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The effect of a positive bound state on the KdV solution: a case study*

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Abstract

We consider a slowly decaying oscillatory potential such that the corresponding 1D Schrödinger operator has a positive eigenvalue embedded into the absolutely continuous spectrum. This potential does not fall into a known class of initial data for which the Cauchy problem for the Korteweg–de Vries (KdV) equation can be solved by the inverse scattering transform. We nevertheless show that the KdV equation with our potential does admit a closed form classical solution in terms of Hankel operators. Comparing with rapidly decaying initial data our solution gains a new term responsible for the positive eigenvalue. To some extent this term resembles a positon (singular) solution but remains bounded. Our approach is based upon certain limiting arguments and techniques of Hankel operators.

Keywords: KdV equation, Wigner–von Neumann potential, inverse scattering transform, Hankel operator, embedded eigenvalues

Mathematics Subject Classification numbers: 34B20, 37K15, 47B35.

1. Introduction

We are concerned with the initial value problem for the Korteweg–de Vries (KdV) equation

$$\begin{aligned} &\partial_t u - 6u\partial_x u + \partial_x^3 u = 0, \qquad -\infty < x < \infty,\ t \geqslant 0, \\ &u(x,0) = q(x). \end{aligned} \tag{1.1}$$

As is a well-known, for smooth rapidly decaying functions q (1.1) was solved in closed form in the short 1967 paper [13] by Gardner–Greene–Kruskal–Miura (GGKM). This seminal paper introduces what we now call the *inverse scattering transform* (IST). Conceptually, it is similar

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to the Fourier transform (see, for example, the classical books [1, 34]) but based on the inverse scattering theory for the Schrödinger operator

$$\mathbb{L}_q = -\partial_x^2 + q(x) \quad \text{on } L^2(\mathbb{R}). \tag{1.2}$$

Moreover, the solution $q(x,t)$ to (1.1) for each $t > 0$ can be obtained by the formula

$$u(x,t) = -2\partial_x^2 \log \tau(x,t), \tag{1.3}$$

where τ is the so-called *Hirota tau-function* introduced in [19] which admits an explicit representation in terms of the scattering data of the pair $\left(\mathbb{L}_q, \mathbb{L}_0\right)$. The solution has a relatively simple and by now well understood wave structure of running (finitely many) solitons accompanied by radiation of decaying waves (see e.g. Grunert–Teschl [16] for a streamlined self-contained exposition). In about 1973, the IST was extended to functions q rapidly approaching different constants $q_\pm$ as $x \to \pm\infty$ (step initial profile). It appeared first in the physical literature [17] and was rigorously treated in 1976 by Hruslov[1] [20]. The formula (1.3) is also available in this case with an explicit representation of the tau-function in terms of certain scattering data. We refer to our recent [15, 37] where (1.3) is extended to essentially arbitrary functions q with a rapid decay only at $+\infty$. The main feature of such initial profiles is infinite sequence of solitons emitted by the initial step. Note that a complete rigorous investigation of all other asymptotic regimes and their generalizations was done only recently by Teschl with his collaborators (see, for example, [4, 11, 12]).

Another equally important and explicitly solvable case is when q is periodic. The periodic IST is quite different from the GGKM one and is actually the *inverse spectral transform* (also abbreviated as IST) since it relies on the Floquet theory for $\mathbb{L}_q$ and analysis of Riemann surfaces and hence is much more complex than the rapidly decaying case. The solution $u(x,t)$ is given essentially by the same formula (1.3), frequently referred to as the *Its–Matveev formula* [22] (see also [10] by Dubrovin–Matveev–Novikov and the 2003 Gesztesy–Holden book [14] where a complete history is given), but τ is a multidimensional[2] theta-function of real hyperelliptic algebraic curves explicitly computed in terms of spectral data of the associated Dirichlet problem for $\mathbb{L}_q$. It is therefore very different from the rapidly decaying case. The main feature of a periodic solution is its quasi-periodicity in time t.

We have outlined two main classes of initial data q in (1.1) for which a suitable form of the IST was found during the initial boom followed by [13]. Such progress was possible due to well-developed inverse scattering/spectral theories for the underlying potentials q. However, while we have proven [15] that no decay at $-\infty$ is required to do the IST but slower than x^{-2} decay at $+\infty$ results in serious complications. The main issue here is that the classical inverse scattering theory, the foundation for the IST, has not been extended beyond short-range potentials, i.e. $q(x) = O\left(|x|^{-2-\varepsilon}\right)$, $x \to \pm\infty$. We emphasize that during the boom in scattering theory there was a number of results on (direct) scattering/spectral theory for a variety of long-range potentials but the inverse scattering theory is a different matter. It was shown in [2] that the short-range scattering data no longer determine the potential uniquely even in the case when $q(x) = O\left(x^{-2}\right)$ and it is not merely a technical issue of adding some extra data. The problem appears to be open even for L^1 potentials [3] for which all scattering quantities are well-defined but may exhibit an erratic behavior at zero energy which is notoriously difficult to analyze and classify. Besides, a possible infinite negative spectrum begets an infinite sequence of norming

[1] Also transcripted as Khruslov.

[2] Infinite dimensional in general.

constants which can be arbitrary. Consequently, it is even unclear how to state a (well-posed) Riemann–Hilbert problem which would solve the inverse scattering problem. Once we leave L^1 then infinite embedded singular spectrum may appear leaving no hope to figure out what true scattering data might be. We note that any attempt to try the inverse spectral transform instead runs into equally difficult problems (see, for example, our [36] and the literature cited therein) as spectral data evolve in time under the KdV flows by a simple law essentially only for the so-called finite gap potentials. In addition, it makes sense to find a suitable IST for (1.1) if (1.1) is actually well-posed. The seminal 1993 paper by Bourgain [7] says that (1.1) is well-posed if q is in L^2 and not much better result should be expected regarding the decay at $+\infty$; also see recent Killip–Vişan [23] where well-posedness is extended to H^{-1}.

In the current paper we look into a specific representative of the important class of continuous potentials asymptotically behaving like

$$q(x) = \left(c/x\right) \sin 2x + O\left(x^{-2}\right), \quad x \to \pm\infty. \tag{1.4}$$

In the half line context such potentials[3] first appeared in 1929 in the famous paper [30] by Wigner–von Neumann where they explicitly constructed a potential of type (1.4) with $c = -8$ which supports bound state $+1$ embedded in the absolutely continuous spectrum. Note that any q of type (1.4) with $|c| > 2$ may support a bound state $+1$ which is rather unstable. On the other hand, it is shown in [8] that the set of $L^1(\mathbb{R})$ perturbations of $W(x) = \left(c/x\right) \sin 2x$, $|c| > 2$, supporting the embedded bound state, is an unbounded differentiable manifold in $L^1(\mathbb{R})$ of codimension one, which is, in a sense, a stability result. Note that $+1$ does not seem to be a bound state for $\mathbb{L}_W$, but is certainly a *Wigner–von Neumann resonance*.

If $|c| > 1/\sqrt{2}$ then the negative spectrum (necessarily discrete) of $\mathbb{L}_q$ is infinite in general [24]. While there is a very extensive literature (see, for example, recent [21]) on potentials of type (1.4) (commonly referred to as Wigner–von Neumann type potentials) but, as Matveev points out in [9]. 'The related inverse scattering problem is not yet solved and the study of the related large times evolution is a very challenging problem'. Observe that since any Wigner–von Neumann potential is clearly in L^2, the Bourgain theorem [7] guarantees well-posedness of (1.1) and the good open problem is if we can solve it by a suitable IST. Our goal here is to investigate a specific case of (1.4) which can be done by the IST. Namely, we consider an even potential $Q(x)$ defined for $x \geqslant 0$ by

$$Q(x) = -2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \log\left(1 + \rho x - \frac{\rho}{2} \sin 2x\right),$$

where ρ is a positive constant. One can easily check that Q is continuous and behaves like (1.4) with $c = -4$. The main feature of Q is that $\mathbb{L}_Q$ admits explicit spectral analysis and consequently the scattering problem for the pair $\left(\mathbb{L}_Q, \mathbb{L}_0\right)$ can also be solved explicitly. In particular, $+1$ is a positive bound state of $\mathbb{L}_Q$ but its negative spectrum consists of just one bound state. We show that for (1.1) with initial data Q the tau-function in (1.3) can be explicitly calculated. The formula however is expressed in the language of Hankel operators (which is not commonly used in integrable systems) and we have to postpone it until section 4. We only mention here that, comparing to the short range case, the tau-function τ gains an extra factor responsible for the positive bound state. Our approach rests on suitable limiting arguments based on certain short range approximations of Q combined with techniques of Hankel operators developed in our [15].

[3] In fact, for 3D radially symmetric potentials.

We, however, were unable to find a direct analog of the IST. The trouble is that both Jost solutions associated with $\mathbb{L}_Q$ have simple poles at ± 1 which makes the corresponding Riemann–Hilbert problem singular. It can however be regularized by introducing an extra condition, similar to the pole condition in the Riemann–Hilbert problem in the short range case, but its solubility is a nontrivial issue. This situation is somewhat similar to the problem that arises in placing the *Peregrine breather* (as well as related higher-order rogue wave) in an inverse scattering context. In the recent [5] Bilman–Miller introduce the *robust IST*, a modification of the standard IST for the *focusing nonlinear Schrödinger (NLS) equation* with nonzero boundary conditions at infinity. The main idea is to transform the original Riemann–Hilbert problem in such a way that all arbitrary-order poles and potentially severe spectral singularities are 'rounded up' in a disk and all necessary information is indirectly encoded in the jump across the circle through the Jost solutions. Since the underlying *AKNS system* has *Schwarz symmetry*, the new Riemann–Hilbert problem is well-posed. For the same reason the robust IST can be readily found for some other integrable systems but not for KdV, which AKNS system is not Schwartz symmetric. Thus [5] is rather an inspiration than a recipe for how to put our solution in the Riemann–Hilbert problem framework. While we are yet to prove well-posedness of our singular Riemann–Hilbert problem, the Bourgain theorem gives an indirect indication that it can be done.

The reader will see that our approach is not restricted to just one initial condition and should work for a whole class of initial data (at least [33] gives some hopes). We however do not make an attempt to be more general for two reasons. First, our consideration would complicate a great deal due to numerous extra technicalities. But the main reason is that the scattering theory, the backbone of our approach, is not developed well enough outside of short-range potentials. (At least not to our satisfaction.) For instance, there are only some results on regularity properties of scattering data for Wigner–von Neumann type potentials (see [25]) but almost nothing is known about their small energy behavior. The latter was posed as an open question in [25] but, to the best of our knowledge, there has been no progress in this direction since then. This is a major impediment to our approach as it requires a careful control of the scattering matrix at all energy regimes.

The paper is organized as follows. In section 2 we review pertinent material from the theory of Hankel operators. In section 3 we construct an explicit short range approximation of our Q, which will be crucial to our consideration. Section 4 is devoted to stating and proving out main result.

2. Our analytic tools

To translate our problem into the language of Hankel operators some common definitions and facts are in order [31, 35].

2.1. Riesz projections

Recall, that a function f analytic in the upper half plane $\mathbb{C}^{\pm} := \{z | \pm \operatorname{Im} z > 0\}$ is in the *Hardy space* $H^2_{\pm}$ of $\mathbb{C}^{\pm}$ if

$$\sup_{h>0} \int_{\mathbb{R}\pm ih} |f(z)|^2 |\mathrm{d}z| < \infty.$$

It is a fundamental fact of the theory of Hardy spaces that any $f \in H^2_{\pm}$ has non-tangential boundary values $f(x \pm \mathrm{i}0)$ for almost every (a.e.) $x \in \mathbb{R}$ and $H^2_{\pm}$ are subspaces of $L^2 := L^2(\mathbb{R})$.

Thus, $H^2_\pm$ are Hilbert spaces with the inner product induced from L^2:

$$\langle f, g\rangle_{H^2_\pm} = \langle f, g\rangle_{L^2} = \langle f, g\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,\overline{g}(x)\,\mathrm{d}x.$$

It is well-known that $L^2 = H^2_+ \oplus H^2_-$, the *orthogonal (Riesz) projection* $\mathbb{P}_\pm$ onto $H^2_\pm$ being given by

$$\begin{aligned}(\mathbb{P}_\pm f)(x) &= \pm\frac{1}{2\pi\mathrm{i}}\lim_{\varepsilon\to 0+}\int_{-\infty}^{\infty}\frac{f(s)\mathrm{d}s}{s-(x\pm\mathrm{i}\varepsilon)}\\ &= \pm\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{f(s)\mathrm{d}s}{s-(x\pm\mathrm{i}0)}.\end{aligned} \tag{2.1}$$

In what follows, we set $H^2_+ = H^2$. Notice that for any $f \in H^2$

$$\mathbb{P}_-\left(\frac{1}{\cdot-\lambda}f\right) = \frac{1}{\cdot-\lambda}f(\lambda), \quad \lambda\in\mathbb{C}^+. \tag{2.2}$$

Besides $H^2_\pm$, we will also use $H^\infty_\pm$, the algebra of uniformly bounded in $\mathbb{C}^\pm$ functions.

2.2. Reproducing kernels

Recall that, a given fixed $\lambda\in\mathbb{C}^\pm$ the function

$$k_\lambda(z) := \frac{\mathrm{i}}{z-\overline{\lambda}}, \quad \lambda\in\mathbb{C}^\pm \tag{2.3}$$

is called the *reproducing* (*or Cauchy–Szego*) *kernel* for $H^2_\pm$. Clearly,

$$\|k_\lambda\| = \sqrt{\langle k_\lambda, k_\lambda\rangle} = \frac{1}{\sqrt{2\,\mathrm{Im}\,\lambda}} \tag{2.4}$$

and hence $k_\lambda \in H^2_\pm$ if $\lambda\in\mathbb{C}^\pm$. The main reason why reproducing kernels are convenient is the following:

$$f\in H^2, \lambda\in\mathbb{C}^+ \Longrightarrow f(\lambda) = \langle f, k_\lambda\rangle \quad \text{(Cauchy's formula)} \tag{2.5a}$$

$$f\in L^2, \lambda\in\mathbb{R} \Longrightarrow (\mathbb{P}_\pm f)(\lambda) = \pm\langle f, k_{\lambda\pm\mathrm{i}0}\rangle. \tag{2.5b}$$

Let B be a *Blaschke product* with finitely[4] many simple zeros $z_n\in\mathbb{C}^+$, i.e.

$$B(z) = \prod_n b_n(z), \quad b_n(z) = \frac{z-z_n}{z-\overline{z_n}}.$$

Introduce

$$K_B = \operatorname{span}\{k_{z_n}\}.$$

It is an easy but nevertheless fundamentally important fact in interpolation of analytic functions, the study of the shift operator, so-called model operators, etc, that

$$K_B = H^2 \ominus BH^2, \quad \text{where } BH^2 := \left\{Bf : f\in H^2\right\}. \tag{2.6}$$

[4] It can also be infinite but it does not concern us.

Lemma 2.1. *The orthogonal projections $\mathbb{P}_B$ of H^2 onto K_B and $\mathbb{P}_B^{\perp} = I - \mathbb{P}_B$ are given by*

$$\mathbb{P}_B = B\mathbb{P}_-\overline{B}, \qquad \mathbb{P}_B^{\perp} = B\mathbb{P}_+\overline{B}. \tag{2.7}$$

Furthermore, if A is a linear bounded operator in H^2 then the matrix of $\mathbb{P}_B A\mathbb{P}_B$ with respect to (k_{z_n}) is given by

$$(\mathbb{P}_B A P_B)_{mn} = \left\langle A k_{z_n}, k_{z_m}^{\perp} \right\rangle, \tag{2.8}$$

where

$$k_{z_n}^{\perp}(z) := \frac{2\,\mathrm{Im}\, z_n}{B_n(z_n)} B_n(z)\, k_{z_n}(z), \quad B_n := B/b_n \tag{2.9}$$

form a bi-orthogonal basis for (k_{z_n}), i.e. $\left\langle k_{z_n}^{\perp}, k_{z_m} \right\rangle = \delta_{nm}$.

Proof. (2.7) are proven in [32]. To show (2.8) we first explicitly evaluate $\mathbb{P}_B$. By (2.1) for $f \in H^2$ we have

$$\mathbb{P}_-\overline{B}f = -\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \frac{f(s)}{B(s)} \frac{\mathrm{d}s}{s - (x - \mathrm{i}0)}$$

and by residues

$$\begin{aligned}
\left(\mathbb{P}_-\overline{B}f\right)(x) &= -\sum_n \mathrm{Res}\left(\frac{f(z)/B(z)}{z - x}, z_n\right) \\
&= \sum_n \frac{2\mathrm{i}\,\mathrm{Im}\, z_n}{B_n(z_n)} \frac{f(z_n)}{x - z_n} \\
&= \sum_n \frac{2\mathrm{i}\,\mathrm{Im}\, z_n}{B_n(z_n)} \frac{\langle f, k_{z_n}\rangle}{x - z_n} (\text{by (2.5a)}).
\end{aligned}$$

Hence, by (2.7),

$$\begin{aligned}
\mathbb{P}_B f &= \sum_n \langle f, k_{z_n}\rangle \frac{2\mathrm{i}\,\mathrm{Im}\, z_n}{B_n(z_n)} B_n \frac{1}{\cdot - \overline{z}_n} \\
&= \sum_n \langle f, k_{z_n}\rangle k_{z_n}^{\perp},
\end{aligned}$$

where $k_{z_n}^{\perp}$ is given by (2.9). It remains to verifies that $\left(k_{z_n}^{\perp}\right)$ forms a bi-orthogonal basis for K_B. Indeed,

$$\begin{aligned}
\left\langle k_{z_n}^{\perp}, k_{z_m}\right\rangle &= \left\langle \frac{2\,\mathrm{Im}\, z_n}{B_n(z_n)} B_n k_{z_n}, k_{z_m}\right\rangle = \frac{2\,\mathrm{Im}\, z_n}{B_n(z_n)} \langle B_n k_{z_n}, k_{z_m}\rangle \\
&= \frac{2\,\mathrm{Im}\, z_n}{B_n(z_n)} B_n(z_m)\, k_{z_n}(z_m).
\end{aligned}$$

If $n \neq m$ then $B_n(z_m) = 0$. If $n = m$ then by (2.4)

$$\left\langle k_{z_n}^{\perp}, k_{z_m}\right\rangle = 2\,\mathrm{Im}\, z_n \quad k_{z_n}(z_n) = 1.$$

The formula (2.8) easily follows now. □

2.3. Hankel operators

A Hankel operator is an infinitely dimensional analog of a Hankel matrix, a matrix whose (j, k) entry depends only on $j + k$. In the context of integral operators the Hankel operator is usually defined as an integral operator on $L^2(\mathbb{R}_+)$ whose kernel depends on the sum of the arguments

$$(\mathbb{H}f)(x) = \int_0^\infty h(x+y)f(y)\mathrm{d}y, \quad f \in L^2(\mathbb{R}_+),\ x \geqslant 0 \tag{2.10}$$

and it is this form that Hankel operators typically appear in the inverse scattering formalism. It is much more convenient for our purposes to consider *Hankel operators* on H^2 (see [31, 35]).

Let

$$(\mathbb{J}f)(x) = f(-x)$$

be the operator of reflection on L^2 and let $\varphi \in L^\infty$. The operators $\mathbb{H}(\varphi)$ defined by

$$\mathbb{H}(\varphi)f = \mathbb{J}\mathbb{P}_-\varphi f, \quad f \in H^2, \tag{2.11}$$

is called the Hankel operator with the symbol φ.

It is clear that $\mathbb{H}(\varphi)$ is bounded from H^2 to H^2 and

$$\mathbb{H}(\varphi + h) = \mathbb{H}(\varphi)\ \text{for any}\ h \in H^\infty. \tag{2.12}$$

It is also straightforward to verify that $\mathbb{H}(\varphi)$ is selfadjoint if $\mathbb{J}\varphi = \bar{\varphi}$.

The following elementary lemma on Hankel operators with analytic symbols will be particularly useful.

Lemma 2.2. *Let a function φ be meromorphic on $\mathbb{C}$ and subject to*

$$\varphi(-\bar{z}) = \bar{\varphi}(z) \quad (\textit{symmetry}). \tag{2.13}$$

If φ has finitely many simple poles $\{z_n\}_{n=-N}^N$ in $\mathbb{C}^+$, is bounded on $\mathbb{R}$, and for any $h \geqslant 0$

$$\varphi(x + \mathrm{i}h) = O\left(x^{-1}\right), \quad x \to \pm\infty, \tag{2.14}$$

then the Hankel operator $\mathbb{H}(\varphi)$ is selfadjoint, trace class, and admits the decomposition

$$\mathbb{H}(\varphi) = \mathbb{H}(\phi) + \mathbb{H}(\Phi), \tag{2.15}$$

where ϕ is a rational function and Φ is an entire function given respectively by

$$\phi(x) = \sum_{-N\leqslant n\leqslant N} \frac{\operatorname{Res}(\varphi, z_n)}{x - z_n},$$

$$\Phi(x) = -\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}+\mathrm{i}h} \frac{\varphi(s)}{s-x}\,\mathrm{d}s, \quad h > \max_n \operatorname{Im} z_n. \tag{2.16}$$

Moreover,

$$\mathbb{H}(\phi) = \sum_{-N\leqslant n\leqslant N} \mathrm{i}\operatorname{Res}(\varphi, z_{-n})\langle\cdot, k_{z_{-n}}\rangle k_{z_n}, \tag{2.17}$$

$$\mathbb{H}(\Phi) = \int_{\mathbb{R}+\mathrm{i}h} \frac{\mathrm{d}z}{2\pi}\varphi(z)\langle\cdot, k_z\rangle k_{-\bar{z}} = \int_{\mathbb{R}+\mathrm{i}h}\frac{\mathrm{d}z}{2\pi}\varphi(-\bar{z})\langle\cdot, k_{-\bar{z}}\rangle k_z, \tag{2.18}$$

where $k_\lambda(z) = \frac{\mathrm{i}}{z-\lambda}$ *is the reproducing kernel of* H^2.

Proof. The selfadjointness follows from (2.13). By (2.12)

$$\mathbb{H}(\varphi) = \mathbb{H}(\mathbb{P}_-\varphi)$$

and hence we have to worry only about $\mathbb{P}_-\varphi$. By the residue theorem ($h > \max_k \operatorname{Im} z_k$), we have

$$\begin{aligned}(\mathbb{P}_-\varphi)(x) &= -\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}} \frac{\varphi(s)}{s-(x-\mathrm{i}0)}\,\mathrm{d}s \\ &= \sum_{-N\leqslant n\leqslant N} \frac{\operatorname{Res}(\varphi, z_n)}{x-z_n} - \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}+\mathrm{i}h}\frac{\varphi(s)}{s-z}\,\mathrm{d}s \\ &= \phi(x) + \Phi(x),\end{aligned}$$

and (2.15) follows. Apparently Φ is analytic (and bounded) below the line $\mathbb{R}+\mathrm{i}h$. Since h is arbitrary, Φ is then entire. Moreover, all derivatives of Φ are bounded on $\mathbb{R}$ and therefore $\mathbb{H}(\Phi)$ is at least trace class (in any Shatten–von Neumann ideal).

It follows from (2.2) that for any $z \in \mathbb{C}^+$

$$\mathbb{H}\left(\frac{1}{\cdot - z}\right) f = \mathrm{i} f(z)\, k_{-\overline{z}}$$

and (2.17) and (2.18) follow. □

Corollary 2.3. *If* φ *has no poles in* $\mathbb{C}^+$ *then* $\mathbb{H}(\varphi) = \mathbb{H}(\Phi)$.

Corollary 2.4. *If* (2.14) *holds uniformly in* $h \geqslant h_0 > \max_n \operatorname{Im} z_n$ *then* $\Phi = 0$.

A very important feature of analytic symbols is that $\mathbb{H}(\varphi)$ is well-defined outside of H^2. In particular, $\mathbb{H}(\varphi)k_{x+\mathrm{i}0}$ is a smooth element of H^2 for any $x \in \mathbb{R}$ while $k_{x+\mathrm{i}0} \notin H^2$. We will need the following statement.

Corollary 2.5. *For every* $x, s \in \mathbb{R}$

$$\begin{aligned}\mathbb{H}(\Phi)k_x(s) &= \lim_{\varepsilon\to 0}\mathbb{H}(\Phi)k_{x+\mathrm{i}\varepsilon}(s) \\ &= -\int_{\mathbb{R}+\mathrm{i}h}\frac{\Phi(z)}{(z-x)(z+s)}\frac{\mathrm{d}z}{2\pi} \\ &= -\int_{\mathbb{R}+\mathrm{i}h}\frac{\varphi(z)}{(z-x)(z+s)}\frac{\mathrm{d}z}{2\pi} \\ &=: K_x(s) \in C^\infty(\mathbb{R})\cap H^2. \end{aligned} \tag{2.19}$$

Moreover, if $\varphi_\varepsilon \to \varphi$ *uniformly on* $\mathbb{R}+\mathrm{i}h$ *then for every* $x, s \in \mathbb{R}$

$$\lim_{\varepsilon\to 0}\mathbb{H}(\Phi_\varepsilon)k_{x+\mathrm{i}\varepsilon}(s+\mathrm{i}\varepsilon) = K_x(s). \tag{2.20}$$

Convergence in (2.19) *and* (2.20) *also holds in* L^2.

Proof. It follows from (2.18) that

$$\begin{aligned}\mathbb{H}(\Phi)k_{x+\mathrm{i}\varepsilon}(s) &= \int_{\mathbb{R}+\mathrm{i}h}\frac{\mathrm{d}z}{2\pi}\varphi(z)\langle k_{x+\mathrm{i}\varepsilon},k_z\rangle k_{-\bar{z}}(s)\\ &= \int_{\mathbb{R}+\mathrm{i}h}\frac{\mathrm{d}z}{2\pi}\varphi(z)\langle k_{x+\mathrm{i}\varepsilon},k_z\rangle k_{-\bar{z}}(s)\,(\text{by }(2.5\text{a}))\\ &= -\int_{\mathbb{R}+\mathrm{i}h}\frac{\mathrm{d}z}{2\pi}\varphi(z)k_{x+\mathrm{i}\varepsilon}(z)k_{-\bar{z}}(s)\\ &\to -\int_{\mathbb{R}+\mathrm{i}h}\frac{\varphi(z)}{(z-x)(z+s)}\frac{\mathrm{d}z}{2\pi} = -\int_{\mathbb{R}+\mathrm{i}h}\frac{\Phi(z)}{(z-x)(z+s)}\frac{\mathrm{d}z}{2\pi},\\ &\varepsilon\to 0,\end{aligned}$$

where we have used two obvious facts: (a) $k_{x+\mathrm{i}\varepsilon}(z)\to k_x(z)$ uniformly on $\mathbb{R}+\mathrm{i}h$, and (b) by the Lebesgue dominated convergence

$$\int_{\mathbb{R}+\mathrm{i}h}\frac{\phi(z)}{(z-x)(z+s)}\frac{\mathrm{d}z}{2\pi} = \lim_{h\to\infty}\int_{\mathbb{R}+\mathrm{i}h}\frac{\phi(z)}{(z-x)(z+s)}\frac{\mathrm{d}z}{2\pi} = 0.$$

Thus (2.19) is proven. (2.20) is proven similarly. □

3. Our explicit potential and its short-range approximation

In this section we explicitly construct a symmetric Wigner–von Neumann type potentials supporting one negative and one positive bound state. Our construction is base upon a classical Gelfand–Levitan example [26] of an explicit potential of a half-line Schrödinger operator which spectral measure has one positive pure point. The symmetric extension of this potential to the whole line will be our initial condition. We then find its explicit short range approximation, which will be crucial to our consideration.

3.1. An explicit Wigner–von Neumann type potential

Consider the function

$$m(\lambda) = \mathrm{i}\sqrt{\lambda} + \frac{2\rho}{1-\lambda}, \quad \operatorname{Im}\lambda \geqslant 0, \tag{3.1}$$

where ρ is some positive number. This is a Herglotz function (i.e. analytic function mapping $\mathbb{C}^+$ to $\mathbb{C}^+$) which coincides with the Titchmarsh–Weyl $m-$function[5] of the (Dirichlet) Schrödinger operators $-\mathrm{d}^2/\mathrm{d}x^2 + q_0(x)$ on $L^2(0,\infty)$ with a Dirichlet boundary condition at 0. The potential q_0 has the following explicit form:

$$q_0(x) = -2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\log\,\tau_0(x), \quad x\geqslant 0, \tag{3.2}$$

where

$$\tau_0(x) = 1 + 2\rho\int_0^x \sin^2\mathrm{d}s = 1 + \rho x - \left(\rho/2\right)\sin 2x. \tag{3.3}$$

[5] We recall that the problem $-\partial_x^2 u + q(x)u = \lambda u, x\in(0,\pm\infty), u(\pm 0,\lambda) = 1$ has a unique square integrable (*Weyl*) solution $\Psi_\pm(x,\lambda)$ for any $\operatorname{Im}\lambda > 0$ for broad classes of q's (called *limit point case*). Define then the (*Titchmarsh–Weyl*) *m-function* $m_\pm$ for $(0,\pm\infty)$ as follows: $m_\pm(\lambda) = \pm\partial_x\Psi_\pm(\pm 0,\lambda)$.

Introduce

$$Q(x) = \begin{cases} q_0(x), & x \geqslant 0 \\ q_0(-x), & x < 0 \end{cases}, \tag{3.4}$$

i.e. Q is an even extension of q_0. One can easily see that the function Q is continuous and $Q(0) = 0$ but not continuously differentiable. In fact, Q is as smooth at $x = 0$ as $|\sin x|$. Moreover, one has

$$Q(x) = -4\frac{\sin 2x}{x} + O\left(\frac{1}{x^2}\right), \quad x \to \pm\infty, \tag{3.5}$$

and hence $Q \in L^2(\mathbb{R})$ but $\left(1 + |x|\right) Q(x)$ is not in $L^1(\mathbb{R})$. Thus, Q is not short-range. Also note that

$$\int_{-\infty}^{\infty} Q(x)\,\mathrm{d}x = 0.$$

The main feature of Q is that $\mathbb{L}_Q$ admits an explicit spectral and scattering theory.

Theorem 3.1. *The Schrödinger operator $\mathbb{L}_Q$ on $L^2(\mathbb{R})$ with Q given by (3.4) has the following properties:*

(*a*) (*Spectrum*) *The spectrum of $\mathbb{L}_Q$ consists of the two fold absolutely continuous part filling $(0, \infty)$, one negative bound state $-\kappa^2$ found from the real solution of*

$$\kappa^3 + \kappa = 2\rho \tag{3.6}$$

and one positive (embedded) bound state $+1$.

(*b*) (*Scattering quantities*) *For the norming constant c of $-\kappa^2$ we have*

$$c = -\mathrm{i}\,\mathrm{Res}\,(T(k), \mathrm{i}\kappa) = -\mathrm{i}\,\mathrm{Res}\,(R(k), \mathrm{i}\kappa) = \frac{2\rho}{3\kappa^2 + 1} \tag{3.7}$$

and for the scattering matrix we have

$$S(k) = \begin{pmatrix} T(k) & R(k) \\ R(k) & T(k) \end{pmatrix}, \quad k \in \mathbb{R}, \tag{3.8}$$

where T and R are, respectively, the transmission and reflection coefficients given by

$$T(k) = \frac{P(k)}{P(k) + 2\mathrm{i}\rho}, \qquad R(k) = \frac{-2\mathrm{i}\rho}{P(k) + 2\mathrm{i}\rho}, \tag{3.9}$$

$$P(k) := k^3 - k.$$

Proof. Due to symmetry $m_- = m_+ = m$ it follows from the general theory [38] that the eigenvalues of the Schrödinger operator $\mathbb{L}_Q$ are the (necessarily simple) poles of m and $1/m$. Thus, $\mathbb{L}_Q$ has one positive bound state $+1$ (the pole of $m(\lambda)$) and one negative bound state $-\kappa^2$ (the zero of $m(\lambda)$). Clearly (3.6) holds. The fact about the absolutely continuous spectrum also follows from the general theory (as well as from (b) below) and therefore (a) is proven.

Turn to (b). By a direct computation one verifies that

$$f_\pm(x, k) = \left\{1 \pm \left(\frac{\mathrm{e}^{\pm\mathrm{i}x}}{k+1} - \frac{\mathrm{e}^{\mp\mathrm{i}x}}{k-1}\right)\frac{\rho \sin x}{1 + \rho|x| - \left(\rho/2\right)\sin 2|x|}\right\}\mathrm{e}^{\pm\mathrm{i}kx},$$

$$\pm x \geqslant 0,$$

solve the Schrödinger equation $\mathbb{L}_Q f = k^2 f$ for $\pm x \geqslant 0$ if $k \neq \pm 1$. Since clearly

$$f_\pm (x, k) = (1 + o(1)) \, \mathrm{e}^{\pm \mathrm{i} k x}, \quad x \to \pm\infty,$$

we can claim that $f_\pm$ are Jost solution corresponding to $\pm\infty$. By the general formulas (see e.g. [18])

$$T(k) = \frac{1}{f_-(k) f_+(k)} \frac{2\mathrm{i}k}{m_+\left(k^2\right) + m_-\left(k^2\right)} \quad \text{(transmission coefficient)}, \tag{3.10}$$

$$R(k) = -\frac{\overline{f_+(k)}}{f_+(k)} \frac{\overline{m_+\left(k^2\right)} + m_-\left(k^2\right)}{m_+\left(k^2\right) + m_-\left(k^2\right)} \quad \text{(right reflection coefficient)}, \tag{3.11}$$

$$L(k) = -\frac{\overline{f_-(k)}}{f_-(k)} \frac{m_+\left(k^2\right) + \overline{m_-\left(k^2\right)}}{m_+\left(k^2\right) + m_-\left(k^2\right)} \quad \text{(left reflection coefficient)} \tag{3.12}$$

and $f_\pm(k) := f_\pm(0, k)$ are Jost functions. Since in our case $m_\pm = m$ and $f_\pm(k) = 1$, we immediately see that $L = R$ and arrive at (3.8).

It remains to demonstrate (3.7). Recall the general fact (see, for example, [3]) that for any short-range q

$$\operatorname{Res}(T, \mathrm{i}\kappa_n) = \mathrm{i}(-1)^{n-1} \sqrt{c_n^+ c_n^-}, \tag{3.13}$$

where $c_n^\pm$ are right/left norming constant associated with the bound states $-\kappa_n^2$ $(n = 1, 2, \ldots)$ enumerated in the increasing order. If q is even then $c_n^+ = c_n^- = c_n$ and hence in our case of a single bound state $-\kappa^2$ we have

$$\operatorname{Res}(T, \mathrm{i}\kappa) = \mathrm{i}c$$

and the first equation in (3.7) follows. The second and third equations in (3.7) can be verified by a direct computation. □

Remark 3.2. Same way as we did in the proof, one can find an analog of theorem 3.1 for the truncated potentials $Q|\mathbb{R}_\pm$. There will be no positive bound state but the formulas (3.10)–(3.12) immediately yield same (3.9) where 2ρ is replaced with ρ. Indeed, for $Q|\mathbb{R}_+$ we have

$$m_+\left(k^2\right) = m\left(k^2\right) = \mathrm{i}k + \frac{2\rho}{1 - k^2}, \quad m_-\left(k^2\right) = \mathrm{i}k, \quad f_\pm(k) = 1,$$

and the claim follows. Moreover, (3.7) also holds for the truncated Q with the same substitution. This demonstrates clearly that the standard triple (R, κ, c) no longer constitutes scattering data.

3.2. Short-range approximation of Q

The simples short range approximation is based upon a truncation but the limiting procedure will not be simple. We instead approximate the scattering data. While much more complicated than truncation, the limiting procedure becomes easier to track.

We start with an elementary observation that will be repeatedly used.

Lemma 3.3. *The cubic equation*

$$k^3 - k + \mathrm{i}\delta = 0, \quad \delta > 0$$

has one purely imaginary solution $k_0 \in \mathbb{C}^+$ and two symmetric solutions $k_\pm \in \mathbb{C}^-$ obeying $k_- = -\overline{k_+}$. Moreover,

$$k_0 = \delta + O\left(\delta^2\right), \qquad k_\pm = \pm 1 - \mathrm{i}\delta/2 + O\left(\delta^2\right), \quad \delta \to 0. \tag{3.14}$$

If you recall the famous characterization due to Marchenko of the scattering matrix [27] of a short-range potential, one of the conditions is that $T(k)$ can vanish on $\overline{\mathbb{C}^+}$ only at $k = 0$. But in our case this occurs if $P(k) = 0$ which happens also for $k = \pm 1$. This prompts us to replace $P(k)$ in $T(k)$ given by (3.9) with $P(k) + \mathrm{i}\varepsilon$ with some small $\varepsilon > 0$. By lemma 3.3, $P(k) + \mathrm{i}\varepsilon$ has three zeros $\mathrm{i}\nu_\varepsilon \in \mathbb{C}^+$ and $\mu_\varepsilon, -\overline{\mu_\varepsilon} \in \mathbb{C}^-$. Form now the Blaschke product B^ε with zeros $z_{-1} = -\mu_\varepsilon, z_1 = \overline{\mu}_\varepsilon, z_0 = \mathrm{i}\nu_\varepsilon$ (all of course in $\mathbb{C}^+$). That is,

$$B^\varepsilon = b_{-1} b_0 b_1, \quad b_n(k) = \frac{k - z_n}{k - \overline{z_n}} .$$

It follows from (3.14) that as $\varepsilon \to 0$

$$z_n = n + \mathrm{i}\varepsilon/2^{|n|} + O\left(\varepsilon^2\right), \quad n = 0, \pm 1.$$

The Blaschke product B^ε will be a building block in our approximation. Apparently, $B^\varepsilon \to 1$ as $\varepsilon \to 0$ uniformly on compacts in $\mathbb{C}^+$ and a.e. on $\mathbb{R}$ (but not uniformly). We are now ready to present our approximation.

Theorem 3.4. *Let*

$$T_\varepsilon(k) = \frac{(P(k) + \mathrm{i}\varepsilon)^2 / b_0^2(k)}{P(k) + \mathrm{i}\rho(1 + a)} \frac{1}{P(k) + \mathrm{i}\rho(1 - a)},$$

$$R_\varepsilon(k) = \frac{-2\mathrm{i}a\rho}{P(k) + \mathrm{i}\rho(1 + a)} \frac{P(k)}{P(k) + \mathrm{i}\rho(1 - a)} \frac{1}{B^\varepsilon(k)},$$

$$a := \sqrt{1 - \left(\varepsilon/\rho\right)^2}, \tag{3.15}$$

with some $0 < \varepsilon < \rho$. Then

(*a*) *The matrix*

$$S_\varepsilon = \begin{pmatrix} T_\varepsilon & R_\varepsilon \\ R_\varepsilon & T_\varepsilon \end{pmatrix}$$

is the scattering matrix of a short-range potential having two bound states $-\left(\kappa_\pm^\varepsilon\right)^2, \kappa_+^\varepsilon > \kappa_-^\varepsilon$, subject to

$$\kappa_+^\varepsilon = \kappa + O\left(\varepsilon^2\right), \qquad \kappa_-^\varepsilon = O\left(\varepsilon^2\right), \quad \varepsilon \to 0. \tag{3.16}$$

(*b*) *If we choose the left and right norming constants associated with $-\left(\kappa_\pm^\varepsilon\right)^2$ equal to each other and to satisfy*

$$c_\pm^\varepsilon = \mp \mathrm{i} \operatorname{Res}\left(T_\varepsilon, \mathrm{i}\kappa_\pm^\varepsilon\right), \tag{3.17}$$

then the unique potential $Q_\varepsilon(x)$ corresponding to the scattering data

$$\left\{R_\varepsilon, \kappa_\pm^\varepsilon, c_\pm^\varepsilon\right\}$$

is even and everywhere

$$Q_\varepsilon(x) \to Q(x), \quad \varepsilon \to 0. \tag{3.18}$$

Proof. To prove part 1 it is enough to show that Marchenko's characterization applies. This amounts to checking eight conditions if we use it in the form given in [29]. We start with showing that S_ε is a unitary matrix. That is to verify that on the real line $T_\varepsilon\overline{R}_\varepsilon + \overline{T}_\varepsilon R_\varepsilon = 0$ and $|T_\varepsilon|^2 + |R_\varepsilon|^2 = 1$. We only show the first one as it is more involved. It follows from (3.15) that

$$\begin{aligned}T_\varepsilon\overline{R}_\varepsilon + \overline{T}_\varepsilon R_\varepsilon &= 2\,\mathrm{Re}\,T_\varepsilon\overline{R}_\varepsilon \\ &= 2\,\mathrm{Re}\,\frac{(P+\mathrm{i}\varepsilon)^2/b_0^2}{|P+\mathrm{i}\rho(1+a)|^2}\frac{2\mathrm{i}a\rho}{|P+\mathrm{i}\rho(1-a)|^2}\frac{P}{\overline{B^\varepsilon}} \\ &= -\frac{4a\rho P}{|P+\mathrm{i}\rho(1+a)|^2|P+\mathrm{i}\rho(1-a)|^2}\,\mathrm{Im}\left(\frac{P+\mathrm{i}\varepsilon}{b_0^2}\right)^2 B^\varepsilon,\end{aligned}$$

where we have used the facts that on the real line P is real and $\overline{B^\varepsilon} = 1/B^\varepsilon$. But

$$\begin{aligned}P(k) + \mathrm{i}\varepsilon = k^3 - k + \mathrm{i}\varepsilon &= (k-\mu_\varepsilon)(k+\overline{\mu_\varepsilon})(k-\mathrm{i}\nu_\varepsilon) \\ &= (k+z_1)(k+z_{-1})(k-z_0)\end{aligned}$$

and hence (recalling that $z_{-1} = -\overline{z_1}, z_0 = -\overline{z_0}$)

$$\begin{aligned}&\mathrm{Im}\,(P+\mathrm{i}\varepsilon)^2 B^\varepsilon/b_0^2 \\ &\quad= \mathrm{Im}\,(k+z_1)^2(k+z_{-1})^2(k-z_0)^2\frac{k-z_1}{k-\overline{z_1}}\frac{k-z_{-1}}{k-\overline{z_{-1}}}\frac{k-z_0}{k-\overline{z_0}}\left(\frac{k-\overline{z_0}}{k-z_0}\right)^2 \\ &\quad= \mathrm{Im}\,(k+z_1)^2(k+z_{-1})^2(k-z_0)^2\frac{k-z_1}{k+z_{-1}}\frac{k-z_{-1}}{k+z_1}\frac{k-z_0}{k-\overline{z_0}}\left(\frac{k-\overline{z_0}}{k-z_0}\right)^2 \\ &\quad= \mathrm{Im}\,(k+z_1)(k-z_1)(k+z_{-1})(k-z_{-1})(k-z_0)(k-\overline{z_0}) \\ &\quad= |k-z_0|^2\,\mathrm{Im}\left(k^2-z_1^2\right)\left(k-z_{-1}^2\right) = |k-z_0|^2\,\mathrm{Im}\left(k^2-z_1^2\right)\left(k-\overline{z}_1^2\right) \\ &\quad= |k-z_0|^2\,\mathrm{Im}\left|k^2-z_1^2\right|^2 = 0.\end{aligned}$$

Thus $T_\varepsilon\overline{R}_\varepsilon + \overline{T}_\varepsilon R_\varepsilon = 0$. One now easily checks the symmetry property: $S_\varepsilon(-k) = \overline{S_\varepsilon(k)}$. Indeed,

$$\begin{aligned}T_\varepsilon(-k) &= \frac{(P(-k)+\mathrm{i}\varepsilon)^2/b_0^2(-k)}{P(-k)+\mathrm{i}\rho(1+a)}\frac{1}{P(-k)+\mathrm{i}\rho(1-a)} \\ &= \frac{(-P(k)+\mathrm{i}\varepsilon)^2/\overline{b_0^2(k)}}{-P(k)+\mathrm{i}\rho(1+a)}\frac{1}{-P(k)+\mathrm{i}\rho(1-a)} \\ &= \frac{(P(k)-\mathrm{i}\varepsilon)^2/\overline{b_0^2(k)}}{P(k)-\mathrm{i}\rho(1+a)}\frac{1}{P(k)-\mathrm{i}\rho(1-a)} \\ &= \overline{\frac{(P(k)+\mathrm{i}\varepsilon)^2/b_0^2(k)}{P(k)+\mathrm{i}\rho(1+a)}\frac{1}{P(k)+\mathrm{i}\rho(1-a)}} = \overline{T_\varepsilon(k)}\end{aligned}$$

and similarly one checks $R_\varepsilon(-k) = \overline{R_\varepsilon(k)}$. Next, by construction, $T_\varepsilon(k)$ is a rational function with simple poles in $\mathbb{C}^+$ determined by

$$P(k) + \mathrm{i}\rho(1 \pm a) = 0. \tag{3.19}$$

Note that z_0, the zero of b_0, is a removable singularity as by the very construction $P(z_0) + \mathrm{i}\varepsilon = 0$. By lemma 3.3 each equation (3.19) has only one (imaginary) solution $\mathrm{i}\kappa_\pm^\varepsilon$ in $\mathbb{C}^+$ and thus $T_\varepsilon(k)$ has two simple imaginary poles $\mathrm{i}\kappa_\pm^\varepsilon$ in $\mathbb{C}^+$. Since also $T_\varepsilon(k) = 1 + O(1/k)$, $k \to \infty$, and has no real zeros, we assert that $T_\varepsilon(k)$ is subject to the conditions of Marchenko's characterization.

Turn now to $R_\varepsilon(k)$. Clearly $R_\varepsilon(k) = O(1/k^3)$ and is a rational function. Therefore one concludes by parts that

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kx} R_\varepsilon(k)\,\mathrm{d}k = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kx}\,\mathrm{i}kR_\varepsilon(k)\,\mathrm{d}k = O\left(\frac{1}{x^3}\right), \quad x \to \pm\infty. \tag{3.20}$$

The latter yields that $\frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kx} R_\varepsilon(k)\,\mathrm{d}k$ is in L^1 along with its first moment which completes verification of all conditions of Marchenko's characterization and the set $\{R_\varepsilon, \kappa_\pm^\varepsilon, c_\pm\}$ is the scattering data for some short-range potential $q_\varepsilon(x)$ for any $c_\pm > 0$. Since by the general theory, $-(\kappa_\pm^\varepsilon)^2$ are bound states and part 1 is proven.

Turn now to part 2. Consider the reflection coefficient R_ε. It follows from (3.15) that

$$R_\varepsilon(k) = -2\mathrm{i}a\rho\left(\frac{b_0(k)}{P(k) + \mathrm{i}\varepsilon}\right)^2 \frac{P(k)}{B^\varepsilon(k)} T_\varepsilon(k).$$

Apparently R_ε has five simple poles: two imaginary poles $\mathrm{i}\kappa_\pm^\varepsilon$ are shared with T_ε plus z_n, $n = 0, \pm 1$, the zeros of $B^\varepsilon(k)$. Note, as we have observed, $P(k) + \mathrm{i}\varepsilon$ has only one zero z_0 in $\mathbb{C}^+$, which is the zero of b_0, and the squared factor in the equation above produces no poles in $\mathbb{C}^+$. One sees

$$\operatorname{Res}\left(R_\varepsilon, \mathrm{i}\kappa_\pm^\varepsilon\right) = -2\mathrm{i}a\rho\left(\frac{b_0\left(\mathrm{i}\kappa_\pm^\varepsilon\right)}{P\left(\mathrm{i}\kappa_\pm^\varepsilon\right) + \mathrm{i}\varepsilon}\right)^2 \frac{P\left(\mathrm{i}\kappa_\pm^\varepsilon\right)}{B^\varepsilon\left(\mathrm{i}\kappa_\pm^\varepsilon\right)} \operatorname{Res}\left(T_\varepsilon, \mathrm{i}\kappa_\pm^\varepsilon\right). \tag{3.21}$$

Recalling that $P(k) + \mathrm{i}\varepsilon = (k + z_1)(k + z_{-1})(k - z_0)$, one has ($k$ is real)

$$P(k) - \mathrm{i}\varepsilon = (k - z_1)(k - z_{-1})(k + z_0)$$
$$P(k) + \mathrm{i}\varepsilon = (k - \overline{z_1})(k - \overline{z_{-1}})(k - z_0)$$

and hence

$$b_0(k)^2/B^\varepsilon(k) = \frac{(k - \overline{z_1})(k - \overline{z_{-1}})(k - z_0)}{(k - z_1)(k - z_{-1})(k + z_0)} = \frac{P(k) + \mathrm{i}\varepsilon}{P(k) - \mathrm{i}\varepsilon}$$

Therefore, since $P\left(\mathrm{i}\kappa_\pm^\varepsilon\right) + \mathrm{i}\rho(1 \pm a) = 0$ one obtains

$$\left(\frac{b_0\left(\mathrm{i}\kappa_\pm^\varepsilon\right)}{P\left(\mathrm{i}\kappa_\pm^\varepsilon\right) + \mathrm{i}\varepsilon}\right)^2 \frac{P\left(\mathrm{i}\kappa_\pm^\varepsilon\right)}{B^\varepsilon\left(\mathrm{i}\kappa_\pm^\varepsilon\right)} = \frac{P\left(\mathrm{i}\kappa_\pm^\varepsilon\right)}{\left(P\left(\mathrm{i}\kappa_\pm^\varepsilon\right) + \mathrm{i}\varepsilon\right)\left(P\left(\mathrm{i}\kappa_\pm^\varepsilon\right) - \mathrm{i}\varepsilon\right)} = \frac{\mathrm{i}\rho(1 \pm a)}{\rho^2(1 \pm a)^2 - \varepsilon^2}.$$

Substituting this into (3.21), yields

$$\begin{aligned}
\operatorname{Res}\left(R_\varepsilon, \mathrm{i}\kappa_\pm^\varepsilon\right) &= -2\mathrm{i}a\rho\frac{\mathrm{i}\rho\,(1\pm a)}{\rho^2(1\pm a)^2-\varepsilon^2}\operatorname{Res}\left(T_\varepsilon, \mathrm{i}\kappa_\pm^\varepsilon\right)\\
&= \frac{2a\rho^2\,(1\pm a)}{\rho^2(1\pm a)^2-\varepsilon^2}\operatorname{Res}\left(T_\varepsilon, \mathrm{i}\kappa_\pm^\varepsilon\right)\\
&= \frac{2a\rho^2\,(1\pm a)}{\rho^2(1\pm a)^2-\rho^2+\rho^2a^2}\operatorname{Res}\left(T_\varepsilon, \mathrm{i}\kappa_\pm^\varepsilon\right)\\
&= \frac{2a\,(1\pm a)}{(1\pm a)^2-(1-a)\,(1+a)}\operatorname{Res}\left(T_\varepsilon, \mathrm{i}\kappa_\pm^\varepsilon\right)\\
&= \frac{2a}{(1\pm a)-(1\mp a)}\operatorname{Res}\left(T_\varepsilon, \mathrm{i}\kappa_\pm^\varepsilon\right) = \pm\operatorname{Res}\left(T_\varepsilon, \mathrm{i}\kappa_\pm^\varepsilon\right).
\end{aligned}$$

Here we have noticed that it follows from (3.15) $\varepsilon^2=\rho^2-\rho^2a^2$. Thus, due to (3.17) we have

$$\operatorname{Res}\left(R_\varepsilon, \mathrm{i}\kappa_\pm^\varepsilon\right) = \pm\operatorname{Res}\left(T_\varepsilon, \mathrm{i}\kappa_\pm^\varepsilon\right) = \mathrm{i}c_\pm^\varepsilon. \tag{3.22}$$

We now solve the inverse scattering problem for the data

$$\left\{R_\varepsilon, \kappa_\pm^\varepsilon, -\mathrm{i}\operatorname{Res}\left(R_\varepsilon, \mathrm{i}\kappa_\pm^\varepsilon\right)\right\},$$

based upon our Hankel operator approach [15]. To this end, form the symbol

$$\varphi_x^\varepsilon(k) = \frac{-\operatorname{Res}\left(R_\varepsilon, \mathrm{i}\kappa_+^\varepsilon\right)}{k-\mathrm{i}\kappa_+^\varepsilon}\,\mathrm{e}^{-2\kappa_+^\varepsilon x} + \frac{-\operatorname{Res}\left(R_\varepsilon, \mathrm{i}\kappa_-^\varepsilon\right)}{k-\mathrm{i}\kappa_-^\varepsilon}\,\mathrm{e}^{-2\kappa_-^\varepsilon x} + R_\varepsilon(k)\,\mathrm{e}^{2\mathrm{i}kx}. \tag{3.23}$$

One immediately sees that φ_x^ε is subject to the conditions of lemma 2.2 with three (symmetric) poles $z_n, n=0,\pm1$. By condition, the left and right scattering data are identical and hence Q_ε must be even and it enough to recover it only on $(0,\infty)$. Therefore we can assume that $x>0$ in (3.23) which by corollary 2.4 implies that the Φ-part of our symbol is zero. By lemma 2.2

$$\mathbb{H}(\varphi_x^\varepsilon) = \sum_{-1\leqslant n\leqslant 1}\mathrm{i}\operatorname{Res}\left(\varphi_x^\varepsilon, z_{-n}\right)\langle\cdot, k_{z_{-n}}\rangle\, k_{z_n}.$$

Thus, our Hankel operator is rank 3 and by the Dyson formula [15] we have

$$Q_\varepsilon(x) = -2\partial_x^2\log\det\left(I+\mathbb{H}(\varphi_x^\varepsilon)\right),\quad x>0.$$

Note that our Q_ε can be explicitly evaluated. We however do not really need it. We will take the limit as $\varepsilon\to0$ in the next section. □

We emphasize that part 2 of theorem 3.4 is essential because, due to non-uniqueness, it is *a priori* unclear if our approximations indeed converges to the original potential.

Remark 3.5. Note, that each $Q_\varepsilon(x)$ has the property that $T_\varepsilon(0)=-1$. Such potentials are called exceptional (or resonant) because generically $T(0)=0$. On the other hand for the limiting potential $Q(x)$ we obviously have $T(0)=0$. This means that $S_\varepsilon(k)\to S(k)$ not uniformly which may look disturbing as theorem 3.4 nevertheless claims that $Q_\varepsilon(x)\to Q(x)$ everywhere. As the reader will see in the next section, uniformity of $S_\varepsilon(k)\to S(k)$ is not to be expected and in fact dealing with this circumstance is the main point of this paper.

Remark 3.6. Continuing in (3.20) integration by parts, we see that the derivative of the Fourier transform of R_ε decays faster than any power (in fact exponentially). This means [27] that $Q_\varepsilon(x)$ decays as fast.

4. Main results

Through this section

$$\xi_{x,t}(k) = \exp\{\mathrm{i}(8k^3t + 2kx)\}.$$

Theorem 4.1. *Let Q be the initial condition (3.4) in the KdV equation (1.1),*

$$\varphi_{x,t}(k) = R(k)\,\xi_{x,t}(k) - \frac{\operatorname{Res}\left(R\xi_{x,t}, \mathrm{i}\kappa\right)}{k - \mathrm{i}\kappa},$$

and $\mathbb{H}_{x,t} := \mathbb{H}\left(\varphi_{x,t}\right)$, the associated Hankel operator. Then (1.1) has the (unique) classical solution given by

$$u(x,t) = u_0(x,t) + u_1(x,t) \tag{4.1}$$

where

$$u_0(x,t) = -2\partial_x^2 \log \det\{I + \mathbb{H}_{x,t}\}, \tag{4.2}$$

and

$$u_1(x,t) = -2\partial_x^2 \log \tau(x,t),$$

$$\tau(x,t) = 1 + \rho(x + 12t) - \frac{\rho}{2}\sin(2x + 8t) + \frac{\rho}{2}\operatorname{Re}\left(I + \mathbb{H}_{x,t}\right)^{-1}\left(\mathbb{H}_{x,t}k_{1+\mathrm{i}0} - \xi_{x,t}(1)\,\mathbb{H}_{x,t}k_{-1+\mathrm{i}0}\right)\Big|_{1+\mathrm{i}0}.$$

Here, as before, $k_\lambda(s) = \frac{\mathrm{i}}{s-\overline{\lambda}}$ is the reproducing kernel.

Proof. Since our approximation $Q_\varepsilon(x)$ decays exponentially, the (classical) solution to the KdV equation can be found in closed form by Dyson's formula

$$Q_\varepsilon(x,t) = -2\partial_x^2 \log \det\left(I + \mathbb{H}\left(\varphi_{x,t}^\varepsilon\right)\right), \tag{4.3}$$

where

$$\varphi_{x,t}^\varepsilon(k) = \frac{-\operatorname{Res}\left(R_\varepsilon, \mathrm{i}\kappa_+^\varepsilon\right)}{k - \mathrm{i}\kappa_+^\varepsilon}\xi_{x,t}\left(\mathrm{i}\kappa_+^\varepsilon\right) + \frac{-\operatorname{Res}\left(R_\varepsilon, \mathrm{i}\kappa_-^\varepsilon\right)}{k - \mathrm{i}\kappa_-^\varepsilon}\xi_{x,t}\left(\mathrm{i}\kappa_-^\varepsilon\right) + R_\varepsilon(k)\,\xi_{x,t}(k). \tag{4.4}$$

Since according to the Bourgain theorem [7] the KdV equation is well-posed in L^2, the limit $\lim_{\varepsilon\to 0} Q_\varepsilon(x,t)$ does exist. However we cannot pass to the limit in (4.3) under the determinant sign since, as we will see later, $\mathbb{H}\left(\varphi_{x,t}^\varepsilon\right)$ does not converge in the trace norm to $\mathbb{H}\left(\varphi_{x,t}\right)$, where

$$\varphi_{x,t}(k) = \frac{-\operatorname{Res}(R, \mathrm{i}\kappa)}{k - \mathrm{i}\kappa}\xi_{x,t}(\mathrm{i}\kappa) + R(k)\,\xi_{x,t}(k).$$

To work around this circumstance we split our determinant as follows. Consider

$$K_\varepsilon = \operatorname{span}\{k_{z_n}\}_{n=-1}^{1},$$

and decompose H^2 into the orthogonal sum (see subsection 2.2)

$$H^2 = K_\varepsilon \oplus K_\varepsilon^\perp, \quad K_\varepsilon^\perp = B^\varepsilon H^2. \tag{4.5}$$

The decomposition (4.5) induces the block representation

$$\mathbb{H}\left(\varphi_{x,t}^\varepsilon\right) = \begin{pmatrix} \mathbb{H}_0 & \mathbb{H}_{01} \\ \mathbb{H}_{01}^* & \mathbb{H}_1 \end{pmatrix},$$

where

$$\mathbb{H}_0 := \mathbb{P}_{B^\varepsilon}\mathbb{H}\left(\varphi_{x,t}^\varepsilon\right)\mathbb{P}_{B^\varepsilon}, \quad \mathbb{H}_1 = \mathbb{P}_{B^\varepsilon}^\perp\mathbb{H}\left(\varphi_{x,t}^\varepsilon\right)\mathbb{P}_{B^\varepsilon}^\perp$$
$$\mathbb{H}_{01} := \mathbb{P}_{B^\varepsilon}^\perp\mathbb{H}\left(\varphi_{x,t}^\varepsilon\right)\mathbb{P}_{B^\varepsilon}, \quad \mathbb{H}_{01}^* = \mathbb{P}_{B^\varepsilon}\mathbb{H}\left(\varphi_{x,t}^\varepsilon\right)\mathbb{P}_{B^\varepsilon}^\perp,$$

and

$$\varphi_{x,t}^\varepsilon(k) = \frac{-\operatorname{Res}\left(R_\varepsilon, \mathrm{i}\kappa_+^\varepsilon\right)}{k - \mathrm{i}\kappa_+^\varepsilon}\xi_{x,t}\left(\mathrm{i}\kappa_+^\varepsilon\right) + \frac{-\operatorname{Res}\left(R_\varepsilon, \mathrm{i}\kappa_-^\varepsilon\right)}{k - \mathrm{i}\kappa_-^\varepsilon}\xi_{x,t}\left(\mathrm{i}\kappa_-^\varepsilon\right)$$
$$+ R_\varepsilon(k)\,\xi_{x,t}(k).$$

Examine the block $\mathbb{H}_1$ first. It follows from (4.4) that the poles of $\varphi_{x,t}^\varepsilon$ coincide with zeros (z_n) of B^ε and therefore by lemma 2.2 $(h > \kappa_+^\varepsilon)$

$$\begin{aligned}\mathbb{H}\left(\varphi_{x,t}^\varepsilon\right) &= \sum_{-1\leqslant n\leqslant 1} \mathrm{i}\operatorname{Res}\left(\xi_{x,t}R_\varepsilon, z_{-n}\right)\langle\cdot, k_{z_{-n}}\rangle\, k_{z_n} \\ &\quad + \int_{\mathbb{R}+\mathrm{i}h}\frac{\mathrm{d}z}{2\pi}\varphi_{x,t}^\varepsilon(z)\,\langle\cdot, k_z\rangle\, k_{-\overline{z}} \\ &= \sum_{-1\leqslant n\leqslant 1} \mathrm{i}\operatorname{Res}\left(\xi_{x,t}R_\varepsilon, z_{-n}\right)\langle\cdot, k_{z_{-n}}\rangle\, k_{z_n} + \mathbb{H}\left(\Phi_{x,t}^\varepsilon\right).\end{aligned}$$

One immediately sees that

$$\mathbb{H}_1 = \mathbb{P}_{B^\varepsilon}^\perp\mathbb{H}\left(\Phi_{x,t}^\varepsilon\right)\mathbb{P}_{B^\varepsilon}^\perp.$$

Since $\varphi_{x,t}^\varepsilon \to \varphi_{x,t}$ uniformly on $\mathbb{R} + \mathrm{i}h$, we obviously have

$$\begin{aligned}\Phi_{x,t}^\varepsilon(s) &= -\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}+\mathrm{i}h}\frac{\varphi_{x,t}^\varepsilon(z)}{z-s}\,\mathrm{d}z \\ &\to -\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}+\mathrm{i}h}\frac{\varphi_{x,t}(z)}{z-s}\,\mathrm{d}z = \Phi_{x,t}(s), \quad \varepsilon \to 0,\end{aligned}$$

in $C^n(\mathbb{R})$ for any n which in turn implies [35] that $\lim_{\varepsilon\to 0}\mathbb{H}\left(\Phi_{x,t}^\varepsilon\right) = \mathbb{H}\left(\Phi_{x,t}\right)$ in the trace norm (in fact in all $\mathfrak{S}_p$, $p > 0$). Since

$$\varphi_{x,t}(k) = \frac{-\operatorname{Res}(R, \mathrm{i}\kappa)}{k - \mathrm{i}\kappa}\xi_{x,t}(\mathrm{i}\kappa) + R(k)\,\xi_{x,t}(k),$$

we see that $\mathrm{i}\kappa$ is a removable singularity for $\varphi_{x,t}$ and hence by corollary 2.3

$$\mathbb{H}\left(\Phi_{x,t}\right)=\mathbb{H}\left(\varphi_{x,t}\right).$$

Since $B^{\varepsilon}\to 1$ a.e., it follows from (2.7) that in the strong operator topology

$$\mathbb{P}^{\perp}_{B^{\varepsilon}}=B^{\varepsilon}\mathbb{P}_{+}\overline{B^{\varepsilon}}\to I,\quad \varepsilon\to 0. \tag{4.6}$$

But [6], if $H_n\to H$ in trace norm, A_n is self-adjoint, $\sup_n\|A_n\|<\infty$, and $A_n\to A$ strongly, then $A_nH_nA_n\to AHA$ in trace norm. Therefore, we can conclude that in trace norm

$$\mathbb{H}_1\to\mathbb{H}\left(\varphi_{x,t}\right),\quad \varepsilon\to 0. \tag{4.7}$$

We now make use of a well-known formula from matrix theory:

$$\det\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}=\det A_{11}\ \det\left(A_{22}-A_{21}A_{11}^{-1}A_{12}\right), \tag{4.8}$$

which yields

$$\begin{aligned}&\det\left(I+\mathbb{H}\left(\varphi^{\varepsilon}_{x,t}\right)\right)\\ &\quad=\det\left\{I+\mathbb{H}_1\right\}\cdot\det\left\{I+\mathbb{H}_0-\mathbb{H}^{*}_{01}(I+\mathbb{H}_1)^{-1}\mathbb{H}_{01}\right\}.\end{aligned} \tag{4.9}$$

Our goal is to study what happens to (4.9) as $\varepsilon\to 0$. The determinants on the right-hand side of (4.9) behave very differently and we treat them separately. It follows from (4.7) that

$$\lim_{\varepsilon\to 0}\det\left\{I+\mathbb{H}_1\right\}=\det\left\{I+\mathbb{H}\left(\varphi_{x,t}\right)\right\}. \tag{4.10}$$

Turn now to the second determinant in (4.9). It is clearly a 3×3 determinant. We are going to show that, in fact, this determinant vanishes as $O(\varepsilon)$. To this end, we explicitly evaluate it in the basis $\left(k_{z_n}\right)$

$$\begin{aligned}&\det\left\{I+\mathbb{H}_0-\mathbb{H}^{*}_{01}(I+\mathbb{H}_1)^{-1}\mathbb{H}_{01}\right\}\\ &\quad=\det\begin{pmatrix}1+h_{-1-1}+\overline{d_{11}} & h_{-10}+d_{-10} & h_{-11}+d_{-11}\\ \overline{h_{0-1}}+\overline{d_{0-1}} & 1+h_{00}+d_{00} & h_{01}+d_{01}\\ \overline{h_{-11}}+\overline{d_{-11}} & h_{10}+d_{10} & 1+\overline{h_{-1-1}}+d_{11}\end{pmatrix},\end{aligned} \tag{4.11}$$

where h_{mn} and d_{mn} are the matrix entries of

$$\sum_{-1\leqslant n\leqslant 1}\mathrm{i}\,\mathrm{Res}\left(\xi_{x,t}R_{\varepsilon},z_{-n}\right)\langle\cdot,k_{z_{-n}}\rangle\,k_{z_n}$$

and

$$\mathbb{P}_B\mathbb{H}\left(\Phi^{\varepsilon}_{x,t}\right)\mathbb{P}_B-\mathbb{H}^{*}_{01}(I+\mathbb{H}_1)^{-1}\mathbb{H}_{01}$$

respectively. By lemma 2.1

$$\begin{aligned} h_{mn} &= \left\langle \sum_{-1\leqslant j\leqslant 1} \mathrm{i}\,\mathrm{Res}\left(\xi_{x,t}R_\varepsilon, z_{-j}\right)\left\langle k_{z_n}, k_{z_{-j}}\right\rangle k_{z_j}, k_{z_m}^{\perp}\right\rangle \\ &= \mathrm{i}\,\mathrm{Res}\left(\xi_{x,t}R_\varepsilon, z_{-m}\right)\left\langle k_{z_n}, k_{z_{-m}}\right\rangle = \mathrm{i}\,\mathrm{Res}\left(\xi_{x,t}R_\varepsilon, z_{-m}\right) k_{z_n}\left(z_{-m}\right) \\ &= \frac{\xi_{x,t}\left(z_{-m}\right)\mathrm{Res}\left(R_\varepsilon, z_{-m}\right)}{\overline{z_m}+\overline{z_n}}. \end{aligned} \tag{4.12}$$

Incidentally, (4.12) implies $h_{1-1} = \overline{h_{-11}}, h_{-1-1} = \overline{h_{11}}$. Recall that z_n are chosen so that $P(z_n) - \mathrm{i}\varepsilon = 0$ if $n = \pm 1$ and $P(z_n) + \mathrm{i}\varepsilon = 0$ if $n = 0$. Rewriting (3.15) as

$$R_\varepsilon(k) = aR(k)\,\frac{P(k)+2\mathrm{i}\rho}{P(k)+\mathrm{i}\rho(1+a)}\,\frac{P(k)}{P(k)+\mathrm{i}\rho(1-a)}\,\frac{1}{B^\varepsilon(k)},$$

for the residues we then have

$$\mathrm{Res}\left(R_\varepsilon, z_n\right) = aR(z_n)\,\frac{P(z_n)+2\mathrm{i}\rho}{P(z_n)+\mathrm{i}\rho(1+a)}\,\frac{P(z_n)}{P(z_n)+\mathrm{i}\rho(1-a)}\,\frac{2\mathrm{i}\,\mathrm{Im}\,z_n}{B_n^\varepsilon(z_n)}.$$

One now readily verifies that

$$\begin{aligned} \frac{P(z_n)+2\mathrm{i}\rho}{P(z_n)+\mathrm{i}\rho(1+a)} &= 1 + O\left(\varepsilon^2\right), \\ \frac{P(z_n)}{P(z_n)+\mathrm{i}\rho(1-a)} &= 1 + (-1)^n\frac{\varepsilon}{2\rho} + O\left(\varepsilon^2\right), \\ B_n^\varepsilon(z_n)^{-1} &= 1 + 5\mathrm{i}n\varepsilon/2 + O\left(\varepsilon^2\right), \end{aligned}$$

and thus

$$\mathrm{Res}\left(R_\varepsilon, z_n\right) = 2\mathrm{i}\,\mathrm{Im}\,z_n R(z_n)\left[1 + \frac{\mathrm{i}\varepsilon}{2}\left(5n + (-1)^n\frac{1}{\rho}\right) + O\left(\varepsilon^2\right)\right]. \tag{4.13}$$

Inserting (4.13) into (4.12) yields

$$h_{mn} = \frac{2\mathrm{i}\,\mathrm{Im}\,z_m}{\overline{z_m}+\overline{z_n}}\,\left(\xi_{x,t}R\right)(z_{-m})\left[1 + \frac{\mathrm{i}\varepsilon}{2}\left(5m + (-1)^m\frac{1}{\rho}\right) + O\left(\varepsilon^2\right)\right].$$

Observe, that $h_{n,m} = O(\varepsilon)$ if $n \neq -m$ and $h_{m,-m}$ does not vanish as $\varepsilon \to 0$ (which is an important fact for what follows). As we will see, only h_{-11} and h_{11} matter. Recalling that $R(1) = -1$ we have

$$h_{-11} = \xi_{x,t}(1)\left\{1 - \frac{\varepsilon}{2}\left[\frac{1}{\rho} + \mathrm{i}\overline{\xi_{x,t}(1)}\left(R\xi_{x,t}\right)'(1) + 5\mathrm{i}\right] + O\left(\varepsilon^2\right)\right\}, \tag{4.14}$$

$$h_{-1-1} = \frac{\mathrm{i}\varepsilon}{2}\xi_{x,t}(1)\left[1 + O(\varepsilon)\right]. \tag{4.15}$$

Similarly, for the matrix (d_{mn}) we have

$$\begin{aligned} d_{mn} &= \left\langle \mathbb{H}\left(\Phi_{x,t}^\varepsilon\right) k_{z_n}, k_{z_m}^{\perp}\right\rangle - \left\langle (I+\ \mathbb{H}_1)^{-1}\mathbb{H}_{01}k_{z_n}, \mathbb{H}_{01}k_{z_m}^{\perp}\right\rangle \\ &= \frac{2\,\mathrm{Im}\,z_m}{B_m^\varepsilon(z_m)}\left\{\left\langle \mathbb{H}\left(\Phi_{x,t}^\varepsilon\right) k_{z_n}, B_m^\varepsilon k_{z_m}\right\rangle\right. \end{aligned}$$

$$- \left\langle (I+\mathbb{H}_1)^{-1}\mathbb{H}_{01}k_{z_n}, \mathbb{H}_{01}B_m^{\varepsilon}k_{z_m}\right\rangle\Big\} .$$
$$= \varepsilon D_{mn} + O(\varepsilon), \tag{4.16}$$

where D_{mn} will be computed later. For the determinant in (4.11) we clearly have

$$\det\left\{I+\mathbb{H}_0 - \mathbb{H}_{01}^*(I+\mathbb{H}_1)^{-1}\mathbb{H}_{01}\right\}$$
$$= (1+h_{00}+d_{00})\det\begin{pmatrix} 1+h_{-1-1}+\overline{d_{11}} & h_{-11}+d_{-11} \\ \overline{h_{-11}+d_{-11}} & 1+\overline{h_{-1-1}+d_{11}}\end{pmatrix} + O\left(\varepsilon^2\right)$$
$$= 2\left\{\left|1+h_{-1-1}+\overline{d_{11}}\right|^2 - |h_{-11}+d_{-11}|^2\right\} + O\left(\varepsilon^2\right) \text{ (by (4.15) and (4.16))}$$
$$= 2\left(1-|h_{-11}|^2 + 2\,\mathrm{Re}\,h_{-1-1}\right) + 2\varepsilon\,\mathrm{Re}\left[\overline{D_{11}} - \xi_{x,t}(1)\overline{D_{-11}}\right] + O\left(\varepsilon^2\right). \tag{4.17}$$

Evaluate each term in the right-hand side of (4.17) separately. By (4.14) and (4.15) one has

$$1-|h_{-11}|^2 + 2\,\mathrm{Re}\,h_{-1-1}$$
$$= \varepsilon\left\{1/\rho + \mathrm{Re}\,\mathrm{i}\overline{\xi_{x,t}(1)}\left[\left(R\xi_{x,t}\right)'(1) - 1\right] + O(\varepsilon)\right\}$$
$$= \frac{2\varepsilon}{\rho}\left\{1+\rho(x+12t) - \frac{\rho}{2}\sin(2x+8t) + O(\varepsilon)\right\} \tag{4.18}$$

and

$$D_{mn} = \lim_{\varepsilon\to 0}\left\{\left\langle \mathbb{H}\left(\Phi_{x,t}^{\varepsilon}\right)k_{z_n}, B_m^{\varepsilon}k_{z_m}\right\rangle - \left\langle (I+\mathbb{H}_1)^{-1}\mathbb{H}_{01}k_{z_n}, \mathbb{H}_{01}B_m^{\varepsilon}k_{z_m}\right\rangle\right\}$$
$$=: D_{mn}^{(1)} + D_{mn}^{(2)}.$$

Since $\mathbb{H}\left(\Phi_{x,t}^{\varepsilon}\right)$ is a self-adjoint operator, by corollary 2.5 we have ($m=\pm 1, n=1$)

$$\overline{D_{mn}^{(1)}} = \lim_{\varepsilon\to 0}\left\langle \mathbb{H}\left(\Phi_{x,t}^{\varepsilon}\right)B_m k_{z_m}, k_{z_n}\right\rangle$$
$$= \lim_{\varepsilon\to 0}\mathbb{H}\left(\Phi_{x,t}^{\varepsilon}\right)B_m k_{z_m}\Big|_{z_n} \text{(by(2.2))}$$
$$= K_m(n), \tag{4.19}$$

where

$$K_m(n) = -\int_{\mathbb{R}+\mathrm{i}h}\frac{\varphi_{x,t}(z)}{(z-m)(z+n)}\frac{\mathrm{d}z}{2\pi}.$$

Similarly, by (4.6), (4.7), and corollary 2.5 we have

$$\overline{D_{mn}^{(2)}} = -\mathbb{H}\left(\varphi_{x,t}\right)\left(I+\mathbb{H}\left(\varphi_{x,t}\right)\right)^{-1}K_m\Big|_{n+\mathrm{i}0}. \tag{4.20}$$

Therefore, combining (4.19) and (4.20) we have

$$\overline{D_{mn}} = K_m(n) - \mathbb{H}\left(\varphi_{x,t}\right)\left(I+\mathbb{H}\left(\varphi_{x,t}\right)\right)^{-1}K_m\Big|_{n+\mathrm{i}0}$$
$$= \left(I+\mathbb{H}\left(\varphi_{x,t}\right)\right)^{-1}K_m\Big|_{n+\mathrm{i}0}.$$

Substituting this and (4.18) into (4.17) yields

$$\frac{\rho}{4\varepsilon}\det\left\{I+\mathbb{H}_0-\mathbb{H}_{01}^{*}(I+\mathbb{H}_1)^{-1}\mathbb{H}_{01}\right\}$$
$$=1+\rho(x+12t)-\sin(2x+8t)$$
$$+\frac{\rho}{2}\operatorname{Re}\left(I+\mathbb{H}\left(\varphi_{x,t}\right)\right)^{-1}\left(K_1-\xi_{x,t}(1)K_{-1}\right)\Big|_{1+\mathrm{i}0}+O(\varepsilon). \quad (4.21)$$

We have now prepared all the ingredients to find the solution to the KdV equation with the initial data Q_ε by the Dyson formula. Indeed,

$$Q_\varepsilon(x,t)=-2\partial_x^2\log\det\left\{I+\mathbb{H}\left(\varphi_{x,t}^{\varepsilon}\right)\right\}\ \ (\text{by}\,(4.17))$$
$$=2\partial_x^2\log\det(I+\mathbb{H}_1)$$
$$-2\partial_x^2\log\det\left\{I+\mathbb{H}_0-\mathbb{H}_{01}^{*}(I+\mathbb{H}_1)^{-1}\mathbb{H}_{01}\right\}$$
$$=-2\partial_x^2\log\det\left\{I+\mathbb{H}\left(\varphi_{x,t}\right)\right\}\ \ (\text{by}\,(4.10)\,\text{and}\,(4.21))$$
$$-2\partial_x^2\log\left\{1+\rho(x+12t)-\frac{\rho}{2}\sin(2x+8t)\right.$$
$$\left.+\frac{\rho}{2}\operatorname{Re}\left(I+\mathbb{H}\left(\varphi_{x,t}\right)\right)^{-1}\left(K_1-\xi_{x,t}(1)K_{-1}\right)\Big|_{1+\mathrm{i}0}\right\}+O(\varepsilon). \quad (4.22)$$

We are now able to fill the gap left in the proof of theorem 3.4, i.e. (3.18). To this end, set $t=0$ in (4.22) and take $x>0$. In this case $\xi_{x,0}\in H^\infty$ and hence $\varphi_{x,0}\in H^\infty$. Therefore, $\mathbb{H}\left(\varphi_{x,t}\right)=0$ and by the Lebesgue dominated convergence theorem (or by corollary 2.5) we also have

$$K_m(s)=-\int_{\mathbb{R}+\mathrm{i}h}\frac{\varphi_{x,0}(z)}{(z-m)(z+s)}\frac{\mathrm{d}z}{2\pi}$$
$$=-\lim_{h\to\infty}\int_{\mathbb{R}+\mathrm{i}h}\frac{\varphi_{x,0}(z)}{(z-m)(z+s)}\frac{\mathrm{d}z}{2\pi}=0.$$

Equation (4.22) simplifies now to read

$$Q_\varepsilon(x,0)=-2\partial_x^2\log\left(1+\rho x-\frac{\rho}{2}\sin 2x\right)+O(\varepsilon),\quad x>0.$$

Recalling (3.2), we conclude that $Q_\varepsilon(x)=Q_\varepsilon(x,0)\to q_0(x)$ for $x>0$. Since $Q_\varepsilon(x)$ is even, (3.18) follows.

Pass now in (4.22) to the limit as $\varepsilon\to 0$. Apparently,

$$\lim_{\varepsilon\to 0}Q_\varepsilon(x,t)=-2\partial_x^2\log\det\left\{I+\mathbb{H}\left(\varphi_{x,t}\right)\right\}$$
$$-2\partial_x^2\log\left\{1+\rho(x+12t)-\frac{\rho}{2}\sin(2x+8t)\right.$$
$$\left.+\frac{\rho}{2}\operatorname{Re}\left(I+\mathbb{H}\left(\varphi_{x,t}\right)\right)^{-1}\left(K_1-\xi_{x,t}(1)K_{-1}\right)\Big|_{1+\mathrm{i}0}\right\}.$$

By the Bourgain theorem $Q(x,t)=\lim_{\varepsilon\to 0}Q_\varepsilon(x,t)$ is the (unique) solution to the KdV equations with data $Q(x)$. Recalling corollary 2.5, we see that

$$K_n=\mathbb{H}(\varphi_{x,t})k_{n+\mathrm{i}0},\quad n=\pm 1.$$

This completes the proof of the theorem. □

Note that the first term $u_0(x,t)$ in the solution (4.1) is given by the same Dyson formula (4.2) as in the short-range case but of course $u_0(x,0)$ is not a short range potential. Thus $u_0(x,t)$ comes from data with the missing embedded eigenvalue. On the other hand, the second term $u_1(x,t)$ in (4.1) is responsible for the bound state $+1$ and if $\rho = 1$ it resembles the so-called positon solution

$$u_{\text{pos}}(x,t) = -2\partial_x^2 \log\left\{1 + x + 12t - \frac{1}{2}\sin 2(x+4t)\right\}. \tag{4.23}$$

Such solutions seem to have appeared first in the late 70s earlier 80s but a systematic approach was developed a decade later by Matveev (see his 2002 survey [28]).

The formula (4.23) readily yields basic properties of one-position solutions. (1) As a function of the spatial variable $u_{\text{pos}}(x,t)$ has a double pole real singularity which oscillates in the $1/2$ neighborhood of the moving point $x = -12t - 1$. (2) For a fixed $t \geqslant 0$

$$u_{\text{pos}}(x,t) = -4\frac{\sin 2(x+4t)}{x} + O\left(x^{-2}\right), \quad x \to \pm\infty. \tag{4.24}$$

Observe that

$$u_{\text{pos}}(x,0) = -2\partial_x^2 \log\left(1 + x - \frac{1}{2}\sin 2x\right),$$

which coincides on $(0,\infty)$ with our $Q(x)$ for $\rho = 1$. Moreover, comparing (3.5) with (4.24) one can see that the asymptotic behaviors for $x \to -\infty$ of our $Q(x)$ with $\rho = 1$ and $u_{\text{pos}}(x,0)$ differ only by $O\left(x^{-2}\right)$. But, of course, $Q(x)$ is bounded on $(-\infty,0)$ while $u_{\text{pos}}(x,0)$ is not. Note also that the positon is somewhat similar to the soliton given by

$$u_{\text{sol}}(x,t) = -2\partial_x^2 \log\cosh(x-4t). \tag{4.25}$$

As opposed to the soliton, the positon has a square singularity (not a smooth hump) moving in the opposite direction three times as fast.

We note that multi-positon as well as soliton-positon solutions have been studied in great detail (see [28] the references cited therein). In [28] Matveev also raises the equation if there is a bounded positon, i.e. a solution having all properties of a positon but is regular. We are unable to tell if our solution is a bounded positon or not.

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