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ABSTRACT

We study networks of coupled oscillators and analyze the role of coupling delays in determining the emergence of cluster synchronization. Given a network topology and a particular arrangement of the coupling delays over the network connections, different patterns of cluster synchronization may emerge. We focus on a simple ring network of six bidirectionally coupled identical oscillators, for which with two different values of the delays, a total of eight cluster synchronization patterns may emerge, depending on the assignment of the delays to the ring connections. We analyze stability of each of the patterns and find that for large enough coupling strength and specific values of the delays, they can all be stabilized. We construct an experimental ring of six bidirectionally coupled Colpitts oscillators, with delayed connections obtained by coupling the oscillators via RF cables of appropriate length. We find that experimental observations of cluster synchronization are in essential agreement with theoretical predictions. We also verify our theory in a fully connected network of fifty nodes for which connections are randomly assigned to be either undelayed or delayed with a given probability.

Many studies have investigated cluster synchronization (CS) in networks of coupled oscillators by focusing on the relationship between the network topology and the emergence and stability of synchronized clusters. However, the role of coupling delays has remained relatively unexplored. Here, we show that in addition to the pattern of connectivity between nodes, communication delays over the network connections can be responsible for the emergence of synchronized clusters.

I. INTRODUCTION

We study networks of coupled oscillators and focus on cluster synchronization (CS), which occurs when the oscillators group into clusters, with the nodes in the same cluster synchronizing but those in different clusters not synchronizing. CS in networks has been studied in a number of papers,[2,3,24] and the issue of stability of the CS solution has been investigated in Refs. 11, 29, 34, 36, and 37.

In this paper, we focus on cluster synchronization in networks in which the interaction between individual systems may not be instantaneous but may be delayed due to finite propagation time of signals. This is quite common in real-life systems: for instance, coupling delays play a significant role on the dynamics of networks of coupled lasers,[35] neuron networks,[15,27] and connected vehicle systems.[16,26] The effects of delays on the emergence of CS in arbitrary networks have so far received little attention, with the exception of one recent paper,[20] which assumes that each node/neuron communicates with others via connections of different types and each type has an associated delay. Here, we lift this assumption and study how coupling delays may affect cluster synchronization in simple networks with nodes all of the same type and connections all of the same type.

Studies on cluster synchronization in networks with delays have investigated symmetry-breaking bifurcations,[12] the role of CS for control purposes,[40] partial relay synchronization of chimera states,[31] cluster and group synchronization and their stability,[9] application to neural systems,[7,25,32,33,42] and symmetric rings of delay-coupled lasers[6] or crystal oscillators.[5] Despite these efforts, to the best of our knowledge, up to now, a general picture about how interaction delays may affect the emergence and the stability of synchronized clusters in networks is lacking.

Cluster synchronization has been studied also experimentally in networks of chaotic oscillators with connections either undelayed[23,39] or delayed.[18] In some sense, using chaotic oscillators makes the analysis simpler than for periodic oscillators, as delay systems generically have families of periodic solutions, which reappear for infinitely many delay times.[13,44] In particular, when the coupled dynamics is periodic, it is possible that signals that propagate with different transmission delays become indistinguishable from each other.[8,43] Networks of periodic oscillators with ring topology and delayed coupling have been studied analytically, numerically, and in some cases also experimentally.[17,30,43] Here, we focus on ring topologies, just because they provide simple regular network topologies with constant degrees of the nodes, and as such they allow the emergence of the complete synchronous solution, in the absence of delays.

Notwithstanding its apparent simplicity, the ring network can produce a variety of cluster synchronization patterns, depending on the assignment of the delays over the ring connections, including the pattern in which all the nodes are in the same cluster and the pattern in which each node is in a cluster by itself. We obtain all the possible CS patterns that are possible in a ring network with six nodes based on the arrangement of two different delays and compute the Lyapunov exponents that determine stability of each pattern. Though not explicitly shown in this paper, our proposed approach is directly generalizable to other more complex network topologies, such as random networks or scale free networks with arbitrary assignments of the delays over the network connections.

The rest of the paper is organized as follows. In Sec. II, we present a general approach to identify CS patterns, which under the assumption of chaotic dynamics, applies to any network and to any arbitrary assignment of the delays over the network connections. As a proof of concept of the proposed approach, in Secs. III and IV, we consider an undirected network of six Colpitts oscillators,[10,21] with ring topology. The individual oscillator parameters are set in order to have chaotic dynamics. The emergence of synchronous clusters is analyzed through numerical simulations in Sec. III and through experimental measurements in Sec. IV. In Sec. V, we present an application of our method to a larger random network with 50 nodes.

II. CLUSTER SYNCHRONIZATION IN NETWORKS WITH DELAYS

We consider networks of coupled dynamical systems described by the set of equations,

$$\dot{\mathbf{x}}_i(t) = \mathbf{F}(\mathbf{x}_i(t)) + \sigma \sum_{j=1}^{N} A_{ij}[\mathbf{H}(\mathbf{x}_j(t-\tau_{ij})) - \mathbf{H}(\mathbf{x}_i(t))], \quad (1)$$

$i = 1, \ldots, N$, where $\mathbf{x}_i(t) \in R^n$ is the state of oscillator i at time t, $\mathbf{F} : R^n \mapsto R^n$ indicates the dynamics of each uncoupled node, and $\mathbf{H} : R^n \mapsto R^n$ indicates the coupling between nodes. The matrix $A = \{A_{ij}\}$ represents the network connectivity; we call $\mathbf{H}(\mathbf{x}_j(t-\tau_{ij}))$ the *signal* received at node i from node j, where τ_{ij} is the coupling delay from j to i. Given the network topology $A = \{A_{ij}\}$ and the coupling delays $\{\tau_{ij}\}$, we determine (i) the existence of cluster synchronization patterns and (ii) their dynamical stability. We now assume each delay can assume a finite set of values $\tau_{ij} \in \{\tau^1, \tau^2, \ldots \tau^K\}$. Then, we rewrite the adjacency matrix $A = A^1 + A^2 + \cdots + A^K$ such that each $A^k = \{A^k_{ij} = A_{ij}\delta(\tau_{ij}, \tau^k)\}$, where $\delta(i, j)$ is the Kronecker delta. We can thus rewrite Eq. (1) as follows:

$$\dot{\mathbf{x}}_i(t) = \mathbf{F}(\mathbf{x}_i(t)) + \sigma \sum_{k=1}^{K}\sum_{j=1}^{N} A^k_{ij}[\mathbf{H}(\mathbf{x}_j(t-\tau^k)) - \mathbf{H}(\mathbf{x}_i(t))], \quad (2)$$

$i = 1, \ldots, N$. Consider a partition of the set of nodes $\mathcal{V} = 1, 2, \ldots, N$ into M clusters $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_M$, $\mathcal{K}_m \cap \mathcal{K}_n = 0$ for $m \neq n$ and $\cup_{m=1}^{M}\mathcal{K}_m = \mathcal{V}$. Say N_m is the number of nodes in cluster $\mathcal{K}_m$, $\sum_{m=1}^{M} N_m = N$. We call a cluster formed of only one node, i.e., with $N_m = 1$, a trivial cluster. Formally, the set of Eq. (2) achieves cluster synchronization when the node dynamics converge on a time evolution such that $\mathbf{x}_i(t) = \mathbf{x}_j(t) = \mathbf{y}_m(t)$ if i, j belong to the same cluster $\mathcal{K}_m$.

Definition 1: A partition of the set of nodes $\mathcal{V}$ into M clusters $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_M$ identifies a coloring $\mathcal{C}$ of the nodes with M colors, where each node in the network is assigned to a color. Then, each cluster also is assigned to one and only one color.

In the rest of this paper, we focus on the case of chaotic dynamics. Then, a necessary condition for the existence of a cluster synchronized solution $\mathbf{x}_i(t) = \mathbf{y}_\ell(t)$ for all nodes $i \in \mathcal{K}_\ell$ that evolve according to Eq. (1) is that each node belonging to cluster $\mathcal{K}_\ell$ receives the same overall input with delay τ^k from the nodes belonging to cluster $\mathcal{K}_m$, for any k, ℓ, m. More compactly, denoting this overall input as $Q^k_{\ell m}$, this condition can be expressed as

$$\sum_{j \in \mathcal{K}_m} A^k_{ij} = Q^k_{\ell m} \quad \forall i \in \mathcal{K}_\ell \quad k = 1, \ldots, K \quad (3)$$

for $\ell, m = 1, \ldots, M$.

Definition 2: Satisfaction of Eq. (3) indicates that the coloring $\mathcal{C}$ is *balanced*, i.e., each node of color ℓ gets the same overall input $Q^k_{\ell m}$ with delay τ^k from the nodes of color m, for all colors ℓ and m and for any delay τ^k with $k = 1, \ldots, K$.

As described in Refs. 13, 14, and 41, the fact that a coloring $\mathcal{C}$ is balanced [corresponding to satisfaction of Eq. (3)] provides a condition for the existence of a CS solution for Eq. (2). We call the M-node network associated with the M-dimensional matrices $Q^k = \{Q^k_{\ell m}\}$, $k = 1, \ldots, K$ the *quotient network*. The CS evolution is, therefore, governed by the dynamics of the quotient network,

$$\dot{\mathbf{y}}_\ell(t) = \mathbf{F}(\mathbf{y}_\ell(t)) + \sigma \sum_{k=1}^{K}\sum_{m=1}^{M} Q^k_{\ell m}[\mathbf{H}(\mathbf{y}_m(t-\tau^k)) - \mathbf{H}(\mathbf{y}_\ell(t))], \quad (4)$$

$\ell = 1, \ldots, M$.

A. Computation of the minimal balanced coloring

Given the adjacency matrix A and the delays τ_{ij} (alternatively, given the set of matrices $A^1, A^2, \ldots, A^K$), we can compute the minimal balanced coloring, that is, the set of clusters of minimum cardinality that can cluster synchronize.[1] We use an existing algorithm.[1,19,22] We proceed under the assumption of chaotic dynamics. This guarantees that a necessary condition for two nodes i, j to receive the same signal from another node k, is that $\tau_{ki} = \tau_{kj}$.

Definition 3: A *minimal balanced coloring* is a balanced coloring with the minimal number of colors.

Definition 4: A coloring $\mathcal{C}' \subseteq \mathcal{C}$ is a *refinement* of a coloring $\mathcal{C}$ if two nodes that have the same color in $\mathcal{C}'$ have the same color also in $\mathcal{C}$; in other words, $\mathcal{C}'$ is a larger set of colors with respect to $\mathcal{C}$.

Definition 5: The *input-driven refinement* $\mathcal{C}'$ of a coloring $\mathcal{C}$ is obtained as follows. Consider a set of nodes that have color ℓ in $\mathcal{C}$, with $\ell \in \{1, \ldots, M\}$. These nodes are assigned a new color if they receive the same number of inputs with delay τ^k from all nodes with the same color in $\mathcal{C}$, for all the possible delays τ^k with $k \in \{1, \ldots, K\}$ [see also condition (3)].

The **algorithm** for computing the minimal balanced coloring consists of three steps:

1. color all nodes with the same color;
2. replace the current coloring $\mathcal{C}$ with the input-driven refinement $\mathcal{C}'$ of the current coloring; and
3. repeat step 2 until no new refinement is obtained.

III. THE CASE OF THE HEXAGON NETWORK

Synchronization in ring topologies in the presence of delays has been studied in many papers, with several of these works focusing on the emergence of periodic rotating waves; see, e.g., Refs. 6, 30, and 43. We here focus on a ring topology but consider the case of chaotic dynamics, for which by delaying some of the ring

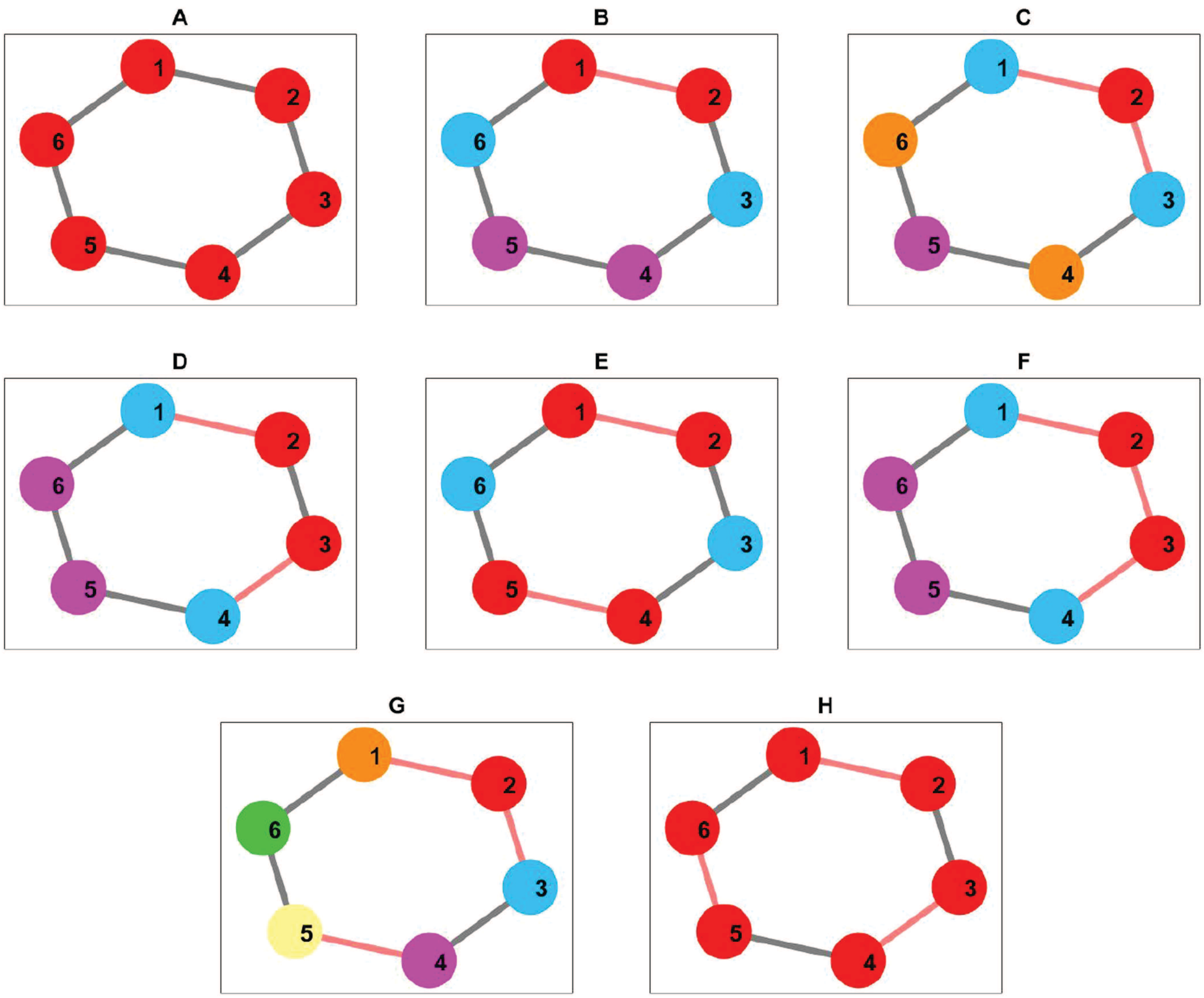

FIG. 1. CS patterns corresponding to the minimal balanced coloring, labeled (a) to (h). Here, a gray edge represents a bi-directional undelayed connection. A red edge represents a bi-directional delayed connection between two nodes. Clusters are represented by nodes of the same color.

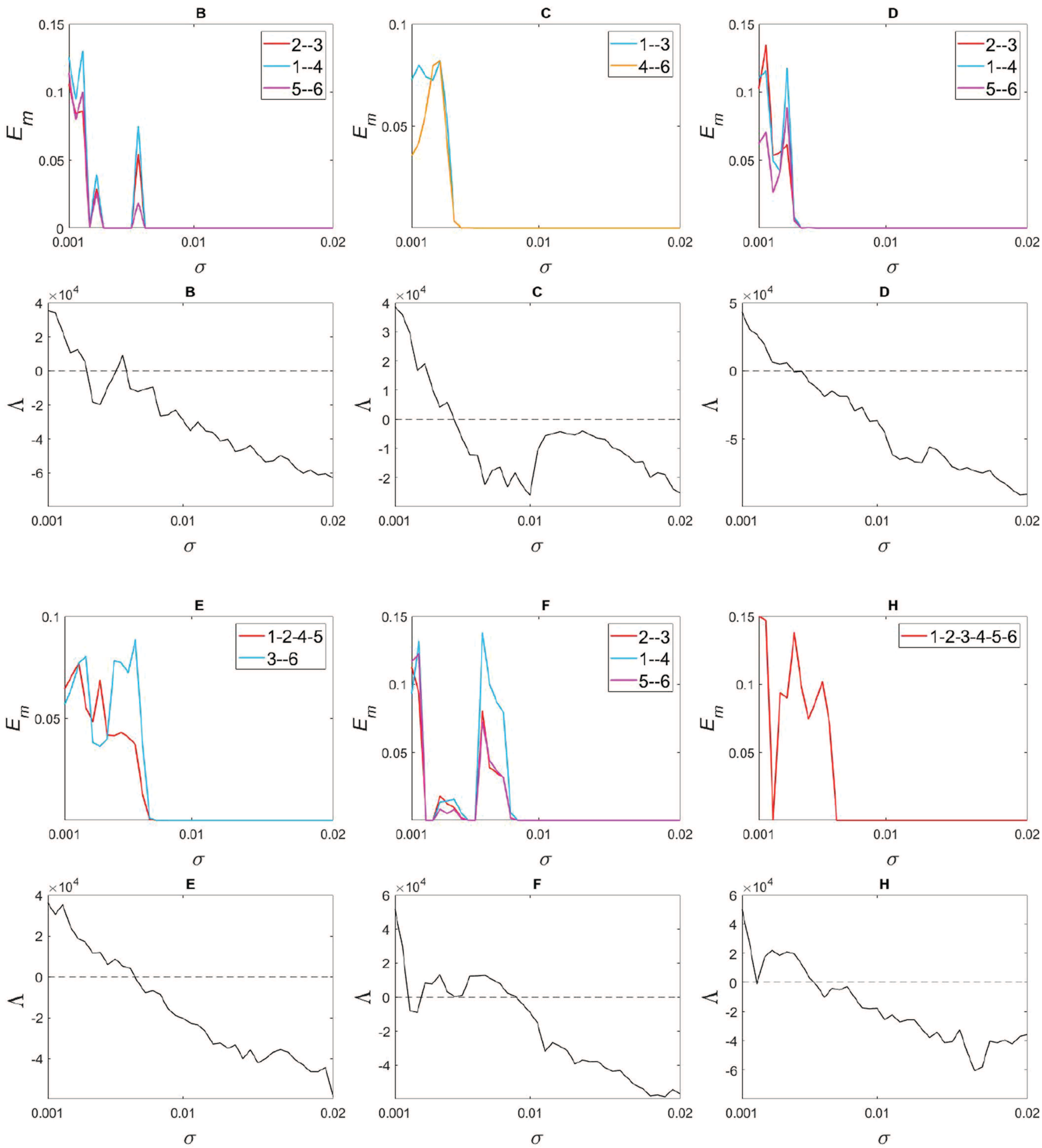

FIG. 2. For each CS pattern [(b)-(f) and (h)], the CS error E_m is compared to largest transverse Lyapunov exponent Λ. For all patterns, the error plot and the Lyapunov exponent plot are placed on top of each other for direct comparison.

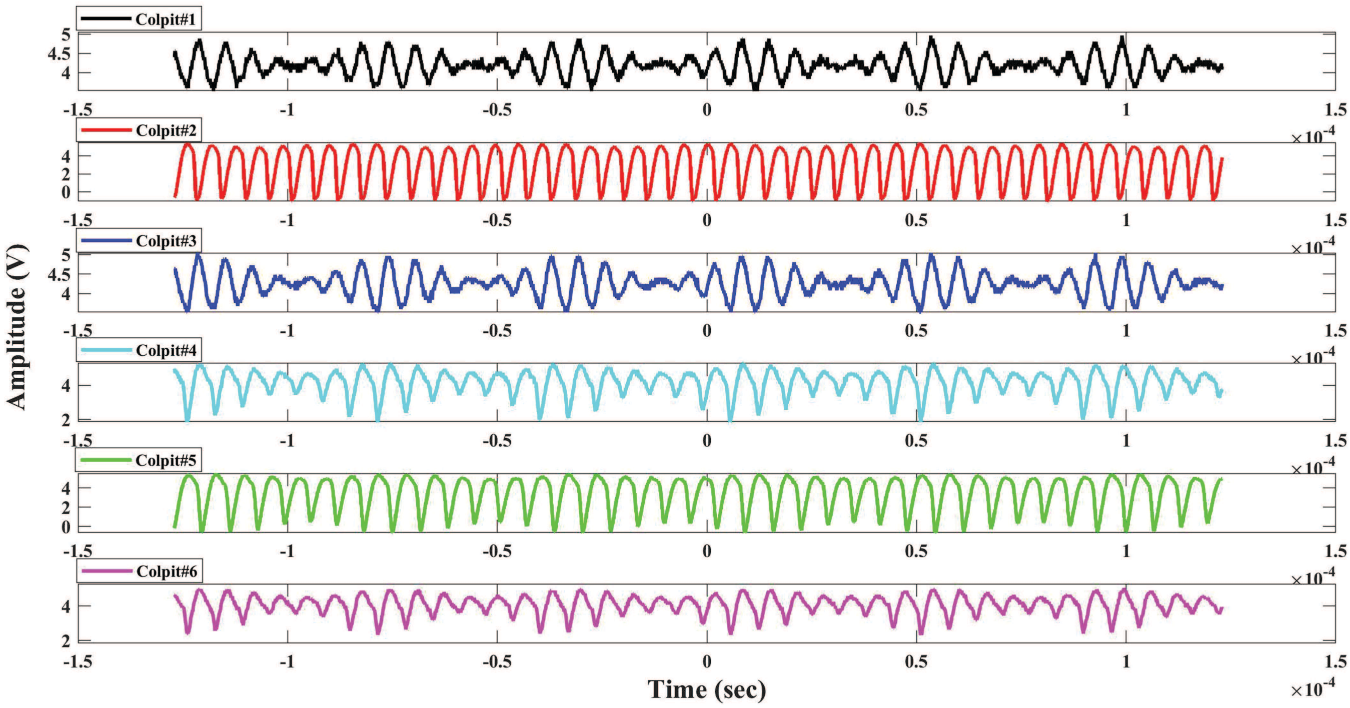

FIG. 3. Experimental waveforms from the ring of coupled Colpitts oscillators coupled via configuration C: $\tau_{12} = \tau_{23} = 615$ ns, and $\sigma = R_C^{-1} = 0.0455\ \Omega^{-1}$.

connections, different patterns of cluster synchronization may arise. We focus on the case study of a hexagon network, represented in Fig. 1(a), with symmetric adjacency matrix $A = \{A_{ij}\}$, $A_{ij} = A_{ji} = \delta(|i-j|, 1) + \delta(|i-j|, 5)\}$ (we recall that $\delta(i,j)$ denotes the Kronecker delta) for which, in the presence of all equal delays $\tau_{ij} = \tau$, a solution exists corresponding to complete synchronization. However, different patterns of cluster synchronization are possible for different choices of the delays. For simplicity, we consider the existence of two types of connections, undelayed $\tau^1 = 0$ and delayed $\tau^2 = \tau$. Accordingly, we define two adjacency matrices, A^{d}, which contains only delayed connections, and A^{u}, which contains only undelayed connections, $A = A^{\mathrm{d}} + A^{\mathrm{u}}$.

Because for the ring topology $\sum_j A_{ij} = 2$ for each i, then we can rewrite Eq. (1) as follows:

$$\dot{\mathbf{x}}_i(t) = \tilde{\mathbf{F}}(x_i(t)) + \sigma \sum_j A_{ij}\mathbf{H}(x_j(t-\tau_{ij})), \quad (5)$$

$i = 1, \ldots, N$, where $\tilde{\mathbf{F}}(x_i(t)) = \mathbf{F}(x_i(t)) - 2\sigma\mathbf{H}(x_i(t))$ and Eq. (2) as follows:

$$\dot{\mathbf{x}}_i(t) = \tilde{\mathbf{F}}(x_i(t)) + \sigma \sum_j A^{\mathrm{u}}_{ij}\mathbf{H}(x_j(t)) + \sigma \sum_j A^{\mathrm{d}}_{ij}\mathbf{H}(x_j(t-\tau)), \quad (6)$$

$i = 1, \ldots, N.$

With six, bi-directional connections, there are 64 possible combinations of delayed and undelayed connections. We refine these 64 combinations into eight unique *configurations*. These eight configurations, shown in Fig. 1, can describe any delay arrangement possible, assuming the position of specific nodes is irrelevant. From here, we predict the existence of CS patterns using the approach described in Sec. II A. A minimal balanced coloring corresponds to each one of the eight different configurations [labeled (a) to (h)], which is shown in Fig. 1. In what follows, we refer to each such coloring as a CS pattern. In general, a CS pattern is not unique to a given configuration. As an example of this, Figs. 1(a) and 1(h) show two different arrangements of delays, to which corresponds the same CS pattern (coloring of the nodes.)

A quotient network corresponds to each one of the cluster patterns in Fig. 1. These are all shown in Sec. 2 in the supplementary material.

A. Stability analysis

We linearize the system of Eq. (6) about the quotient network dynamics,

$$\delta\dot{\mathbf{x}}_i(t) = D\tilde{\mathbf{F}}(x_i(t))\delta\mathbf{x}_i(t) + \sigma \sum_j A^{\mathrm{u}}_{ij}D\mathbf{H}(x_j(t))\delta\mathbf{x}_j(t) + \sigma \sum_j A^{\mathrm{d}}_{ij}D\mathbf{H}(x_j(t-\tau))\delta\mathbf{x}_j(t-\tau), \quad (7)$$

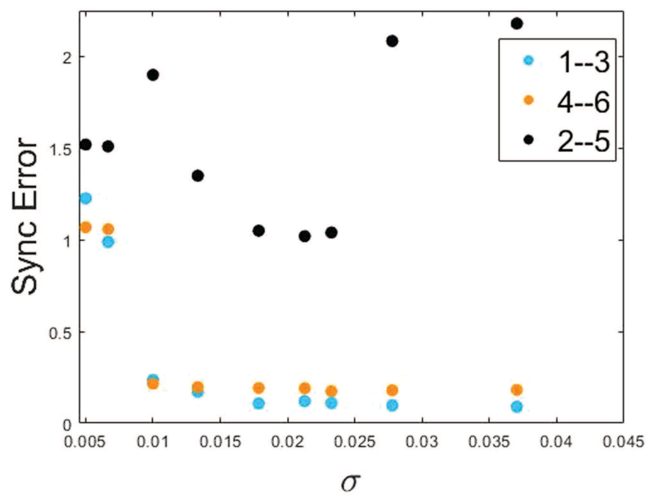

FIG. 4. Experimentally computed synchronization errors between Colpitts pairs 1 and 3, 4 and 6, and 2 and 5, vs σ for configuration C. The synchronization errors between Colpitts 1 and 3 and Colpitts 4 and 6 become small for $\sigma \geq 0.01$. In comparison, the error between Colpitts 2 and 5 (not synchronized, as expected) is substantially larger and does not decrease for increasing σ.

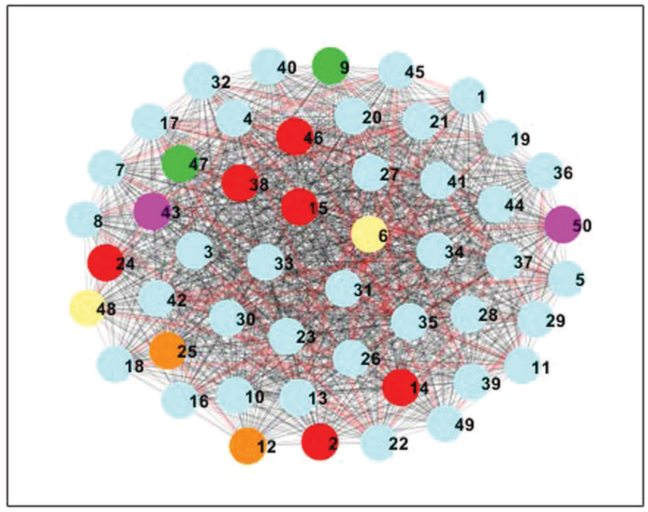

FIG. 5. All-to-all coupled network with $N = 50$ nodes. A red edge represents a bi-directional delayed connection and a gray edge represents a bi-directional undelayed connection. Clusters are represented by nodes of the same color. All trivial clusters are represented by a light blue color.

$i = 1, \ldots, N$. The above set of equations can be rewritten in compact form,

$$\delta\dot{\mathbf{x}}(t) = \Big[\sum_{k=1}^{M} J_k \otimes D\tilde{\mathbf{F}}(x_k(t)) + \sigma A^{\mathrm{u}} \sum_{k=1}^{M} J_k \otimes D\tilde{\mathbf{H}}(x_k(t))\Big]\delta\mathbf{x}(t) + \sigma A^{\mathrm{d}} \sum_{k=1}^{M} J_k \otimes D\tilde{\mathbf{H}}(x_k(t-\tau))\delta\mathbf{x}(t-\tau) \tag{8}$$

in the vector $\delta\mathbf{x} = [\delta\mathbf{x}_1^T, \delta\mathbf{x}_2^T, \ldots, \delta\mathbf{x}_N^T]^T$, where the entries of the diagonal matrix J_k are equal to

$$J_{kii} = \begin{cases} 1 & \text{if } i \in C_k, \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

and $\sum_{k=1}^{M} J_k = I_N$ is the identity matrix.

In order to reduce the dimensionality of Eq. (8), we introduce an N-dimensional transformation matrix T.[29,37] By applying the transformation matrix T to Eq. (8), we obtain

$$\dot{\mathbf{z}}(t) = \Big[\sum_{k=1}^{M} G_k \otimes D\tilde{\mathbf{F}}(x_k(t)) + \sigma B^{\mathrm{u}} \otimes I_N \sum_{k=1}^{M} G_k \otimes D\tilde{\mathbf{H}}(x_k(t))\Big]\mathbf{z}(t) + \Big[\sigma B^{\mathrm{d}} \otimes I_N \sum_{k=1}^{M} G_k D\tilde{\mathbf{H}}(x_k(t-\tau))\Big]\mathbf{z}(t-\tau), \tag{10}$$

where $\mathbf{z} = (T \otimes I_n)\delta\mathbf{x}$, the matrix T for each pattern can be derived as described in Refs. 29 and 37, and the matrices G_k are equal to $G_k = TJ_kT^T$. The matrices $B^{\mathrm{u}} = TA^{\mathrm{u}}T^T$ and $B^{\mathrm{d}} = TA^{\mathrm{d}}T^T$ share the same block-diagonal structure. Of these blocks, there is one that we call "parallel," which describes the motion in the CS manifold and a number of blocks which we call "transverse" that describe motion orthogonal to the manifold. We are interested in the growth or decay of the perturbations associated with these orthogonal blocks when integrated in time. For each transverse block $s = 1, \ldots, S$, given an initial time t_i and a final time $t_f \gg t_i$, we compute the maximum transverse Lyapunov exponent

$$\lambda_s = \ln\left(\frac{\|\mathbf{z}_s(t_f)\|}{\|\mathbf{z}_s(t_i)\|}\right)\frac{1}{t_f - t_i} \tag{11}$$

to examine growth or decay. The condition for stability of the given CS pattern is that all the transverse Lyapunov exponents $\lambda_1, \lambda_2, \ldots, \lambda_S$ are negative, or alternatively, that the largest transverse Lyapunov exponent, $\Lambda = \max_{s=1,\ldots,S} \lambda_s < 0$.

B. Colpitts oscillators

We choose the Colpitts oscillator to describe the dynamics of each individual oscillator. The Colpitts oscillator is an electronic oscillator with versatile characteristics, the most important of which is its ability to oscillate chaotically. Colpitts oscillators provide a good test bed for the study of cluster synchronization in networks.[4] Following Ref. 4, we then consider Eq. (2) with the choice of the adjacency matrix corresponding to the hexagon

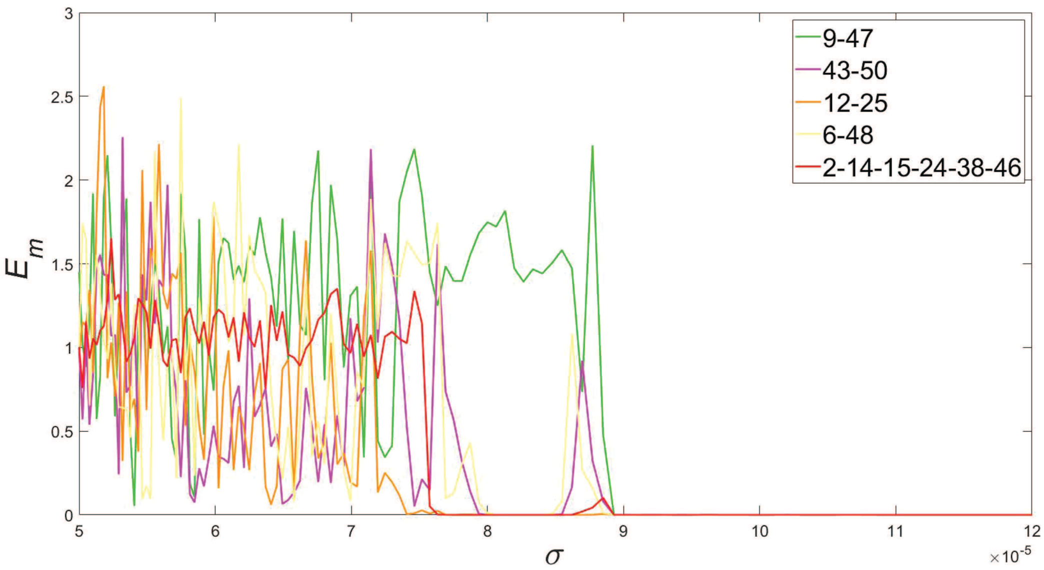

FIG. 6. CS error, E_m, for each of the five nontrivial clusters, against the coupling strength, σ. The set of nodes corresponding to each cluster are listed in the legend. Coloring is consistent with Fig. 5.

network and

$$\mathbf{x}_i(t) = \begin{bmatrix} V_{ce,i}(t) \\ V_{be,i}(t) \\ I_{L,i}(t) \end{bmatrix}$$

$$\mathbf{F}(x_i) = \begin{bmatrix} \frac{1}{C_1}(I_{L,i}(t) - I_C[V_{be,i}(t)]) \\ \frac{1}{C_2}\left(\frac{-(V_{ee}+V_{be,i}(t))}{R_{ee,i}} - I_B[V_{be,i}(t)] - I_{L,i}(t)\right) \\ \frac{1}{L}(V_{cc} - V_{ce,i}(t) + V_{be,i}(t) - I_{L,i}(t)R_L) \end{bmatrix}$$

$$\mathbf{H}(x_i) = \frac{1}{R_C}\begin{bmatrix} 0 \\ \frac{1}{C_2}V_{be} \\ 0 \end{bmatrix}, \tag{12}$$

with

$$I_B = \begin{cases} 0, & V_{be} \le V_{th}, \\ \frac{V_{be}-V_{th}}{R_{on}}, & V_{be} > V_{th}, \end{cases} \tag{13}$$

$$I_C = \beta I_B,$$

where L is the inductance, C_1 and C_2 are the capacitances of the circuit components (see Fig. 1 in the supplementary material), V_{ce} is the voltage drop between the collector and the emitter of the transistor, and V_{be} is the voltage drop between the base and the emitter. V_{cc} and V_{ee} are voltage sources or power supplies, and I_B and I_c are the current of the base and the collector, respectively. These two currents are the nonlinear terms in the system; they are zero below a threshold voltage and increase linearly above this cutoff. In a bipolar junction transistor (BJT) these currents are related through $\beta = \Delta I_c/\Delta I_b \approx I_c/I_b$, where β is the BJT amplification factor.

In our experimental setup, we have $V_{th} = 0.75$ V, $R_{on} = 200\,\Omega$, $\beta = 220$, $V_{cc} = 4.5$ V, $R_{ee} = 220\,\Omega$, $V_{ee} = -4.5$ V, $C_1 = C_2 = 47$ nF, $L = 56\,\mu$F, and transistor 2N222A.

For pattern A, we have verified that for $R_L = 23\,\Omega$ and for a low enough value of the resistance R_C (namely, a large enough value of the coupling strength σ), e.g., $R_C = 50\,\Omega$, all six oscillators converge to a chaotic trajectory corresponding to the dynamics of an uncoupled oscillator.[38] However, in what follows, we omit to present our study of pattern A (none of the connections delayed), since that can be performed using the classic master stability function approach of Ref. 28.

In order to analyze the emergence of cluster synchronization for patterns (b)–(h), we define for each pattern and each cluster $m = 1, \ldots, M$ the CS error below,

$$E_m = \left((N_m^2 - N_m)t_f\right)^{-1} \sum_{i=1}^{N_m}\sum_{j=1}^{N_m}\int_{\tau}^{\tau+t_f} |V_{ce,i}(t) - V_{ce,j}(t)|\,dt, \tag{14}$$

$m = 1, \ldots, M$.

C. Stability of the CS patterns

We next use Eq. (11) to compute stability for each CS pattern as a function of the coupling strength σ for a fixed value of the delay $\tau = 615$ ns. Figure 2 shows for each pattern the numerically computed CS errors $E_m(\sigma)$ (top) and the largest transverse Lyapunov exponent $\Lambda(\sigma)$ (bottom) for direct comparison. For both cases, we perform the computation by decreasing σ from 0.02 to 0 in steps of 0.0005. The matrices T as well as the block-diagonal matrices B for each one of the configurations [labeled (a) to (h)] are reported in Sec. 3 in the supplementary material. For all patterns, we see excellent agreement between the CS error and the largest transverse Lyapunov exponent. Note that each pattern, with the only exception of pattern G, which does not allow cluster synchronization, becomes stable for σ exceeding a threshold value, which is pattern specific.

IV. EXPERIMENTAL VALIDATION

We constructed an experimental ring of six identical Colpitts oscillators bidirectionally coupled via resistors. We focused on the configuration C in Fig. 1, with two delayed connections. The two delays are induced by connecting Colpitts 1 and 2 and Colpitts 2 and 3 with 400 ft RF cable (RG-58). The cables induce $\tau = 615.8$ ns delay and 22 Ω resistance, and the loss along the cable is about 6.6%. The voltages from the collector port of each Colpitts oscillator are monitored simultaneously using two synchronized oscilloscopes (TEK TDS2024 B, four channels; TEK TDS1001B, two channels).

The experimental scheme for an individual Colpitts in the ring is shown in Fig. 1 of the supplementary material. The parameter values are nominally the same we used to run our simulations in Fig. 2, and for which the dynamics of the six Colpitts oscillators is found to be chaotic. The waveforms of the six coupled Colpitts for $\sigma = R_C{}^{-1} = 0.0455$ are shown in Fig. 3, from which we see the emergence of the expected cluster synchronization pattern, corresponding to the minimal balanced coloring: cluster $\mathcal{C}_1$ with Colpitts 1 and 3, cluster $\mathcal{C}_2$ with Colpitts 4 and 6, and the remaining two Colpitts in trivial clusters. The waveforms of Colpitts 2 and 5 are clearly distinguishable from the others; however from the figure, they also appear to be qualitatively similar to each other. The synchronization errors E_m for $\mathcal{C}_1$ and $\mathcal{C}_2$ vs σ are plotted in Fig. 4, from which we see that synchronization is achieved approximately for $\sigma \geq 0.01$. This is in qualitative agreement with the results shown in Fig. 2, with the difference that the transition in Fig. 2 occurs for a lower value of σ. However, a higher threshold for synchronization is expected due to the presence of noise and non-identicalities between the individual oscillators in the experimental setup. In Fig. 4, we also show, for comparison, the synchronization error between Colpitts 2 and 5, computed by setting $N_m = 2$, $i = 2$, and $j = 5$ in Eq. (14), from which we see that though the waveforms for these two oscillators appear similar by inspection, they do not synchronize for any σ. This is in agreement with the emergence of CS pattern C, with a cluster formed of oscillators 1 and 3, another cluster formed of oscillators 4 and 6, and oscillators 2 and 5 in trivial clusters. A direct comparison of the experimental waveforms for Colpitts 1 and 3, Colpitts 4 and 6, and Colpitts 2 and 5 is included in Sec. 4 in the supplementary material.

V. A LARGER NETWORK

We now apply the method developed in this paper to a larger all-to-all coupled network with $N = 50$ nodes and a random assignment of undelayed and delayed connections. We randomly assign each connection to be either undelayed or delayed with probability 0.02 of being delayed. Figure 5 shows a realization of such a network. Nodes are colored according to the minimum balanced coloring, obtained as described in Sec. II A.

We see from Fig. 5 that there are a total of five clusters composed of multiple nodes. We now simulate the dynamics of this network using Eq. (1) and choosing the individual nodes to be Colpitts oscillators as described previously in this paper. We set the delay $\tau = 445$ ns. Analogously to Fig. 2, we see that all the Lyapunov exponents become negative for large enough σ, indicating that all five nontrivial clusters will synchronize for large enough σ. Figure 6 shows the CS errors E_m defined in Eq. (14) vs σ for all five nontrivial clusters. All E_m approach zero for σ large enough, which is in accordance with our prediction.

VI. CONCLUSIONS

In this paper, we have studied how coupling delays can determine the emergence of cluster synchronization in networks of coupled chaotic oscillators. We considered a situation in which all the systems are described by the same dynamics, and delays could be chosen out of a finite set. Our main conclusion is that in addition to the pattern of connectivity between nodes, communication delays can be responsible for the emergence of synchronized clusters in networks of coupled oscillators.

We focused on a simple ring network of coupled Colpitts oscillators, which admits full synchronization in the absence of delays and showed that by adding up to three delayed connections, we could attain various CS patterns, which we were able to predict by computing the "minimal balanced coloring" for each configuration. We fixed the value of the coupling delay and found that for all the configurations, the predicted CS pattern could be stabilized for a sufficiently large value of the coupling strength. We validated our theoretical predictions for configuration C in an experimental ring of six bidirectionally coupled Colpitts oscillators, with delayed connections obtained by coupling the oscillators via RF cables of appropriate length. The experimental setting showed that the pattern could be stabilized for large enough coupling $\sigma = R_C{}^{-1}$, though the critical σ was found to be higher than in simulations. Finally, we also verified the theory in a fully connected network of 50 nodes for which connections were randomly assigned to be either undelayed or delayed with a certain probability.

Our work is generalizable to networks with arbitrary connectivity and distribution of the delays over the connections. For each delay configuration, we applied the stability analysis to the pattern corresponding to the minimal balanced coloring, but the same analysis can be applied to other non-minimal balanced patterns.[34] A main limitation of this work is that it only applies to chaotic dynamics; the case of periodic dynamics requires further investigation.

SUPPLEMENTARY MATERIAL

See the Supplementary material for a description of the experimental scheme and setup, a presentation of all the quotient networks corresponding to different configurations of the hexagon network, the transformation into block-diagonal form for each one of the CS patterns, and time traces from the experimental system.

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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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