

# THE INVISIBLE HAND OF LAPLACE: THE ROLE OF MARKET STRUCTURE IN PRICE CONVERGENCE AND OSCILLATION

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ABSTRACT. The “invisible hand” of the free market is remarkably effective at producing near-equilibrium prices. This is difficult to quantify, however, in the absence of an agreed model for out-of-equilibrium trade. Short of a fully reductionist model, a useful substitute would be a scaling law relating equilibration time and other market parameters. Even this, however, is missing in the literature.

We make progress in this direction. We examine a class of Arrow-Debreu markets with price signaling driven by continuous-time proportional-tâtonnement. We show that the connectivity among the participants in the market determines quite accurately a scaling law for convergence time of the market to equilibrium, and thus determines the effectiveness of the price signaling. To our knowledge this is the first characterization of price stability in terms of market connectivity. At a technical level, we show how convergence in our class of markets is determined by a market-dependent Laplacian matrix.

If a market is not isolated but, rather, subject to external noise, equilibrium theory has predictive value only to the extent to which that noise is counterbalanced by the price equilibration process. Our model quantifies this predictive value by providing a scaling law that relates the connectivity of the market with the variance of its prices.

## 1. INTRODUCTION

**Dynamics.** In a free market, the rise or fall of prices signal excess demand or supply. Ideally, this signaling causes goods to clear and prices to restore to equilibrium; or, after a shock, to

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restore to a possibly new near-equilibrium zone;<sup>1</sup> this is the rationale for studying equilibrium theories.<sup>2</sup> However, markets in general do not have central auctioneers; instead, price signaling occurs through adjustments among agents who are actually trading. The purpose of this paper is to analyze the significance of this geometry to the equilibration process. We focus upon a simple setting that nonetheless allows rich geometry: Arrow-Debreu markets in which agents have constant-elasticity-of-substitution (CES) utilities in the gross substitutes regime.

Thanks to the seminal works [43, 6, 4] it is well known that in these conditions an equilibrium exists and is unique, so there is no question about statics—only about dynamics. The same works established that simple tâtonnement dynamics converge to the equilibrium. To be specific, throughout this paper we employ Samuelson’s [55] continuous-time proportional-tâtonnement (henceforth Samuelson dynamics), well-known to converge to equilibrium [4]:

$$(1.1) \quad \dot{p}_j = \frac{d_j - s_j}{s_j} p_j.$$

(Here  $j$  is a good;  $s_j$  is the fixed endowment of this good;  $p_j$  is its time-varying price;  $d_j$  is its time-varying demand; and we write  $\dot{p}_j = \partial p_j / \partial t$  to suppress the time variable  $t$ .)

In showing that tâtonnement dynamics converge to equilibrium, these classic works established that the equilibrium is stable—under the proviso that these are plausible dynamics for an economic market. We believe this proviso is reasonable for a market close to equilibrium. To begin with, plausibility of tâtonnement was its *raison d’être* in the foundational writings of Walras.<sup>3</sup> Apart from Walras’s informal argument, two more justifications for studying tâtonnement, near equilibrium, might be offered.

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<sup>1</sup>F. M. Fisher, 1983 [22]: “The view that equilibria are the points of interest must logically rest on two underlying properties about the dynamics of what happens out of equilibrium. First, the system must be *stable*; that is, it must converge to some equilibrium. Second, such convergence must take place relatively quickly. If the predictions of comparative statics are to be interesting in a world in which conditions change, convergence to equilibrium must be sufficiently rapid that the system, reacting to given parameter shift, gets close to the predicted new equilibrium before parameters shift once more.”

<sup>2</sup>F. A. Hayek, 1948 [29]: “This appears to me one of the most important of the points where the starting point of the theory of competitive equilibrium assumes away the main task which only the process of competition can solve.”

<sup>3</sup>L. Walras, 1874 [70], e.g., p. 251: “This groping takes place naturally in the services market under a system of free competition, since, under such a system, the price of services rises when demand exceeds offer and falls when offer exceeds demand.” See also McKenzie [44] §2.4.

The first justification is merely to note that very little has actually been assumed here—mainly (a) that dynamics are invariant to simultaneous scaling of all prices, and (b) that prices exhibit first-order response to excess demand. Point (b) is significant: we do not of course imagine that “microscopically,” agents implement Samuelson dynamics. Rather, we think that the details should matter little to a network-wide effect such as convergence time.

The second, more phenomenological justification is that sellers of a product typically maintain a *buffer* (inventory), and suffer significant loss if the buffer empties or overflows. In this setting, short-term pricing response may primarily be designed to prevent buffer failures. Such a response can be implemented without even explicitly estimating current demand, as follows. Each time the buffer shrinks by one, multiply the current price by a factor  $e^{\varepsilon/s_j}$ , slightly greater than 1; each time the buffer grows by one, multiply the price by  $e^{-\varepsilon/s_j}$ . This is a discrete-time implementation of Samuelson’s  $\dot{p}_j/p_j = \varepsilon(d_j - s_j)/s_j$ . In control theory (see PID controllers in, e.g., [64]) this is called “anticipatory” or “derivative” control.

Far from equilibrium, higher derivatives of the response curve would matter, as well as possible concerns about agent look-ahead. Indeed we do not have reason to think that first-order dynamics are plausible far from equilibrium, and consequently, we do not study rates of *global* convergence in this paper.

If one nonetheless considers the tâtonnement process initialized far from equilibrium, then due to the analysis [4] we know it will eventually approach equilibrium closely enough for our linearized analysis to become relevant; the duration of this “burn-in” period will depend on the initial condition as well as on the magnitude of quadratic correction terms around the equilibrium.

**Meaningfulness of equilibrium.** In view of these justifications, *if* one neglects the time scale for convergence, then one might consider that the theorems of the 1950s on tâtonnement establish a satisfactory picture (in WGS markets), in which the restorative forces justify the focus of the theory upon equilibrium prices.

The caveat about convergence time, however, is serious. Every mathematical model of a market is imperfect; if the convergence force is weak, then so is the claim for relevance of the equilibrium point, as other effects might dominate.

Thus the question of convergence rate is really a question about the justification for equilibrium theory. Or, perhaps more importantly and leaving theory aside, a question about why in typical free markets, much of the time, prices are fairly steady.

In this paper we show that this justification is actually quite sensitive to the connectivity of the market. This is because price updates must propagate like waves through the market.

The above-cited classical convergence results for tâtonnement dynamics hold only in an abstract “auctioneer” model in which there is perfect instantaneous communication; when price updates must spread through interaction, the strength and geometry of that web of interactions is critical to stability of equilibrium.

**Statics in open systems.** Our results on convergence *rate* lead in turn to conclusions about *statics*, because any real-world market is continually subject to small noise perturbations. This being the case, the market cannot be modeled as residing at a perfect equilibrium price vector—but rather, in a stationary distribution over price vectors. Only if that distribution is tightly concentrated around the perfect equilibrium point, does the equilibrium acquire real-world meaning.<sup>4</sup> Fortunately, we will be able to show, at least for a restricted class of markets and a simple noise model, precisely how bounds on the convergence rate of the dynamics imply bounds on price variance in the stationary distribution.

Our work establishes a quantitative foundation for the following appealing intuition: a well-connected market (one which *cannot* be partitioned into two blocs with little trade between them), converges rapidly, almost as if equilibration were performed by a centralized auction; whereas a poorly-connected market (for an extreme example one may think of agents arranged in a cycle and valuing only their immediate neighbors’ goods) will only very slowly equilibrate. Correspondingly, in the presence of noise, the first market will have reliably steady prices whereas the latter will see large price swings. Figure 1 (to be explained in more detail in Section 4.4.1) illustrates two markets, with identical *local* parameters—both have ten participants, each of whom value some three goods equally. Solely because of the different global structure of the markets, the one on the left converges to equilibrium more than four times as rapidly as the one on the right.

Useful quantification of the “invisible hand” must include information about rates of convergence. A truly quantitative theory cannot be one-sided (bounding the rate of convergence only from above or only from below) because bounds from both above and below are necessary if we wish to compare two markets, or analyze in a single market whether observed changes are due to external stimulation vs. being perhaps long-lived oscillations of the system. All results of this paper are in this sense truly quantitative. Having said this we hasten to

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<sup>4</sup>As expressed by A. Marshall, 1890 [42]: “This is the real drift of that much quoted, and much-misunderstood doctrine of Adam Smith and other economists that the normal, or ‘natural,’ value of a commodity is that which economic forces tend to bring about in the long run. It is the average value which economic forces would bring about if the general conditions of life were stationary for a run of time long enough to enable them all to work out their full effect.”

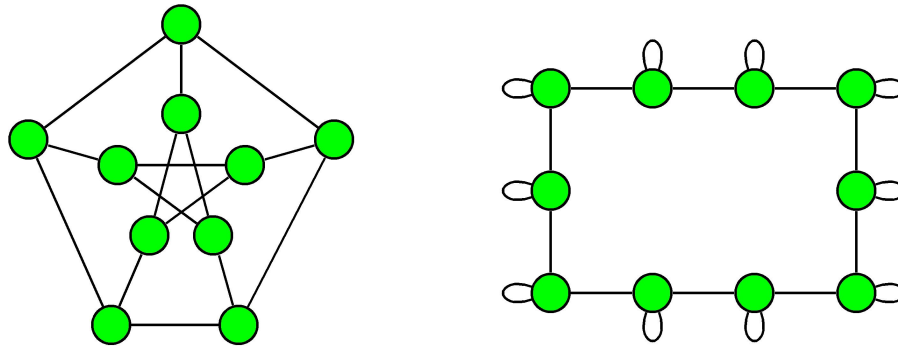


FIGURE 1. Rapidly (Petersen) vs. slowly (Cycle) equilibrating markets, both with  $n = 10$  and  $k = 3$ .

add that in order to obtain rigorous results we have had to make restrictions on the markets. Nevertheless, we think the tradeoff of connectivity with price stability should prevail more generally. To our knowledge our results are the first, in any market model, to characterize the effect of market connectivity upon the stability of prices.

**Results and Structure of the Paper.** In Section 2 we give a rigorous description of the market model, along with some necessary technical preliminaries, culminating in a generator matrix for the dynamics.

In Section 3, first, we provide the fundamental mathematical analysis of the dynamics, showing that they are of second order in the matrices describing the local interactions, and representable in terms of a market “Laplacian;” second, we relate the convergence rate of the dynamics to the quality of connectivity of the market as expressed by its edge expansion, a notion from Markov chain theory. We also explain why these results are robust to small errors in measurement of the market properties.

In Section 4 we address a special class of markets in which stronger results may be obtained. In these so-called circulation-free markets, (a) We give a tighter bound on convergence rate, in terms of the spectral gap of the Laplacian. (b) We give a stronger robustness guarantee (e.g., we can characterize market dynamics fairly accurately knowing only the prices and purchasing patterns). (c) Perhaps most interestingly, we will be able to exactly relate the convergence rate of the market on the one hand, with the variance of prices under conditions when the market is not in isolation but is continually perturbed by external noise. This last is an analogue of what in physics is called a fluctuation-dissipation theorem.

Tâtonnement-type processes arise naturally in several disciplines. We will be taking advantage throughout the paper of some of the theory that has been previously developed. We

will also briefly discuss in Section 5 some of the non-market dynamics where similar processes arise. We also raise there some questions for future work.

## 2. MARKET MODEL AND DYNAMICS

**2.1. Agents and utilities.** We consider an Arrow-Debreu market on  $n$  ( $n \geq 2$ ) participants in which each participant  $i$  values an allocation  $x_{ij}$  of the goods  $j$  according to a CES utility function [5]  $u_i(x) = \left( \sum_j C_{ij}^{1-\rho} x_{ij}^\rho \right)^{1/\rho}$  where  $C_{ij}$  are nonnegative coefficients, and the parameter  $\rho$  is in  $(0, 1)$ , the gross substitutes regime. It will simplify later expressions to replace the customary  $\rho$  by the monotonically-related parameter  $\delta = \frac{\rho}{1-\rho}$  which ranges in  $(0, \infty)$ . ( $1 + \delta$  is the *elasticity of substitution*.) So the utilities are

$$(2.1) \quad u_i(x) = \left( \sum_j C_{ij}^{\frac{1}{1+\delta}} x_{ij}^{\frac{\delta}{1+\delta}} \right)^{\frac{1+\delta}{\delta}}$$

What is it that gives our markets *geometry*? Partly, that the utility functions  $u_i$  vary from agent to agent, but this alone is not sufficient: e.g., it includes the case of a Fisher market, which can be viewed as an Arrow-Debreu market in which all participants are endowed with goods in the same proportions. What is needed is also that endowments vary among participants. In order for the geometry to emerge most sharply, we take this to the extreme and assume a bijection between agents and goods, with agent  $i$  being the sole agent endowed with good  $i$ , in the fixed quantity  $s_i > 0$ .

In the Arrow-Debreu model, at prices  $p = (p_1, \dots, p_n)$  (not all 0), participant  $i$  has budget  $b_i = s_i p_i$ , which is then allocated to goods  $j$  so as to optimize basket utility; this results in the following demand by participant  $i$  for good  $j$ :

$$(2.2) \quad d_{ij}(p) = \frac{b_i C_{ij}}{p_j^{1+\delta} \sum_k C_{ik}/p_k^\delta} = \frac{s_i p_i C_{ij}}{p_j^{1+\delta} \sum_k C_{ik}/p_k^\delta}$$

Let  $d_j = \sum_i d_{ij}$  be the total demand for good  $j$ . Prices  $p_j$  are an equilibrium if all  $d_j = s_j$  (i.e., if demands match endowments).

(Incidentally, the price-taking assumption will be strained if demand for a good is concentrated at very few buyers. We would be skeptical of applying our analysis to such situations, although there is no formal obstacle to writing down markets with such demands.)

It is worth noting that the model is essentially unaffected if agents are endowed with multiple goods, provided the set of such goods at any agent  $i$  is disjoint from the goods at the other agents. Writing  $S_i$  for the goods in agent  $i$ 's endowment, the budget at agent  $i$  becomes  $b_i = \sum_{j \in S_i} s_j p_j$ ; in (2.2) we have  $d_{ij}(p) = \frac{C_{ij} \sum_{\ell \in S_i} s_\ell p_\ell}{p_j^{1+\delta} \sum_k C_{ik}/p_k^\delta}$ . Since the last expression

is linear in the endowments  $s_\ell$ , the dynamics are equivalent to those of a market in which each participant  $i$  is split into several participants having identical utility functions, each endowed with a distinct good from  $S_i$ , in the same fixed quantity.

Since the parameters  $C_{ij}$  are nonnegative, we regard them as edge-weights of a directed graph (0 means the edge is absent). We make the following assumption:

- (A) **Connectedness:** for every  $i, j$  there is a directed path from  $i$  to  $j$ , namely, there are  $i = i_0, i_1, \dots, i_k = j$  such that  $\prod_{\ell=1}^k C_{i_{\ell-1}i_\ell} > 0$ .

This is not a restrictive assumption. In equilibrium, the payments entering and leaving each agent are equal. We see from (2.2) that for any  $i, j$  s.t.  $C_{ij} > 0$ , payments are made in equilibrium from agent  $i$  to agent  $j$ . If there exist  $i$  and  $j$  with a directed path from  $i$  to  $j$  but not from  $j$  to  $i$ , payment equilibrium is impossible. This leaves only the case that the agents partition into disjoint sets, each of which is connected in the sense just defined. But in this case, each connected component of the market evolves separately. So it suffices to study the connected case.

**2.2. Dynamics in terms of log-prices; rescalings.** Let  $r = (r_1, \dots, r_n)$  denote the equilibrium prices of the system (which as noted, are known to exist and be unique). Working with log-prices

$$(2.3) \quad \alpha_j = \log(p_j/r_j),$$

the Samuelson dynamics (1.1) become

$$(2.4) \quad \dot{\alpha}_j = \dot{p}_j/p_j = -1 + d_j/s_j.$$

Simply by rescaling units, we may suppose that all endowments  $s_i = 1$ . Specifically, consider the “primed” system with  $C'_{ij} = C_{ij}s_j^\delta$ , and  $s'_j = 1$ . We claim that the linear transformation  $p'_j = p_j s_j$  commutes with the dynamics of the original and primed systems; that is,  $\dot{p}'_j|_{p'} = s_j \dot{p}_j|_p$ . Equivalently, working with the logarithms of prices, and using the transformation  $\alpha'_j = \alpha_j + \log s_j$ , the claim establishes that this additive shift in the space commutes with the dynamics:  $\dot{\alpha}'_j|_{\alpha'} = \dot{\alpha}_j|_\alpha$ . To show this claim, in view of (1.1) or (2.4), we simply need to establish that  $d'_j/s'_j = d_j/s_j$ . One has only to substitute (2.2) for each system; we omit the calculation.

Introduce the functions  $P_i$  of the prices  $p$ :

$$(2.5) \quad P_i(p) = \sum_k \frac{C_{ik}}{p_k^\delta}.$$

After the rescaling of the endowments, the demands in a system with unit endowments (dropping the “primes”) can be written:

$$(2.6) \quad d_{ij}(p) = \frac{p_i C_{ij}}{p_j^{1+\delta} \sum_k C_{ik}/p_k^\delta} = \frac{p_i C_{ij}}{p_j^{1+\delta} P_i(p)}$$

(Subsequently we generally omit the argument  $p$  of  $d_{ij}(p)$ .)

The dynamics (2.4) depend only on the demands and endowments, and the demands (2.2) of each agent  $i$  are unchanged if all  $C_{ij}$  are multiplied by any common positive number; furthermore the demands generated by agent  $i$  are not affected by any  $C_{i'j}, i' \neq i$ . So we may without loss of generality rescale the  $C_{ij}$ 's so that for every  $i$ ,

$$(2.7) \quad \min_{j: C_{ij} > 0} C_{ij} = 1.$$

**2.3. Generator of the dynamics.** After the rescalings of the previous section, we have that at the equilibrium prices  $r$ ,  $d_j(r) = 1$  for all  $j$ . It will be convenient to abbreviate  $R_i = P_i(r)$  (thus,  $R_i = \sum_k C_{ik}/r_k^\delta$ ); then the demand by  $i$  for  $j$  at equilibrium is

$$(2.8) \quad d_{ij}(r) = \frac{r_i C_{ij}}{r_j^{1+\delta} R_i}$$

and the equilibrium condition is the following system of equations, ranging over all  $j$ :

$$(2.9) \quad 1 = \frac{1}{r_j^{1+\delta}} \sum_i \frac{r_i C_{ij}}{R_i}.$$

In view of the rescaling of the endowments, and following (1.1), (2.4), the dynamics are

$$(2.10) \quad \dot{\alpha}_j = d_j - 1.$$

*Conventions:* all vectors will be column vectors unless otherwise noted; for scalar  $c$ ,  $\vec{c}$  is the column vector with all entries  $c$ ;  $v^*$  is the conjugate transpose of vector or matrix  $v$ ; for vector  $v$ ,  $\text{diag}(v)$  is the diagonal matrix with  $\text{diag}(v)_{ii} = v_i$ . Vector norms are denoted  $\|v\|_p = (\sum v_i^p)^{1/p}$ , with  $\|v\| := \|v\|_2$ .

By definition the equilibrium (i.e., the point at which  $\dot{\alpha} = \vec{0}$ ) is  $\alpha = \vec{0}$ . It is clear from the form of the demands that  $\dot{\alpha}$  is continuously differentiable at any  $\alpha$ . Consequently,  $\dot{\alpha}$  can be expressed to first order around the origin as

$$(2.11) \quad \dot{\alpha} = D\alpha$$

where  $D$  is the Jacobian of the demands w.r.t.  $\alpha$  at  $\vec{0}$ ,

$$(2.12) \quad D_{ji} = \left. \frac{\partial d_j}{\partial \alpha_i} \right|_{\vec{0}}$$



Any scaling of the equilibrium price vector  $r$  is also an equilibrium (i.e., there is a ray, not a single point, of equilibrium prices), so from (2.11) we can deduce

$$(2.13) \quad \vec{0} = D\vec{1}$$

and so,  $D$  has a 0 eigenvalue. (Moreover, due to uniqueness of the equilibrium, the right-kernel of  $D$  consists solely of  $\text{span } \vec{1}$ .) This particular eigenvalue is irrelevant to our considerations: it merely expresses that the model is scale-invariant in the prices. (An alternative, less natural approach would have been to introduce an  $(n + 1)$ 'st good as numéraire.) We express this scale-invariance in the dynamics by writing the log-prices vector  $\alpha(t)$  as

$$(2.14) \quad \alpha(t) = \vec{c} + \bar{\alpha}(t)$$

(note  $\bar{\alpha}$  is a vector), for real  $c$  determined by the condition

$$(2.15) \quad \sum_{i=1}^n r_i \bar{\alpha}_i(t) = 0 \quad \text{equivalently} \quad \sum r_i \alpha_i(t) = c \sum r_i$$

In (2.14) the term  $\vec{c}$  is unchanging in time as we see from (2.11) and (2.13).

It turns out that many expressions simplify if we use the following diagonal matrix  $B$  to rescale (i.e., make a change of basis in the log-prices space):

$$(2.16) \quad B = \text{diag}(\sqrt{r_i})$$

$$(2.17) \quad \beta = B\alpha$$

and rewrite the dynamics (2.11) accordingly:

$$(2.18) \quad \tilde{D} = BDB^{-1}$$

$$(2.19) \quad \dot{\beta} = \tilde{D}\beta$$

In view of (2.19) we refer to  $\tilde{D}$  as the generator of the dynamics (for  $\beta$ ). The decomposition (2.14) is equivalent to decomposing  $\beta$  as

$$(2.20) \quad \beta_i = c\sqrt{r_i} + \bar{\beta}_i \quad \text{for vector } \bar{\beta} \text{ determined by the condition } \sum \sqrt{r_i} \bar{\beta}_i(t) = c \sum r_i.$$

### 3. THE MARKET LAPLACIAN, DYNAMICS SEMIGROUP AND CONVERGENCE TIME

Our task is now to understand the dynamics matrix  $\tilde{D}$ . Toward this end we recall the (normalized) Laplacian matrix of an Eulerian graph. As before we regard a matrix  $A$  with nonnegative entries as the weighted adjacency matrix of a weighted directed graph,  $A_{ij}$  being the weight of edge  $i \rightarrow j$ . The vector of weighted outdegrees is  $\zeta(A) := A\vec{1}$ , and we assume that all entries of  $\zeta(A)$  are positive (otherwise some of the agents would have trivial roles in

the market). We say the graph or weighted adjacency matrix is Eulerian if  $\zeta(A) = \zeta(A^*)$ . E.g., an undirected weighted graph without isolated vertices is necessarily Eulerian. The Laplacian matrix corresponding to  $A$  is defined to be<sup>5</sup>

$$(3.1) \quad \Delta(A) = I - \text{diag}(\zeta(A))^{-1/2} \cdot A \cdot \text{diag}(\zeta(A))^{-1/2}.$$

We will be discussing the Laplacians of several different, but related, weighted graphs. The Laplacian which exactly controls the price dynamics we call  $L_C$ . In order to define it, we start with the weighted adjacency matrix

$$(3.2) \quad K_{ij} = \frac{C_{ij}r_i^{1/2}}{R_i r_j^{1/2+\delta}}.$$

$K$  is nonnegative and (by assumption (A)) irreducible, and consequently, by the Perron-Frobenius theorem, has a unique nonnegative eigenvector (henceforth called the PF vector); the corresponding eigenvalue is real, and strictly larger in norm than any other eigenvalue. In our case the PF vector is the same,  $B\vec{1}$ , on both the right and the left, because:

$$(3.3) \quad KB\vec{1} = B\vec{1} \quad \text{from the definition of } R$$

$$(3.4) \quad K^*B\vec{1} = B\vec{1} \quad \text{because } r \text{ is at equilibrium}$$

In particular this means that if we define a weighted adjacency matrix

$$(3.5) \quad W = BKB,$$

then  $W$  is Eulerian and  $B = \text{diag}(\zeta(W))^{1/2}$ ; thus, applying the formalism (3.1), we have the *market Laplacian*

$$(3.6) \quad L_{\hat{C}} = \Delta(W) = I - K.$$

For both of the matrices  $\Delta(W)$  and  $\Delta(W)^*$ , the right-kernel contains  $B\vec{1}$ , as can be verified from the calculations above; moreover both right-kernels are spanned by this vector, due to the uniqueness part of the PF theorem. (As noted after (2.13), characterization of the right-kernel of  $\Delta(W)$  is also implied by the known uniqueness of the equilibrium of the dynamics.)

**Remark 1.**  $W$  carries significant meaning about trade in the market: combining (2.8) and (3.5) we see that  $W_{ij}$  is equal to the payment made, in equilibrium, by agent  $i$  to agent  $j$  for agent  $j$ 's good. (This is generally of course different from the *net* payment between the two agents.)

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<sup>5</sup>Some authors use  $I - \text{diag}(\zeta(A))^{-1} \cdot A$  which has the advantage that  $\text{diag}(\zeta(A))^{-1} \cdot A$  is row-stochastic; our convention has the advantage that, in the circulation-free case discussed in Sec. 4,  $\Delta(A)$  is symmetric.

The tool that makes our results possible is the following theorem:

**Theorem 2** (Quadratic expansion of the dynamics in terms of local interactions). *The dynamics generator  $\tilde{D}$  is given by the following expressions in  $K$  or in its Laplacian  $L_{\hat{C}}$ :*

$$(3.7) \quad \tilde{D} = -(1 + \delta)I + K^* + \delta K^*K$$

$$(3.8) \quad = -\delta L_{\hat{C}} - (1 + \delta)L_{\hat{C}}^* + \delta L_{\hat{C}}^* L_{\hat{C}}$$

*Proof.* We start by calculating  $D$ , the generator of the dynamics (2.11). First we consider entries  $D_{jk}$ ,  $k \neq j$ . Applying (2.12) gives:

$$\begin{aligned} D_{jk} &= \left. \frac{\partial}{\partial \alpha_k} \left[ \sum_i d_{ij} \right] \right|_{\vec{0}} = \left. \frac{\partial}{\partial \alpha_k} \left[ \sum_i \frac{p_i C_{ij}}{p_j^{1+\delta} \sum_h C_{ih}/p_h^\delta} \right] \right|_{\vec{0}} && \text{from (2.5) and (2.6)} \\ &= \left. \frac{\partial}{\partial \alpha_k} \left[ \frac{p_k C_{kj}}{p_j^{1+\delta} \sum_h C_{kh}/p_h^\delta} + \sum_{i \neq k} \frac{p_i C_{ij}}{p_j^{1+\delta} \sum_h C_{ih}/p_h^\delta} \right] \right|_{\vec{0}} \\ &= \left. \frac{\partial}{\partial \alpha_k} \left[ \frac{r_k e^{\alpha_k} C_{kj}}{r_j^{1+\delta} (C_{kk} e^{-\delta \alpha_k} / r_k^\delta + \sum_{h \neq k} C_{kh} / r_h^\delta)} + \sum_{i \neq k} \frac{r_i C_{ij}}{r_j^{1+\delta} (C_{ik} e^{-\delta \alpha_k} / r_k^\delta + \sum_{h \neq k} C_{ih} / r_h^\delta)} \right] \right|_{\vec{0}} \\ &= \frac{(r_j^{1+\delta} R_k + \delta r_j^{1+\delta} C_{kk} / r_k^\delta) r_k C_{kj}}{r_j^{2+2\delta} R_k^2} + \sum_{i \neq k} \frac{\delta r_i C_{ij} r_j^{1+\delta} C_{ik} / r_k^\delta}{r_j^{2+2\delta} R_i^2} \\ &= \frac{r_k C_{kj}}{r_j^{1+\delta} R_k} + \delta \sum_i \frac{r_i C_{ij} C_{ik}}{r_k^\delta r_j^{1+\delta} R_i^2} \end{aligned}$$

Applying (3.2), we calculate  $(K^*K)_{ij} = \sum_k \frac{C_{ki} C_{kj} r_k}{R_k^2 r_i^{1/2+\delta} r_j^{1/2+\delta}}$ . Using the change of basis (2.17) we have

$$\begin{aligned} \tilde{D}_{jk} = r_j^{1/2} r_k^{-1/2} D_{jk} &= \frac{r_k^{1/2} C_{kj}}{r_j^{1/2+\delta} R_k} + \delta \sum_i \frac{r_i C_{ij} C_{ik}}{r_k^{1/2+\delta} r_j^{1/2+\delta} R_i^2} \\ &= K_{jk}^* + \delta (K^*K)_{jk} \end{aligned}$$

Next we calculate entries  $\tilde{D}_{jj}$ . Again applying (2.12),

$$\begin{aligned} \tilde{D}_{jj} = D_{jj} &= \left. \frac{\partial}{\partial \alpha_j} \left[ \sum_i d_{ij} \right] \right|_{\vec{0}} = \left. \frac{\partial}{\partial \alpha_j} \left[ \sum_i \frac{p_i C_{ij}}{p_j^{1+\delta} \sum_k C_{ik}/p_k^\delta} \right] \right|_{\vec{0}} \\ &= \left. \frac{\partial}{\partial \alpha_j} \left[ \frac{p_j C_{jj}}{p_j^{1+\delta} \sum_k C_{jk}/p_k^\delta} + \sum_{i \neq j} \frac{p_i C_{ij}}{p_j^{1+\delta} \sum_k C_{ik}/p_k^\delta} \right] \right|_{\vec{0}} \end{aligned}$$

$$\begin{aligned}
&= \frac{\partial}{\partial \alpha_j} \left[ \frac{e^{-\delta \alpha_j} C_{jj}}{C_{jj} e^{-\delta \alpha_j} + r_j^\delta \sum_{k \neq j} C_{jk} / r_k^\delta} + \sum_{i \neq j} \frac{r_i C_{ij}}{r_j e^{\alpha_j} C_{ij} + r_j^{1+\delta} e^{(1+\delta) \alpha_j} \sum_{k \neq j} C_{ik} / r_k^\delta} \right] \Big|_{\vec{0}} \\
&= \frac{-\delta r_j^\delta R_j C_{jj} + \delta C_{jj}^2}{r_j^{2\delta} R_j^2} + \sum_{i \neq j} \frac{-r_i C_{ij} (-\delta r_j C_{ij} + (1+\delta) r_j^{1+\delta} R_i)}{r_j^{2+2\delta} R_i^2} \\
&= \frac{C_{jj}}{r_j^\delta R_j} + \delta \sum_i \frac{r_i r_j C_{ij}^2}{r_j^{2+2\delta} R_i^2} - (1+\delta) \sum_i \frac{C_{ij} r_i}{r_j^{1+\delta} R_i} \\
&= K_{jj} + \delta (K^* K)_{jj} - (1+\delta) r_j^{-1/2} \sum_i r_i^{1/2} K_{ij} \\
&= K_{jj} + \delta (K^* K)_{jj} - (1+\delta) I \quad \text{using (3.4)}
\end{aligned}$$

Putting these two calculations together,  $\tilde{D} = -(1+\delta)I + K^* + \delta K^* K$  as desired.  $\square$

Now let us see what Theorem 2 tells us about market stability. Start by setting  $\bar{K} = \frac{1}{1+\delta}(K^* + \delta K^* K)$ . Using Theorem 2 the dynamics (2.19) solve to

$$(3.9) \quad \beta(t) = e^{(1+\delta)(\bar{K}-I)t} \beta(0)$$

For a set of agents  $S$ , let  $\mathbf{1}_S$  (the indicator vector of  $S$ ) be the vector which is one on elements of  $S$  and zero elsewhere; let  $\mathbf{1} = \mathbf{1}_{[n]}$  for  $[n]$ . We require two definitions from the theory of Markov chains:

**Definition 3.** Let  $A$  be a nonnegative matrix whose right PF vector  $u$  is the transpose of its left PF vector. The edge expansion (a.k.a. conductance) of  $A$  is

$$(3.10) \quad \phi(A) = \min_S \left\{ \frac{\mathbf{1}_S^* \text{diag}(u) A \text{diag}(u) \mathbf{1}_{\bar{S}}}{\min\{\mathbf{1}_S^* \text{diag}(u) A \text{diag}(u) \mathbf{1}, \mathbf{1}_{\bar{S}}^* \text{diag}(u) A \text{diag}(u) \mathbf{1}\}} \right\}$$

which in the special case that  $A$  has PF eigenvalue 1, simplifies to

$$(3.11) \quad \phi(A) = \min_{S: 0 < \sum_{i \in S} u_i^2 \leq \sum_{i \notin S} u_i^2} \left\{ \frac{\mathbf{1}_S^* \text{diag}(u) A \text{diag}(u) \mathbf{1}_{\bar{S}}}{\sum_{i \in S} u_i^2} \right\}$$

Edge expansion is a well-known combinatorial characterization of the smallest bottleneck in the connectivity of a weighted graph. One way to think of it is to note that, with  $A, u$  as above,  $\text{diag}(u)^{-1} A \text{diag}(u)$  is a row-stochastic matrix (thus a Markov chain) with invariant measure  $u_i^2$  on states  $i$ ; for a set of states  $S$ , the right-hand side of (3.11) is the probability, in the invariant measure, of leaving  $S$  in the next step, conditional on being in  $S$ .  $\phi(A)$  is determined by the set minimizing this quantity, i.e., which represents a ‘‘bottleneck’’ to spreading throughout the state space. To quantify this we recall the concept of mixing time, which is

the time required for the Markov chain to almost forget its starting point. (For technical reasons we write the definition in terms of the continuous-time Markov chain associated with  $A$ , for which the time- $\tau$  transition matrix is  $\exp(\tau(\text{diag}(u)^{-1}A \text{diag}(u) - I))$ .)

**Definition 4.** Let  $A$  be a nonnegative irreducible matrix with PF eigenvalue  $p > 0$ , whose right PF vector  $u$  is the transpose of its left PF vector. Normalize so that  $u^*u = 1$ , and let  $r$  be the column vector  $r_i := u_i^2$ . The mixing time  $\tau_\epsilon(A)$  is the least  $\tau$  such that

$$\|z(\exp(\tau \cdot (A - pI)) - uu^*) \text{diag}(u)\|_1 \leq \epsilon$$

for all vectors  $z$  s.t.  $\|z \text{diag}(u)\|_1 = 1$ . Equivalently,  $\tau_\epsilon(A)$  is the least  $\tau$  such that

$$\|z'(\exp(\tau(\text{diag}(u)^{-1}A \text{diag}(u) - pI)) - \mathbf{1}r^*)\|_1 \leq \epsilon$$

for all vectors  $z'$  s.t.  $\|z'\|_1 = 1$ . Here  $r^*$  represents the steady state, and  $z'$  the initial condition, of the Markov chain.

**Theorem 5.** The mixing time for the dynamics (3.9) is

$$(3.12) \quad \Omega\left(\frac{1}{(1+\delta)\phi(\bar{K})}\right) \leq \tau_\epsilon((1+\delta)\bar{K}) \leq O\left(\frac{\ln\left(\frac{n}{\epsilon \min_i r_i}\right)}{(1+\delta)\phi^2(\bar{K})}\right).$$

(With  $\Omega, O$  denoting lower, resp. upper, bounds within constant factors.)

Thus, we find that a market converges rapidly if and only if it has strong connectivity in the sense of high edge expansion.

*Proof.* From (3.3), (3.4) we see that  $B\vec{\mathbf{1}}$  is the right and left PF vector of  $\bar{K}$ , with  $\bar{K}B\vec{\mathbf{1}} = \bar{K}^*B\vec{\mathbf{1}} = B\vec{\mathbf{1}}$ . So Definition 3, for  $\bar{K}$ , becomes

$$(3.13) \quad \phi(\bar{K}) = \min_{S:0 < \sum_{i \in S} r_i \leq \sum_{i \notin S} r_i} \left\{ \frac{\mathbf{1}_S^* B \bar{K} B \mathbf{1}_{\bar{S}}}{\sum_{i \in S} r_i} \right\}.$$

Now combine Theorem 2 with the following theorem of Mihail, with  $A = \bar{K}$ . Observe that the dynamics (3.9) amount to speeding up the dynamics  $e^{(\bar{K}-I)t}$  by a factor of  $1 + \delta$ .

**Theorem 6** (Mihail [48], as adapted in [45]). Let  $A, u$  be as in Definition 4, and with  $A$  having PF eigenvalue 1. Then:

$$(3.14) \quad \Omega\left(\frac{1}{\phi(A)}\right) \leq \tau_\epsilon(A) \leq O\left(\frac{\ln\left(\frac{n}{\epsilon \min_i u_i^2}\right)}{\phi^2(A)}\right).$$

□

**Remark 7** (Structural interpretation). Recall from Remark 1 that  $W_{ij}$  is equal to the payment made, in equilibrium, by agent  $i$  to agent  $j$ ; and so if  $S, T$  are sets of agents,  $W_{S,T} = \sum_{i \in S, j \in T} W_{ij}$  equals the total payments in equilibrium from agents in  $S$  to agents in  $T$ . This gives us a concrete interpretation what it means for a market to have a “bottleneck.” Examining  $\phi(\bar{K})$  in (3.13) we see that the numerator equals  $\frac{1}{1+\delta}(W_{\bar{S},S} + \delta \mathbf{1}_S B K^* K B \mathbf{1}_{\bar{S}})$ : due to the first term, any partition of the agents into two blocks with large trade between them, has large edge expansion in  $\bar{K}$ . The contrapositive is that *slow price equilibration implies the existence of two blocks which trade very little with each other*.

Interestingly, the other term which can hasten equilibration is the quadratic term in  $\phi(\bar{K})$ , which represents indirect price interaction between sellers who share a common buyer.

**Remark 8** (Computation). There are efficient algorithms to find an approximately-worst bottleneck in a graph (see [65] for an exposition); these can be used to diagnose a connectivity flaw in a market which is observed to be slowly-converging.

**Remark 9** (Robustness). Measurement of the activity in a market will necessarily be imprecise, so one should ask whether the bounds of Theorem 5 are robust to perturbation in the market parameters. Fortunately, a positive answer is implicit in the form of the bounds. Any error in the stationary prices  $r_i$  affects the right-hand side of (3.12) directly only through a logarithm; while errors in the  $r_i$  and in the  $C_{ij}$  affect  $K$  (and therefore  $\bar{K}$ ) with exponents respectively at most  $2 + 2\delta$  and 1. The edge conductance is just a sum of entries of a submatrix, so its sensitivity to error is similar.

#### 4. CIRCULATION-FREE MARKETS

4.1. **A further condition.** We now turn our attention to markets that, beside the innocuous assumption (A), satisfy an additional condition:

(B) Circulation-free: let  $i_0, i_1, \dots, i_k = i_0$  be any cycle through the vertices. Then 
$$\prod_{\ell=1}^k C_{i_{\ell-1}i_\ell} = \prod_{\ell=1}^k C_{i_\ell i_{\ell-1}}.$$

(It follows that  $C_{ij} = 0$  if and only if  $C_{ji} = 0$ . Write  $i \sim j$  if  $C_{ji} > 0$ .) This assumption is strong. As we will show in Theorem 10, it implies that the net payment between any two agents, in steady state, is 0. That is to say, *when* the market is in steady state, all trade could be achieved by barter. Payments and prices are still essential however, since they drive the Samuelson dynamics.

The reason for imposing condition (B) is that it enables spectral techniques which considerably tighten the general bounds given earlier (Sec. 4.4). This does not mean that spectral

estimates are necessarily be bad when (B) is not satisfied; there are markets violating the assumption for which the general bounds cannot be improved, but these are exceptional. The gap between spectral and edge-expansion estimates is most noticeable when the market Laplacian is not diagonalizable or the diagonalizing matrix has poor (i.e., large) condition number.

**4.2. Quantifying local and global disparity.** Two “disparity” parameters of a circulation-free market emerge as useful. To define these, start by defining for every two agents  $i, j$  the value

$$\psi_{i,j} = \prod_{\ell=1}^k \frac{C_{i_{\ell-1}i_{\ell}}}{C_{i_{\ell}i_{\ell-1}}},$$

where  $i = i_0, i_1, \dots, i_k = j$  is a path in the graph; to see this is well defined, consider any other path  $i = i_0, i'_1, \dots, i'_{k'} = j$ , form the cycle  $i_0, i_1, \dots, i_{k-1}, j, i'_{k'-1}, \dots, i'_1, i_0$ , and apply the circulation-free property. Consequently we can fix  $i_0$  to be a vertex such that  $\psi_{i_0,j} \geq 1$  for all  $j$ , and simplify notation by writing

$$(4.1) \quad \psi_j = \psi_{i_0,j}$$

(with  $\psi_{i_0} = 1$ ). For future reference, observe that we have for any edge  $i \sim j$  the identity

$$(4.2) \quad \psi_i C_{ij} = \psi_j C_{ji}.$$

Now set:

- (1)  $\psi = \max_j \psi_j$ . Recall that  $\min_j \psi_j = 1$ , so  $\psi$  is a measure of the global disparity in the desirability of various goods in the market. (We will later also show it roughly measures the global disparity in prices.)
- (2)  $\gamma = \max_i \sum_j C_{ij}$ . This is a measure of the local (at an agent) disparity among the utilities of the goods to which that agent assigns positive utility. (Recall we normalized so that each nonzero  $C_{ij}$  is at least 1.)

### 4.3. Properties of Equilibrium.

**4.3.1. Existence, uniqueness and detailed balance.** The equilibrium equations (2.9) are homogeneous of degree 1 in  $r$ , so (as already noted) any scalar multiple of an equilibrium vector  $r$  is also an equilibrium vector. Subsequently when we discuss uniqueness, “up to scaling” is implied. Due to connectedness of the market, no price can be 0 at an equilibrium. The existence of an equilibrium in our setting is a corollary of the theorem of Arrow and Debreu [6] and McKenzie [43] (improving on an earlier argument of Wald, see [32, 20]). (We remind that this is in the regime  $\delta > 0$ , among the other assumptions detailed in Section 2.)

In fact, as the utility functions are strongly concave and twice continuously differentiable, the equilibrium is unique (up to an overall scale factor). For convenience a short proof of uniqueness is provided in Appendix C.

We now show that the circulation-free condition implies an important property of the market. Adopting a term from statistical physics, we say that a market is in *detailed balance* at prices  $\pi$  if for every  $i, j$ , the payments from  $i$  to  $j$  equal those from  $j$  to  $i$ . The payment from  $i$  to  $j$  is  $d_{ij}\pi_j$  so (applying (2.6)) the detailed balance conditions are

$$(4.3) \quad \frac{\pi_i C_{ij}}{\pi_j^\delta P_i(\pi)} = \frac{\pi_j C_{ji}}{\pi_i^\delta P_j(\pi)} \quad (= \text{total payments in each direction across edge } ij),$$

or

$$(4.4) \quad \frac{\pi_i^{1+\delta} C_{ij}}{P_i(\pi)} = \frac{\pi_j^{1+\delta} C_{ji}}{P_j(\pi)}$$

**Theorem 10.** *At equilibrium  $r$ , a market satisfying the circulation-free condition (B) is in detailed balance.*

*Proof.*

**Lemma 11.** *There exist prices  $\pi$  satisfying the detailed balance conditions (4.4).*

*Proof.* For  $p$  a price vector, let  $p_{\max} = \max_j p_j$ ,  $p_{\min} = \min_j p_j$ , and  $\tilde{p} = p_{\max}/p_{\min}$ . Let  $|\cdot|$  denote geometric mean, so  $|p| = \prod_1^n p_i^{1/n}$ .

Let  $\mathbb{K} = \{p : |p| = 1, \tilde{p} \leq \gamma\psi\}$ . Let  $f^0 : \mathbb{K} \rightarrow \mathbb{R}^n$ ,

$$(f^0(p))_j = (\psi_j P_j(p))^{\frac{1}{1+\delta}}.$$

Let  $f : \mathbb{K} \rightarrow \mathbb{R}^n$ ,  $(f(p))_j = (f^0(p))_j / |f^0(p)|$ . By (4.2), a fixed point of  $f$  is a solution of (4.4).

We now claim that  $f$  maps  $\mathbb{K}$  into  $\mathbb{K}$ . By construction,  $|f(p)| = 1$ ; what we have to show is that  $\widetilde{f(p)} \leq \gamma\psi$ . This is equivalent to showing that  $\widetilde{f^0(p)} \leq \gamma\psi$ . We have

$$\begin{aligned} (f^0(p))_j &\leq \left( \frac{\psi\gamma}{p_{\min}^\delta} \right)^{\frac{1}{1+\delta}} \\ (f^0(p))_j &\geq \left( \frac{\min_i \psi_i}{p_{\max}^\delta} \right)^{\frac{1}{1+\delta}} = \left( \frac{1}{p_{\max}^\delta} \right)^{\frac{1}{1+\delta}} \end{aligned}$$

so for  $p \in \mathbb{K}$ ,

$$\widetilde{f^0(p)} \leq (\psi\gamma\tilde{p}^\delta)^{\frac{1}{1+\delta}} \leq (\psi\gamma(\gamma\psi)^\delta)^{\frac{1}{1+\delta}} = \gamma\psi$$

justifying the claim. Since  $\mathbb{K}$  is compact and convex and  $f$  is continuous on  $\mathbb{K}$ , the Brouwer fixed point theorem ensures  $f$  has a fixed point in  $\mathbb{K}$ .  $\square$



Now, using the demand functions (2.6), we compute the total demand at  $j$  given detailed-balance prices  $\pi$ :

$$(4.5) \quad \sum_i d_{ij}(\pi) = \sum_i \frac{\pi_i C_{ij}}{\pi_j^{1+\delta} P_i(\pi)} = \sum_i \frac{\pi_i^{-\delta} \pi_j^{1+\delta} C_{ji}}{\pi_j^{1+\delta} P_j(\pi)} = \frac{1}{P_j(\pi)} \sum_i \frac{C_{ji}}{\pi_i^\delta} = 1$$

where in the second equality we have applied the detailed balance conditions (4.4).

Thus, prices  $\pi$  satisfying detailed balance (Lemma 11 has shown these exist) necessarily satisfy the conditions (2.9) characterizing an equilibrium  $r$ . Since this equilibrium is known from classical results (or, to make this self-contained, from Lemma 27) to be unique, it follows that the equilibrium prices  $r$  satisfy detailed balance.  $\square$

On the basis of Theorem 10 we can now rewrite identity (4.4) as

$$(4.6) \quad r_j^{1+\delta} C_{ji} R_i = r_i^{1+\delta} C_{ij} R_j$$

Substituting this expression into the definition (3.2) gives the remarkable fact that for markets satisfying the circulation-free condition (B),  $K$  is symmetric, and can be rewritten

$$(4.7) \quad K_{ij} = \sqrt{\frac{C_{ij} C_{ji}}{R_i r_i^\delta R_j r_j^\delta}}.$$

4.3.2. *Bound on the equilibrium prices.* We stated earlier that  $\psi$  can be regarded as a rough measure of the price disparity in the market. We justify this in Proposition 13.

**Lemma 12.**  $\max_{i \sim j} \frac{r_i}{r_j} \leq \gamma$ .

*Proof.* Let  $i \sim j$  be such that  $\mu = \frac{r_i}{r_j} = \max_{i' \sim j'} \frac{r_{i'}}{r_{j'}}$ . Applying (4.6):

$$\mu^{1+\delta} = \frac{r_i^{1+\delta}}{r_j^{1+\delta}} = \frac{C_{ji} R_i}{C_{ij} R_j} = \frac{C_{ji} \sum_k \frac{C_{ik}}{r_k^\delta}}{C_{ij} \sum_\ell \frac{C_{j\ell}}{r_\ell^\delta}} \leq \left( \frac{\mu^2 r_j}{r_i} \right)^\delta \frac{C_{ji} \sum_k C_{ik}}{C_{ij} \sum_\ell C_{j\ell}} = \frac{\mu^\delta C_{ji} \sum_k C_{ik}}{C_{ij} \sum_\ell C_{j\ell}} \leq \frac{\mu^\delta \gamma C_{ji}}{\sum_\ell C_{j\ell}} \leq \mu^\delta \gamma.$$

$\square$

**Proposition 13.** *For the equilibrium prices  $r$ ,*

$$\frac{1}{\gamma} \left( \frac{\psi_j}{\psi_i} \right)^{\frac{1}{1+2\delta}} \leq \frac{r_j}{r_i} \leq \gamma \left( \frac{\psi_j}{\psi_i} \right)^{\frac{1}{1+2\delta}}$$

*Proof.* First we note the inequalities

$$(4.8) \quad r_i^{-\delta} \gamma^{-\delta} \leq R_i \leq r_i^{-\delta} \gamma^{1+\delta}$$

For the lower bound on  $R_i$ , we apply Definition 2.5, in the numerator using that there is a  $C_{i\ell} \geq 1$ , and in the denominator using Lemma 12. The upper bound is similar, simply

using the definition of  $\gamma$ . Now to show the proposition, observe that (4.2) and (4.4) yield by telescoping product another expression for  $\psi_j$ :

$$(4.9) \quad \psi_j = \frac{r_j^{1+\delta} R_{i_0}}{r_{i_0}^{1+\delta} R_j}$$

so  $\frac{\psi_j}{\psi_i} = \frac{r_j^{1+\delta} R_i}{r_i^{1+\delta} R_j}$ . The proposition follows by application of (4.8) to  $R_i$  and  $R_j$ .  $\square$

**4.4. Spectral bounds on the convergence rate.** For circulation-free markets we can obtain more precise bounds on convergence rate than for general markets. In order to do this we define:

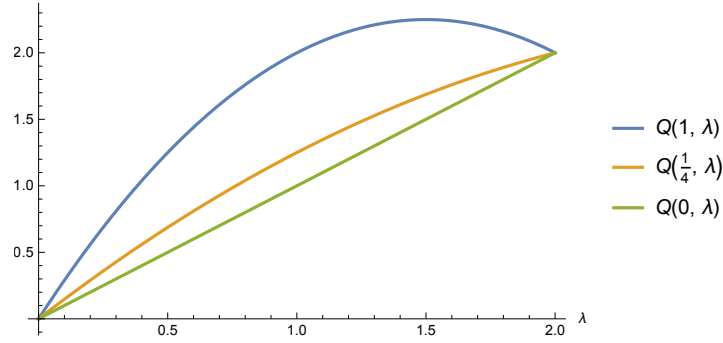
**Definition 14.** *The convergence rate  $\chi(C)$  of market  $C$  (at a fixed value of the parameter  $\delta$ , which is suppressed), is defined to be (where  $\|\cdot\|_*$  is any vector norm—the choice does not affect the definition):*

$$\chi(C) = \sup_{\bar{\beta}(0)} \lim_{t \rightarrow \infty} \frac{1}{t} \log \left( \frac{\|\bar{\beta}(0)\|_*}{\|\bar{\beta}(t)\|_*} \right).$$

It will follow from Theorem 15 that the convergence rate is nonnegative and that rate zero can occur only in a market that partitions into disconnected submarkets, a case excluded by assumption (A). We can therefore confine attention to markets with positive convergence rate. We call the inverse of the convergence rate the *convergence time*: roughly, this is the time within which any perturbation from equilibrium will shrink in norm by a constant factor. One may think of it as the “half-life” of perturbations from equilibrium.

There is a well-developed theory in which the connectivity of an undirected graph is usefully measured by a single number, its *algebraic connectivity*, defined as the second eigenvalue  $\lambda$  of the Laplacian matrix associated with the graph. (This is also equal to the *spectral gap* since the first eigenvalue of the Laplacian is always 0.) We emphasize that rapid equilibration can occur even in very sparsely connected markets. (For example, sparse but random connections almost always generate a rapidly-equilibrating network.) It is not local degree, but global quality of interconnection, that matters.

For a symmetric matrix  $A$  let  $\lambda_{\downarrow i}(A)$  denote the  $i$ 'th-largest, and  $\lambda_{\uparrow i}(A)$  the  $i$ 'th-smallest, eigenvalue; recall  $A$  has an orthogonal basis of eigenvectors. Recalling the formation of Laplacians in (3.1), it is well known that for any symmetric nonnegative  $A$ ,  $0 = \lambda_{\uparrow 1}(\Delta(A)) \leq \dots \leq \lambda_{\uparrow n}(\Delta(A)) \leq 2$ ; and that the rank of  $\ker(\Delta(A))$  is the number of connected components of  $A$ . Note that  $\Delta$  is invariant to scaling of its argument, or homogeneous of degree 0. As argued in Section 4.3.1, under condition (B),  $K$  is symmetric, and therefore so are  $W$  and

FIGURE 2.  $Q(\delta, \lambda)$ : linear in  $\delta$ , quadratic in  $\lambda$ 

$L_{\widehat{C}}$ . Taking  $A = W$  we have from Eqn. 3.6 that  $\lambda_{\uparrow i}(L_{\widehat{C}}) = 1 - \lambda_{\downarrow i}(K)$ ; applying Theorem 2, (3.8), we find that  $\tilde{D} = -(1 + 2\delta)L_{\widehat{C}} + \delta L_{\widehat{C}}^2$ . Therefore, defining  $Q$  to be

$$(4.10) \quad Q(\delta, \lambda) = (1 + 2\delta)\lambda - \delta\lambda^2,$$

we conclude that our dynamics generator  $\tilde{D}$  has a real spectrum, determined as follows by  $L_{\widehat{C}}$ : for every eigenvalue  $\lambda$  of  $L_{\widehat{C}}$ ,  $-Q(\delta, \lambda)$  is an eigenvalue of  $\tilde{D}$ , with the same eigenvector. Figure 2 shows  $Q$  for  $\lambda$  in  $[0, 2]$  for several values of  $\delta$ . The slowest-converging mode of our system (recall the dynamics are (2.19)) is the eigenvector corresponding to the eigenvalue  $\lambda_{\downarrow 2}(\tilde{D})$ , because (with  $\|\cdot\|$  denoting Euclidean norm):

- (i) For all  $\bar{\beta}(0)$  satisfying (2.20),  $\|\bar{\beta}(t)\| \leq \|\bar{\beta}(0)\| \cdot e^{\lambda_{\downarrow 2}(\tilde{D})t}$
- (ii) For some  $\bar{\beta}(0) \neq 0$  satisfying (2.20),  $\|\bar{\beta}(t)\| = \|\bar{\beta}(0)\| \cdot e^{\lambda_{\downarrow 2}(\tilde{D})t}$

The quadratic dependence of the dynamics on the Laplacian marks an interesting change from the usual linear dependence that arises in the context of continuous-time Markov chains. Qualitatively however, predictions are similar because for  $\lambda$  near 0,  $Q$  scales to first order proportionally to  $\lambda$ .

Based on Theorem 2, Assumption (B), Properties (i),(ii) and Definition 14 we have:

**Theorem 15** (Convergence Rate Characterization for Circulation-Free Markets).

$$\chi(C) = -\lambda_{\downarrow 2}(D) = \lambda_{\uparrow 2}(Q(\delta, L_C)).$$

We note that the kernel of  $L_C$  equals  $\text{span}(B\vec{1})$ ; this is an immediate consequence of Theorem 2 and our comment following (2.13) regarding the right-kernel of  $D$ .

Only for markets with *extremely* large Laplacian spectral gaps can the largest nonzero eigenvalue of  $\tilde{D}$  not correspond to the smallest nonzero eigenvalue of  $L_{\widehat{C}}$ . If we want to express Theorem 15 directly in terms of the spectrum of the Laplacian (i.e., exchange the  $Q$

and  $\lambda_{\uparrow 2}$  operations) then this possibility creates a small technical complication, accounted for in Theorem 16 by the “2” within the min:

**Theorem 16** (Convergence Rate for Circulation-Free Markets, version 2).  $Q(\delta, \lambda_{\uparrow 2}(L_C)) \geq \chi(C) \geq \min\{Q(\delta, \lambda_{\uparrow 2}(L_C)), 2\}$ .

*Proof.* Due to Theorem 15 every eigenvalue  $\lambda$  of  $L_C$  maps to an eigenvalue  $-Q(\delta, \lambda) = -(1 + 2\delta)\lambda + \delta\lambda^2$  of  $\tilde{D} = BDB^{-1}$ . The mapping  $Q$  is monotone increasing in  $\lambda$  throughout  $[0, 1 + \frac{1}{2\delta}]$ ; if  $1 + \frac{1}{2\delta} < 2$  it then descends, symmetrically, to 2 at  $\lambda = 2$ . Also,  $Q(\delta, \frac{1}{\delta}) = 2$ . (See Figure 2.) Thus, sufficient conditions that  $\lambda_{\downarrow 2}(\tilde{D}) = -Q(\delta, \lambda_{\uparrow 2}(L_C))$  include (a) that the spectrum of  $L_C$  is contained in  $[0, 1 + \frac{1}{2\delta}]$ , or (b) that  $\lambda_{\uparrow 2}(L_C) \leq \frac{1}{\delta}$ .

Clause (a) will occur if  $W$  is “far from bipartite,” in particular if there is sufficient local consumption of goods (i.e., the coefficients  $C_{jj}$  are large enough). Clause (b) is in particular guaranteed if  $\delta \leq 1/2$ .

Even outside these favorable cases, note that for any  $\lambda > 1 + \frac{1}{2\delta}$ ,  $Q(\delta, \lambda) \geq 2$ . Consequently in all cases:

$$(4.11) \quad -Q(\delta, \lambda_{\uparrow 2}(L_C)) \leq \lambda_{\downarrow 2}(\tilde{D}) \leq \max\{-Q(\delta, \lambda_{\uparrow 2}(L_C)), -2\}$$

□

**Remark 17.** A famous pair of inequalities connects between combinatorial notion of connectivity, and the algebraic one which refines it. These are the discrete Cheeger inequalities (see e.g., [13, 2, 1, 60]). They show for any weighted graph  $W$  that

$$(4.12) \quad \phi(W)^2/2 \leq \lambda_{\uparrow 2}(\Delta(W)) \leq 2\phi(W)$$

where  $\phi(W)$  is the edge expansion, defined in (3.13).

4.4.1. *A simple application.* We can apply Theorem 16 to contrast the two markets illustrated in Figure 1. With  $C_{ij} = 1$  for every edge here (including self-loops), every participant has the same “degree,” that is, there is a  $k$  such that  $|\{j : C_{ij} = 1\}| = |\{j : C_{ji} = 1\}| = k$  for all  $i$ ; consequently equilibrium prices are uniform. Specifically, each market has ten participants and locally the markets have the same parameters: each participant is interested in some three goods (equally), and not at all in any other good. Where the markets differ is in their global connectivity. The market on the right is connected only by a long cycle; price disturbances have to propagate all the way around the cycle. That on the left (known as the Petersen graph) has smaller diameter (every two agents are within distance 2, compared with 5 on the right), and between any two agents one may find 3 edge-disjoint paths (compared

with only 2 on the right). Thus both in terms of distances and in terms of resilience to disconnection, the market on the left is better connected. One should expect prices in the market on the left to converge more rapidly, and Theorem 16 affirms this intuition. For the Petersen market,  $\lambda_{\uparrow 2}(L_C) = 2/3$ ; while for the cycle market,  $\lambda_{\uparrow 2}(L_C) = (3 - \sqrt{5})/6 \cong 0.1273$ . We can compare the convergence rates at, say,  $\delta = 1/4$ : we find for the Petersen market  $Q(1/4, \lambda_{\uparrow 2}(L_C)) = 8/9 \cong 0.8889$  and for the cycle market  $Q(1/4, \lambda_{\uparrow 2}(L_C)) \cong 0.1869$ . Thus the convergence time for prices in the cycle market is over *four times* longer than it is in the Petersen market, demonstrating that the impact of market structure upon convergence time can be very substantial, even in small markets.

**4.5. Market comparisons; stability of the Laplacian.** The dependence of  $L_{\hat{C}}$  on  $C$  is somewhat indirect: one must first obtain the stationary prices  $r$ , then combine this with the market parameters  $C$  to form  $K$  as in (3.2), and finally apply (3.6). It is worth asking, therefore, whether it is possible to obtain useful bounds with less detailed information, e.g., if only the stationary prices (which might easily be observed, with some noise) or only the trading patterns are known. Likewise, even if in principle we can collect full information about  $L_{\hat{C}}$ , in practice we will know less. What are the implications for the bounds on dynamics and convergence time? This is a continuation of the question we posed for general markets in Remark 9. In the circulation-free case we can give even more precise answers thanks to Lemma 18 which establishes stability of symmetric matrix Laplacians. (This lemma seems to be new. Analogous lemmas are known, which apply in more general circumstances but yield weaker conclusions [21].) First, a definition: For two  $n \times n$  symmetric weighted adjacency matrices  $W, \tilde{W}$ , let

$$(4.13) \quad \nu = \nu(W, \tilde{W}) = \left( \max_{i,j} \frac{W_{ij}}{\tilde{W}_{ij}} \right) \cdot \left( \max_{i,j} \frac{\tilde{W}_{ij}}{W_{ij}} \right),$$

with the ratios taken as 1 when both numerator and denominator are 0. (Thus  $\nu \geq 1$ , with  $\nu = 1$  only if there is a  $c \neq 0$  s.t.  $W = c\tilde{W}$ .)

**Lemma 18** (Laplacian Stability). *For two  $n \times n$  symmetric weighted adjacency matrices  $W, \tilde{W}$ ,*

$$\lambda_{\uparrow 2}(\Delta(\tilde{W})) \leq \nu(W, \tilde{W}) \cdot \lambda_{\uparrow 2}(\Delta(W)).$$

*For all values of  $\nu$  this bound is best possible.*

(Proof in Appendix A.)

In order to apply the lemma we need to bound  $\nu$ ; for this we will rely on the market disparity measures  $\gamma$  and  $\psi$ . The first part of the comparison theorem assumes knowledge only of  $\gamma, \psi$ , and the underlying graph, which we describe with the matrix  $U$ :

$$(4.14) \quad U_{ij} = 1 \text{ if } C_{ij} > 0, \quad U_{ij} = 0 \text{ if } C_{ij} = 0.$$

The second part of the comparison theorem assumes that we know the graph, and  $\gamma$ , and the equilibrium prices. If one is studying a functioning market near equilibrium then likely one can observe the prices. Let  $E$  denote the weighted adjacency matrix

$$(4.15) \quad E_{ij} = \sqrt{r_i r_j} \text{ on edges } i \sim j \text{ of the network (and 0 elsewhere).}$$

**Theorem 19** (Market Comparison Bounds). *In the following expression, (1) For bounds using the unweighted Laplacian, set  $L = \Delta(U)$  and  $u = \psi\gamma^{2+\delta}$ ; (2) For bounds using the equilibrium prices Laplacian, set  $L = \Delta(E)$  and  $u = \gamma^{1+\delta}$ .*

$$Q(\delta, \min\{u\lambda_{\uparrow 2}(L), 1 + \frac{1}{2\delta}, \frac{n}{n-1}\}) \geq \chi(C) = -\lambda_{\downarrow 2}(\tilde{D}) \geq \min\{Q(\delta, \frac{\lambda_{\uparrow 2}(L)}{u}), 2\}.$$

Before beginning the proof we make several notes upon this theorem. First, in order to apply either bounds (1) or (2), one must know a little more than just the current prices in the market: one must also know at least which allocations (of good  $j$  to participant  $i$ ) are nonzero. Second, (2) is stated in terms of the equilibrium prices, which in principle are unknown, but this is not a significant limitation since all our results pertain to the regime in which dynamic prices are a small perturbation of equilibrium. Third, the ‘‘min’’ expressions complicate the bounds in the theorem, but the main cases of interest for the theorem are those in which  $u$  is small (so not much is lost by the eigenvalue bound) and  $\lambda_{\uparrow 2}(L)$  is not too large (so it, and not  $\lambda_{\uparrow n}(L)$ , is decisive in the dynamics). In these situations, the expression simplifies to

$$Q(\delta, u\lambda_{\uparrow 2}(L)) \geq \chi(C) \geq Q(\delta, \frac{\lambda_{\uparrow 2}(L)}{u}).$$

A sufficient condition for this simplification is that  $\delta \leq 1$  and  $u\lambda_{\uparrow 2}(L) \leq 1$ ; see Remark 23.

Also before proving Theorem 19 let us exhibit that  $\psi$  can be exponential in the network size even if  $\gamma$  is bounded. It is therefore highly advantageous to know the equilibrium prices (i.e., use bound (2)) when applying these results.

**Example 20.** *Take  $\delta = 1$ . Fix any  $a > 1$  and create a market among participants  $1, \dots, n$  arranged in a chain, as follows.  $C_{ij}$  is nonzero only for  $|i - j| \leq 1$ . For such  $i, j$ ,  $C_{ij} = a^{j-i}$ . Up to some edge-effects, prices in this network are proportional to  $a^{2i}$ . Thus  $\psi \in \Theta(a^{2n})$ .*

Toward proving Theorem 19 we start with a lemma about the equilibrium prices:

**Lemma 21.** For any  $i \sim j$ ,  $\gamma^{-2-2\delta} \leq \frac{C_{ij}C_{ji}}{r_i^\delta R_i r_j^\delta R_j} \leq 1$ . That is,  $\gamma^{-1-\delta} \leq K_{ij} \leq 1$ .

*Proof.* The upper bound follows by dropping most terms in the denominator, leaving only  $r_i^\delta \frac{C_{ij}}{r_j^\delta} r_j^\delta \frac{C_{ji}}{r_i^\delta}$ .

For the lower bound we apply the upper bound in (4.8) to both  $R_i$  and  $R_j$ ; this upper bounds the denominator by  $\gamma^{2+2\delta}$ . The numerator is lower bounded by 1.  $\square$

*Proof of Theorem 19:* The proof breaks into two lemmas. The first is a general bound on the convergence rate of our market (with weighted adjacency matrix  $W$  and Laplacian  $L_C = \Delta(W)$ ), in terms of two features of any other weighted adjacency matrix  $W'$ : the spectrum of its Laplacian, and  $\nu(W, W')$ . The second lemma bounds  $\nu(W, U)$  and  $\nu(W, E)$ ; applying the first lemma with each of these bounds then yields each of the parts of the theorem.

**Lemma 22.** Let  $\nu = \nu(W, W')$ . Then

$$\begin{aligned} -Q(\delta, \min\{\nu\lambda_{\uparrow 2}(\Delta(W')), 1 + \frac{1}{2\delta}, 1 + \frac{1}{n-1}\}) &\leq \lambda_{\downarrow 2}(\tilde{D}) \\ &\leq -Q(\delta, \min\{\lambda_{\uparrow 2}(\Delta(W'))/\nu, 1/\delta\}). \end{aligned}$$

*Proof.* For the first inequality in the Lemma, recall  $\lambda_{\downarrow 2}(\tilde{D}) \geq -Q(\delta, \lambda_{\uparrow 2}(L_C))$  from (4.11); also note that  $Q(\delta, \lambda)$  is monotone increasing in  $\lambda$  until the global maximum at  $1 + \frac{1}{2\delta}$ . We have two upper bounds on  $\lambda_{\uparrow 2}(L_C)$ :  $\lambda_{\uparrow 2}(L_C) \leq \nu\lambda_{\uparrow 2}(\Delta(W'))$  from Lemma 18, and  $\lambda_{\uparrow 2}(L_C) \leq 1 + \frac{1}{n-1}$  because  $\text{Tr}(L_C) \leq n$  (see (3.1)) and  $\lambda_{\uparrow 1}(L_C) = 0$ . The first inequality follows.

For the second inequality, since  $Q(\delta, \frac{1}{\delta}) = 2$ , the upper bound equals  $\max\{-Q(\delta, \lambda_{\uparrow 2}(\Delta(W'))/\nu), -2\}$ . Now recall  $\lambda_{\downarrow 2}(\tilde{D}) \leq \max\{-Q(\delta, \lambda_{\uparrow 2}(L_C)), -2\}$  from (4.11). If  $\lambda_{\downarrow 2}(\tilde{D}) > -2$  then necessarily  $Q(\delta, \lambda_{\uparrow 2}(L_C)) < 2$ , and then we must have  $\lambda_{\uparrow 2}(L_C) < \min\{1/\delta, 2\}$ . This implies  $Q(\delta, x)$  is monotone for  $x \in [0, \lambda_{\uparrow 2}(L_C)]$ ; then applying  $\lambda_{\uparrow 2}(\Delta(W'))/\nu \leq \lambda_{\uparrow 2}(L_C)$  from Lemma 18, we find  $Q(\delta, \lambda_{\uparrow 2}(L_C)) \geq Q(\delta, \lambda_{\uparrow 2}(\Delta(W'))/\nu)$ .  $\square$

**Remark 23.** In many cases of interest the bounds in Lemma 22 will be determined by the  $\lambda_{\uparrow 2}(\Delta(W'))$  term. Specifically, this is the case if  $\nu\lambda_{\uparrow 2}(\Delta(W')) \leq \min\{1, 1/\delta\}$ . (Note,  $Q(\delta, x)$  is monotone increasing and  $\leq 2$  for  $x \in [0, 1/\delta]$ . This demonstrates the case  $\delta \geq 1$  because then  $1/\delta \leq 1 + \frac{1}{2\delta}, 1 + \frac{1}{n-1}$ . For  $\delta < 1$ , use that  $\max_{0 \leq x \leq 1} Q(\delta, x) = \min_{1 \leq x \leq 2} Q(\delta, x)$ .) More generally, this will also be the case in any network with  $\lambda_{\uparrow 2}(\Delta(W'))$  small enough (i.e., a network that is not an excellent expander), unless  $\nu$  is large (but if  $\nu$  is large then of course one cannot expect much benefit from the comparison theorem).

Nevertheless it is worth pointing out an example, even with  $\nu = 1$ , in which the bound is not determined by  $\lambda_{\uparrow 2}(\Delta(W'))$ . Take the complete bipartite graph  $K_{2,2}$  with uniform edge weights. Its Laplacian spectrum is  $0, 1, 1, 2$ . For  $\delta > 1/2$  the critical eigenvalue here is not  $\lambda_{\uparrow 2}(\Delta(K_{2,2})) = 1$ , but  $\lambda_{\uparrow 4}(\Delta(K_{2,2})) = 2$ , and correspondingly the convergence rate is 2.

**Lemma 24.**  $\nu(W, U) \leq \psi\gamma^{2+\delta}$  and  $\nu(W, E) \leq \gamma^{1+\delta}$ .

*Proof.* Consider from (3.5) the entries of the weighted adjacency matrix. Applying (4.7) and

(2.16) we have  $W_{ij} = \sqrt{\frac{C_{ij}C_j r_i^{1-\delta} r_j^{1-\delta}}{R_i R_j}}$ , and using Lemma 21 gives

$$(4.16) \quad \sqrt{r_i r_j} \gamma^{-1-\delta} \leq W_{ij} \leq \sqrt{r_i r_j}.$$

Earlier (Section 4.3) we bounded the variation in prices in terms of  $\gamma\psi$ , and so we have that if  $W_{ij} \neq 0$  then for any  $i', j'$ :  $W_{i'j'}/W_{ij} \leq \psi\gamma^{2+\delta}$ . This implies the first bound in the lemma. The second bound in the lemma follows immediately from (4.16).  $\square$

This completes the proof of Theorem 19.  $\square$

**Remark 25** (on the necessity of the dependences on  $\gamma$  and  $\psi$  in Theorem 19). First, concerning  $\gamma$ : it is clear that the bounds in the theorem must depend on  $\gamma$  because very “weak” edges, those expressing little interest of participant  $i$  in good  $j$ , are in the unweighted graph indistinguishable from any other edge. Weak edges express themselves in our parameters by forcing  $\gamma$  to be large.

Next, concerning  $\psi$ : the main difference in the strengths of Parts 1 and 2 of the theorem is that in the latter we do not lose the factor of  $\psi$  due to the disparity in desirability of goods. One might ask whether the dependence on  $\psi$  in the bound of Part 1 is an artifact of the analysis. The answer is that it is not: it is unavoidable. In markets with very unbalanced prices, even if  $\gamma$  is bounded, the Laplacian  $\Delta(U)$  of the unweighted graph can be an exponentially poor proxy for the actual market Laplacian  $L_C$ . We show this in Appendix B.

**4.6. Markets subject to noise.** The entire discussion above—like most work in general equilibrium theory—has presumed that we are discussing the market as a “closed system” (to use terminology from physics) and, moreover, that the Samuelson dynamics are implemented perfectly by the agents. However, in economics no less than in physics, it is necessary to consider open systems, subject to random noise from external sources. Moreover the agents themselves might not be perfectly reliable and deterministic actors.



In this setting one can no longer write down the state of the market as a particular vector  $\beta$ . Instead, the state at time  $t$  is a probability distribution with density  $\rho(t, \beta)$  over vectors  $\beta$ . This is analogous to what is called in physics a thermal state.

There is a well-known framework for extending the linear dynamics (2.19) for a symmetric matrix  $\tilde{D}$ , to the noisy case. This is the Ornstein-Uhlenbeck stochastic differential equation [66]

$$(4.17) \quad \dot{\beta} = \tilde{D}\beta + V\dot{\Omega}$$

where  $\Omega$  is a Wiener process and  $V$  is any real matrix (which shapes how the Wiener process impacts the otherwise deterministic dynamics). We recollect that  $B\vec{1}$  is in the right-kernel of  $\tilde{D}$ , which represents the fact that there is no force in the market counterbalancing overall inflation or deflation of prices. In order to discuss a normalized thermal state we therefore need to project the dynamics into the orthogonal,  $(n-1)$ -dimensional subspace. Set  $J$  to be the all-ones matrix,  $\Pi = I - \frac{1}{n}J$ , and consider  $P = B\Pi B^{-1}$  as a projection operator on column vectors; the invariant subspace of this projection is  $\text{Im } P = (B^{-1}\vec{1})^\perp$ . Set  $\beta_P = P\beta$ . Then the noise-free dynamics (2.19) within the subspace  $(B^{-1}\vec{1})^\perp$  are  $\dot{\beta}_P = \tilde{D}\beta_P$ . For the noise process we restrict ourselves to isotropic diffusion  $V = \sqrt{2}I$ . So (4.17) is instantiated by

$$(4.18) \quad \dot{\beta}_P = \tilde{D}\beta_P + \sqrt{2}P\dot{\Omega}$$

which again leaves  $(B^{-1}\vec{1})^\perp$  invariant. Under these time dynamics, from an initial condition, we have at time  $t$  a density  $\rho_P(t, \beta_P)$  on  $(B^{-1}\vec{1})^\perp$ . Let  $\phi$  be the following quadratic “confining potential” on the subspace  $(B^{-1}\vec{1})^\perp$ :

$$\phi(\beta_P) = -\frac{1}{2}\beta_P^* \tilde{D} \beta_P$$

Note  $\phi$  is nondegenerate (i.e., 0 only at the origin). Now the stochastic time evolution (4.18) can be written as a Fokker-Planck equation [54] for the density:

$$(4.19) \quad \frac{\partial \rho_P}{\partial t} = \nabla \cdot (\nabla \rho_P + \rho_P \nabla \phi)$$

Here  $\nabla$  is the gradient operator w.r.t.  $\beta$  and  $\nabla \cdot \nabla = \sum \frac{\partial^2}{\partial^2 \beta_i}$  is the “analytic Laplacian” [35] (which is negative semi-definite). These dynamics diagonalize in the basis of eigenvectors of  $\tilde{D}$ . It is known [66, 54, 41] that  $\rho_P$  converges for large  $t$  to the Gaussian density

$$(4.20) \quad \left( \prod_{i=2}^n \frac{-\lambda_{\downarrow i}(\tilde{D})}{2\pi} \right)^{1/2} e^{-\phi}$$

(One can readily verify that this density is stationary for (4.19).) Thus the stationary distribution along the  $i$ 'th eigenvector is Gaussian about the origin with variance  $-1/\lambda_{\downarrow i}(\tilde{D})$ .

Applying the preceding discussion to the slowest mode (i.e.,  $\lambda_{\downarrow 2}(\tilde{D})$ ) of the system, we can now make a concrete conclusion regarding the “thermal” steady state:

**Theorem 26** (Steady State Distribution). *In steady state, for a market subject to the infinitesimal noise model described above, the distribution on  $\beta_P$  is multivariate normal, with maximum directional variance proportional to  $1/\chi(C)$ . (Equivalently, proportional to the half-life of perturbations in the isolated market.)*

We can state this theorem as a scaling law. (Our proof shows the law only for infinitesimal white Gaussian noise. But we believe it is robust.)

*The following two quantities are proportional to each other:*

- (1) *The convergence time of prices to equilibrium—for the market in isolation (i.e., under the pure dynamics (2.11)).*
- (2) *The variance of prices—for the market in steady-state distribution at a fixed level of external noise.*

This connection between two properties of a system with restorative and diffusive forces—its speed of response to an externally-forced disturbance, vs. the variance of its properties in a thermal state—is a classic one in physics, known as a fluctuation-dissipation theorem.

We conclude with a note on the magnitude of the effect under discussion. Earlier we already illustrated with a concrete example how strongly the connectivity of a graph can affect its leading Laplacian eigenvalue. In fact, it is well understood in the graph theory literature what the extreme possible values of this eigenvalue are—with the following consequence (in view of Theorem 26). Fix any  $k \geq 3$  and fix the CES parameter  $\delta$ . For connected networks of  $n$  agents in which all  $C_{ij}$  are 0 or 1, and every node has  $k$  neighbors,  $\chi(C)$  may range from a constant (independent of  $n$ , the number of agents) down to a constant times  $1/n^2$ . Thus even in markets of such limited form, the variance of the price distribution is sensitive to the market structure by a factor as large as  $O(n^2)$ .

## 5. DISCUSSION

### 5.1. Some context for our work.

*Markov Chain Monte Carlo (MCMC)*. Some of the key tools we have been relying on were developed by theoretical computer scientists and probabilists for the study of so-called MCMC algorithms. Such algorithms have their roots in the early 1950s [46], but deeper

understanding, and in the particular of the importance of edge connectivity in convergence rates, awaited the late 1980s [60]. This is still an active field [59, 49, 45]. The preponderance of work has focused on reversible processes but [48] is a key exception.

*Distributed computing and control theory.* In a consensus problem, multiple agents, communicating over a (usually fixed but sometimes dynamic) graph, start out with individual inputs and are trying to compute a consensus value. Sometimes this is called sensor fusion. In an alignment problem, agents are trying to physically align themselves, somewhat like birds within a flock. For both of these problems (and many similar variants), researchers have investigated a variety of message-passing distributed protocols such as gossip algorithms and belief propagation. By and large these are first-order iterative adjustment methods—namely, varieties of tâtonnement; and the Laplacian of the communication graph plays a key role in the convergence analysis. See [52] for a survey of this literature.

*Network effects.* There has been substantial recent interest in analyzing how network structures affect resilience and contagion in economies; for a broad overview see [12]. Interestingly, the primary connectivity parameter considered in the economics literature has been vertex degrees, a purely local measure that has almost no bearing on the global connectivity (as measured by expansion and in the symmetric case by spectral gap) that has been essential to our work.

*Formation of trading links.* For some game-theoretic issues, see e.g., [38, 36, 11].

*Modeling price dynamics.* A variety of both non-trading (abstract auctioneer) and trading processes have been offered as models for Walrasian tâtonnement—even if only in order to justify market stability.<sup>6</sup> The question has been particularly studied in the context of Arrow-Debreu markets: although it was shown early [4, 7, 67] that for gross substitutes, continuous-time tâtonnement converges to an equilibrium, Scarf [57] famously showed that without this assumption, it may not. This spurred the study of alternative forms of tâtonnement [58, 39, 61, 69, 37, 30, 34, 40, 9], with stronger convergence properties. (The connection between convergence of tâtonnement and of Newton’s method also led to Smale’s study [63] of algorithmic complexity over  $\mathbb{R}$ .) Work on the stability of tâtonnement began with Hicks [31], who discussed its local stability under some conditions; Samuelson [55, 56]

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<sup>6</sup>F. M. Fisher, 2011 [23]: “Whether or not the actual economy is stable, we largely lack a convincing theory of why that should be so. Lacking such a theory, we do not have an adequate theory of value, and there is an important lacuna in the center of microeconomic theory... To only look at situations where the Invisible Hand has finished its work cannot lead to a real understanding of how that work is accomplished.”

showed that the Hicksian conditions are neither sufficient nor necessary for stability, and Metzler [47] showed that the Hicksian conditions are sufficient in the gross substitutes case.

Not all study of market dynamics has been on non-trading processes; out-of-equilibrium trade models have also been developed and shown to converge, such as the Edgeworth process [68], [50] (and see [44] §2.9), the Hahn process [27, 28], or an exchange model of Smale [62]; however, these are less relevant to our study for several reasons, including that the first requires coordination of large coalitions; the second and third converge to an equilibrium that generally does not agree with the specified endowments; and the third is non-deterministic, as it depends upon the sequence of trade encounters. For more see, e.g., [22, 16]. Simply put, although it would be desirable to pursue our topic in a trading process, no model with all the needed properties has yet been settled on.

It should be said that neither tâtonnement nor the existing trading models are fully reductionist theories; that is, we do not have a model of individual strategic transactions from which emerges at the market level an Arrow-Debreu market with the stated dynamics and which equilibrates to the given endowments. Nonetheless, if one seeks quantitative statements, one must work in a definite model. In the last two decades laboratory evidence has accumulated in support of tâtonnement dynamics [53, 3, 33, 26, 19] even in markets such as Scarf's or Gale's [25] where it makes surprising predictions. And it is notable that the tâtonnement process is predictive of experimental trading dynamics, despite formally involving no trade. (For more on these dynamics see [44, 51].) Because of its combination of laboratory support and mathematical clarity, we have focused on tâtonnement and specifically Samuelson dynamics in this paper.

*Markets and algorithms.* Recent work in computer science [17, 24, 18, 10, 15, 14, 8] focuses on tâtonnement and related processes [71] as algorithms. These papers propose several discrete-time interpretations of Samuelson dynamics, and establish global *upper* bounds on the convergence time. The highlight of this line of work from our perspective is the paper [14] that relates discrete-time tâtonnement to the gradient descent optimization method, and upper bounds its convergence time across the CES spectrum (including the complements case,  $\rho < 0$ ). However, none of these papers quantify rate of convergence in terms of the market structure—in fact their framework does not consider the market structure at all, and they provide only one-sided (upper) bounds on the convergence time.

**5.2. Limitations of our work, and future directions.** Since the intended application is to systems (markets) whose properties will never be exactly known (or even constant in time),

a key component of the work has been stability theorems showing that our characterization of the market convergence rate is insensitive to inaccuracies in the market parameters; and that rate bounds can even be obtained without measuring the market parameters but only observing prices and knowing which allocations are nonzero.

Nevertheless, we regard these results as essentially a proof-of-concept for the project of quantifying the stability of prices in a market, as a function of the structure of trade. An incomplete list of challenges follows.

- (1) Our results apply to utilities in CES form. It is likely that they can be generalized to nested CES functions or other more flexible classes of utilities.
- (2) Our results hold only in an infinitesimal neighborhood of equilibrium because we have made no assumption about the form of the dynamics away from equilibrium. It is plausible to make one more step without assuming a concrete expression for the dynamics away from equilibrium, by assuming a bound on the quadratic correction terms. With such a bound, one might perhaps be able to extend our convergence rate bounds to a correspondingly-sized finite (as opposed to infinitesimal) neighborhood of equilibrium; and one might be able to obtain an analogue of our price variance result, if the Ornstein-Uhlenbeck solution can be mimicked for sufficiently similar PDEs. (E.g., either of the terms on the RHS of (4.17) might be multiplied by some slowly changing function of  $\beta$ .)
- (3) The restrictive technical assumption (B) makes possible both treatment of markets-with-noise, as well as tighter bounds on convergence rate of isolated markets. It would be very valuable to obtain some analogue of the first of these without assumption (B). As regards the second, this goes to the question of whether the Cheeger inequalities hold up for directed graphs. Recently Mehta and the second author [45] were able to show that one of these inequalities breaks down badly. However, the counterexamples are rather delicate, and it is conceivable that reasonable assumptions in the market setting (e.g., bounds on asymmetry in trade) would restore the inequalities (perhaps weakened) and enable application of spectral methods.
- (4) A natural next step may be a similar study of economic models with production. Possibly, oscillatory phenomena may occur which do not in our markets.

## APPENDIX A. PROOF OF LEMMA 18 (LAPLACIAN STABILITY)

Recall that the lemma states that for any two weighted, symmetric adjacency matrices  $W, \tilde{W}$ ,

$$\lambda_{\uparrow 2}(\Delta(\tilde{W})) \leq \nu(W, \tilde{W}) \cdot \lambda_{\uparrow 2}(\Delta(W))$$

and that for all values of  $\nu$  this bound is best possible.

We may assume that  $W$  is connected, as otherwise its spectrum is simply the union, with multiplicity, of the spectra of the connected components.

Note that there is always a  $c > 0$  s.t.  $W_{ij} \leq \nu^{1/2} c \tilde{W}_{ij} \leq \nu W_{ij}$  for all  $i, j$  (and this serves as an alternative definition of  $\nu$ ). Recalling that  $\Delta$  is invariant under rescaling of its argument, we may assume that  $\tilde{W}$  has been scaled so that  $W_{ij} \leq \nu^{1/2} \tilde{W}_{ij} \leq \nu W_{ij}$  for all  $i, j$ .

For brevity in this section let  $\zeta = \zeta(W) = W\vec{1}$  and  $\tilde{\zeta} = \zeta(\tilde{W})$ . Let  $Z = (\text{diag}(\zeta))^{1/2}$  and  $\tilde{Z} = (\text{diag}(\tilde{\zeta}))^{1/2}$ . (So also  $\zeta = Z^2 \vec{1}$  and  $\tilde{\zeta} = \tilde{Z}^2 \vec{1}$ .) Let  $L = \Delta(W)$  and  $\tilde{L} = \Delta(\tilde{W})$ .

We know (comments following (3.6)) that  $\ker L = \text{span } Z\vec{1}$  and  $\ker \tilde{L} = \text{span } \tilde{Z}\vec{1}$ . By the spectral theorem,

$$\lambda_{\uparrow 2}(L) = \inf_{x: x^* Z\vec{1}=0} \frac{x^* L x}{x^* x}$$

and applying the linear transformation  $b = Z^{-1} x$  we have

$$(A.1) \quad \lambda_{\uparrow 2}(L) = \inf_{b: b^* \zeta=0} \frac{b^* Z L Z b}{b^* Z^2 b}$$

Note that

$$\frac{b^* Z L Z b}{b^* Z^2 b} = \frac{\sum_{i<j} W_{ij} (b_i - b_j)^2}{\sum_i b_i^2 \zeta_i} =: R_W(b)$$

$R_W(b)$  is known as the Raleigh quotient of  $b$  in  $W$ .

Let  $b$  be a vector achieving (A.1), that is to say, a second-to-least eigenvector of  $L$ . So  $b^* \zeta = 0$  and  $\lambda_{\uparrow 2}(L) = \frac{b^* Z L Z b}{b^* Z^2 b}$ .

We use  $b$  to produce a proxy  $\hat{b}$  for a second eigenvector of  $\tilde{L}$ :

$$\hat{b} = b - \frac{b^* \tilde{\zeta}}{\vec{1}^* \tilde{\zeta}} \vec{1}$$

This satisfies  $\hat{b}^* \tilde{\zeta} = 0$ . So

$$\lambda_{\uparrow 2}(\tilde{L}) \leq R_{\tilde{W}}(\hat{b}) = \frac{\sum_{i<j} \tilde{W}_{ij} (\hat{b}_i - \hat{b}_j)^2}{\sum_i \hat{b}_i^2 \tilde{\zeta}_i} = \frac{\sum_{i<j} \tilde{W}_{ij} (b_i - b_j)^2}{\sum_i \hat{b}_i^2 \tilde{\zeta}_i}$$

Upper bounding the entries of  $\tilde{W}$ , we have

$$\dots \leq \nu^{1/2} \frac{\sum_{i<j} W_{ij} (b_i - b_j)^2}{\sum_i \hat{b}_i^2 \tilde{\zeta}_i}$$

and lower bounding the entries of  $\tilde{\eta}$ , we have

$$\begin{aligned} \dots &\leq \nu \frac{\sum_{i<j} W_{ij}(b_i - b_j)^2}{\sum_i \hat{b}_i^2 \zeta_i} = \nu \frac{\sum_{i<j} W_{ij}(b_i - b_j)^2}{\sum_i b_i^2 \zeta_i} \frac{\sum_i b_i^2 \zeta_i}{\sum_i \hat{b}_i^2 \zeta_i} \\ &= \nu R_W(b) \frac{\sum_i b_i^2 \zeta_i}{\sum_i \hat{b}_i^2 \zeta_i} = \nu \lambda_{\uparrow 2}(L) \frac{\sum_i b_i^2 \zeta_i}{\sum_i \hat{b}_i^2 \zeta_i} \end{aligned}$$

We need to lower bound the last denominator. Recall that there is a  $t$  s.t.  $\hat{b}_i = b_i - t$ . Let  $f(t) = \sum_i \zeta_i (b_i - t)^2$ . Then  $f$  is a quadratic in  $t$  with positive leading coefficient (recall the  $\zeta_i$  are positive), and  $\partial f / \partial t = 2t \sum_i \zeta_i - 2 \sum_i b_i \zeta_i = 2t \sum_i \zeta_i$ ; so  $f$  achieves its global minimum at  $t = 0$ . Consequently,  $\frac{\sum_i b_i^2 \zeta_i}{\sum_i \hat{b}_i^2 \zeta_i} \leq 1$  and therefore

$$\lambda_{\uparrow 2}(\tilde{L}) \leq \nu \lambda_{\uparrow 2}(L).$$

Turning to optimality of the lemma: a tight example must focus the “ $W$ ” weight away from the “ $b$ ” weight, so that large jumps in  $b$  occur only across weakly-weighted edges. This is achieved in a chain  $W$  of three edges in which the middle edge has weight 1 and the outside edges have weight  $x$ . One may calculate that  $\lambda_{\uparrow 2}(\Delta(W)) = \frac{1}{1+x}$ . Now consider  $\tilde{W}$  in which the outside edges have weight  $x/\nu$ . Then  $\lambda_{\uparrow 2}(\Delta(\tilde{W})) / \lambda_{\uparrow 2}(\Delta(W)) = \nu \frac{1+x}{\nu+x}$ . Fixing any  $\nu$  and taking the limit of large  $x$  we see that the supremum of this ratio is  $\nu$ .  $\square$

#### APPENDIX B. EXAMPLE SHOWING EXPONENTIAL RATIO BETWEEN THE CONVERGENCE RATES OF $L_C$ AND $\Delta(U)$ .

In the example we use  $\delta = 1$ . Fix any constant  $a > 1$  and create a market among participants  $-n, \dots, n$  arranged in a chain, so that  $C_{ij} > 0$  if and only if  $|i - j| \leq 1$ . We will show how to set the coefficients  $C_{ij}$  in a bounded range so that  $r_i = a^{|i|}$ . We describe the  $C_{ij}$ ’s for  $i \geq 0$ ; the construction will be symmetric about the origin. Set all  $C_{ii} = 1$ ; then (4.6) is, first for the edge  $(0, 1)$ , then for  $(i, i + 1)$ ,  $1 \leq i \leq n - 2$ , and then for the edge  $(n - 1, n)$ :

$$(B.1) \quad \frac{C_{01}}{1 + 2C_{01}a^{-1}} = \frac{a^2 C_{10}}{C_{10} + a^{-1} + C_{12}a^{-2}}$$

$$(B.2) \quad \frac{C_{i,i+1}a^{2i}}{a^{1-i}C_{i,i-1} + a^{-i} + C_{i,i+1}a^{-i-1}} = \frac{a^{2i+2}C_{i+1,i}}{a^{-i}C_{i+1,i} + a^{-i-1} + C_{i+1,i+2}a^{-i-2}}$$

$$(B.3) \quad \frac{C_{n-1,n}a^{2n-2}}{a^{2-n}C_{n-1,n-2} + a^{-n+1} + C_{n-1,n}a^{-n}} = \frac{a^{2n}C_{n,n-1}}{a^{-n+1}C_{n,n-1} + a^{-n}}$$

Next specialize to taking all  $C_{i,i+1} = a$ . Then (B.2) becomes

$$\frac{a^{2i+1}}{a^{1-i}C_{i,i-1} + 2a^{-i}} = \frac{a^{2i+2}C_{i+1,i}}{a^{-i}C_{i+1,i} + 2a^{-i-1}}$$

It turns out that  $C_{i+1,i}$  converges rapidly to  $a^{-2}$  which we can see from writing  $C_{i,i-1} = a^{-2} + \alpha_i$ ,  $C_{i+1,i} = a^{-2} + \alpha_{i+1}$ , and deriving the recurrence  $\alpha_{i+1} = g(\alpha_i)$  where

$$g(x) := \frac{-x}{a^2 + 2a}$$

maps the interval  $(-a^{-2}, 1)$  into itself.

It remains only to show that the boundary conditions can be satisfied consistent with these choices. In (B.1), which becomes  $a/3 = a^2 C_{10}/(C_{10} + 2/a)$ , we have  $C_{10} = \frac{2}{3a^2 - a}$ , which lies in the interval  $(-a^{-2}, 1)$  for any  $a > 1$ . We must verify that there is a positive solution to (B.3): this becomes  $a^{n-1}/(a^{3-n} + 2a^{1-n}) = a^n C_{n,n-1}/(a^{1-n} C_{n,n-1} + a^{-n})$  and is solved by  $C_{n,n-1} = \frac{1}{a^4 + 2a^2 - a}$  which is indeed bounded away from 0.

Now that we have such a simple representation for the equilibrium prices, we can examine the weighted graph. Note that all nonzero entries of  $K$  (recall (4.7)) are within a constant factor of 1. From (2.16) we have  $B_{ii} = a^{|i|/2}$ , and then from (3.5) that the weight of edge  $W_{i,i+1}$  is within a constant factor of  $a^{|i|}$ . Therefore, splitting this graph about the origin, we see that its conductance is proportional to  $a^{-n}$ . From the discrete Cheeger inequalities we can conclude that the algebraic connectivity, too, is exponentially small in  $n$ . The algebraic connectivity of the unweighted chain, by contrast, is far larger, being proportional to  $1/n^2$ .

Thus, in this market, price equilibration is exponentially slower than that of a market that has the same connectivity structure but in which all goods have the same price.

### APPENDIX C. UNIQUENESS OF THE MARKET EQUILIBRIUM

**Lemma 27.** *There can be at most one equilibrium price vector.*

*Proof.* Suppose there are two distinct vectors  $r, r'$  solving the equilibrium equations (2.9), with  $j$  a vertex minimizing  $r'_j/r_j$  and having an in-neighbor (that is, an  $i$  s.t.  $C_{ij} > 0$ ) which does not minimize this ratio. Rescale  $r'$  so  $r'_j = r_j$ ,  $r'_i \geq r_i$  for all  $i$ , and  $r'_i > r_i$  for some in-neighbor  $i$  of  $j$ . Observe that since  $\delta > 0$ , the quantity  $r_i/R_i$  is a nondecreasing function of the price vector  $r$ , and moreover strictly increasing in  $r_i$  and in any  $r_k$  for  $k$  an out-neighbor of  $i$ . Then applying (2.9):

$$r_j^{1+\delta} = \sum_i \frac{r'_i C_{ij}}{R'_i} > \sum_i \frac{r_i C_{ij}}{R_i} = r_j^{1+\delta}$$

a contradiction. □



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