (3). More details can be found in [14, Lemma 1.4.2]. □

The analogous statement for ω classes is rather trivial.

Lemma 2.9 (ω-string) *Let $n > 0$, and consider the forgetful morphism $\pi_{n+1} : \overline{\mathrm{M}}_{g,n+1} \to \overline{\mathrm{M}}_{g,n}$. Let $K \in \mathbb{Z}^n$ denote the vector $(k_1, \ldots k_n)$ and $\omega^K = \prod \omega_i^{k_i}$. Then*

$$\pi_{n+1*}(\omega^K) = 0.$$

Proof The map π_{n+1} has positive dimensional fibers, and by definition

$$\omega^K = \pi^*_{n+1}(\omega^K),$$

which implies the vanishing of the push-forward. $\square$

Lemma 2.10 *For any g, n with $2g-2+n > 0$,*

$$\kappa_0 = (2g-2+n)[1]_{\overline{\mathcal{M}}_{g,n}}. \tag{4}$$

Proof The space $\overline{\mathcal{M}}_{g,n}$ is proper, connected, and irreducible, hence the class κ_0 must be a multiple of the fundamental class. Recall that by definition $\kappa_0 = \pi_{n+1*}(\psi_{n+1})$. Fixing a moduli point $\mathbf{m} = [(C; p_1, \ldots, p_n)] \in \overline{\mathcal{M}}_{g,n}$, one may verify from the definitions that $\sigma^*_{n+1}(\omega_\pi)_{|\pi^*_{n+1}(\mathbf{m})} = \omega_C(p_1 + \cdots + p_n)$. Using projection formula one computes:

$$\kappa_0 \cdot [\mathbf{m}] = \pi_{n+1*}(\psi_{n+1} \cdot \pi^*_{n+1}([\mathbf{m}])) = \pi_{n+1*}(c_1(\omega_C(p_1 + \cdots + p_n))). \tag{5}$$

Since $\deg(\omega_C(p_1 + \cdots + p_n)) = 2g-2+n$, equation (4) is established. $\square$

Lemma 2.11 (Dilaton) *Consider the forgetful morphism $\pi_{n+1} : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$. Let $K \in \mathbb{Z}^n$ denote a vector of non-negative integers $(k_1, \ldots, k_n) = (0, \ldots, 0)$. Then*

$$\pi_{n+1*}(\psi^K \psi_{n+1}) = (2g-2+n)\psi^K.$$

Proof Observe that $\psi_{n+1} D_{i,n+1} = 0$ for dimension reasons. By Lemma 2.5, we have

$$\psi^K \psi_{n+1} = \pi^*_{n+1}(\psi^K)\psi_{n+1}.$$

The proof is concluded by applying projection formula and using Lemma 2.10. $\square$

Lemma 2.12 (ω-dilaton) *Let $g+n \geq 2$, and consider the forgetful morphism $\pi_{n+1} : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$. Let $K \in \mathbb{Z}^n$ denote a vector of non-negative integers $(k_1, \ldots, k_n)$. Then*

$$\pi_{n+1*}(\omega^K \omega_{n+1}) = (2g-2)\omega^K. \tag{6}$$

Proof First assume $g \geq 2$, and consider the following commutative diagram:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n+1} & \xrightarrow{\pi_{n+1}} & \overline{\mathcal{M}}_{g,n} \\ \downarrow{\scriptstyle \rho_{n+1}} & & \downarrow{\scriptstyle F} \\ \overline{\mathcal{M}}_{g,\{n+1\}} & \xrightarrow{\pi_{n+1}} & \overline{\mathcal{M}}_{g}. \end{array} \tag{7}$$

We have $\pi_{n+1*}(\omega_{n+1}) = \pi_{n+1*}\rho^*_{n+1}(\psi_{n+1}) = F^*\pi_{n+1*}(\psi_{n+1}) = (2g-2)[1]_{\overline{\mathcal{M}}_{g,n}}$. We remark that the second equality holds even if diagram (7) is not Cartesian: one may check it with a computation analogous to (5). Equation (6) then follows by projection formula.

When $g = 1$, any monomial in ω classes of degree greater than one vanishes because $\omega_i = \lambda_1$ and $\lambda_1^2 = 0$ [18, (5.4)]. Similarly, $\pi_{n+1*}(\omega_i) = 0$, since λ_1 is pulled-back from $\overline{M}_{1,1}$. ⊓⊔

The proof of Lemma 2.12 generalizes to give a natural relation between ω and κ classes.

Lemma 2.13 ([5, Lemma 3.3]) *Let $g \geq 2$, and consider the total forgetful morphism $F : \overline{M}_{g,n} \to \overline{M}_g$. Let $K \in \mathbb{Z}^n$ denote a vector of non-negative integers $(k_1, \ldots, k_n)$ and $\mathbf{1} = (1, 1, \ldots, 1)$. Then*

$$F_*(\omega^K) = \kappa_{K-\mathbf{1}}.$$

In [5] a graph formula is used to express an arbitrary monomial in ω classes in terms of a special class of boundary strata appropriately decorated with ψ classes. Denote by $P \vdash [n]$ an unordered partition of the set $[n]$, i.e., a collection of pairwise disjoint non-empty subsets $P_1, \ldots, P_r$ such that

$$P_1 \cup \cdots \cup P_r = [n].$$

Given $P \vdash [n]$, when $|P_i| = 1$ denote by $\bullet_i$ the element of the singleton P_i. For $|P_i| > 1$, introduce new labels $\bullet_i$ and $\star_i$. The *pinwheel stratum* Δ_P is the image of the gluing morphism

$$gl_P : \overline{M}_{g,\{\bullet_1,\ldots,\bullet_r\}} \times \prod_{|P_i|>1} \overline{M}_{0,\{\star_i\}\cup P_i} \to \overline{M}_{g,n}$$

that glues together each $\bullet_i$ with $\star_i$ when for each $|P_i| > 1$. The class of the stratum equals the push-forward of the fundamental class via gl_P.

Proposition 2.14 ([5, Theorem 2.2]) *For $1 \leq i \leq n$, let k_i be a non-negative integer, and let $K = \sum_{i=1}^n k_i$. For any partition $P = \{P_1, \ldots, P_r\} \vdash [n]$, define $\alpha_j := \sum_{i\in P_j} k_i$. With notation as in the previous paragraph, the following formula holds in $R^K(\overline{M}_{g,n})$:*

$$\prod_{i=1}^n \omega_i^{k_i} = \sum_{P\vdash[n]} [\Delta_P] \prod_{j=1}^r \frac{\psi_{\bullet_j}^{\alpha_j}}{(-\psi_{\bullet_j} - \psi_{\star_j})^{1-\delta_j}},$$

where $\delta_j = \delta_{1,|P_j|}$ is a Kronecker delta and we follow the convention of considering negative powers of ψ equal to 0.

Specializing to monomials of top degree and using projection formula one can deduce a formula relating intersection numbers of κ classes on $\overline{M}_g$ with intersection numbers of ψ classes.

Corollary 2.15 ([1, Corollary 7.10], [5, Theorem 3.1, Corollary 3.5]) *For* $1 \le i \le n$, *let* k_i *be a non-negative integer, and let* $\sum_{i=1}^{n} k_i = 3g-3+n$. *For any partition* $P = \{P_1, \ldots, P_r\} \vdash [n]$, *define* $\alpha_j := \sum_{i \in P_j} k_i$.

$$\int_{\overline{M}_g} \prod_{i=1}^{n} \kappa_{k_i-1} = \int_{\overline{M}_{g,n}} \prod_{i=1}^{n} \omega_i^{k_i} = \sum_{P \vdash [n]} (-1)^{n+r} \int_{\overline{M}_{g,r}} \prod_{i=1}^{r} \psi_i^{\alpha_i - |P_i| + 1}. \tag{8}$$

3 Generating functions and differential operators

In this section we introduce generating functions for intersection numbers of ψ, ω and κ classes, and we show that Corollary 2.15 gives rise to functional equations relating these potentials. We first introduce the generating function for intersection numbers of ψ classes, also known as the Gromov–Witten potential of a point.

Consider a countably infinite dimensional vector space H^+ with basis given by vectors τ_i, with linear dual coordinates t_i. Then $\tau := \sum_0^\infty t_i \tau_i$ is the expression in coordinates of the identity function of H^+. With a common abuse of notation, we think of τ as the expression for a general vector in H^+, keeping in mind that for any vector of H^+ only finitely many coordinates evaluate to non-zero numbers. The Witten brackets, depending on g, n, are defined as

$$\langle \tau_0^{n_0} \cdots \tau_m^{n_m} \rangle_{g,n} = \int_{\overline{M}_{g,n}} \prod_{i=n_0+1}^{n_0+n_1} \psi_i \prod_{i=n_0+n_1+1}^{n_0+n_1+n_2} \psi_i^2 \cdots \prod_{i=n_0+n_1+\cdots+n_{m-1}+1}^{n} \psi_i^m, \tag{9}$$

where $n = \sum_{j=0}^{m} n_j$ and $3g-3+n = \sum_{j=0}^{m} j\, n_j$.

Since monomials in the classes τ_i form a basis for $\mathrm{Sym}^\bullet(H^+)$, one may use (9) to define by multilinearity a (formal[2]) function on H^+, i.e., a power series in the variables t_i; this allows for the following compact definition of the Gromov–Witten potential of a point (see also [10, Definition 26.5.1]).

Definition 3.1 The *genus g Gromov–Witten potential of a point* is defined to be

$$F^g(t_0, t_1, \ldots) = \langle e^\tau \rangle_g = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \tau, \ldots, \tau \rangle_{g,n}.$$

The *total Gromov–Witten potential* is obtained by summing over all genera and adding a formal variable u keeping track of genus (more precisely, of Euler characteristic):

$$F(u; t_0, t_1, \ldots) = \sum_{g=0}^{\infty} u^{2g-2} F^g(t_0, t_1, \ldots).$$

[2] We are not concerned with issues of convergence here; see [7] for a discussion.

Remark 3.2 Unpacking the combinatorics of Definition 3.1, one sees that the Gromov–Witten potential $\mathcal{F}$ is an exponential generating function for intersection numbers of ψ classes. Because such intersection numbers are invariant under the action of the symmetric group permuting the marks, the variable t_i does not refer to the insertion at the i-th mark, but rather to an insertion (at some mark) of ψ^i. For example, the intersection number

$$\int_{\overline{M}_{2,6}} \psi_1^6\psi_2\psi_3\psi_4 = \int_{\overline{M}_{2,6}} \underbrace{\psi_1^6}_{1}\,\underbrace{\psi_2^1\psi_3^1\psi_4^1}_{3}\,\underbrace{\psi_5^0\psi_6^0}_{2}$$

is the coefficient of the monomial

$$u^2\frac{1}{6!}\binom{6}{3,2,1}t_0^2t_1^3t_6^1 = u^2\frac{t_0^2}{2!}\frac{t_1^3}{3!}t_6^1$$

in $\mathcal{F}$.

We define similar generating functions for ω classes. Denote $\sigma := \sum_0^\infty s_i\sigma_i$, and

$$\langle\sigma_0^{n_0}\cdots\sigma_m^{n_m}\rangle_{g,n} = \int_{\overline{M}_{g,n}} \prod_{i=n_0+1}^{n_0+n_1}\omega_i \prod_{i=n_0+n_1+1}^{n_0+n_1+n_2}\omega_i^2 \cdots \prod_{i=n_0+n_1+\ldots+n_{m-1}+1}^{n}\omega_i^m.$$

Definition 3.3 The *genus g ω-potential of a point* is defined to be

$$\mathcal{S}^g(s_0,s_1,\ldots) = \langle e^\sigma\rangle_g = \sum_{n=0}^\infty \frac{1}{n!}\langle\sigma,\ldots,\sigma\rangle_{g,n}.$$

The *total ω-potential* is

$$\mathcal{S}(u;s_0,s_1,\ldots) = \sum_{g=1}^\infty u^{2g-2}\mathcal{S}^g(s_0,s_1,\ldots).$$

The analysis of push-forwards of ω classes yields some immediate results about the structure of $\mathcal{S}$.

Lemma 3.4

$$\mathcal{S}^g = e^{(2g-2)s_1}\tilde{\mathcal{S}}^g(s_2,s_3,\ldots),$$

where $\tilde{\mathcal{S}}^g$ is some function depending only on variables s_i, with $i \geq 2$.

Proof Lemma 2.9 implies that $\mathcal{S}$ is constant in s_0; the statement of Lemma 2.12 is equivalent to $\mathcal{S}^g$ satisfying the first order differential equation

$$\frac{\partial\mathcal{S}^g}{\partial s_1} = (2g-2)\,\mathcal{S}^g. \tag{10}$$

One sees this by observing the coefficients of any monomial $s^K/K!$ in (10):

$$\int_{\overline{M}_{g,n+1}} \omega^K \omega_{n+1} = (2g-2)\int_{\overline{M}_{g,n}} \omega^K; \tag{11}$$

the equality in (11) follows immediately from Lemma 2.12.

Solving (10) by separating variables one obtains that S^g is the product of $e^{(2g-2)s_1}$ and some function in the remaining variables. □

Finally, we introduce a potential for intersection numbers of κ classes on $\overline{M}_g$.

Definition 3.5 Let $p_0, p_1, \ldots$ be a countable set of formal variables, and $\mu = (\mu_0, \mu_1, \ldots)$ denote a vector of integers such that $w(\mu) := \sum i\mu_i = 3g-3$. Then the κ-*potential* is

$$K(u; p_0, p_1, \ldots) = \frac{p_0}{24} + \sum_{\substack{g \geq 2 \\ w(\mu)=3g-3}} \left(\int_{\overline{M}_g} \prod \kappa_i^{\mu_i} \right) u^{2g-2} \prod \frac{p_i^{\mu_i}}{\mu_i!}.$$

The potentials K and S are very closely related.

Lemma 3.6

$$K(u; p_0, p_1, \ldots) = S(u; 0, p_0, p_1, \ldots) = S(ue^{p_0}; 0, 0, p_1, \ldots). \tag{12}$$

Proof Lemma 2.13 implies that K is obtained from S by setting $u = 1$ and shifting variables so that $s_{i+1} = p_i$. Then the structure statement of Lemma 3.4 implies the second equality in (12). □

The equalities (12) motivate the introduction of the "unstable" term $p_0/24$, which corresponds to assigning value $1/24$ to κ_0 on $\overline{M}_1$ (which is not a Deligne–Mumford stack).

The combinatorial formulae of Corollary 2.15 can be rephrased as the existence of a differential operator which acts on the Gromov–Witten potential to produce the ω-potential.

Theorem 3.7 *Define the* fork *operator as the formal operator*

$$L := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \sum_{i_1,\ldots,i_n=0}^{\infty} s_{i_1} \cdots s_{i_n} \partial_{t_{i_1+\cdots+i_n+1-n}}. \tag{13}$$

Denote by $F = \sum_{g=1}^{\infty} u^{2g-2} F_g$ *the positive genus part of the Gromov–Witten potential. Then*

$$(e^{L} F)_{|\mathbf{t}=\mathbf{0}} = S. \tag{14}$$

Remark 3.8 Before we embark in a formal proof of Theorem 3.7, we provide some intuition. Consider the summand $\left\langle \int_{\overline{M}_{2,5}} \omega_1^2\omega_2^2\omega_3^2\omega_4\omega_5 \right\rangle u^2 \frac{s_1^2}{2!}\frac{s_2^3}{3!}$ in the generating function S; consider the operator $s_1 s_2 \partial_{t_2}$ (up to sign, a term of L). Let us assume that there is a generating function G satisfying the differential equation

$$S = s_1 s_2 \partial_{t_2} G. \tag{15}$$

We want to understand what kind of information we get about G by equating the coefficients of the monomial $\frac{s_1^2}{2}\frac{s_2^3}{3!}$ in (15). There is a unique monomial, namely $u^2 t_2 s_1 \frac{s_2^2}{2!}$, that is transformed into the monomial $\frac{s_1^2}{2!}\frac{s_2^3}{3!}$ after the action of $s_1 s_2 \partial_{t_2}$. If we denote by γ the coefficient of $u^2 t_2 s_1 \frac{s_2^2}{2!}$ in G, then it is a calculus exercise to verify that

$$\int_{\overline{M}_{2,5}} \omega_1^2\omega_2^2\omega_3^2\omega_4\omega_5 = 6\gamma . \tag{16}$$

We now give a diagrammatic representation of relation (16). We associate decorated graphs to every summand of the generating functions S and G as depicted in Fig. 1: the graph Γ_1 is associated to the monomial $\left\langle \int_{\overline{M}_{2,5}} \omega_1^2\omega_2^2\omega_3^2\omega_4\omega_5 \right\rangle u^2 \frac{s_1^2}{2!}\frac{s_2^3}{3!}$ of S and the graph Γ_2 is associated to the monomial $\gamma u^2 t_2 s_1 \frac{s_2^2}{2!}$ of G. The following technical detail is important: the lower indexing of the ω and ψ classes is irrelevant as the corresponding intersection numbers are invariant under permutations of the lower indices; so graphs that are equal up to a permutation of the lower indices are to be considered equivalent.

We now interpret applying the operator $s_1 s_2 \partial_{t_2}$ to G as follows: it consists of removing a leg labeled ψ^2 and introducing two legs labeled ω^1 and ω^2. However, we want to fix a monomial of S and see what monomials in G are transformed to it via $s_1 s_2 \partial_{t_2}$, so we need to reverse the diagrammatic process just described: we want to fix the graph Γ_1 and see in how many ways one obtains a graph equivalent to Γ_2 by replacing two legs labeled ω^1 and ω^2 with one leg labeled ψ^2. As shown in Fig. 2, this can be done in six distinct ways, keeping in account the distinct lower indices of the ω classes. Hence Fig. 2 is a diagrammatic representation of relation (16).

In general, an operator of the form $s^I \partial_{t_k}$ compares the coefficient of S corresponding to a graph Γ to coefficients of G obtained by grabbing in all possible ways a collection of ω-legs of Γ decorated according to the multi-index I, and replacing them by a unique leg decorated by ψ^k. The operator L runs over all possibilities of grabbing a number of ω-legs and replacing them with a unique leg, decorated with a power of ψ determined by the number and the powers of ω classes that have been grabbed. Applying the exponential of the operator L then corresponds to grabbing multiple groups of ω-legs and replacing each group with a ψ-leg, not caring about the order of the grabbing. At this point it should be possible to recognize, when applying e^L to the Gromov–Witten potential F, that the coefficients of monomials are given by the rightmost term in equation (8). The operator L is constructed precisely to reproduce the formula based on the summation over all partitions of the index set in (8).

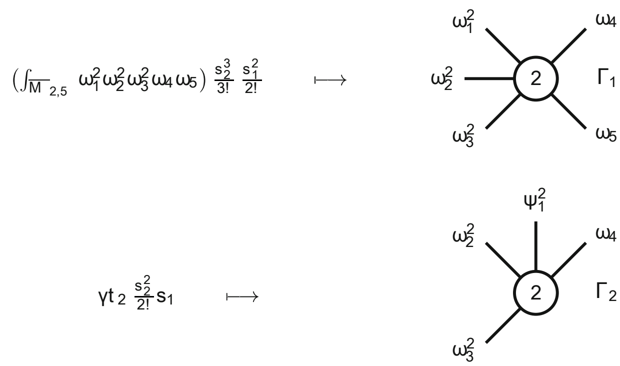

Fig. 1 Decorated graphs are associated to monomials in s and t variables. Such graphs also keep track of the coefficients of the monomial being an intersection number of ψ and ω classes

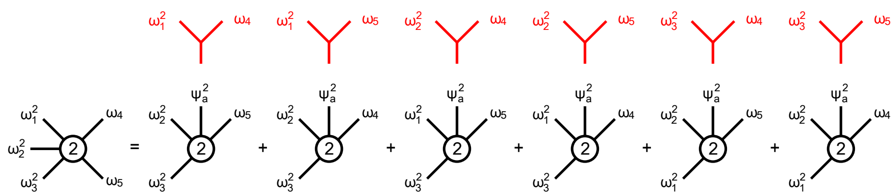

Fig. 2 The comparison of one of the coefficients of S and G given by the differential equation $S = s_1 s_2 \partial_{t_2} G$. There are six ways to "grab" a leg decorated by ω_i and a leg decorated by ω_j^2 (in red the two legs that are pulled off are put together in a tripod; there is a reason for that, but it is not important for the current discussion) and replacing them with a leg decorated ψ_a^2

Proof A formal proof of Theorem 3.7 is an exercise in bookkeeping: for any given monomial, we show that the coefficients on either side of (14) agree, by using formula (8).

Fix a monomial $u^{2g-2}\frac{s_1^{k_1}}{k_1!}\cdots\frac{s_n^{k_n}}{k_n!}$; we let $K = 1^{k_1}, 2^{k_2}, \ldots, m^{k_m}$ denote a multi-index where the first k_1 indices have value 1, and so on. We do not care about the order of the values, so all multi-indices arising in this proof can be assumed normalized so that the values are non-decreasing. The coefficient for this monomial on the right-hand side of (14) is $\int_{\overline{M}_{g,n}} \omega^K$, where we use multi-index notation in the natural way. Formula (8) gives an expression for this quantity in terms of a weighted sum over all partitions of the index set, so let us fix a partition $P = P_1, \ldots, P_r$ of the index set. Each part P_i of the partition P produces a multi-index by looking at the powers of ω classes supported on the points that belong to P_i. It is possible that different parts give rise to the same multi-index. We denote $\underline{J} = J_1^{\nu_1}, \ldots, J_t^{\nu_t}$ the collection of the multi-indices arising from the parts of P, intending that the multi-index J_1 arises ν_1 times, and so on. For i from 1 to t, we denote $J_i = 1^{j_{i,1}}, \ldots, m^{j_{i,m}}$. Finally, from $\underline{J}$, we can produce a multi-index $\alpha = \alpha_1^{\nu_1}, \ldots, \alpha_t^{\nu_t}$, where $\alpha_i = 1 + j_{i,2} + 2j_{i,3} + \cdots + (m-1)j_{i,m}$ (note that this is one plus the sum of the values minus the number of the parts of J_i, as in the definition of the exponents of the ψ classes in (8)). The multi-index α is the exponent vector of the monomial in ψ classes corresponding to the partition P in (8).

It is possible that different partitions of the set of indices give rise to the same multi-index α: in fact there are exactly $\prod k_i!/\prod\prod j_{i,l}!^{v_i}\prod v_i!$ distinct partitions of the indices that will produce α. We can rewrite (8) as a summation over the combinatorial data given by $\underline{J}$:

$$\int_{\overline{M}_{g,n}} \omega^K = \sum_{\underline{J}} (-1)^{n+\ell(\alpha)} \frac{\prod k_i!}{\prod\prod j_{i,l}!^{v_i}\prod v_i!} \int_{\overline{M}_{g,\ell(\alpha)}} \psi^{\alpha}. \tag{17}$$

We exhibit a summand m_α in $\mathcal{F}$, and a term $L_{\underline{J}}$ in the differential operator $e^{\mathcal{L}}$ so that $L_{\underline{J}} m_\alpha$ is a multiple of $u^{2g-2}s^K/K!$; such multiple may be shown to be $\prod k_i!/\prod\prod j_{i,l}!^{v_i}\prod v_i!$. Define

$$m_\alpha = \left(\int_{\overline{M}_{g,\ell(\alpha)}} \psi^{\alpha}\right) u^{2g-2} \frac{t_{\alpha_1}^{v_1}}{v_1!}\cdots\frac{t_{\alpha_t}^{v_t}}{v_t!},$$

$$L_{\underline{J}} = \frac{(-1)^{n+\ell(\alpha)}}{\prod v_i!} \prod \left(\frac{s_1^{j_{i,1}}}{j_{i,1}!}\cdots\frac{s_m^{j_{i,m}}}{j_{i,m}!}\partial_{t_{\alpha_i}}\right)^{v_i}.$$

One may show that all pairs of terms $m \in \mathcal{F}$, $L \in e^{\mathcal{L}}$ such that Lm is a multiple of $u^{2g-2}\frac{s^K}{K!}$ arise in this fashion. It follows that the coefficient of $u^{2g-2}\frac{s^K}{K!}$ in $e^{\mathcal{L}}\mathcal{F}$ equals the right-hand side of (17), and therefore it agrees with the coefficient of the same monomial in $\mathcal{S}$. This concludes the proof of Theorem 3.7. □

4 Change of variables

Theorem 3.7 states that $\mathcal{S}$ is the restriction of a function obtained by applying the exponential[3] of a vector field to the function $\mathcal{F}$. It follows that the two generating functions are related by a change of variables. In this section, we derive explicitly this change of variables, and then deduce an equivalent statement for the potential for κ classes $\mathcal{K}$.

Lemma 4.1 *Denote* $\mathcal{L} = \sum_{i=0}^{\infty} f_i(\mathbf{s})\partial_{t_i}$, *where* $f_i(\mathbf{s})$ *encodes the coefficient on* ∂_{t_i} *in* (13), *and let* $\mathbf{t} + \mathbf{f}(\mathbf{s}) = (t_0 + f_0(\mathbf{s}), t_1 + f_1(\mathbf{s}), \ldots)$ *then*

$$e^{\mathcal{L}}\mathcal{F}(\mathbf{t}) = \mathcal{F}(\mathbf{t} + \mathbf{f}(\mathbf{s})).$$

Proof This is a formal manipulation reminiscent of the exponential flow in differential geometry. For the benefit of readers who may not be familiar with this technique, we provide the sketch of an elementary proof. In one variable it is a Taylor expansion

[3] It is important to note that the coefficients of the vector field are functions in variables that commute with the ∂_{t_i}.

argument: imagine that $\mathsf{L} = f(s)\partial_t$ and F is a function of one variable t, then

$$
\begin{aligned}
e^{\mathsf{L}}\mathsf{F}(t) &= \left(1 + f(s)\partial_t + \frac{1}{2} f^2(s)\partial_t^2 + \cdots\right)\mathsf{F}(t) \\
&= \sum_{n=0}^{\infty} \frac{\mathsf{F}^{(n)}(t)}{n!}\,(f(s))^n \\
&= \mathsf{F}(t + f(s)) \text{ expanded at } t.
\end{aligned}
$$

In countably many variables one has

$$
e^{\mathsf{L}}\mathsf{F}(\mathbf{t}) = e^{\sum f_i(\mathbf{s})\partial_{t_i}}\mathsf{F}(\mathbf{t}) = \prod e^{f_i(\mathbf{s})\partial_{t_i}}\mathsf{F}(\mathbf{t}) = \mathsf{F}(\mathbf{t} + \mathbf{f}(\mathbf{s})),
$$

where the variables have been translated one at a time. □

Corollary 4.2 *The potential* S *is obtained from* F *via the change of variables encoded in the following generating function:*

$$
\sum_{i=0}^{\infty} t_i z^i = \left[z\left(1 - e^{-\sum_{k=0}^{\infty} s_k z^{k-1}}\right)\right]_+, \tag{18}
$$

where the subscript $+$ *denotes the truncation of the expression in brackets to terms with non-negative exponents for the variable* z.

Proof Combining Theorem 3.7 and Lemma 4.1, we obtain that $\mathsf{S}(u;\mathbf{s}) = \mathsf{F}(u; f_0(\mathbf{s}), f_1(\mathbf{s}), \ldots)$. Resumming the coefficients to express L in the form $\sum_{i=0}^{\infty} f_i(\mathbf{s})\partial_{t_i}$ one obtains

$$
\begin{aligned}
t_0 &= f_0(\mathbf{s}) = s_0 - s_0 s_1 + \frac{s_0^2}{2!} s_2 + s_0\frac{s_1^2}{2!} - \frac{s_0^3}{3!} s_3 - \frac{s_0^2}{2!} s_1 s_2 - s_0\frac{s_1^3}{3!} + \cdots, \\
t_1 &= f_1(\mathbf{s}) = s_1 - s_0 s_2 - \frac{s_1^2}{2!} + \frac{s_0^2}{2!} s_3 + s_0 s_1 s_2 + \frac{s_1^3}{3!} - \frac{s_0^3}{3!} s_4 - \cdots, \\
t_2 &= f_2(\mathbf{s}) = s_2 - s_0 s_3 - s_1 s_2 + \frac{s_0^2}{2!} s_4 + s_0 s_1 s_3 + s_0\frac{s_2^2}{2!} + \frac{s_1^2}{2!} s_2 - \cdots, \\
t_3 &= f_3(\mathbf{s}) = s_3 - s_0 s_4 - s_1 s_3 - \frac{s_2^2}{2!} + \frac{s_0^2}{2!} s_5 + s_0 s_1 s_4 \\
&\qquad + s_0 s_2 s_3 + \frac{s_1^2}{2!} s_3 + s_1\frac{s_2^2}{2!} - \cdots, \\
&\;\;\cdots \\
t_n &= f_n(\mathbf{s}) = \sum_{\substack{m\in\mathbb{N} \\ \alpha\in\mathbb{N}^{m+1} \\ \sum i\alpha_i = n + (\sum \alpha_i - 1)}} (-1)^{1+\sum \alpha_i}\left(\frac{s_0^{\alpha_0} s_1^{\alpha_1}\cdots s_m^{\alpha_m}}{\alpha_0!\,\alpha_1!\cdots\alpha_m!}\right).
\end{aligned}
$$

On the other hand, expanding the right-hand side of (18), we have

$$\begin{aligned}
\left(z\left(1-e^{-\sum_{k=0}^{\infty} s_k z^{k-1}}\right)\right)_+ &= \left(z\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell+1}}{\ell!}\left(\sum_{k=0}^{\infty} s_k z^{k-1}\right)^{\ell}\right)_+\\
&= \left(\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell+1}}{\ell!}\frac{1}{z^{\ell-1}}\left(\sum_{k=0}^{\infty} s_k z^{k}\right)^{\ell}\right)_+\\
&= \left(\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell+1}}{\ell!}\frac{1}{z^{\ell-1}}\left(\sum_{\substack{m\in\mathbb{N}\\ \alpha\in\mathbb{N}^{m+1}\\ \sum\alpha_i=\ell}}\binom{\ell}{\alpha_0,\alpha_1,\ldots,\alpha_m}\prod_{i=0}^{m}(s_i z^i)^{\alpha_i}\right)\right)_+\\
&= \left(\sum_{\ell=1}^{\infty}\left(\sum_{\substack{m\in\mathbb{N}\\ \alpha\in\mathbb{N}^{m+1}\\ \sum\alpha_i=\ell}}(-1)^{1+\sum\alpha_i}\left(\frac{s_0^{\alpha_0}s_1^{\alpha_1}\cdots s_m^{\alpha_m}}{\alpha_0!\,\alpha_1!\cdots\alpha_m!}\right)z^{\sum i\alpha_i-(\ell-1)}\right)\right)_+ .
\end{aligned}$$

The coefficient on z^n is then given when $\ell=\sum_{i=0}^{m} i\alpha_i-(n-1)$ and is computed to be

$$\sum_{\substack{m\in\mathbb{N}\\ \alpha\in\mathbb{N}^{m+1}\\ \sum\alpha_i=\sum i\alpha_i-(n-1)}}(-1)^{1+\sum\alpha_i}\left(\frac{s_0^{\alpha_0}s_1^{\alpha_1}\cdots s_m^{\alpha_m}}{\alpha_0!\,\alpha_1!\cdots\alpha_m!}\right),$$

as desired. $\square$

We have at this point all the technical information necessary to give our proof of [17, Theorem 4.1]. Before doing so, we establish one final piece of notation.

Definition 4.3 For $i \geq 0$ and a countable set of variables $\mathbf{p} = p_1, p_2, \ldots$, we define the polynomials $S_i(\mathbf{p})$ via

$$\sum_{i=0}^{\infty} S_i(\mathbf{p})z^i = e^{\sum_{k=1}^{\infty} p_k z^k}.$$

Remark 4.4 The functions S_i appear in some literature (especially in the school of integrable systems) as *elementary Schur polynomials*. The reason is that by interpreting the variables p_k as (normalized) power sums,

$$p_k = \sum \frac{x_j}{j},$$

then S_i is the *complete symmetric polynomial of degree* k in the variables x_j, which also is the Schur polynomial associated to the one part partition k.

Theorem 4.5 *The generating functions* $\mathsf{K}(u; p_0, \mathbf{p})$ *and* $\mathsf{F}(u; t_0, t_1, \ldots)$ *agree when restricting the domain of* F *to the subspace* $t_0 = t_1 = 0$ *and applying the following transformation to the remaining variables:*

$$u = ue^{p_0},$$
$$t_i = -S_{i-1}(-\mathbf{p}).$$

Proof By the ω-string relation (Lemma 2.9), the function S is constant in s_0, therefore, one may restrict the domain to the hyperplane $s_0 = 0$. Imposing this restriction on the transformation given in (18), one deduces that the domain of the function F is restricted to the hyperplane $t_0 = 0$. After reindexing, one obtains

$$\sum_{i=0}^{\infty} t_{i+1} z^i = 1 - e^{\sum_{k=0}^{\infty} -s_{k+1} z^k}.$$

For $k \geq 0$, define $s_{k+1} = p_k$, to obtain the change of variables:

$$t_1 = 1 - e^{-p_0},$$
$$t_i = -e^{-p_0} S_{i-1}(-p_1, -p_2, \ldots), \quad \text{for } i \geq 2. \tag{19}$$

Combining the statement of Corollary 18 with Lemma 3.6, one obtains

$$\begin{aligned}\mathsf{K}(u; p_0, p_1, \ldots) &= \mathsf{S}(u; 0, p_0, p_1, \ldots) = \mathsf{S}(ue^{p_0}; 0, 0, p_1, \ldots)\\ &= \mathsf{F}(u; 0, 1 - e^{-p_0}, -e^{-p_0} S_1(-p_1, -p_2, \ldots), -e^{-p_0} S_2(-p_1, -p_2, \ldots), \ldots)\\ &= \mathsf{F}(ue^{p_0}; 0, 0, S_1(-p_1, -p_2, \ldots), S_2(-p_1, -p_2, \ldots), \ldots).\end{aligned}$$

□

5 Virasoro relations for potential

The goal of this section is to obtain a countable number of differential equations that annihilate the potential K, and determine recursively all intersection numbers of κ classes on $\overline{\mathsf{M}}_g$ from the unstable term $\frac{1}{24} p_0$. We start from the Virasoro constrains annihilating and determining the Gromov–Witten potential F.

Theorem 5.1 (Witten's conjecture, Kontsevich's theorem) *Consider the differential operators* L_n *defined as follows:*

$$L_{-1} = -\partial_{t_0} + \sum_{i=0}^{\infty} t_{i+1} \partial_{t_i} + \frac{t_0^2}{2}, \tag{20}$$

$$L_0 = -\frac{3}{2} \partial_{t_1} + \sum_{i=0}^{\infty} \frac{(2i+1)}{2} t_i \partial_{t_i} + \frac{1}{16}, \tag{21}$$

and for all positive n,

$$L_n = -\frac{(2n+3)!!}{2^{n+1}}\partial_{t_{n+1}} + \sum_{i=0}^{\infty}\left(\frac{(2i+2n+1)!!}{(2i-1)!!\,2^{n+1}}\right)t_i\partial_{t_{i+n}} + \frac{u^2}{2}\sum_{i=0}^{n-1}\left(\frac{(2i+1)!!\,(2n-2i-1)!!}{2^{n+1}}\right)\partial_{t_i}\partial_{t_{n-1-i}}. \tag{22}$$

For all $n \geq -1$, we have

$$L_n(e^{\mathcal{F}}) = 0. \tag{23}$$

Further, equations (23) determine recursions among the coefficients of $\mathcal{F}$ that recover uniquely the Gromov–Witten potential from the initial condition $\mathcal{F} = t_0^3/3! +$ higher order terms.

Equation (20) is equivalent to the string recursion (Lemma 2.8), equation (21) to dilaton (Lemma 2.11). For positive values of n, there is no geometric interpretation for the recursions on the coefficients of $\mathcal{F}$ given by (22).

Theorem 4.5 gives an explicit change of variables that equates the potential $\mathcal{K}$ with the restriction of $\mathcal{F}$ to the linear subspace $t_0 = 0$, $u = 1$. Essentially, since $\mathcal{F}$ is annihilated by the operators L_n, we produce operators annihilating $\mathcal{K}$ by rewriting the L_n's in the variables p_i. There are a couple subtle points to address:

- The function $\mathcal{F} = \mathcal{F} + \mathcal{F}_0/u^2$, where $\mathcal{F}_0$ is a multiple of t_0^3. Therefore,

$$(L_n e^{\mathcal{F}})_{|t_0=0} = (L_n e^{\mathcal{F}})_{|t_0=0},$$

  since for any n, L_n has at most a quadratic term in ∂_{t_0}.
- The operations of applying the differential operator L_n and restricting to the hyperplane $t_0 = 0$ do not commute. In Sect. 5.2, we use the string equation L_{-1} to replace the operator ∂_{t_0} with a differential operator in the remaining variables.

The computation is long and technical; we break it down into several subsections to give the reader a chance to isolate segments of it that may be of interest.

5.1 Auxiliary variables and notation

In this section we make some elementary changes of variables and introduce some notation, in order to simplify the computation. Let

$$\hat{t}_0 = -(t_1 - 1), \quad \hat{t}_i = -t_{i+1} \text{ for } i = -1 \text{ and } i \geq 1, \quad \hat{p}_k = -p_k \text{ for all } k \geq 0.$$

The translation of the variable t_1 by 1 is the well-known *dilaton shift* in Gromov–Witten theory, which has the effect of making the operators L_n homogeneous quadratic. The remaining reindexing and signs are chosen to simplify the change of variables (19) to

$$\sum_{i=0}^{\infty} \hat{t}_i z^i = e^{\sum_{k=0}^{\infty} \mathfrak{p}_k z^k}, \tag{24}$$

or equivalently

$$\hat{t}_i = e^{\mathfrak{p}_0} S_i(\mathfrak{p}_1, \mathfrak{p}_2, \ldots).$$

In the hatted variables, the operators L_n become

$$L_{-1} = \sum_{i=-1}^{\infty} \hat{t}_{i+1} \partial_{\hat{t}_i} + \frac{\hat{t}_{-1}^2}{2},$$

$$L_0 = \sum_{i=-1}^{\infty} \frac{(2i+3)}{2} \hat{t}_i \partial_{\hat{t}_i} + \frac{1}{16},$$

$$L_n = \sum_{i=-1}^{\infty} \left(\frac{(2i+2n+3)!!}{(2i+1)!!\, 2^{n+1}} \right) \hat{t}_i \partial_{\hat{t}_{i+n}} + \frac{u^2}{2} \sum_{i=-1}^{n-2} \left(\frac{(2i+3)!!\,(2n-2i-3)!!}{2^{n+1}} \right) \partial_{\hat{t}_i} \partial_{\hat{t}_{n-3-i}}.$$

5.2 Removing $\hat{t}_{-1} = -t_0$

In this section we derive operators $\tilde{L}_n$, for all $n \geq 0$ that annihilate the restriction $e^{F}|_{\hat{t}_{-1}=0}$. From the operator L_{-1} we have

$$\partial_{\hat{t}_{-1}} e^{F} = -\left(\sum_{i=0}^{\infty} \frac{\hat{t}_{i+1}}{\hat{t}_0} \partial_{\hat{t}_i} + \frac{\hat{t}_{-1}^2}{2\hat{t}_0} \right) e^{F}. \tag{25}$$

We may write the operators L_n making sure that any monomial containing $\partial_{\hat{t}_{-1}}$ has it as the rightmost term. Then replacing $\partial_{\hat{t}_{-1}}$ with the right-hand side of (25) produces an operator $\bar{L}_n$ still annihilating e^{F} and not containing any derivative in $\hat{t}_{-1}$. The application of $\bar{L}_n$ commutes with restriction to the hyperplane $\hat{t}_{-1} = 0$, so we may define $\tilde{L}_n = (\bar{L}_n)|_{\hat{t}_{-1}=0}$. Up to some tedious but straightforward computation we have proved the following.

Lemma 5.2 *For all $n \geq 0$, the operators $\hat{L}_n$ defined below annihilate $(e^{F})_{|\hat{t}_{-1}=0}$:*

$$\hat{L}_0 = \sum_{i=0}^{\infty} \frac{2i+3}{2}\hat{t}_i\partial_{\hat{t}_i} + \frac{1}{16},$$

$$\hat{L}_1 = \sum_{i=0}^{\infty} \frac{(2i+3)(2i+5)}{4}\hat{t}_i\partial_{\hat{t}_{i+1}} + \frac{u^2}{8\hat{t}_0^2}\left(\sum_{i=0}^{\infty}\left(\hat{t}_{i+2} - \frac{\hat{t}_1\hat{t}_{i+1}}{\hat{t}_0}\right)\partial_{\hat{t}_i} + \sum_{i,j=0}^{\infty}\hat{t}_{i+1}\hat{t}_{j+1}\partial_{\hat{t}_i}\partial_{\hat{t}_j}\right),$$

$$\hat{L}_2 = \sum_{i=0}^{\infty} \frac{(2i+3)(2i+5)(2i+7)}{8}\hat{t}_i\partial_{\hat{t}_{i+2}} + \frac{3u^2}{8\hat{t}_0^2}\left(\sum_{i=0}^{\infty}\left(\hat{t}_{i+1}(1-\hat{t}_0\partial_{\hat{t}_0})\right)\partial_{\hat{t}_i}\right),$$

$$\hat{L}_n = \sum_{i=0}^{\infty}\left(\frac{(2i+2n+3)!!}{(2i+1)!!\,2^{n+1}}\right)\hat{t}_i\partial_{\hat{t}_{i+n}} + \frac{u^2}{2}\sum_{i=0}^{n-3}\left(\frac{(2i+3)!!\,(2n-2i-3)!!}{2^{n+1}}\right)\partial_{\hat{t}_i}\partial_{\hat{t}_{n-3-i}} - \frac{u^2}{\hat{t}_0}\frac{(2n-1)!!}{2^{n+1}}\left(\partial_{\hat{t}_{n-3}} + \sum_{i=0}^{\infty}\hat{t}_{i+1}\partial_{\hat{t}_i}\partial_{\hat{t}_{n-2}}\right).$$

The need to treat the first three cases separately comes from the fact that the variable $\hat{t}_0$ and the remaining variables appear asymmetrically in the right-hand side of (25). We now rewrite these operators in the variables $\hat{p}_k$. We separate the computation in two parts, corresponding to the coefficients of u^0 and u^2 in the operators. We start by carrying out explicitly the change of variable for the first operator.

5.3 Warm-up: $\hat{L}_0$

We first focus on the term

$$\sum_{i=0}^{\infty}\hat{t}_i\partial_{\hat{t}_i} = \sum_{i=0}^{\infty}\hat{t}_i\sum_{j=0}^{\infty}\frac{\partial\hat{p}_j}{\partial\hat{t}_i}\partial_{\hat{p}_j}. \tag{26}$$

Consider the change of variables (24). By taking a partial derivative with respect to a variable $\hat{t}_i$, we obtain

$$\sum_{j=0}^{\infty}\frac{\partial\hat{p}_j}{\partial\hat{t}_i}z^{j-i} = e^{-\sum_{k=0}^{\infty}\hat{p}_k z^k}. \tag{27}$$

We observe that (27) holds for any choice of $\hat{t}$, which implies that the partial derivatives $\frac{\partial \tilde{p}_j}{\partial \hat{t}_i}$ depend only on the difference $j - i$. In other words, the Jacobian of the change of variables may be thought as an infinite upper triangular matrix which is constant along translates of the diagonal. It follows that the coefficient of $\partial_{\tilde{p}_m}$ in (26) is the coefficient of z^m in the product of generating series:

$$\sum_{i=0}^{\infty} \hat{t}_i \partial_{\hat{t}_i} = \sum_{m=0}^{\infty} \left[\left(\sum_{i=0}^{\infty} \hat{t}_i z^i \right) \left(\sum_{j=0}^{\infty} \frac{\partial \tilde{p}_j}{\partial \hat{t}_i} z^{j-i} \right) \right]_{z^m} \partial_{\tilde{p}_m}$$

$$= \sum_{m=0}^{\infty} \left(e^{\sum_{k=0}^{\infty} \tilde{p}_k z^k} e^{-\sum_{k=0}^{\infty} \tilde{p}_k z^k} \right)_{z^m} \partial_{\tilde{p}_m} = \partial_{\tilde{p}_0}.$$

We return to the change of variables (24) and apply the operator $z\partial_z$ to obtain

$$\sum_{i=0}^{\infty} i \hat{t}_i z^i = e^{\sum_{k=0}^{\infty} \tilde{p}_k z^k} \sum_{k=0}^{\infty} k \tilde{p}_k z^k.$$

With an analogous argument to the previous case, we obtain

$$\sum_{i=0}^{\infty} i \hat{t}_i \partial_{\hat{t}_i} = \sum_{m=0}^{\infty} \left[\left(e^{\sum_{k=0}^{\infty} \tilde{p}_k z^k} \sum_{k=0}^{\infty} k \tilde{p}_k z^k \right) \left(e^{-\sum_{k=0}^{\infty} \tilde{p}_k z^k} \right) \right]_{z^m} \partial_{\tilde{p}_m}$$

$$= \sum_{m=0}^{\infty} m \tilde{p}_m \partial_{\tilde{p}_m}.$$

We may now express the operator $\tilde{L}_0$ in the variables $p_i = -\tilde{p}_i$ as

$$\tilde{L}_0 = \sum_{i=0}^{\infty} \frac{(2i+3)}{2} \hat{t}_i \partial_{\hat{t}_i} + \frac{1}{16} = -\frac{3}{2} \partial_{p_0} + \sum_{m=0}^{\infty} m p_m \partial_{p_m} + \frac{1}{16}.$$

One may recognize that $\tilde{L}_0(e^{K}) = 0$ (which implies $\tilde{L}_0(K) = 1/24$), is equivalent to the statement that κ_0 on $\overline{M}_g$ has degree $2g - 2$ (Lemma 2.10).

5.4 Bell polynomials

We now focus on the $\hbar$ coefficients of the operators $\tilde{L}_n$, with $n > 0$. As a preliminary step, we change variables to expressions of the form

$$\sum_{i=0}^{\infty} i^d \hat{t}_i z^i = (z\partial_z)^d \left(\sum_{i=0}^{\infty} \hat{t}_i z^i \right).$$

Iterated applications of the operator $z\,\partial_z$ to the right-hand side of (24) are described by a variation of the classical *Faà di Bruno formula* [3,8], giving the result in terms of Bell polynomials [4].

Definition 5.3 For $d \geq 0$, the *d-th Bell polynomial* $B_d(x_1,\dots,x_d)$ is defined by

$$\sum_{d=0}^{\infty} B_d(x_1,\dots,x_d)\frac{y^d}{d!} = e^{\left(\sum_{j=1}^{\infty} x_j \frac{y^j}{j!}\right)}.$$

The first few Bell polynomials are $B_0 = 1$, $B_1 = x_1$, $B_2 = x_2 + x_1^2$, $B_3 = x_3 + 3x_2x_1 + x_1^3$, $B_4 = x_4 + 4x_3x_1 + 3x_2^2 + 6x_2x_1^2 + x_1^4$. The coefficient of a monomial $\prod x_j^{m_j}$ in the Bell polynomial B_n (with $n = \sum jm_j$) counts the number of ways to partition a set with n elements into a collection of m_1 unlabeled subsets of cardinality 1, m_2 unlabeled subsets of cardinality 2, etc.

Lemma 5.4 *For all $i \geq 0$, let*

$$\tilde{q}_i = \sum_{k=0}^{\infty} k^i \tilde{p}_k z^k.$$

Then,

$$(z\partial_z)^d e^{\tilde{q}_0} = e^{\tilde{q}_0} B_d(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_d).$$

Proof This statement is a ready adaptation of the classical Faà di Bruno formula, which expresses the successive derivatives of a composite function $f(g(z))$ in terms of derivatives of f and of g. In our case the function f is an exponential function, hence all of its derivatives are equal to itself, and since we are replacing the derivation operator with $z\,\partial_z$ the $\tilde{q}_i$'s play the role of the successive derivatives of g. While it is at this point an exercise to complete the proof this way, we also provide a brief sketch of a bijective proof, which we find more conceptually satisfactory. Note that $z\partial_z\tilde{q}_i = \tilde{q}_{i+1}$, and that applying $z\partial_z$ to $e^{\tilde{q}_0}\tilde{q}_{i_1}\dots\tilde{q}_{i_n}$ results in a sum of terms, where one term is obtained by multiplying m by $\tilde{q}_1$ (aka "producing a new $\tilde{q}_1$"), the other terms are obtained by raising by one of the indices of one of the $\tilde{q}_{i_j}^1 s$. Now let us imagine performing successive applications of $z\partial_z$ to $e^{\tilde{q}_0}$ maintaining at all times all summands distinct, and let us call "level i" the i-th application of the operator. We want to put in bijection the number of times a term of the form $e^{\tilde{q}_0}\tilde{q}_{i_1}\dots\tilde{q}_{i_n}$ appears (necessarily at level $d = i_1 + \dots + i_n$) with the number of partitions of the set $[d]$ in n subsets of cardinality $i_1,\dots,i_n$. For each variable $\tilde{q}_{i_j}$, there are i_j distinct levels where the index of such variable has been increased. We associate to such variable the subset of such levels, and by running over all variables that are appearing we obtain a partition of $[d]$. It is at this point easy to see that this construction realizes the bijection we desire. We illustrate our proof in a simple example. Consider $(z\partial_z)^3 e^{\tilde{q}_0} = 3e^{\tilde{q}_0}\tilde{q}_2\tilde{q}_1 + \cdots$ and let us observe that 3 is obtained as the ways to partition the set $[3]$ into two subsets of size 2 and 1. First we write out all levels up to 3:

$$
\begin{aligned}
&\text{Level 1:}\quad e^{\tilde q_0}\tilde q_1\\
&\text{Level 2:}\quad e^{\tilde q_0}\overline{\tilde q}_1\tilde q_1 + e^{\tilde q_0}\tilde q_2\\
&\text{Level 3:}\quad e^{\tilde q_0}\overline{\overline{\tilde q}}_1\overline{\tilde q}_1\tilde q_1 + e^{\tilde q_0}\overline{\tilde q}_2\tilde q_1 + e^{\tilde q_0}\overline{\tilde q}_1\tilde q_2 + e^{\tilde q_0}\overline{\overline{\tilde q}}_1\tilde q_2 + e^{\tilde q_0}\tilde q_3.
\end{aligned}
$$

The bars over the variables are purely combinatorial decorations that keep track of at which level a given variable appears. We then observe that the monomials that reduce to $e^{\tilde q_0}\tilde q_2\tilde q_1$ when forgetting the decorations are in bijection with two part non-trivial partitions of $[3]$ where the singleton corresponds to the level where the variable $\tilde q_1$ appears, and the two part subset corresponds to two levels: the level where the variable first appears as a $\tilde q_1$ and the level where its index is raised by one. □

Lemma 5.5 *With notation as in Lemma 5.4, for any nonnegative integers n, d we have*

$$\sum_{i=0}^{\infty} i^d\,\hat t_i\,\partial_{\hat t_{i+n}} = \sum_{m=0}^{\infty}\,[B_d(\tilde q_1,\dots,\tilde q_d)]_{z^m}\,\partial_{\tilde p_{m+n}}.$$

Proof In analogy to the computations in Sect. 5.3, we have

$$
\begin{aligned}
\sum_{i=0}^{\infty} i^d\,\hat t_i\,\partial_{\hat t_{i+n}} &= \sum_{m=0}^{\infty}\left[(z\partial_z)^d e^{\tilde q_0}\left(\sum_{j=0}^{\infty}\frac{\partial\tilde p_j}{\partial\hat t_{i+n}}z^{j-i-n}\right)\right]_{z^{m-n}}\partial_{\tilde p_m}\\
&= \sum_{m=0}^{\infty}\left[e^{\tilde q_0}B_d(\tilde q_1,\dots,\tilde q_d)\,e^{-\tilde q_0}\right]_{z^{m-n}}\partial_{\tilde p_m},
\end{aligned}
$$

and the result follows from canceling the exponential terms and reindexing the summation. □

Lemma 5.5 allows to perform the change of variables for the u^0 coefficient of the operators $\tilde{\mathcal{L}}_n$.

Lemma 5.6 *For $n \geq 0$, let $\left(i+\frac{3}{2}\right)\left(i+\frac{5}{2}\right)\cdots\left(i+\frac{2n+3}{2}\right) = \sum_{d=0}^{n+1}\alpha_{n,d}\,i^d$. Then*

$$[\tilde{\mathcal{L}}_n]_{u^0} = \sum_{d=0}^{n+1}\alpha_{n,d}\left(\sum_{m=0}^{\infty}[B_d(\tilde q_1,\dots,\tilde q_d)]_{z^m}\,\partial_{\tilde p_{m+n}}\right).$$

We now turn our attention to the u^2 coefficient.

5.5 The u^2 coefficient: $n\,!\,2$

We recall that from (27) we can express

$$\partial_{\hat{t}_i} = \sum_{j=0}^{\infty} \frac{\partial p_j}{\partial \hat{t}_i} \partial_{p_j} = e^{-!p_0} \sum_{j=0}^{\infty} S_{j-i}(-!\mathbf{p}) \partial_{p_j}. \tag{28}$$

It will be useful to take the partial derivative of (27) with respect to a variable p_m and divide by z^m to obtain

$$\sum_{j=0}^{\infty} \frac{\partial^2 p_j}{\partial p_m \partial \hat{t}_i} z^{j-i-m} = - \; e^{-\sum_{k=0}^{\infty} p_k z^k}.$$

We may now use generating functions to perform the change of variables for an operator of second order

$$\begin{aligned} \partial_{\hat{t}_l}\partial_{\hat{t}_i} &= \sum_{m=0}^{\infty} \frac{\partial p_m}{\partial \hat{t}_l} \partial_{p_m} \sum_{j=0}^{\infty} \frac{\partial p_j}{\partial \hat{t}_i} \partial_{p_j} \\ &= \sum_{j,m=0}^{\infty} \frac{\partial p_m}{\partial \hat{t}_l} \frac{\partial p_j}{\partial \hat{t}_i} \partial_{p_m}\partial_{p_j} + \sum_{m=0}^{\infty} \frac{\partial p_m}{\partial \hat{t}_l} \sum_{j=0}^{\infty} \frac{\partial^2 p_j}{\partial p_m \partial \hat{t}_i} \partial_{p_j} \\ &= \sum_{j,m=0}^{\infty} e^{-2p_0} S_{m-l}(-!\mathbf{p}) S_{j-i}(-!\mathbf{p}) \partial_{p_m}\partial_{p_j} \\ &\qquad + \sum_{j=0}^{\infty} \left(\left[e^{-\sum_{k=0}^{\infty} p_k z^k} \right] \left[- e^{-\sum_{k=0}^{\infty} p_k z^k} \right] \right)_{z^{j-i-l}} \partial_{p_j} \\ &= e^{-2p_0} \left[\sum_{j,m=0}^{\infty} S_{m-l}(-!\mathbf{p}) S_{j-i}(-!\mathbf{p}) \partial_{p_m}\partial_{p_j} - \sum_{j=0}^{\infty} S_{j-i-l}(-2\mathbf{p}) \partial_{p_j} \right]. \end{aligned} \tag{29}$$

The last expression that needs to be addressed to complete the change of variables for $L_n, n\,!\,3$, is

$$\begin{aligned}
\sum_{i=0}^{\infty}\hat t_{i+1}\partial_{\hat t_i}\partial_{\hat t_{n-2}} &= \sum_{i=0}^{\infty}\left(\sum_{m=0}^{\infty}\hat t_{i+1}\frac{\partial p_m}{\partial \hat t_i}\partial_{p_m}\right)\left(\sum_{j=0}^{\infty}\frac{\partial p_j}{\partial \hat t_{n-2}}\partial_{p_j}\right)\\
&= \sum_{i,j,m=0}^{\infty}\hat t_{i+1}\frac{\partial p_m}{\partial \hat t_i}\frac{\partial p_j}{\partial \hat t_{n-2}}\partial_{p_m}\partial_{p_j} + \sum_{i,m=0}^{\infty}\hat t_{i+1}\frac{\partial p_m}{\partial \hat t_i}\sum_{j=0}^{\infty}\frac{\partial^2 p_j}{\partial p_m\partial \hat t_{n-2}}\partial_{p_j}\\
&= \sum_{j,m=0}^{\infty}\left[\left(e^{\sum_{k=0}^{\infty}p_k z^k}-e^{p_0}\right)\left(e^{-\sum_{k=0}^{\infty}p_k z^k}\right)\right]_{z^{m+1}} e^{-p_0}S_{j-n+2}(-\mathbf{p})\partial_{p_m}\partial_{p_j}\\
&\quad+\sum_{j=0}^{\infty}\left[\left(e^{\sum_{k=0}^{\infty}p_k z^k}-e^{p_0}\right)\left(e^{-\sum_{k=0}^{\infty}p_k z^k}\right)\left(-e^{-\sum_{k=0}^{\infty}p_k z^k}\right)\right]_{z^{j-n+3}}\partial_{p_j}\\
&= e^{-p_0}\left(\sum_{j,m=0}^{\infty}-S_{m+1}(-\mathbf{p})S_{j-n+2}(-\mathbf{p})\partial_{p_m}\partial_{p_j}\right.\\
&\qquad\left.+\sum_{j=0}^{\infty}\left(S_{j-n+3}(-2\mathbf{p})-S_{j-n+3}(-\mathbf{p})\right)\partial_{p_j}\right).
\end{aligned}\tag{30}$$

To deal with the case $n=2$, we compute

$$\begin{aligned}
\sum_{i=0}^{\infty}\hat t_{i+1}\partial_{\hat t_i} &= \sum_{i=0}^{\infty}\left[\sum_{m=0}^{\infty}\hat t_{i+1}\frac{\partial p_m}{\partial \hat t_i}\partial_{p_m}\right]\\
&= \sum_{m=0}^{\infty}\left[\left(e^{\sum_{k=0}^{\infty}p_k z^k}-e^{p_0}\right)\left(e^{-\sum_{k=0}^{\infty}p_k z^k}\right)\right]_{z^{m+1}}\partial_{p_m}\\
&= \sum_{m=0}^{\infty}-S_{m+1}(-\mathbf{p})\partial_{p_m}.
\end{aligned}$$

Combining (28), (29), (30), (31) and appropriately reindexing we obtain the following.

Lemma 5.7 *For $n\geq 2$,*

$$\begin{aligned}
[\tilde L_n]_{u^2} = -\frac{e^{-2p_0}}{2}&\left(\sum_{i=0}^{n-1}\frac{(2i+1)!!\,(2n-2i-1)!!}{2^{n+1}}\right)\left(\sum_{j=0}^{\infty}S_j(-2\mathbf{p})\partial_{p_{j+n-3}}\right)\\
+\frac{e^{-2p_0}}{2}&\left(\sum_{i=0}^{n-1}\frac{(2i+1)!!\,(2n-2i-1)!!}{2^{n+1}}\right.\\
&\left.\cdot\sum_{j,m=0}^{\infty}S_m(-\mathbf{p})S_j(-\mathbf{p})\partial_{p_{m+n-2-i}}\partial_{p_{j+i-1}}\right).
\end{aligned}$$

5.6 The u^2 coefficient: $n = 1$

In order to compute $\widehat{L}_1$ we must perform the following computations:

$$
\begin{aligned}
\sum_{i=0}^{\infty} \hat{t}_{i+2}\partial_{\hat{t}_i} &= \sum_{i=0}^{\infty}\left[\sum_{m=0}^{\infty} \hat{t}_{i+2}\frac{\partial \tilde{p}_m}{\partial \hat{t}_i}\right]\partial_{\tilde{p}_m} \\
&= \sum_{m=0}^{\infty}\left(\left[e^{\sum_{k=0}^{\infty}\tilde{p}_k z^k} - e^{\tilde{p}_0}(1+\hbar p_1 z)\right]\left[e^{-\sum_{k=0}^{\infty}\tilde{p}_k z^k}\right]\right)_{z^{m+2}}\partial_{\tilde{p}_m} \\
&= \sum_{m=0}^{\infty} S_1(-\hbar\mathbf{p})\,S_{m+1}(-\hbar\mathbf{p}) - S_{m+2}(-\hbar\mathbf{p})\partial_{\tilde{p}_m},
\end{aligned}
\tag{31}
$$

$$
\sum_{i=0}^{\infty}\frac{\hat{t}_1\hat{t}_{i+1}}{\hat{t}_0}\partial_{\hat{t}_i} = \sum_{m=0}^{\infty} S_1(-\hbar\mathbf{p})\,S_{m+1}(-\hbar\mathbf{p})\partial_{\tilde{p}_m},
\tag{32}
$$

$$
\begin{aligned}
\sum_{i,j=0}^{\infty}\hat{t}_{i+1}\hat{t}_{j+1}\partial_{\hat{t}_i}\partial_{\hat{t}_j} &= \sum_{i,j=0}^{\infty}\hat{t}_{i+1}\hat{t}_{j+1}\left[\sum_{l=0}^{\infty}\frac{\partial\tilde{p}_l}{\partial\hat{t}_i}\partial_{\tilde{p}_l}\right]\left[\sum_{m=0}^{\infty}\frac{\partial\tilde{p}_m}{\partial\hat{t}_j}\partial_{\tilde{p}_m}\right] \\
&= \sum_{i,j,l,m=0}^{\infty}\hat{t}_{i+1}\hat{t}_{j+1}\frac{\partial\tilde{p}_l}{\partial\hat{t}_i}\frac{\partial\tilde{p}_m}{\partial\hat{t}_j}\partial_{\tilde{p}_l}\partial_{\tilde{p}_m} \\
&\quad + \sum_{i,j,l=0}^{\infty}\hat{t}_{i+1}\hat{t}_{j+1}\frac{\partial\tilde{p}_l}{\partial\hat{t}_i}\sum_{m=0}^{\infty}\frac{\partial^2\tilde{p}_m}{\partial\tilde{p}_l\partial\hat{t}_j}\partial_{\tilde{p}_m} \\
&= \sum_{l,m=0}^{\infty}\left(\left[e^{\sum_{k=0}^{\infty}\tilde{p}_k z^k} - e^{\tilde{p}_0}\right]\left[e^{-\sum_{k=0}^{\infty}\tilde{p}_k z^k}\right]\right)_{z^{l+1}} \\
&\qquad\cdot\left(\left[e^{\sum_{k=0}^{\infty}\tilde{p}_k z^k} - e^{\tilde{p}_0}\right]\left[e^{-\sum_{k=0}^{\infty}\tilde{p}_k z^k}\right]\right)_{z^{m+1}}\partial_{\tilde{p}_l}\partial_{\tilde{p}_m} \\
&\quad + \sum_{m=0}^{\infty}\left(\left[e^{\sum_{k=0}^{\infty}\tilde{p}_k z^k} - e^{\tilde{p}_0}\right]^2\left[e^{-\sum_{k=0}^{\infty}\tilde{p}_k z^k}\right]\left[-e^{-\sum_{k=0}^{\infty}\tilde{p}_k z^k}\right]\right)_{z^{m+2}}\partial_{\tilde{p}_m} \\
&= \left[\sum_{l,m=0}^{\infty} S_{l+1}(-\hbar\mathbf{p})S_{m+1}(-\hbar\mathbf{p})\partial_{\tilde{p}_l}\partial_{\tilde{p}_m} + \sum_{m=0}^{\infty}(2S_{m+2}(-\hbar\mathbf{p}) - S_{m+2}(-2\hbar\mathbf{p}))\partial_{\tilde{p}_m}\right].
\end{aligned}
\tag{33}
$$

Combining the results of (31), (32), (33), we obtain the following.

Lemma 5.8

$$
\begin{aligned}
[\widehat{L}_1]_{u^2} = \frac{e^{-2\tilde{p}_0}}{8}\Bigg[\sum_{m=0}^{\infty}&(S_{m+2}(-\hbar\mathbf{p}) - S_{m+2}(-2\hbar\mathbf{p}))\partial_{\tilde{p}_m} \\
&+ \sum_{l,m=0}^{\infty} S_{l+1}(-\hbar\mathbf{p})\,S_{m+1}(-\hbar\mathbf{p})\partial_{\tilde{p}_l}\partial_{\tilde{p}_m}\Bigg].
\end{aligned}
$$

5.7 Proof of Theorem 1.2

The first part of Theorem 1.2 is proved by combining the results of Lemmas 5.2, 5.6, 5.7 and 5.8, and switching back to the unhatted variables $\hat{p}_k = -!\, p_k$. To prove that the recursions obtained from the vanishing of coefficients of $L_n(e^{K})$ reconstruct K from the "unstable" term $1/24\, p_0$, one observes that each $\hat{L}_n$ has a term of the form $A\,\partial_{p_n}$, with $A \in \mathbb{Q} \neq 0$; this means that the vanishing of a coefficient of $L_n(e^{K})$ compares the intersection number of a monomial m containing κ_n with a combination of other intersection numbers determined by the remaining terms of $\hat{L}_n$. A direct analysis of the remaining terms shows that the monomials that are compared to m are either strictly shorter than m, or they correspond to intersection numbers on lower genus. This proves that any monomial can inductively be computed from the "genus one" term $1/24\, p_0$. We illustrate this strategy for $g = 2, 3$ in the next section.

6 Recursions for κ classes

In this section we collect some of the relations among κ classes that are produced by the vanishing of coefficients of $\hat{L}_n(e^{K})$. We choose to exhibit a set of relations that inductively reconstruct all intersection numbers in genus 2 and 3. Throughout this section we denote $[\kappa^I]_g := \int_{\overline{\mathcal{M}}_g} \kappa^I$. We extend this notation to the unstable term, and define $[\kappa_0]_1 := 1/24$. For $g \geq 2$, we only consider monomials with no factor of κ_0, since $[\kappa^I \kappa_0^n]_g = (2g-2)^n [\kappa^I]_g$. In genus 2, we have the following:

$$[\hat{L}_3(e^{K})]_1 : \quad [\kappa_3]_2 = \frac{13}{630}[\kappa_0]_1 + \frac{1}{210}[\kappa_0]_1^2 = \frac{1}{1152},$$

$$[\hat{L}_1(e^{K})]_{p_2} : \quad [\kappa_2\kappa_1]_2 = \frac{48}{15}[\kappa_3]_2 + \frac{1}{30}[\kappa_0]_1 = \frac{1}{240},$$

$$[\hat{L}_1(e^{K})]_{p_1^2} : \quad [\kappa_1^3]_2 = \frac{8}{3}[\kappa_2\kappa_1]_2 - \frac{8}{15}[\kappa_3]_2 + \frac{1}{15}[\kappa_0]_1^2 + \frac{1}{10}[\kappa_0]_1 = \frac{43}{2880}.$$

A set of reconstructing relations in genus 3 is given by

$$[\hat{L}_6(e^{K})]_1 : \quad [\kappa_6]_3 = \frac{1}{99}[\kappa_3]_2 + \frac{1}{1287}[\kappa_2\kappa_1]_2 + \frac{1}{715}[\kappa_3]_2[\kappa_0]_1,$$

$$[\hat{L}_1(e^{K})]_{p_5} : \quad [\kappa_5\kappa_1]_3 = 12[\kappa_6]_3 + \frac{1}{30}[\kappa_3]_2,$$

$$[\hat{L}_2(e^{K})]_{p_4} : \quad [\kappa_4\kappa_2]_3 = \frac{136}{7}[\kappa_6]_3 + \frac{4}{35}[\kappa_3]_2 + \frac{1}{35}[\kappa_3]_2[\kappa_0]_1,$$

$$[\hat{L}_3(e^{K})]_{p_3} : \quad [\kappa_3^2]_3 = \frac{136}{7}[\kappa_6]_3 + \frac{38}{315}[\kappa_3]_2 + \frac{1}{63}[\kappa_2\kappa_1]_2 - [\kappa_3]_2^2 + \frac{31}{630}[\kappa_3]_2[\kappa_0]_1 + \frac{1}{210}[\kappa_3]_2[\kappa_0]_1^2,$$

$$[\hat{L}_1(e^{K})]_{p_4 p_1} : \quad [\kappa_4\kappa_1^2]_3 = -\frac{32}{15}[\kappa_6]_3 + \frac{128}{15}[\kappa_5\kappa_1]_3 + \frac{4}{3}[\kappa_4\kappa_2]_3$$

$$+ \frac{7}{30}[\kappa_3]_2 + \frac{1}{30}[\kappa_2\kappa_1]_2 + \frac{1}{15}[\kappa_3]_2[\kappa_0]_1,$$

$$[\not{L}_1(e^{\kappa})]_{p_3p_2}: \quad [\kappa_3\kappa_2\kappa_1]_3 = -\frac{16}{5}[\kappa_6]_3 + \frac{28}{5}[\kappa_4\kappa_2]_3 + \frac{16}{5}[\kappa_3^2]_3 + \frac{1}{6}[\kappa_3]_2 + \frac{1}{10}[\kappa_2\kappa_1]_2 + \frac{16}{5}[\kappa_3]_2^2 - [\kappa_3]_2[\kappa_2\kappa_1]_2 + \frac{1}{30}[\kappa_3]_2[\kappa_0]_1,$$

$$[\not{L}_2(e^{\kappa})]_{p_2^2}: \quad [\kappa_2^3]_3 = -\frac{288}{35}[\kappa_6]_3 + \frac{56}{5}[\kappa_4\kappa_2]_3 + \frac{6}{35}[\kappa_3]_2 + \frac{2}{7}[\kappa_2\kappa_1]_2 + \frac{1}{35}[\kappa_3]_2[\kappa_0]_1 + \frac{2}{35}[\kappa_2\kappa_1]_2[\kappa_0]_1,$$

$$[\not{L}_1(e^{\kappa})]_{p_2^2p_1}: \quad [\kappa_2^2\kappa_1^2]_3 = -\frac{32}{15}[\kappa_5\kappa_1]_3 - \frac{32}{15}[\kappa_4\kappa_2]_3 + \frac{32}{5}[\kappa_3\kappa_2\kappa_1]_3 + \frac{4}{3}[\kappa_2^3]_3 + \frac{11}{30}[\kappa_3]_2 + \frac{5}{6}[\kappa_2\kappa_1]_2 + \frac{1}{15}[\kappa_1^3]_2 + \frac{32}{5}[\kappa_3]_2[\kappa_2\kappa_1]_2 - 2[\kappa_2\kappa_1]_2^2 + \frac{1}{15}[\kappa_3]_2[\kappa_0]_1 + \frac{1}{5}[\kappa_2\kappa_1]_2[\kappa_0]_1,$$

$$[\not{L}_1(e^{\kappa})]_{p_3p_1^2}: \quad [\kappa_3\kappa_1^3]_3 = -\frac{16}{5}[\kappa_5\kappa_1]_3 - \frac{8}{15}[\kappa_3^2]_3 + \frac{28}{5}[\kappa_4\kappa_1^2]_3 + \frac{8}{3}[\kappa_3\kappa_2\kappa_1]_3 + \frac{29}{30}[\kappa_3]_2 + \frac{8}{15}[\kappa_2\kappa_1]_2 + \frac{1}{30}[\kappa_1^3]_2 - \frac{8}{15}[\kappa_3]_2^2 + \frac{8}{3}[\kappa_3]_2[\kappa_2\kappa_1]_2 - [\kappa_3]_2[\kappa_1^3]_2 + \frac{1}{2}[\kappa_3]_2[\kappa_0]_1 + \frac{2}{15}[\kappa_2\kappa_1]_2[\kappa_0]_1 + \frac{1}{15}[\kappa_3]_2[\kappa_0]_1^2,$$

$$[\not{L}_1(e^{\kappa})]_{p_2p_1^3}: \quad [\kappa_2\kappa_1^4]_3 = -\frac{16}{5}[\kappa_4\kappa_1^2]_3 - \frac{8}{5}[\kappa_3\kappa_2\kappa_1]_3 + \frac{16}{5}[\kappa_3\kappa_1^3]_3 + 4[\kappa_2^2\kappa_1^2]_3 + \frac{9}{10}[\kappa_3]_2 + \frac{19}{5}[\kappa_2\kappa_1]_2 + \frac{29}{30}[\kappa_1^3]_2 - \frac{8}{5}[\kappa_3]_2[\kappa_2\kappa_1]_2 + \frac{16}{5}[\kappa_3]_2[\kappa_1^3]_2 + 8[\kappa_2\kappa_1]_2^2 - 4[\kappa_2\kappa_1]_2[\kappa_1^3]_2 + \frac{1}{5}[\kappa_3]_2[\kappa_0]_1 + \frac{17}{10}[\kappa_2\kappa_1]_2[\kappa_0]_1 + \frac{7}{30}[\kappa_1^3]_2[\kappa_0]_1 + \frac{1}{5}[\kappa_2\kappa_1]_2[\kappa_0]_1^2,$$

$$[\not{L}_1(e^{\kappa})]_{p_1^5}: \quad [\kappa_1^6]_3 = -\frac{16}{3}[\kappa_3\kappa_1^3]_3 + \frac{20}{3}[\kappa_2\kappa_1^4]_3 + \frac{17}{10}[\kappa_3]_2 + \frac{35}{6}[\kappa_2\kappa_1]_2 + 12[\kappa_1^3]_2 - \frac{16}{3}[\kappa_3]_2[\kappa_1^3]_2$$

$$+\frac{80}{3}[\kappa_2\kappa_1]_2[\kappa_1^3]_2 - 10[\kappa_1^3]_2^2 + \frac{1}{3}[\kappa_3]_2[\kappa_0]_1 + \frac{4}{3}[\kappa_2\kappa_1]_2[\kappa_0]_1 + \frac{17}{3}[\kappa_1^3]_2[\kappa_0]_1 + \frac{2}{3}[\kappa_1^3]_2[\kappa_0]_1^2.$$

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