SIAM J. APPL. MATH.
Vol. 81, No. 5, pp. 1847–1869


# SOME NOVEL RESULTS IN TWO SPECIES COMPETITION*

RANA D. PARSHAD†, KWADWO ANTWI-FORDJOUR‡, AND ERIC M. TAKYI†

**Abstract.** We investigate certain two species ODE and PDE Lotka–Volterra competition models, where one of the competitors could potentially go extinct in finite time. We show that in this setting, various novel dynamics are possible. In particular competitive exclusion can be *avoided*, and the slower diffusing competitor may *not* win. Numerical simulations are performed to corroborate our analytical findings.

**Key words.** two species competition, finite time extinction, spatially inhomogeneous resources, reaction diffusion equation

**AMS subject classifications.** 35K57, 92D40, 34C12

**DOI.** 10.1137/20M1387274

## 1. Introduction.

The two species Lotka–Volterra competition model and its variants have been rigorously investigated in the last few decades. They represent a simplified scenario of two competing species, taking into account growth and inter/intraspecific competition [1, 2]. They predict well-observed states in ecology of co-existence, competitive exclusion of one competitor, and bistability and find immense applications in applied mathematics, population ecology, invasion science, evolutionary biology, and economics, to name a few areas [2, 3, 4, 5]. The equilibrium states are achieved only asymptotically as is the case in many differential equation population models.

In the current manuscript we consider competitive systems when one of the competitors has the potential to go extinct in finite time. There are various motivations to study finite time extinction mechanisms (FTEM) in population dynamics. In modeling predator-pest densities such as in classical biological control, a very low pest density need not indicate essential extinction—and pest populations can *rebound* from levels as close to extinction as one pleases [6]. An explosive increase in population is well observed with soybean aphids (*Aphis glicines*), the chief invasive pest on soybean crops, particularly in the midwestern United States [7]. Therein the arrival of aphids in very low density ($<< 1$) could lead to population levels of several thousand on a soybean plant in a matter of few months [8].

Another motivation to study the FTEM is its applicability to epidemics. This is timely and useful in studying the dynamics of the coronavirus disease (COVID19) global pandemic to curb a mortality increase [9]. Recent work has considered a large class of susceptible-infected models with nonsmooth incidence functions that can lead to host extinction in finite time yet are seen to be better fits to data than their smooth counterparts [10, 11]. This has been observed in disease transmitted by rhanavirus

*Received by the editors December 21, 2020; accepted for publication (in revised form) June 24, 2021; published electronically September 9, 2021.
https://doi.org/10.1137/20M1387274
**Funding:** The work of the first and third authors was partially supported by the National Science Foundation via grant DMS 1839993. The work of the second author was partially supported by Samford Faculty Development Grant FDG084.
†Department of Mathematics, Iowa State University, Ames, IA 50011 USA (rparshad@iastate.edu, takyiem@iastate.edu).
‡Department of Mathematics and Computer Science, Samford University, Birmingham, AL 35229 USA (kantwifo@samford.edu).

among amphibian populations [12] and disease transmission in host-parasitoid models [13], as well as in virus transmission in gypsy moths [14]. Non-smooth responses have been considered analytically in the predator-prey literature as well [15, 16, 17]; a careful analysis of this splitting of phase, initial condition dependent extinction, and all of the rich dynamics and bifurcations involved therein has been made in [18, 19]. They have also been considered a fair bit in the applied sense, due to the fit they provide to various data [20, 21, 22, 23]. A two species competition model with nonsmooth response has been considered in [24], in which one population is highly socialized, living and wandering in a herd.

The following are demonstrated in the current manuscript:

- A two species ODE Lotka–Volterra competition model with an FTEM can lead to bistability, via Lemma 2.2 and Theorem 2.7; also see Figure 1. FTEM can also enable a competitor to avoid competitive exclusion and persist for various regimes of initial conditions. This is seen via Theorem 2.4 and Lemma 2.5; also see Figure 2.
- FTEM in the spatially homogeneous PDE model can cause diffusion induced recovery, as opposed to diffusion induced extinction seen in the classical case, via Theorem 3.2; see Figure 8.
- FTEM in the spatially inhomogeneous PDE model can cause the slower diffuser to *lose*, via Theorem 3.6; see Figure 11. It can also bring about a rich bifurcation structure in the space of diffusion parameters; see Figure 10.

## 2. The ODE case.

**2.1. The extinction/competitive exclusion case.** Consider the classical two species Lotka–Volterra competition model,

$$\begin{cases} \dfrac{du}{dt} = u(a_1 - b_1 u - c_1 v), \\ \dfrac{dv}{dt} = v(a_2 - b_2 v - c_2 u), \end{cases} \tag{2.1}$$

where $u$ and $v$ are the population densities of two competing species, $a_1$ and $a_2$ are the intrinsic (per capita) growth rates, $b_1$ and $b_2$ are the intraspecific competition rates, and $c_1$ and $c_2$ are the interspecific competition rates. All parameters considered are positive.

We consider first the competitive exclusion case,

$$\frac{a_1}{a_2} > \max\left\{\frac{b_1}{c_2}, \frac{c_1}{b_2}\right\}. \tag{2.2}$$

In this setting as $t \to \infty$, the solutions $(u(t), v(t))$ converge uniformly to $(a_1/b_1, 0)$ irrespective of initial conditions. Without loss of generality (WLOG) we consider the case when $(a_1/b_1, 0)$ is globally asymptotically stable, thus $u$ is the stronger competitor and drives $v$ to extinction, and $v$ is said to be competitively excluded [25].

We posit that including FTEM can alter the above classical dynamics. To this end consider

$$\begin{cases} \dfrac{du}{dt} = a_1 u - b_1 u^2 - c_1 u^p v, \ 0 < p \leq 1, \\ \dfrac{dv}{dt} = a_2 v - b_2 v^2 - c_2 u v^q, \ 0 < q \leq 1. \end{cases} \tag{2.3}$$

We see that the classical model is a special case of the above, when $p = q = 1$. Note that $0 < p < 1, q = 1$, allows for *finite* time extinction of $u$, and $p = 1, 0 < q < 1$, allows for *finite* time extinction of $v$.

*Remark* 2.1. There is also the more complex case when $0 < p < 1, 0 < q < 1$. This case is not within the scope of the current work; however, it is discussed in some detail in the conclusion section.

We state the following result.

LEMMA 2.2. *Consider* (2.3), $q = 1$, *and* $\frac{a_1}{b_1} > \frac{a_2}{c_2}$ *hold; then for any* $p \in (0, 1)$, *there exists an interior equilibrium, which is a saddle under the parametric restriction* (A.8).

*Proof.* The $u$ and $v$ nullclines are given by

$$v = f(u) = u^{1-p}\left(\frac{a_1}{c_1} - \frac{b_1}{c_1}u\right), \quad v = g(u) = \frac{a_2}{b_2} - \frac{c_2}{b_2}u. \tag{2.4}$$

Now the graph of $f(u)$ is parabolic shaped with zeroes at $u = 0, u = \frac{a_1}{b_1}$. Also we have $\frac{a_1}{b_1} > \frac{a_2}{c_2}$ with $g(\frac{a_1}{b_1}) < 0, g(0) > 0$. Thus continuity of $f$ and the intermediate value theorem will ensure that the curve $f$ intersects $g$ or the oblique boundary of the triangle, at some value of $u \in [0, \frac{a_1}{b_1}]$ creating an interior equilibrium. Standard linear analysis proves this is a saddle; see section A. ∎

*Remark* 2.3. Note the condition $\frac{a_1}{b_1} > \frac{a_2}{c_2}$ as used in the proof of Lemma 2.2 is necessary for (2.2). Thus one can now compare the models (2.1) and (2.3). If (2.2) is satisfied, then $u$ is the stronger competitor and $v$ is competitively excluded. However, if $u$ possesses FTEM, $0 < p < 1, q = 1$, then for initial conditions lying above the stable manifold of the interior saddle equilibrium $E_3$, it is $u$ that goes extinct; see Figure 1. This is perhaps expected as we are reducing the competitive advantage of $u$ by speeding up the process to its extinction, via the FTEM. Also since this result

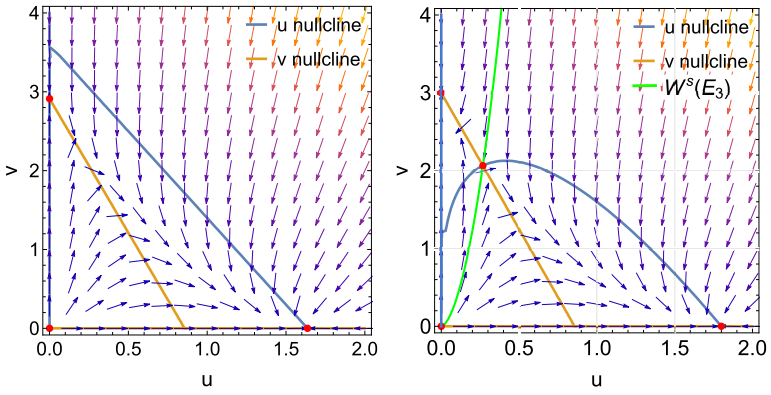



FIG. 1. *Here we see the competitive exclusion case, that is, if* (2.2) *is assumed in model* (2.3). *We choose* $a_1 = 1.8$, $a_2 = 3$, $b_1 = 1.1$, $b_2 = 1.03$, $c_1 = 0.5$, $c_2 = 3.5$. (a) *The classical result when* $p = 1$, $q = 1$, *that is,* $(u^*, 0)$ *is globally asymptotically stable, and* $v$ *is said to be competitively excluded.* (b) *An FTEM in* $u$ *is assumed with* $p = 0.7$, $q = 1$. *We see that this mechanism can induce co-existence. Here* $W^s(E_3)$ *denotes the stable manifold of the interior saddle equilibrium* $E_3$.

is qualitatively similar for *any* value of $p \in (0,1)$, we can infer that any increase in the extinction rate of $u$, as long as it happens in finite time, will yield bistability.

We next present results where $v$ possesses the finite time extinction dynamic.

THEOREM 2.4. *Consider* (2.3), *with* $p = 1$, $\frac{a_1}{c_1} > \frac{a_2}{b_2}$, $(a_2)^2 b_1 + 2a_2 c_1 c_2 > 4a_1 b_2 c_2$. *Then there exists a* $q^* \in (0,1)$ *s.t. for any* $q^* < q < 1$ *there is no interior equilibrium, for* $q = q^*$ *there is a unique nonhyperbolic equilibrium, and for* $0 < q < q^*$ *there exist two interior equilibria, a saddle and a nodal sink.*

*Proof.* We consider the nullclines as functions of $v$. These are given by

$$u = f_1(v) = \frac{a_1}{b_1} - \frac{c_1}{b_1} v, \; u = g_1(v) = v^{1-q}\left(\frac{a_2}{c_2} - \frac{b_2}{c_2} v\right). \tag{2.5}$$

$f_1$ is a straight line, with $v$ and $u$ intercepts at $\frac{a_1}{b_1}$ and $\frac{a_1}{c_1}$, and the graph of $g_1$ is parabolic shaped with zeroes at 0 and $\frac{a_2}{b_2}$. A sufficient condition for the occurrence of two equilibria is if there exists a $0 \le v \le \frac{a_2}{b_2}$ where

$$\sup_{0 \le v \le \frac{a_2}{b_2}} g_1(v) \ge f_1(v). \tag{2.6}$$

The maximum of $g_1$ occurs when

$$g_1'(v) = v^{1-q}\left(\frac{-b_2}{c_2}\right) + (1-q)\left(\frac{a_2}{c_2} - \frac{b_2}{c_2} v\right) v^{-q} = 0, \tag{2.7}$$

and this maximum is attained at $v = \frac{(1-q)a_2}{(2-q)b_2}$. Thus,

$$\begin{aligned} g_1\left(\frac{(1-q)a_2}{(2-q)b_2}\right) &= \left(\frac{(1-q)a_2}{(2-q)b_2}\right)^{1-q}\left(\frac{a_2}{c_2} - \frac{b_2}{c_2}\left(\frac{(1-q)a_2}{(2-q)b_2}\right)\right) \\ &\ge f_1\left(\frac{(1-q)a_2}{(2-q)b_2}\right) = \left(\frac{a_1}{b_1} - \frac{c_1}{b_1}\left(\frac{(1-q)a_2}{(2-q)b_2}\right)\right) \end{aligned} \tag{2.8}$$

will suffice to have two equilibria. Note $(a_2)^2 b_1 + 2a_2 c_1 c_2 > 4a_1 b_2 c_2 <=> g_1(\frac{a_2}{2b_2}) > f_1(\frac{a_2}{2b_2})$, and so this condition ensures that we have two interior equilibria, as $q \to 0$.

Next we need to show that $g_1$ will decrease sufficiently as $q \nearrow$ s.t. $g_1$ and $f_1$ touch to form one equilibrium. We proceed by contradiction. Assume not, then

$$g_1(v) \ge f_1(v), \; v = \frac{(1-q)a_2}{(2-q)b_2} \; \forall \; q \in (0,1). \tag{2.9}$$

Note $\frac{a_2}{c_2} < \frac{a_1}{b_1}$, since when $q = 1$ we are in the classical case. Thus there exists a $\delta > 0$ s.t. $\frac{a_2}{c_2} \le \frac{a_1}{b_1} - \delta$. Now consider $q = 1 - \epsilon$, for any given $0 < \epsilon << 1$, s.t.

$$\epsilon < \min\left(\frac{b_2}{a_2 + b_2}, \frac{a_2 c_1 c_2}{a_2 c_1 c_2 + b_2(a_1 c_2 - b_1 a_2)}\right). \tag{2.10}$$

We obtain

$$\begin{aligned} g_1\left(\frac{(1-q)a_2}{(2-q)b_2}\right) &= g_1\left(\frac{(\epsilon)a_2}{(1-\epsilon)b_2}\right) \\ &= \left(\frac{(\epsilon)a_2}{(1-\epsilon)b_2}\right)^{\epsilon}\left(\frac{a_2}{c_2}-\frac{b_2}{c_2}\left(\frac{(\epsilon)a_2}{(1-\epsilon)b_2}\right)\right) \\ &\le (1)^{\epsilon}\left(\frac{a_2}{c_2}-\frac{b_2}{c_2}\left(\frac{(\epsilon)a_2}{(1-\epsilon)b_2}\right)\right) \le \frac{a_2}{c_2} \le \frac{a_1}{b_1}-\delta \\ &= \frac{a_1}{b_1}-\left(\frac{(\epsilon)a_2}{(1-\epsilon)b_2}\left(\frac{c_1}{b_1}\right)\right) = f_1\left(\frac{(\epsilon)a_2}{(1-\epsilon)b_2}\right) \\ &= f_1\left(\frac{(1-q)a_2}{(2-q)b_2}\right). \end{aligned} \tag{2.11}$$

This follows from (2.10) as $\epsilon < \frac{b_2}{a_2+b_2}$ and if we choose $\delta = (\frac{(\epsilon)a_2}{(1-\epsilon)b_2}(\frac{c_1}{b_1}))$. Note from (2.10) we have $\epsilon < \frac{a_2c_1c_2}{a_2c_1c_2+b_2(a_1c_2-b_1a_2)}$ and so $\delta = (\frac{(\epsilon)a_2}{(1-\epsilon)b_2}(\frac{c_1}{b_1})) < \frac{a_1}{b_1} - \frac{a_2}{c_2}$. Thus we arrive at a contradiction. So there must exist a $q \in (0,1)$ where $g_1(v) < f_1(v)$ for a $v = \frac{(1-q)a_2}{(2-q)b_2}$. Thus by continuity of $g_1$ and the intermediate value theorem, there must exist a $q^*$ where $f_1$ and $g_1$ meet to form one nonhyperbolic equilibrium. Also by continuity for $q^* < q < 1$, there will be no equilibrium. ∎

We next provide a bound on $q^*$ in terms of the parameters in the problem. We state the following result.

LEMMA 2.5. *Consider* (2.3) *with* $p = 1$, $\frac{a_1}{c_1} > \frac{a_2}{b_2}$, $(a_2)^2b_1 + 2a_2c_1c_2 > 4a_1b_2c_2$, $a_2 > 2b_2$. *The critical* $q = q^*$ *where there is a unique nonhyperbolic equilibrium satisfies* $0 < q^* < \frac{a_2-2b_2}{a_2-b_2} < 1$.

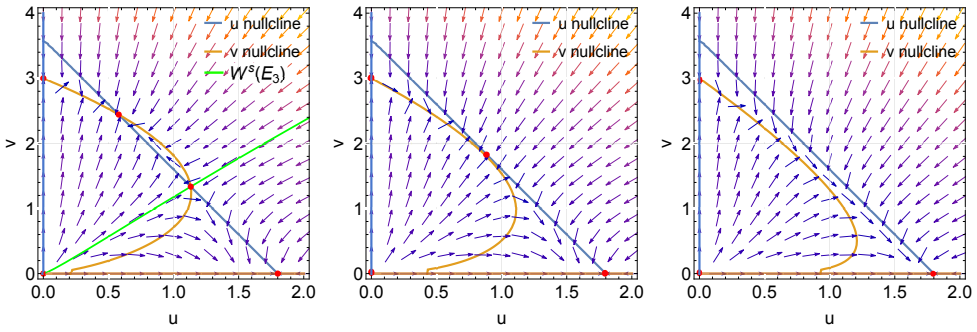
*Proof.* Note

$$\frac{d}{dq}\left(g_1\left(\frac{(1-q)a_2}{(2-q)b_2}\right)\right) = \frac{a_2}{c_2(q-2)}\log\left(\frac{(1-q)a_2}{(2-q)b_2}\right)\left(\frac{(1-q)a_2}{(2-q)b_2}\right)^{1-q} < 0 \tag{2.12}$$

when $q \in (0, \frac{a_2-2b_2}{a_2-b_2})$ and so $g_1 \searrow$ for $q \in (0, \frac{a_2-2b_2}{a_2-b_2})$, whereas we have shown via Theorem 2.4 that there exists a $q^*$ where $g_1$ and $f_1$ touch to form one equilibrium.

For $q \in (\frac{a_2-2b_2}{a_2-b_2}, 1)$, $\frac{d}{dq}(g_1(\frac{(1-q)a_2}{(2-q)b_2})) > 0$. Thus $g_1 \nearrow$ for $q \in (\frac{a_2-2b_2}{a_2-b_2}, 1)$, and $g_1$ would never touch $f_1$ to form one equilibrium. This would then imply a contradiction to Theorem 2.4. Thus we must have $q^* < \frac{a_2-2b_2}{a_2-b_2}$. ∎

*Remark* 2.6. Note the condition $\frac{a_1}{b_1} > \frac{a_2}{c_2}$ as used in the proof of Theorem 2.4 is necessary for (2.2). Thus one can now compare the models (2.1) and (2.3). If (2.2) is satisfied and $p = q = 1$, then $u$ is the stronger competitor and $v$ is competitively excluded. If, however, we let $v$ possess FTEM, $p = 1, 0 < q < 1$, and $(a_2)^2b_1 + 2a_2c_1c_2 > 4a_1b_2c_2$, then for initial conditions lying above the stable manifold of the interior saddle $E_3^1$, $v$ could coexist with $u$; see Figure 2. This is counterintuitive, as $v$ can seemingly *improve* its competitive ability and avoid competitive exclusion for various regimes of initial conditions by *speeding up* the process to its own extinction. Note, however, that an advantage to the weaker competitor or the co-existence result occurs only if $0 < q < q^* < 1$. Thus we can infer that co-existence depends on increasing the extinction rate of $v$ to be sufficiently fast. This is qualitatively different from placing FTEM in the stronger competitor, where any increase of extinction rate will yield bistability—hence an advantage to the weaker competitor.




(a) (b) (c)

FIG. 2. *Here we explore the effect of FTEM in v, in the competitive exclusion case. We choose* $a_1 = 1.8$, $a_2 = 3$, $b_1 = 1$, $b_2 = 1$, $c_1 = 0.5$, $c_2 = 1.8$, $p = 1$. *We note that depending on the value of q, various dynamics are possible.* (a) *FTEM when* $q = 0.3$. *We see that co-existence is now possible. The saddle interior equilibrium when it occurs is denoted by* $E_3^1$, *and the locally stable interior equilibrium when it occurs is denoted by* $E_3^2$. *Here* $W^s(E_3^1)$ *denotes the stable manifold of* $E_3^1$. (b) *When* $q = q^* = 0.48946$. (c) *When* $q = 0.8$, *thus for values of* $q > q^*$, *the FTEM in v cannot induce co-existence.*

*Example* 1. We provide a numerical example to corroborate Lemma 2.5 by using the following set of parameter values: $a_1 = 1.8$, $a_2 = 3$, $b_1 = 1$, $b_2 = 1$, $c_1 = 0.5$, $c_2 = 1.8$, and $p = 1$. We note that $\frac{a_1}{c_1} = 3.6 > 3 = \frac{a_2}{b_2}$, $(a_2)^2 b_1 + 2a_2 c_1 c_2 = 14.4 > 12.96 = 4a_1 b_2 c_2$, and $a_2 = 3 > 2 = 2b_2$.

Additionally, from Figure 2(b), we obtain that $q^* = 0.48946 < 0.5 = \frac{a_2 - 2b_2}{a_2 - b_2}$.

Note that when $0 < p, q < 1$ the kinetic terms are nonsmooth, causing issues for uniqueness. Linearization at interior equilibrium is not affected. However, standard linearization methods do not work for boundary equilibria due to the nonsmoothness. WLOG if $p = 1, 0 < q < 1$, $(u^*, 0)$ would be attained by $v \to 0$ in a finite time, followed by an asymptotic rate of attraction to $u^*$. So initial data taken on the $u$-axis can lead to nonuniqueness backward in time. This can be circumvented if we avoid data on the $u$-axis. Such and related issues have been dealt with in [6, 10, 18, 19]. We thus derive an explicit sufficient condition on the initial data that yields finite time extinction of the stronger competitor $u$. This is stated and proved via the following theorem.

THEOREM 2.7. *Consider the competition model given by* (2.3) *for any* $0 < p < 1$, $q = 1$, *and suppose* $\frac{a_1}{b_1} > \frac{a_2}{c_2}$ *holds. Then for positive initial conditions* $(u(0), v(0))$ *s.t.* $u(0) \leq \frac{a_1}{b_1}$, *and*

$$v(0) > \left( \frac{a_1 c_2 + (1-p) a_1 b_1 + a_2 b_1}{(1-p) c_1 b_1} \right) (u(0))^{1-p}, \tag{2.13}$$

*u will go extinct in finite time and trajectories will approach* $(0, a_2/b_2)$. *However, if* $(u(0), v(0))$ *lies below the stable manifold* $W^s(E_3)$ *of the interior saddle equilibrium* $E_3$, *then all trajectories initiating from them will approach* $(a_1/b_1, 0)$ *asymptotically.*

*Proof.* Consider the equation for initial condition $0 < u(0) \leq \frac{a_1}{b_1}$. Then

$$\frac{dv}{dt} \geq -b_2 v^2 - c_2 uv \geq -b_2 \left( \frac{a_2}{b_2} \right) v - c_2 \left( \frac{a_1}{b_1} \right) v. \tag{2.14}$$

This follows as $u \leq \frac{a_1}{b_1}$, $v \leq \frac{a_2}{b_2}$, by comparison to the logistic equation. Thus,

$$v(t) \geq v(0)e^{-\left(a_2+\frac{a_1c_2}{b_1}\right)t} \tag{2.15}$$

for all time $t \geq 0$.

Note that

$$\begin{aligned} \frac{du}{dt} &= a_1u - b_1u^2 - c_1u^pv, \\ &\leq a_1u - c_1u^pv. \end{aligned} \tag{2.16}$$

Now we can divide the above by $u^p$ since $u$ is positive to obtain

$$\frac{du}{dt}\frac{1}{u^p} \leq a_1u^{1-p} - c_1v. \tag{2.17}$$

Using the lower bound on $v$ yields

$$\frac{d}{dt}(u^{1-p}) \leq (1-p)a_1u^{1-p} - (1-p)c_1v(0)e^{-\left(a_2+\frac{a_1c_2}{b_1}\right)t}. \tag{2.18}$$

Multiplying both sides by the integrating factor $e^{-(1-p)a_1t}$, and subsequently integrating the above in the time interval $[0,t]$, we obtain

$$\begin{aligned} &e^{-(1-p)a_1t}u^{1-p} \\ &\leq (u(0))^{1-p} - \left(\frac{(1-p)c_1v(0)}{(1-p)a_1+a_2+\frac{a_1c_2}{b_1}}\right)\left(1-e^{-\left((1-p)a_1+a_2+\frac{a_1c_2}{b_1}\right)t}\right), \end{aligned} \tag{2.19}$$

which then implies the finite time extinction of $u$ if

$$(u(0))^{1-p} < \left(\frac{(1-p)c_1v(0)}{(1-p)a_1+a_2+\frac{a_1c_2}{b_1}}\right). \tag{2.20}$$

Thus it suffices to choose initial data such that $v(0) > (\frac{(1-p)a_1b_1+a_2b_1+a_1c_2}{(1-p)c_1b_1})(u(0))^{1-p}$ for $u$ to go extinct in finite time. This proves the theorem. ∎

We provide some simulation next to elucidate.

*Remark* 2.8. We can similarly compare the situation with FTEM to the classical case, when there is coexistence or the weak competition case, that is, if

$$\frac{b_1}{c_2} > \frac{a_1}{a_2} > \frac{c_1}{b_2}. \tag{2.21}$$

The classical theory for $p = 1$, $q = 1$ is that all initial conditions would be attracted to an interior equilibrium. In this setting the competitors $u$ and $v$ coexist. In the case of FTEM, $0 < p < 1, q = 1$, or $0 < q < 1, p = 1$, the situation can change qualitatively. We provide some simulations in Figure 4 to elucidate. Note in all of the numerical simulations in Figures 1–5, we are interested only in the first quadrant, as this is what is most relevant for population dynamic applications. Furthermore standard theory yields positivity for the systems under consideration [19].

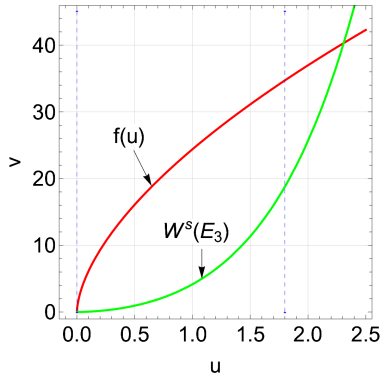



FIG. 3. *Here we demonstrate our results for Theorem* 2.7. *Here* $a_1 = 1.8$, $a_2 = 3$, $b_1 = 1$, $b_2 = 1$, $c_1 = 0.5$, $c_2 = 1.8$, $p = 0.4$, $q = 1$. *Here* $W^s(E_3)$ *denotes the stable manifold of the interior saddle equilibrium* $E_3$. *Also* $f(u) = (\frac{a_1c_2+a_2b_1+(1-p)a_1b_1}{(1-p)c_1b_1})u^{1-p}$ *and the dotted blue line on the right is* $u = \frac{a_1}{b_1}$, *the boundary of Theorem* 2.7. *The dotted blue line does not affect the larger result regardless of the theorem's limitations. Note any initial data lying above* $W^s(E_3)$ *is attracted to* $(0, v^*)$; *however, we are only able to prove this for initial data lying above* $f(u)$. *Thus there is some wiggle room between all of the initial data that will be driven to* $(0, v^*)$ *and the data that we can prove will be driven to* $(0, v^*)$ *at present.*

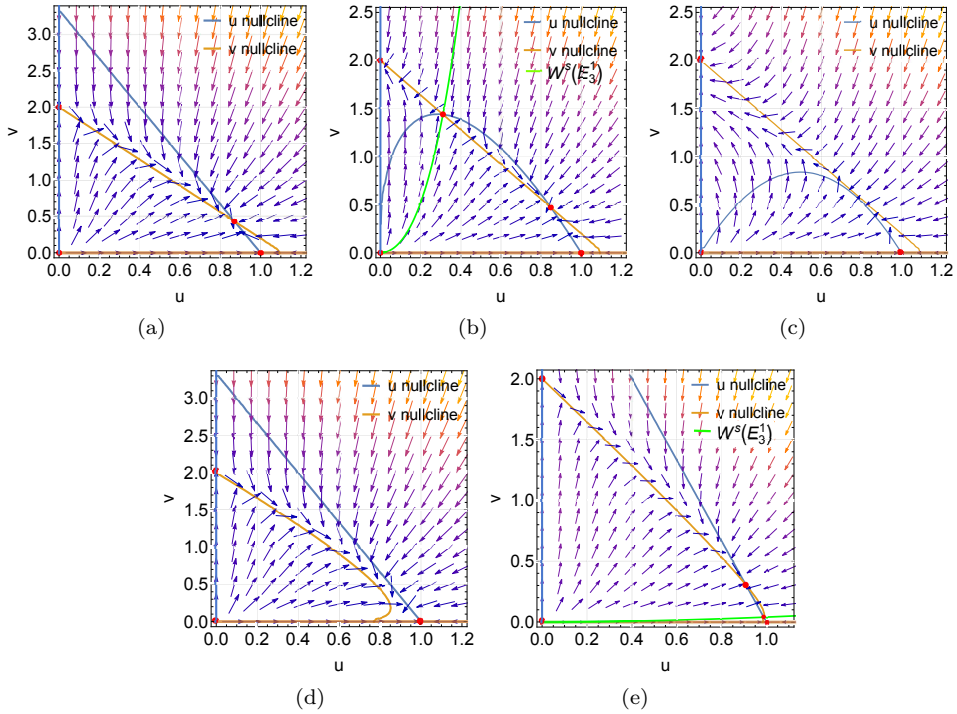



FIG. 4. *Here we explore the effect of FTEM in* $u$ *or* $v$ *in the weak competition or co-existence case. So* (2.21) *is assumed in model* (2.3). *We choose* $a_1 = 1$, $a_2 = 2$, $b_1 = 1$, $b_2 = 1$, $c_1 = 0.3$, $c_2 = 1.8$. *The saddle interior equilibrium when it occurs is denoted by* $E_3^1$, *and the locally stable interior equilibrium when it occurs is denoted by* $E_3^2$. *Here* $W^s(E_3^1)$ *denotes the stable manifold of* $E_3^1$. *We notice that depending on the values of* $p$ *or* $q$ *various dynamics are possible:* (a) *classical case when* $p = 1$, $q = 1$, (b) *FTEM when* $p = 0.6$, $q = 1$, (c) $p = 0.01$, $q = 1$, (d) *FTEM when* $p = 1$, $q = 0.9$, (e) $p = 1$, $q = 0.97$.

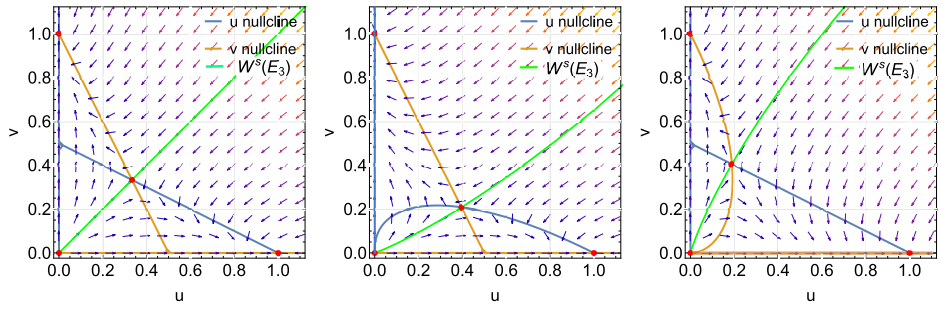



FIG. 5. *Here we explore the effect of FTEM in the strong competition case, that is,* (3.3) *is assumed in model* (2.3). *We choose* $a_1 = 1,\ a_2 = 1,\ b_1 = 1,\ b_2 = 1,\ c_1 = 2,\ c_2 = 2$. *The saddle interior equilibrium is denoted by* $E_3$. *Here* $W^s(E_3)$ *denotes the stable manifold of the interior saddle equilibrium* $E_3$. *We observe the qualitative behavior of the system cannot be changed by FTEM in* $u$ *or* $v$. *That is, there are two locally stable boundary equilibria and one saddle interior equilibrium whether* $p = 1, 0 < q < 1$ *or* $0 < p < 1, q = 1$. (a) *Classical when* $p = 1,\ q = 1$. (b) *FTEM when* $p = 0.6,\ q = 1$. (c) *FTEM when* $p = 1,\ q = 0.5$.

## 3. The PDE case.

**3.1. The case of strong competition.** The spatially explicit two species competition model has been intensely investigated [3, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36]. We consider a generalized version

$$\begin{cases} u_t &= d_1\Delta u + a_1 u - b_1 u^2 - c_1 u^p v, 0 < p \le 1, \\ v_t &= d_2\Delta v + a_2 v - b_2 v^2 - c_2 u v^q, 0 < q \le 1, \end{cases} \tag{3.1}$$

$$\nabla u \cdot n = \nabla v \cdot n = 0, on\ \partial\Omega\ ,\ u(x,0) = u_0(x) > 0,\ v(x,0) = v_0(x) > 0, \tag{3.2}$$

where we consider a bounded domain $\Omega \subset \mathbb{R}^n,\ n = 1, 2$. Under the strong competition setting,

$$\frac{b_1}{c_2} < \frac{a_1}{a_2} < \frac{c_1}{b_2}. \tag{3.3}$$

When $p = q = 1$, classical results show that in the absence of diffusion, there is a stable manifold of the saddle equilibrium (separatrix) denoted as $h$ that splits the phase space into two regions, $W_B$ the region above the separatrix. Note that for initial data $(u(x,0), v(x,0)) \in W_B\ \forall x \in \Omega$, the solution converges to $(0, \frac{a_2}{b_2})$. Likewise, $W_A$ is the region below the separatrix, and for initial data $(u(x,0), v(x,0)) \in W_A\ \forall x \in \Omega$, the solution converges to $(\frac{a_1}{b_1}, 0)$. We recap a classical result from [26, 37] of diffusion induced extinction.

THEOREM 3.1. *Let* $(u, v)$ *be a solution of* (3.1)–(3.2), $\frac{b_1}{c_2} < \frac{a_1}{a_2} < \frac{c_1}{b_2}$, *and* $p = q = 1$. *Suppose that* $h'' \le 0$. *Then there exists initial data such that* $(u(x,0), v(x,0)) \in W_B\ \forall x \in \Omega$, *but the solution initiating from this data converges uniformly to* $(\frac{a_1}{b_1}, 0)$.

These dynamics can change when FTEM are present in $u$ and we can have finite time extinction induced recovery. To this end we state and prove the following result.

THEOREM 3.2. *Consider* (3.1)–(3.2) *with* $\frac{b_1}{c_2} < \frac{a_1}{a_2} < \frac{c_1}{b_2}$. *Then* $\exists\ 0 < p < 1, q = 1$, *and initial data* $(u_0(x), v_0(x)) \in W_B\ \forall x \in \Omega$, *that converges uniformly to* $(0, \frac{a_2}{b_2})$, *as long as*

$$\left(\frac{a_1c_2+(1-p)a_1b_1+a_2b_1}{(1-p)c_1b_1}\right)\leq\left(\frac{c_2a_1-b_1a_2}{c_1a_2-a_1b_2}\right)\left(\frac{c_1a_2-a_1b_2}{c_2c_1-b_1b_2}\right)^p.$$

*Proof.* For the constant coefficient case as we are dealing with herein, solutions are spatially homogeneous [26, 37], thus our system is reduced to

$$\begin{cases} u_t &= a_1u-b_1u^2-c_1u^pv,\\ v_t &= a_2v-b_2v^2-c_2uv. \end{cases} \tag{3.4}$$

Standard estimates as in Theorem 2.7 yield the finite time extinction of $u$ for initial data chosen s.t.

$$\left(\frac{a_1c_2+(1-p)a_1b_1+a_2b_1}{(1-p)c_1b_1}\right)(u_0(x))^{1-p}=f_1(u_0)\leq v_0(x). \tag{3.5}$$

Note that analysis of the ODE/kinetic system, via a simple modification of Lemma 2.2 (see Figure 1), clearly shows that when $p<1$, the interior equilibrium is lowered and so is the separatrix. We refer to the separatrix for the $0<p<1$ case as $h_1$. Now consider when $p=1$, initial data $(u_0(x),v_0(x))\in W_B$, for which diffusion induced extinction occurs. Since $(u(x,0),v(x,0))\in W_B$, it lies above the separatrix $h$, and by the concavity assumption on $h$, $h$ lies above the line segment connecting $(0,0)$ and $(u^*,v^*)$, which is given by the equation

$$v_0(x)=f_2(u_0)=\left(\frac{c_2a_1-b_1a_2}{c_1a_2-a_1b_2}\right)u_0(x). \tag{3.6}$$

Thus if we choose $p$ s.t. $h_1$ is lowered enough s.t. $f_1(u_0)<f_2(u_0)$, for certain $u_0^*$, then there exists data $(u_0^*,v_0^*)\in W_B$ (for which diffusion induced extinction occurs if $p=1$), but that lies above the separatrix $h$, which in turn lies above $f_2(u_0)$, which by the appropriate choice of $p<1$ lies above $f_1(u_0)$, which lies above $h_1$—and so will converge uniformly to $(0,v^*)$—and diffusion induced extinction does not occur when $p<1$. To this end it is sufficient that $f_1(u_0(x))\leq v_0(x)\leq f_2(u_0(x))$, or

$$\left(\frac{a_1c_2+(1-p)a_1b_1+a_2b_1}{(1-p)c_1b_1}\right)(u_0(x))^{1-p}\leq v_0(x)\leq\left(\frac{c_2a_1-b_1a_2}{c_1a_2-a_1b_2}\right)u_0(x) \tag{3.7}$$

and

$$u_0(x)\leq u^*=\left(\frac{c_1a_2-a_1b_2}{c_2c_1-b_1b_2}\right). \tag{3.8}$$

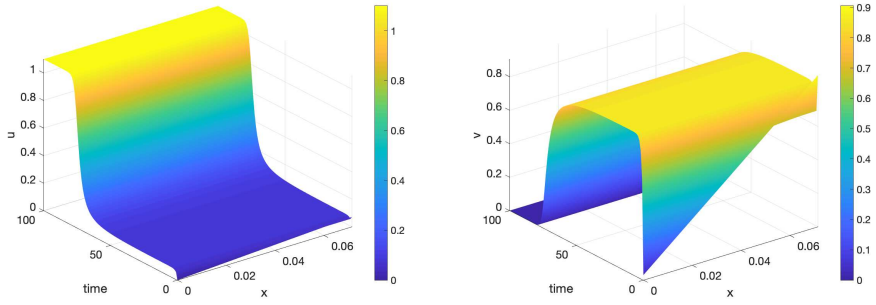
A sufficient parametric restriction for which the above is true is given by

$$\left(\frac{a_1c_2+(1-p)a_1b_1+a_2b_1}{(1-p)c_1b_1}\right)\leq\left(\frac{c_2a_1-b_1a_2}{c_1a_2-a_1b_2}\right)\left(\frac{c_1a_2-a_1b_2}{c_2c_1-b_1b_2}\right)^p. \tag{3.9}$$

This proves the theorem. ∎

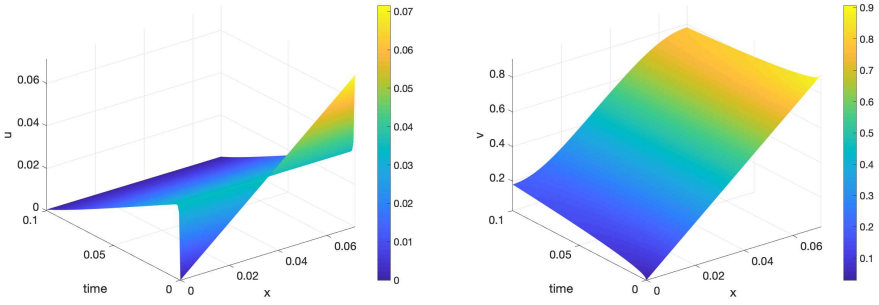
*Remark* 3.3. This enables us to compare the model with $p<1$, when there is an FTEM, to the classical case. One sees that we can construct initial data s.t. when $p=1$, solutions initiating from this data converge to $(\frac{a_1}{b_1},0)$, but when $p<1$, they converge to $(0,\frac{a_2}{b_2})$. See Figures 6–8.

*Remark* 3.4. We note similar results are possible when there is an FTEM in $v$. Herein initial data lying above the separatrix can go to a coexistence state, or one could have diffusion induced extinction. See Figure 9.




(a) (b)

FIG. 6. *Diffusion induced extinction is seen. Here we consider parameters* $a_1 = 1.1, b_1 = 1, c_1 = 1.2, a_2 = 1, b_2 = 1, c_2 = 2, p = 1, d_1 = 1, d_2 = 0.001$ *for* (3.1)–(3.2). *We choose* $\Omega = [0, 0.071429]$. *The initial data is chosen as per the estimates of Theorem* 3.2*; see Figure* 8.




(a) (b)

FIG. 7. *Finite time extinction induced recovery of superior species. We choose* $a_1 = 1.1, b_1 = 1, c_1 = 1.2, a_2 = 1, b_2 = 1, c_2 = 2, p = 0.1, d_1 = 1, d_2 = 0.001$, *and* $\Omega = [0, 0.071429]$ *for* (3.1)–(3.2). *The initial data is chosen as per the estimates of Theorem* 3.2*; see Figure* 8.

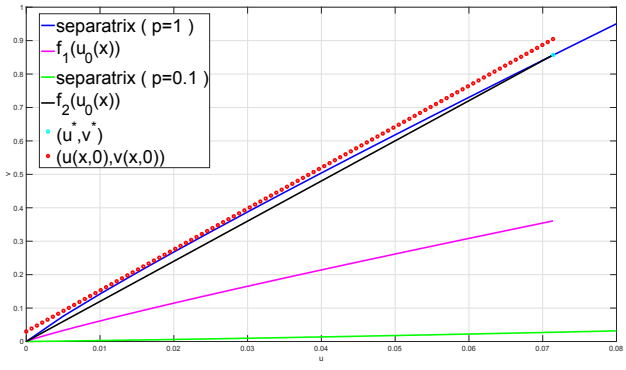



FIG. 8. *Plot corroborating Theorem* 3.2. *Parameters used are* $a_1 = 1.1, b_1 = 1, c_1 = 1.2, a_2 = 1, b_2 = 1, c_2 = 2, p = 0.1$. *We choose* $\Omega = [0, 0.071429]$. *Here,* $(u^*, v^*) = (0.071429, 0.85714)$. *The red dots are the data that lie above the separatrix when* $p = 1$*; we see diffusion induced extinction occur, that is, we approach* $(\frac{a_1}{b_1}, 0)$ *in this case—see Figure* 6. *When* $p < 1$, *the same data converges to* $(0, \frac{a_2}{b_2})$*; see Figure* 7.

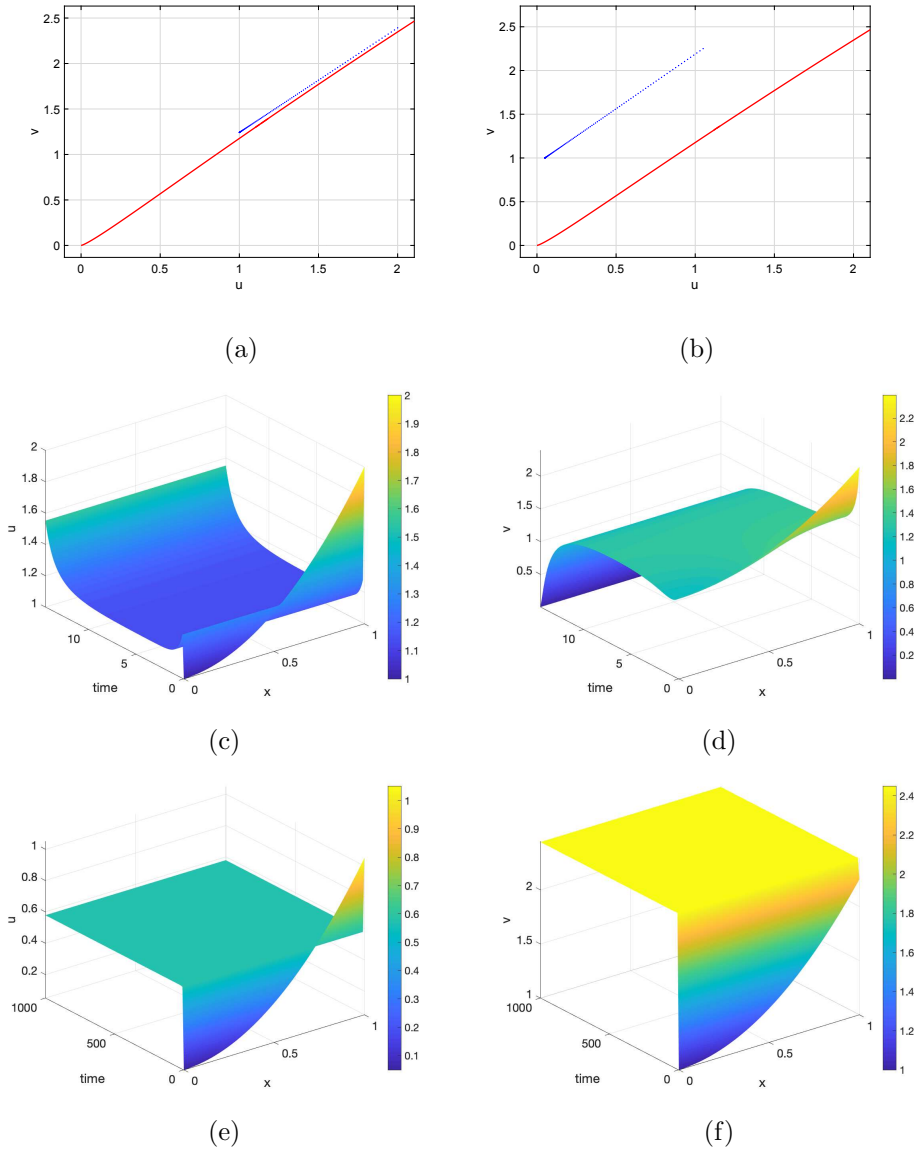



FIG. 9. *Here we consider parameters for the extinction case in Figure* 2 *with* $q = 0.3$. *We choose our domain as* $\Omega = [0, 1]$. (a) *Initial data (blue)* $(u_0(x), v_0(x)) = (x^2 + 1, 1.15x^2 + 1.242)$ *chosen above the separatrix (red).* (b) *Initial data (blue)* $(u_0(x), v_0(x)) = (x^2 + 0.05, 1.25x^2 + 1)$ *above separatrix (red). We see solutions converging to* $(u^*, 0)$ *in* (c) *and* (d) *with initial data in* (a) *and diffusion coefficients chosen as* $d_1 = 15, d_2 = 0.03$. *Plots* (e) *and* (f) *show solutions converging to* $(u^*, v^*)$ *with initial data chosen from* (b) *and diffusion coefficients* $d_1 = 0.5, d_2 = 1$.

**3.2. The spatially inhomogeneous problem.** The spatially inhomogeneous problem has been intensely investigated in the past two decades [3, 27, 28, 31, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49]. The premise here is that $u, v$ do not have resources that are uniformly distributed in space; rather there is a spatially dependent resource function $m(x)$. We consider again a normalized generalization of the classical formulation, where there are two parameters $b$ and $c$ for inter/intraspecific kinetics

as opposed to six kinetic parameters in (3.1) from earlier. The parameter choice $0 < p < 1$ enables an FTEM in $u$.

$$\begin{cases} u_t = d_1\Delta u + m(x)u - u^2 - cu^p v, 0 < p \leq 1, \\ v_t = d_2\Delta v + m(x)v - v^2 - buv, \end{cases} \tag{3.10}$$

$$\nabla u \cdot n = \nabla v \cdot n = 0 \text{ on } \partial\Omega\ ,\ u(x,0) = u_0(x) > 0,\ v(x,0) = v_0(x) > 0. \tag{3.11}$$

Note that $p = 1$ is the classical case. We consider $m$ to be nonnegative on $\Omega$ and bounded. We recap a seminal classical result [50, 51], which shows that the slower diffuser wins.

THEOREM 3.5. *Consider* (3.10)–(3.11)*, when* $b = c = p = 1$*, and* $d_1 < d_2$*, solutions initiating from any positive initial data* $(u_0(x), v_0(x))$ *converge uniformly to* $(u^*(x), 0)$.

That is, the slower diffuser wins, in the case of equal kinetics. However, a difference in the interspecific kinetics via FTEM can cause the slower diffuser to *lose*, depending on the initial conditions. We now state the following result in one spatial dimension.

THEOREM 3.6. *Consider* (3.10)–(3.11)*, where* $\Omega \subset \mathbb{R}$*, is a bounded domain, and when* $b = c = p = 1$*,* $d_1 < d_2$*. There exists positive initial data* $(u_0(x), v_0(x))$*, for which solutions converge to* $(u^*(x), 0)$*, but solutions with the same diffusion coefficients, initiating from the same data, will converge to* $(0, v^*(x))$ *in finite time, for a sufficiently chosen* $p \in (0, 1)$.

*Proof.* Via comparison with the logistic equation [3], we see that $u \leq Cm(x)$, $\forall x, t \in \Omega \times [0, \infty)$. Now from the equation for $v$ in (3.10), we have

$$v_t = d_2 v_{xx} + m(x)v - v^2 - uv \geq d_2 v_{xx} - v^2 - C||m||_\infty v, \tag{3.12}$$

and via comparison we have

$$v(x,t) \geq C_1 v_0(x) e^{-C_2 t}, \tag{3.13}$$

where $C_1, C_2$, are independent of $u$. We now multiply the $u$ equation in (3.10) by $u$ and integrate by parts to obtain

$$\frac{1}{2}\frac{d}{dt}||u||_2^2 + d_1||u_x||_2^2 + \int_\Omega u^{1+p} v dx + \int_\Omega u^3 dx = \int_\Omega m(x)u^2 dx. \tag{3.14}$$

Using the estimate on $v$ from (3.13) we obtain

$$\frac{1}{2}\frac{d}{dt}||u||_2^2 + d_1||u_x||_2^2 + C_3 e^{-C_2 t}\int_\Omega u^{1+p} dx + \int_\Omega u^3 dx \leq \int_\Omega m(x)u^2 dx. \tag{3.15}$$

Here $C_3 = C_1 \min v_0(x) > 0$; then it follows that

$$\frac{1}{2}\frac{d}{dt}||u||_2^2 + \min(d_1, C_3)e^{-C_2 t}\left(||u_x||_2^2 + \int_\Omega u^{1+p} dx\right) + \int_\Omega u^3 dx \leq \int_\Omega m(x)u^2 dx.$$

Thus we have that

$$\frac{1}{2}\frac{d}{dt}||u||_2^2 + C_4 e^{-C_2 t}\left(||u_x||_2^2 + \int_\Omega u^{1+p} dx\right) + \int_\Omega u^3 dx \leq ||m(x)||_\infty ||u||_2^2 dx.$$

Our goal is to show that

$$\left(||u||_2^2\right)^{\alpha} \leq C_4 \left(||u_x||_2^2 + \int_{\Omega} u^{1+p} dx\right), \tag{3.16}$$

where $0 < \alpha < 1$; then we will have the finite time extinction of $u$ in analogy with the ODE

$$\frac{dy}{dt} = C_5 y - C_4 e^{-C_2 t} y^{\alpha}, 0 < \alpha < 1, C_2, C_4, C_5 > 0. \tag{3.17}$$

Now recall the Gagliardo–Nirenberg–Sobolev (GNS) inequality [52],

$$||\phi||_{W^{k,p'}(\Omega)} \leq C||\phi||^{\theta}_{W^{m,q'}(\Omega)} ||\phi||^{1-\theta}_{L^q(\Omega)} \tag{3.18}$$

for $\phi \in W^{m,q}(\Omega)$ provided $p', q', q \geq 1, 0 \leq \theta \leq 1$, and

$$k - \frac{n}{p'} \leq \theta \left(m - \frac{n}{q'}\right) - (1-\theta)\frac{n}{q}. \tag{3.19}$$

Now consider exponents s.t.

$$W^{k,p'}(\Omega) = L^2(\Omega),\ W^{m,q'}(\Omega) = W^{1,2}(\Omega),\ L^q(\Omega) = L^{1+p}(\Omega) \tag{3.20}$$

for $0 < p < 1$. This yields

$$||u|_{L^2(\Omega)} \leq C||u||^{\theta}_{W^{1,2}(\Omega)} ||u||^{1-\theta}_{L^q(\Omega)}, \tag{3.21}$$

as long as

$$\frac{2-q}{2+q} \leq \theta \leq 1. \tag{3.22}$$

We raise both sides of (3.21) to the power of $l$, $0 < l < 2$, to obtain

$$\left(\int_{\Omega} u^2 dx\right)^{\frac{l}{2}} \leq C \left(\int_{\Omega} (u_x)^2 dx\right)^{\frac{l\theta}{2}} \left(\int_{\Omega} u^q dx\right)^{\frac{l(1-\theta)}{q}}. \tag{3.23}$$

Using Young's inequality on the right-hand side (for $ab \leq \frac{a^r}{r} + \frac{b^m}{m}$), with $r = \frac{2}{l\theta},\ m = \frac{q}{l(1-\theta)}$, yields

$$\left(\int_{\Omega} u^2 dx\right)^{\frac{l}{2}} \leq C \left(\int_{\Omega} (u_x)^2 dx + \int_{\Omega} u^q dx\right). \tag{3.24}$$

We notice that given any $1 < q < 2$, it is always possible to choose $0 < l < 2$, s.t. $\frac{1}{r} + \frac{1}{m} = 1$,

$$\frac{1}{r} + \frac{1}{m} = \frac{l\theta}{2} + \frac{l(1-\theta)}{q} = 1, \tag{3.25}$$

by choosing

$$\theta = \frac{\frac{1}{l} - \frac{1}{q}}{\frac{1}{q} - \frac{1}{2}} = \frac{2(q-l)}{l(2-q)}, \tag{3.26}$$

thus we need to choose $l$ s.t.

$$\frac{2(q-l)}{l(2-q)} \geq \frac{2-q}{2+q} \tag{3.27}$$

or

$$\frac{1}{l} \geq \frac{(2-q)^2}{2q(2+q)} + \frac{1}{2q}. \tag{3.28}$$

This enables the application of Young's inequality above, within the required restriction (3.22), enforced by the GNS inequality.

Thus we have

$$\frac{1}{2}\frac{d}{dt}||u||_2^2 + C_4 e^{-C_2 t}\left(||u||_2^2\right)^{\frac{l}{2}} \leq C_5||u||_2^2 dx.$$

Let $\alpha = \frac{l}{2} < 1$. We have that $||u||_2^2 \to 0$ as $t \to T^* < \infty$, for appropriately chosen initial data, in analogy with the ODE

$$\frac{dy}{dt} = C_5 y - C_4 e^{-C_2 t} y^\alpha, 0 < \alpha < 1, C_2, C_4, C_5 > 0. \tag{3.29}$$

We set $y = g(t)e^{C_5 t}$ to obtain

$$\frac{dg}{dt} = -C_4 e^{-C_6 t}(g(t))^\alpha, 0 < \alpha < 1, C_6, C_4, C_5 > 0. \tag{3.30}$$

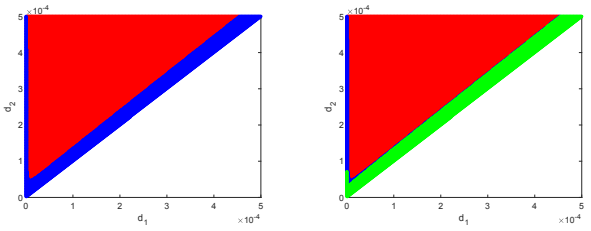
Solving (3.30) yields

$$g(t) = \left(\frac{(1-\alpha)C_4 e^{-C_6 t}}{C_6} + K\right)^{\frac{1}{1-\alpha}}, K \text{ a constant.} \tag{3.31}$$

Here $K = (g_0)^{(1-\alpha)} - \frac{((1-\alpha))C_4}{C_6}$. Thus for initial data chosen such that $g(0) < (\frac{(1-\alpha)C_4}{C_6})^{\frac{1}{1-\alpha}}$, then $g$ goes extinct at finite time $T^* = \ln(\frac{C_6}{(1-\alpha)C_4 - C_6(g_0)^{1-\alpha}})$, and so does $y(t)$. Thus we need to choose the initial data such that $||u_0||_2^2 < (\frac{(1-\alpha)C_4}{C_6})^{\frac{1}{1-\alpha}}$. Since $L^2(\Omega)$ convergence implies uniform convergence on $\Omega$, which is closed and bounded, we see that for sufficiently chosen data $(u, v) \to (0, v^*(x))$ uniformly, and this occurs in finite time. However, if $p = 1$, classical results [50] would imply the same data would have converged to $(u^*(x), 0)$. This completes the proof. ∎

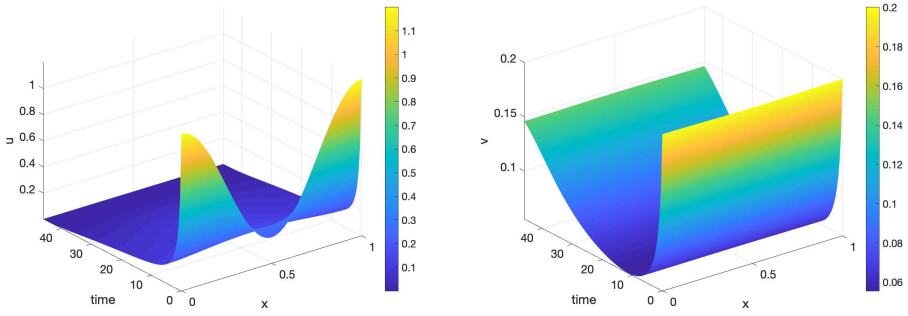
*Remark* 3.7. In the event that $0 < b, c < 1$ in (3.10)–(3.11), and $p = 1$, we are in the weak competition case. Herein, if $d_1 < d_2$, the slower diffuser could win or coexistence can occur [41, 42]. Once we bring in FTEM, that is, $0 < p < 1$, numerical simulations illustrate interesting scenarios in the bifurcation plots in $(d_1, d_2)$ space. See Figure 10(a) for the classical result [41, 42]—however, when $0 < p < 1$, the bifurcation plot changes qualitatively, in that one can have the additional dynamic where the slower diffuser could lose; see Figure 10(b).

**4. Discussions and conclusions.** The current manuscript considers the two species ODE and PDE Lotka–Volterra competition model, where one competitor possesses the dynamic of finite time extinction. As mentioned this is of immense interest currently to mathematicians and ecologists alike; in particular there is effort to understand in what capacity species will "optimize" [35, 53, 54, 55]. We see that bringing in



(a) p=1, q=1 (b) p=0.999, q=1

FIG. 10. *Plots of $d_1$ versus $d_2$ for* (3.10)–(3.11). *We choose* $\Omega = [0,1]$ *and* $m(x) = x(1-x)$. *We use the following parameters:* $b = c = 0.999$. *The red region shows that u prevails, i.e.,* $(u^*, 0)$, *the blue region shows coexistence* $(u^*, v^*)$, *and the green region shows v prevailing, i.e.,* $(0, v^*)$. *The classical result* [41, 42] *is seen in* (a); *however, FTEM show different results in* (b)—*here we see that the faster diffuser could also prevail. The initial condition used for each simulation is* $(u_0(x), v_0(x)) = (0.363027 + 0.1(\cos(\pi x))^2, 0.290947 + 0.1(\cos(\pi x))^2)$.



(a) (b)

FIG. 11. *Simulation for* (3.10)–(3.11) *with* $\Omega = [0,1]$ *and* $m(x) = x(1-x)$. *Simulations in* (a) *and* (b) *show solutions converging to* $(0, v^*)$ *in finite time with parameters* $b = c = 1, d_1 = 0.01, d_2 = 1, p = 0.7$ *and initial data* $(u_0(x), v_0(x)) = (0.199 + (\cos(\pi x))^2, 0.199 + 0.001(\cos(\pi x))^2)$. *The slower diffuser loses herein.*

FTEM can change (albeit counterintuitively) certain classical ecological scenarios, see also Figure 11. Most notably in the ODE case, we see that the weaker competitor can avoid competitive exclusion with FTEM present—this is counterintuitive as it posits that speeding up its extinction enables it to turn the tables on a stronger competitor and coexist for various regimes of initial data. Note that a possible ecological explanation for this seemingly counterintuitive phenomenon is that increasing the weaker competitors rate to extinction may cause its population to fall rapidly, decreasing intraspecific competition among the weaker competitor, but increasing intraspecific competition among the stronger competitor—as the stronger competitor has *fewer* of the weaker competitors to contend with. Theorem 2.4 shows this under the sufficient restriction $(a_2)^2 b_1 + 2a_2 c_1 c_2 > 4a_1 b_2 c_2$. It would be interesting to investigate weakening this condition and also to think about this condition in ecological settings. Note that such counterintuitive ecological mechanisms, such as the "hydra effect," have been considered in the ecology literature [56, 57].

These results suggest interesting consequences for biocontrol applications [58], as well as motivate the use of such mechanisms in insect resistance management

strategies, where two competing biotypes of a pest species are preferred to coexist [8]—our results could be used to develop tactics in these directions. Note that from an applied point of view, the FTEM can be engineered by self-regulating mechanisms or external control as well as via (4.1); thus a future direction could be a detailed investigation of such models. Also interesting would be considering models where the stronger competitor counters the FTEM in the weaker competitor with its own FTEM dynamic. Here one would consider (2.3) with both $0 < p < 1, 0 < q < 1$. Here, various further interesting dynamics could be possible. For example, if one considers the classical case of strong competition, where there is an internal equilibrium, which is a saddle, so there is always initial condition dependent extinction (see Figure 5(a)), the FTEM in one competitor does not change the dynamics in the phase qualitatively (see Figure 5(b)–(c)). However, if an FTEM mechanism were added in both competitors, one sees geometrically that two interior equilibria may form with one being a saddle and the other a stable equilibrium. Thus co-existence now may be possible in certain initial data regimes. Proving this rigorously is a future objective. Note also, via Theorem 2.7, we are only able to prove that initial data lying above $f(u)$ will be driven to $(0, v^*)$. However, any data lying above $W^s(E_3)$, the stable manifold of the interior saddle equilibrium $E_3$, is attracted to $(0, v^*)$. Bridging the gap between $W^s(E_3)$ and $f(u)$ via analytical estimates remains a goal for future investigations. Note that we believe that the interior equilibrium formed in Lemma 2.2 is always a saddle simply if $0 < p < 1, q = 1$. Proving this in general via planar dynamical systems methods remains a future endeavor.

We also aim to consider various self-regulating or external mechanisms of control in future work. We could consider the case where some proportion of the weaker competitor is harvested by an external controller or self-regulates its population by an action such as cannibalism [58]. We ask if this "strategy" might make it possible for stabilization of a weaker population. We choose parametric restrictions according to the extinction case. Let $v = dv + ev$, where $d + e = 1$, and $e$ is the proportion of the population that will possess the FTEM dynamic. If $e = 0, d = 1$, we are in the competitive exclusion case (2.2). This leads us to the model

$$\begin{cases} \dfrac{du}{dt} = a_1 u - b_1 u^2 - c_1 auv, \\ \dfrac{dv}{dt} = a_2 v - b_2 v^2 - c_2 duv - c_2 ev^q. \end{cases} \tag{4.1}$$

We see that even in this setting $v$ can avoid competitive exclusion and persist, thus coexist with the stronger competitor $u$—see Figure 12.

In the PDE case, Figure 10 is immensely interesting from both a mathematical and an evolutionary point of view. Mathematically we aim to focus to prove in full generality the results seen in Figure 10(b). From an evolutionary point of view, what we see is that the FTEM takes away some of the competitive advantage the slower diffuser has, in that if $d_1 < d_2$, but close to $d_2$, the faster diffuser may win and thus be selected for. This is the "green" band seen in Figure 10(b). It would be interesting if we could prove that one can move from a $(u^*, 0)$ state to a state of coexistence, when $d_1 < d_2$, and $p = 1$, and $q < 1$, that is, in the weak competition case. This would show that FTEM in the weaker competitor ($v$ in this case) could actually be competitively advantageous for $v$, leading to coexistence. We see similar results in the ODE case; see Figure 4(e).

Note that the FTEM could hinder well-posedness due to the nonsmooth term $u^p$, $0 < p < 1$, in (3.1). Two species semilinear reaction diffusion systems have

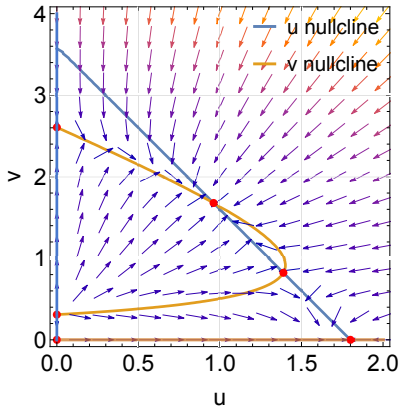



FIG. 12. *External mechanisms of control* (4.1)*: extinction case* (2.2) *for* $a_1 = 1.8,\ a_2 = 3,\ b_1 = 1,\ b_2 = 1,\ c_1 = 0.5,\ c_2 = 1.7,\ a = 1,\ d = 0.45,\ e = 0.55,\ q = 0.1.$

been considered in [59], where there are nonsmooth terms in *one* of the equations—such as in our case. The key tool used to show existence of bounded global in time, classical solutions is a weak comparison principle method [59]. This is for the Dirichlet boundary condition, however. Recently such problems have also been investigated in the case of more complicated boundary conditions [60]. In general, there could be data that lead to nonunique solutions; however, for certain given data, one has weak/classical solutions to the class of problems considered herein [61], even for the Neumann problem. Our goal is not to demonstrate well (or ill) posedness here but more to focus on the dynamical changes that the $0 < p < 1$ can bring about, and the many ecological consequences therein. Another worthwhile future direction will be an extensive numerical simulation across a broader parameter range to investigate how these dynamics might be affected. Also, such results may/may not hold in time varying environments [32]; this is also worthy of future investigations in light of the finite time extinction dynamic.

## Appendix A.

**A.1. Equilibrium analysis.** We present analytic guidelines in this section to analyze the equilibria of model (2.3). Consider the solutions to the steady state equations:

$$u\left[a_1 - b_1 u - c_1 u^{p-1} v\right] = 0, \tag{A.1}$$

$$v\left[a_2 - b_2 v - c_2 u v^{q-1}\right] = 0. \tag{A.2}$$

The above equations, (A.1) and (A.2), have four types of nonnegative equilibria:

(i) $E_0(0,0)$,
(ii) $E_1(a_1/b_1, 0)$,
(iii) $E_2(0, a_2/b_2)$,
(iv) $E_3(u^*, v^*)$.

For $q = 1$, we have

$$u^* = \frac{1}{c_2}\left[a_2 - b_2 v^*\right], \tag{A.3}$$

$$v^* = \frac{1}{c_1}\left[a_1 (u^*)^{1-p} - b_1 (u^*)^{2-p}\right], \tag{A.4}$$

and for $p = 1$, we have

$$v^* = \frac{1}{c_1}\left[a_1 - b_1 u^*\right], \tag{A.5}$$

$$u^* = \frac{1}{c_2}\left[a_2 (v^*)^{1-q} - b_2 (v^*)^{2-q}\right]. \tag{A.6}$$

The possible existence of a unique interior or multiple equilibria is shown in Figures 2, 4, and 5.

Now we discuss the local stability of an interior equilibrium point. The Jacobian matrix $\mathbf{J}$ of the model (2.3) evaluated at any of the possible interior equilibria $E_3(u^*, v^*)$ is

$$\mathbf{J} = \begin{bmatrix} a_1 - 2b_1 u^* - pc_1 u^{*p-1} v^* & -c_1 u^{*p} \\ -c_2 v^{*q} & a_2 - 2b_2 v^* - qc_2 u^* v^{*q-1} \end{bmatrix}. \tag{A.7}$$

The characteristic equation corresponding to $\mathbf{J}$ is given by

$$\lambda^2 - \operatorname{tr}(\mathbf{J})\lambda + \det(\mathbf{J}) = 0,$$

where

$$\operatorname{tr}(\mathbf{J}) = a_1 + a_2 - 2b_1 u^* - 2b_2 v^* - pc_1 u^{*p-1} v^* - qc_2 u^* v^{*q-1}$$

and

$$\det(\mathbf{J}) = \left(a_1 - 2b_1 u^* - pc_1 u^{*p-1} v^*\right)\left(a_2 - 2b_2 v^* - qc_2 u^* v^{*q-1}\right) - c_1 c_2 u^{*p} v^{*q}.$$

Here, $\operatorname{tr}(\mathbf{J})$ and $\det(\mathbf{J})$ represent the trace and determinant of the Jacobian matrix. Hence the stability of $E_3(u^*, v^*)$ is determined by the sign of $\det(\mathbf{J})$ and $\operatorname{tr}(\mathbf{J})$.

The above results are summarized in the following theorem.

THEOREM A.1. *The interior equilibrium $E_3(u^*, v^*)$ of model (2.3) is a saddle equilibrium if*

$$\left(\frac{a_1}{b_1}\right)^{1-p} + \frac{c_1 c_2 (1-p)}{b_1 b_2} < \frac{a_2 c_1 (1-p)}{a_1 b_2} + \frac{c_1 c_2}{b_1 b_2}.$$

*Proof.* We consider the case $p < 1, q = 1$; then

$$\begin{aligned}
&\det(\mathbf{J}) \\
&= \left(a_1 - 2b_1 u^* - pc_1 u^{*p-1} v^*\right)(a_2 - 2b_2 v^* - c_2 u^*) - c_1 c_2 u^{*p} v^* \\
&= \left(a_1 - 2b_1 u^* - pc_1 u^{*p-1} v^*\right)(a_2 - b_2 v^* - c_2 u^* - b_2 v^*) - c_1 c_2 u^{*p} v^* \\
&= \left(a_1 - b_1 u^* - b_1 u^* - pc_1 u^{*p-1} v^*\right)(-b_2 v^*) - c_1 c_2 u^{*p} v^* \\
&= \left(c_1 v^* u^{*p-1} - b_1 u^* - pc_1 u^{*p-1} v^*\right)(-b_2 v^*) - c_1 c_2 u^{*p} v^* \\
&= \left((1-p)c_1 v^* u^{*p-1} - b_1 u^*\right)(-b_2 v^*) - c_1 c_2 u^{*p} v^*.
\end{aligned}$$

This follows from using the equilibrium equations (A.3)–(A.4). Now dividing the above by $b_1 b_2 u^{*p} v^*$, using the comparisons $u^* < \frac{a_1}{b_1}$, thus $\frac{1}{u^*} > \frac{b_1}{a_1}$, and the equilibrium equation for $v^*$ via (A.3) we obtain

$$
\begin{aligned}
&\left(\frac{1}{b_1 b_2 u^{*p} v^*}\right) \det(\mathbf{J}) \\
&= (u^*)^{1-p} - \frac{(1-p)c_1}{b_1}\frac{v^*}{u^*} - \frac{c_1 c_2}{b_1 b_2} \\
&\leq \left(\frac{a_1}{b_1}\right)^{1-p} - \frac{(1-p)c_1}{b_1}\frac{b_1 v^*}{a_1} - \frac{c_1 c_2}{b_1 b_2} \\
&= \left(\frac{a_1}{b_1}\right)^{1-p} - \frac{(1-p)c_1}{b_1}\frac{b_1\left(\frac{1}{b_2}\left[a_2 - c_2 u^*\right]\right)}{a_1} - \frac{c_1 c_2}{b_1 b_2} \\
&= \left(\frac{a_1}{b_1}\right)^{1-p} + \frac{(1-p)c_1}{b_1}\frac{b_1}{a_1}\frac{c_2}{b_2}u^* - \frac{(1-p)c_1}{b_1}\frac{b_1}{a_1}\frac{a_2}{b_2} - \frac{c_1 c_2}{b_1 b_2} \\
&\leq \left(\frac{a_1}{b_1}\right)^{1-p} + \frac{(1-p)c_1}{a_1}\frac{c_2 a_1}{b_2 b_1} - \frac{(1-p)c_1 a_2}{a_1 b_2} - \frac{c_1 c_2}{b_1 b_2} \\
&< 0
\end{aligned}
$$

as long as

$$
\left(\frac{a_1}{b_1}\right)^{1-p} + \frac{c_1 c_2(1-p)}{b_1 b_2} < \frac{a_2 c_1(1-p)}{a_1 b_2} + \frac{c_1 c_2}{b_1 b_2}. \tag{A.8}
$$

Thus the above is a sufficient condition for $\det(\mathbf{J}) < 0$ which yields a saddle equilibrium. This proves the theorem. ∎

*Example* 2. We provide a numerical example for Theorem A.1 by using the following set of parameter values: $a_1 = 1.8,\ a_2 = 3,\ b_1 = 1.1,\ b_2 = 1.03,\ c_1 = 0.5,\ c_2 = 3.5,\ p = 0.7,\ q = 1$. The interior equilibrium point $E_3(0.26182, 2.02294)$ emerges and the Jacobian $\mathbf{J}$ is given by

$$
\mathbf{J} = \begin{bmatrix} 0.16560 & -0.19569 \\ -7.08029 & -2.08363 \end{bmatrix}.
$$

Here, $\operatorname{tr}(\mathbf{J}) = -1.91803 < 0$ and $\det(\mathbf{J}) = -1.73017 < 0$, thus conditions for the saddle are satisfied. In addition, note here that

$$
\left(\frac{a_1}{b_1}\right)^{1-p} + \frac{c_1 c_2(1-p)}{b_1 b_2} = 1.62 < 1.79 = \frac{a_2 c_1(1-p)}{a_1 b_2} + \frac{c_1 c_2}{b_1 b_2}.
$$

Thus, the parametric restriction (A.8) is satisfied. We provide simulation in Figure 1(b) to corroborate.

**A.2. Lambert *W*-function.** Note there is a numerical approach to calculating the critical $q^*$. At $q = q^*$, the slopes of the tangents of the $u$ and $v$ nullclines coincide instantaneously, and by the implicit function theorem, $\det(\mathbf{J}) = 0$ (see (A.7) for the Jacobian matrix $J$). Thus,

$$
(a_1 - 2b_1 u - c_1 v)\left(a_2 - 2b_2 v - q^* c_2 u v^{q^*-1}\right) - c_1 c_2 u v^{q^*} = 0.
$$

With the help of Mathematica software, we obtain

$$
q^* = \frac{1}{\ln(v)} W\left(\frac{(a_2 - 2b_2 v)\, v^{\frac{a_1 - 2b_1 u}{a_1 - 2b_1 u - c_1 v}} \ln(v)}{c_2 u}\right) - \frac{c_1 v}{a_1 - 2b_1 u - c_1 v}, \tag{A.9}
$$

where $W$ is the Lambert $W$-function or product logarithm (see [62] and references therein for a definition of the Lambert $W$-function). Thus one can obtain the critical $q^*$ by plugging the nonhyperbolic equilibrium into the above expression.

*Example* 3. We provide an example to substantiate $q^*$ given in (A.9) by using the following set of parameter values: $a_1 = 1.8$, $a_2 = 3$, $b_1 = 1$, $b_2 = 1$, $c_1 = 0.5$, $c_2 = 1.8$, and $p = 1$. We note that the unique nonhyperbolic equilibrium point occurs at $(u, v) = (0.89465, 1.81070)$. Thus

$$\begin{aligned} q^* &= \frac{1}{\ln(1.81070)} W\left(\frac{(-0.62140)(0.99292)\ln(1.81070)}{1.61037}\right) + 1.01196 \\ &= \frac{1}{\ln(1.81070)} W\left(-0.22748\right) + 1.01196 \\ &= \frac{1}{\ln(1.81070)} \left(-0.310214\right) + 1.01196 \\ &= 0.48946. \end{aligned}$$

The critical $q^*$ obtained here is the same as in Figure 2(b).

**Conflict of interest.** The authors declare there is no conflict of interest in this paper.

## REFERENCES

[1] P. N. Brown, *Decay to uniform states in ecological interactions*, SIAM J. Appl. Math., 38 (1980), pp. 22–37.

[2] Y. Lou, *Some challenging mathematical problems in evolution of dispersal and population dynamics*, in Tutorials in Mathematical Biosciences. IV, Lecture Notes in Math. 1922, Springer, Berlin, 2008, pp. 171–205.

[3] R. S. Cantrell and C. Cosner, *Spatial Ecology via Reaction-Diffusion Equations*, Ser. Math. Comput. Biol., John Wiley and Sons, Chichester, UK, 2003.

[4] R. S. Cantrell, C. Cosner, and Y. Lou, *Multiple reversals of competitive dominance in ecological reserves via external habitat degradation*, J. Dynam. Differential Equations, 16 (2004), pp. 973–1010.

[5] R. S. Cantrell, C. Cosner, and X. Yu, *Dynamics of populations with individual variation in dispersal on bounded domains,* J. Biol. Dyn., 12 (2018), pp. 288–317.

[6] R. D. Parshad, S. Wickramsooriya, and S. Bailey, *A remark on "Biological control through provision of additional food to predators: A theoretical study"* [Theor. Popul. Biol. 72 (2007), pp. 111–120], Theor. Popul. Biol., 132 (2020), pp. 60–68.

[7] D. W. Ragsdale, D. A. Landis, J. Brodeur, G. E. Heimpel, and N. Desneux, *Ecology and management of the soybean aphid in North America,* Annu. Rev. Entomology, 56 (2011), pp. 375–399.

[8] M. E. O'Neal, A. J. Varenhorst, and M. C. Kaiser, *Rapid evolution to host plant resistance by an invasive herbivore: Soybean aphid (Aphis glycines) virulence in North America to aphid resistant cultivars,* Current Opinion in Insect Science, 26 (2018), pp. 1–7.

[9] P. Mehta, D. F. McAuley, M. Brown, E. Sanchez, R. S. Tattersall, and J. J. Manson, *COVID-19: Consider cytokine storm syndromes and immunosuppression,* The Lancet, 395 (2020), pp. 1033–1034.

[10] A. P. Farrell, J. P. Collins, A. L. Greer, and H. R. Thieme, *Do fatal infectious diseases eradicate host species?*, J. Math. Biol., 77 (2018), pp. 2103–2164.

[11] A. P. Farrell, J. P. Collins, A. L. Greer, and H. R. Thieme, *Times from infection to disease-induced death and their influence on final population sizes after epidemic outbreaks*, Bull. Math. Biol., 80 (2018), pp. 1937–1961.

[12] A. L. Greer, C. J. Briggs, and J. P. Collins, *Testing a key assumption of host-pathogen theory: Density and disease transmission*, Oikos, 117 (2008), pp. 1667–1673.

[13] A. Fenton, J. P. Fairbairn, R. Norman, and P. J. Hudson, *Parasite transmission: Reconciling theory and reality*, J. Animal Ecology, 71 (2002), pp. 893–905.

[14] G. Dwyer, J. S. Elkinton, and J. P. Buonaccorsi, *Host heterogeneity in susceptibility and disease dynamics: Tests of a mathematical model,* American Naturalist, 150 (1997), pp. 685–707.
[15] P. A. Braza, *Predator–prey dynamics with square root functional responses*, Nonlinear Anal. Real World Appl., 13 (2012), pp. 1837–1843.
[16] J. Sugie, R. Kohno, and R. Miyazaki, *On a predator-prey system of Holling type*, Proce. Amer. Math. Soc., 125 (1997), pp. 2041–2050.
[17] J. Sugie and M. Katayama, *Global asymptotic stability of a predator-prey system of Holling type*, Nonlinear Anal., 38 (1999), pp. 105–121.
[18] N. Beroual and T. Sari, *A predator-prey system with Holling-type functional response,* Proc. Amer. Math. Soc., 148 (2020), pp. 5127–5140, doi:10.1090/proc/15166.
[19] K. Antwi-Fordjour, R. D. Parshad, and M. A. Beauregard, *Dynamics of a predator-prey model with generalized functional response and mutual interference,* Math. Biosci., 326 (2020), 108407, doi:10.1016/j.mbs.2020.108407.
[20] H. W. McKenzie, E. H. Merrill, R. J. Spiteri, and M. A. Lewis, *How linear features alter predator movement and the functional response,* Interface Focus, 2 (2012), pp. 205–216.
[21] G. D. Ruxton, *Increasing search rate over time may cause a slower than expected increase in prey encounter rate with increasing prey density,* Biology Lett., 1 (2005), pp. 133–135.
[22] C. C. Ioannou, G. D. Ruxton, J. Krause, *Search rate, attack probability, and the relationship between prey density and prey encounter rate,* Behavioral Ecology, 19(2008), pp. 842–846, https://doi.org/10.1093/beheco/arn038.
[23] C. M. Mols, K. van Oers, L. M. Witjes, C. M. Lessells, P. J. Drent, and M. E. Visser, *Central assumptions of predator–prey models fail in a semi-natural experimental system,* Proc. Roy. Soc. London B Biol. Sci., 271 (2004), pp. S85–S87.
[24] V. Ajraldi, M. Pittavino, and E. Venturino, *Modeling herd behavior in population systems*, Nonlinear Anal. Real World Appl., 12 (2011), pp. 2319–2338.
[25] J. D. Murray, *Mathematical Biology,* Springer, New York, 1993.
[26] H. Ninomiya, *Separatrices of competition-diffusion equations*, J. Math. Kyoto Univ., 35 (1995), pp. 539–567.
[27] S. S. Chen and J. P. Shi, *Global dynamics of the diffusive Lotka-Volterra competition model with stage structure*, Calc. Var. Partial Differential Equations, 59 (2020), 33.
[28] D. DeAngelis, W.-M. Ni, and B. Zhang, *Dispersal and spatial heterogeneity: single species*, J. Math. Biol., 72 (2016), pp. 239–254.
[29] Y. Lou and B. Wang, *Local dynamics of a diffusive predator-prey model in spatially heterogeneous environment*, J. Fixed Point Theory Appl., 19 (2017), pp. 755–772.
[30] I. Mazari, G. Nadin, and Y. Privat, *Optimal location of resources maximizing the total population size in logistic models*, J. Math. Pure. Appl., 134 (2020), pp. 1–35.
[31] S. Liang, and Y. Lou, *On the dependence of the population size on the dispersal rate*, Discrete Contin. Dyn. Syst. Ser. B, 17 (2012), pp. 2771–2788.
[32] V. Hutson, K. Mischaikow, and P. Poláčik, *The evolution of dispersal rates in a heterogeneous time-periodic environment*, J. Math. Biol., 43 (2001), pp. 501–533.
[33] C. Y. Kao, Y. Lou, and E. Yanagida, *Principal eigenvalue for an elliptic problem with indefinite weight on cylindrical domains*, Math. Biosci. Eng., 5 (2008), pp. 315–335.
[34] W.-M. Ni, *The Mathematics of Diffusion*, CBMS-NSF Regional Conf. Ser. in Appl. Math. 82, SIAM, Philadelphia, 2011.
[35] W. Ni, J. Shi, and M. Wang, *Global stability of nonhomogeneous equilibrium solution for the diffusive Lotka-Volterra competition model*, Calc. Var., 59 (2020), 132.
[36] W.-M. Ni, *Complete Dynamics in a Heterogeneous Competition-Diffusion System*, East China Normal University and University of Minnesota, 2012.
[37] I. Masato, M. Tatsuya, N. Hirokazu, and E. Yanagida, *Diffusion-induced extinction of a superior species in a competition system*, Jpn. J. Ind. Appl. Math., 15 (1998), pp. 233–252
[38] X. He, K.-Y. Lam, Y. Lou, and W.-M. Ni, *Dynamics of a consumer-resource reaction-diffusion model: Homogeneous versus heterogeneous environments*, J. Math. Biol., 78 (2019), pp. 1605–1636.
[39] X. He and W.-M. Ni, *The effects of diffusion and spatial variation in Lotka-Volterra competition-diffusion system* I*: Heterogeneity vs. homogeneity*, J. Differential Equations, 254 (2013), pp. 528–546.
[40] X. He and W.-M. Ni, *The effects of diffusion and spatial variation in Lotka-Volterra competition-diffusion system* II*: The general case*, J. Differential Equations, 254 (2013), pp. 4088–4108.
[41] X. He, and W.-M. Ni, *Global dynamics of the Lotka-Volterra competition-diffusion system: Diffusion and spatial heterogeneity* I, Comm. Pure. Appl. Math., 69 (2016), pp. 981–1014.

[42] X. He and W.-M. Ni, *Global dynamics of the Lotka-Volterra competition-diffusion system with equal amount of total resources* II, Calc. Var. Partial Differential Equations, 55 (2016), 25.
[43] V. Hutson, Y. Lou, K. Mischaikow, and P. Poláčik, *Competing species near the degenerate limit*, SIAM J. Math. Anal., 35 (2003), pp. 453–491.
[44] K.-Y. Lam and W.-M. Ni, *Uniqueness and complete dynamics of the Lotka-Volterra competition diffusion system*, SIAM J. Appl. Math., 72 (2012), pp. 1695–1712.
[45] R. Li and Y. Lou, *Some monotone properties for solutions to a reaction-diffusion model*, Discrete Contin. Dyn. Syst. Ser. B, 24 (2019), pp. 4445–4455.
[46] Y. Lou, *On the effects of migration and spatial heterogeneity on single and multiple species*, J. Differential Equations, 223 (2006), pp. 400–426.
[47] Y. Lou, S. Martinez, and P. Poláčik, *Loops and branches of coexistence states in a Lotka-Volterra competition model*, J. Differential Equations, 230 (2006), pp. 720–742.
[48] K. Nagahara, Y. Lou, and E. Yanagida, *Maximizing the total population with logistic growth in a patchy environment*, J. Math. Biol., 82 (2021), 2, DOI:10.1007/s00285-021-01565-7.
[49] K. Nagahara and E. Yanagida, *Maximization of the total population in a reaction-diffusion model with logistic growth*, Calc. Var. Partial Differential Equations, 57 (2018), 80.
[50] J. Dockery, V. Hutson, K. Mischaikow, and M. Pernarowski, *The evolution of slow dispersal rates: A reaction-diffusion model*, J. Math. Biol., 37 (1998), pp. 61–83.
[51] A. Hastings, *Can Spatial variation alone lead to selection for dispersal?*, Theor. Popul. Biol., 24 (1983), pp. 244–251.
[52] G. R. Sell and Y. You, *Dynamics of Evolutionary Equations*, Appl. Math. Sci. 143, Springer, New york, 2020.
[53] X. Bai, X. He and F. Li, *An optimization problem and its application in population dynamics*, Proc. Amer. Math. Soc., 144 (2016), pp. 2161–2170.
[54] F. Caubet, T. Deheuvels, and Y. Privat, *Optimal location of resources for biased movement of species: The* 1*D Case*, SIAM J. Appl. Math., 77 (2017), pp. 1876–1903.
[55] W. Ding, H. Finotti, S. Lenhart, Y. Lou, and Q. Ye, *Optimal control of growth coefficient on a steady-state population model*, Nonlinear Anal. Real World Appl., 11 (2010), pp. 688–704.
[56] P. A. Abrams, *When does greater mortality increase population size? The long history and diverse mechanisms underlying the hydra effect*, Ecology Lett., 12 (2009), pp. 462–474.
[57] M. Sieber and F. M. Hilker, *The hydra effect in predator–prey models,* J. Math. Biol., 64 (2012), pp. 341–360.
[58] J. Lyu, P. J. Schofield, K. M. Reaver, M. Beauregard, and R. D. Parshad, *A comparison of the Trojan Y chromosome strategy to harvesting models for eradication of nonnative species,* Natural Resource Modeling, 33 (2020), e12252.
[59] N. Bedjaoui and P. Souplet, *Critical blowup exponents for a system of reaction-diffusion equations with absorption,* Z. Angew. Math. Phys., 53 (2002), pp. 197–210.
[60] K. Fellner, E. Latos, and B. Q. Tang, *Well-posedness and exponential equilibration of a volume-surface reaction–diffusion system with nonlinear boundary coupling,* Ann. Inst. H. Poincaré Anal. Non Linéaire, 35 (2018), pp. 643–673.
[61] M. F. Bidaut-Véron, M. García-Huidobro, and C. Yarur, *On a semilinear parabolic system of reaction–diffusion with absorption,* Asympto. Anal., 36 (2003), pp. 241–283.
[62] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, *On the Lambert W function*, Adv. Comput. Math., 5 (1996), pp. 329–356.