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A Thurston boundary for infinite-dimensional Teichmüller spaces

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Abstract
For a compact surface X_0, Thurston introduced a compactification of its Teichmüller space $\mathcal{T}(X_0)$ by completing it with a boundary $\mathcal{PML}(X_0)$ consisting of projective measured geodesic laminations. We introduce a similar bordification for the Teichmüller space $\mathcal{T}(X_0)$ of a noncompact Riemann surface X_0, using the technical tool of geodesic currents. The lack of compactness requires the introduction of certain uniformity conditions which were unnecessary for compact surfaces. A technical step, providing a convergence result for earthquake paths in $\mathcal{T}(X_0)$, may be of independent interest.

The Teichmüller space of a Riemann surface X_0 is the space of quasiconformal deformations of the complex structure of X_0. When X_0 is compact of genus at least 2, Thurston famously introduced a compactification of $\mathcal{T}(X_0)$ by adding a boundary at infinity consisting of projective measured foliations [13,14,34] or, equivalently, projective measured geodesic laminations [5,32]. In this paper, we introduce a similar construction of a boundary for the Teichmüller space of a noncompact surface X_0.

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In addition to the fact that Teichmüller spaces of noncompact Riemann surfaces are fundamental objects in complex analysis, our motivation here is to put in evidence the hidden features that underlie Thurston's construction, by tying it more closely to the quasiconformal geometry of X_0 and less to the purely topological considerations that suffice for compact surfaces.

Like Thurston, we restrict attention to Riemann surfaces X_0 that are conformally hyperbolic, in the sense that the conformal structure of X_0 can be realized by a complete hyperbolic metric. This is equivalent to the property that the universal cover $\widetilde{X}_0$ is biholomorphically equivalent to the unit disk $\mathbb{D} \subset \mathbb{C}$. This condition only excludes the cases where X_0 is an elliptic surface, diffeomorphic to the torus, or is the Riemann sphere minus 0, 1 or 2 points. A case of particular interest is that of the disk $\mathbb{D}$, in which case the Teichmüller space $\mathcal{T}(\mathbb{D})$ is Bers's *Universal Teichmüller Space* [3].

Thurston's original length spectrum approach [13,34] is not available here, and we follow the strategy introduced in [5] by embedding the Teichmüller space $\mathcal{T}(X_0)$ in the space $\mathcal{C}(X_0)$ of geodesic currents. These are defined as those measures on the space $G(\widetilde{X}_0)$ of Poincaré geodesics of the universal cover $\widetilde{X}_0$ which are invariant under the action of the fundamental group $\pi_1(X_0)$. When X_0 is compact, these are purely topological objects, which were introduced in [4] as a completion of the set of free homotopy classes of closed curves on the surface; in fact, geodesic currents can be described [6] solely in terms of the algebraic structure of $\pi_1(X_0)$. The definition of geodesic currents was motivated by Thurston's definition of measured foliations and measured geodesic laminations, introduced as a way to complete the set of isotopy classes of simple closed curves on the surface [13,14,32,33]. The topological nature of geodesic currents and measured geodesic laminations becomes much weaker for noncompact surfaces, and this requires the consideration of uniformity conditions which were taken for granted in the compact case.

More precisely, if X_0 is a conformally hyperbolic Riemann surface and if its universal cover $\widetilde{X}_0$ is endowed with the Poincaré metric, the space $G(\widetilde{X}_0)$ of complete geodesics of $\widetilde{X}_0$ comes with a preferred measure, the *Liouville measure* $L_{\widetilde{X}_0}$. If we have a quasiconformal deformation of the complex structure of X_0, represented by a quasiconformal diffeomorphism $f: X_0 \to X$ from X_0 to another Riemann surface X, we can then use f to pull back the Liouville measure $L_{\widetilde{X}}$ of $G(\widetilde{X})$ to a $\pi_1(X_0)$-invariant measure on $G(\widetilde{X}_0)$, namely to a geodesic current in X_0.

This enables us to define what we call the *Liouville embedding*

$$\mathbf{L}: \mathcal{T}(X_0) \to \mathcal{C}(X_0)$$

of the Teichmüller space, which associates the Liouville current L_f to each element $[f] \in \mathcal{T}(X_0)$ represented by a quasiconformal diffeomorphism $f: X_0 \to X$.

There is nothing new so far. But a challenge arises when the surface X_0 is noncompact: Find a "good" topology on the space $\mathcal{C}(X_0)$ of geodesic currents for which the Liouville embedding $\mathbf{L}$ is really a topological embedding, namely restricts to a homeomorphism $\mathcal{T}(X_0) \to \mathbf{L}\big(\mathcal{T}(X_0)\big)$. The natural topology on $\mathcal{T}(X_0)$ is the Teichmüller topology, defined by the Teichmüller metric; see Sect. 1. As a space of measures, $\mathcal{C}(X_0)$ is traditionally endowed with the weak* topology (see Sect. 2). However, this topology fails to take into account the many symmetries of the universal cover $\widetilde{X}_0$

coming from the group $\mathbf{H}(\widetilde{X}_0) \cong \mathrm{PSL}_2(\mathbb{R})$ of all biholomorphic diffeomorphisms of $\widetilde{X}_0$.

This leads us to restrict attention to *bounded* geodesic currents, which satisfy a certain boundedness property with respect to the action of $\mathbf{H}(\widetilde{X}_0)$, and to introduce the *uniform weak* topology* on the space $\mathcal{C}_{\mathrm{bd}}(X_0)$ of bounded geodesic currents. See Sect. 2 for precise definitions. When the surface X_0 is compact, every geodesic current is bounded and the uniform weak* topology coincides with the usual weak* topology on $\mathcal{C}(X_0) = \mathcal{C}_{\mathrm{bd}}(X_0)$ (Proposition 5). See [23,25–27] for earlier (and slightly different) incarnations of the uniform weak* topology.

Theorem 1 *The Liouville embedding $\mathbf{L}\colon \mathcal{T}(X_0) \to \mathcal{C}(X_0)$ is valued in the space $\mathcal{C}_{\mathrm{bd}}(X_0)$ of bounded geodesic currents, and restricts to a homeomorphism $\mathcal{T}(X_0) \to \mathbf{L}\big(\mathcal{T}(X_0)\big) \subset \mathcal{C}_{\mathrm{bd}}(X_0)$ when $\mathcal{C}_{\mathrm{bd}}(X_0)$ is endowed with the uniform weak* topology. In addition, the image $\mathbf{L}\big(\mathcal{T}(X_0)\big)$ is closed in $\mathcal{C}_{\mathrm{bd}}(X_0)$, and the embedding $\mathbf{L}\colon \mathcal{T}(X_0) \to \mathcal{C}_{\mathrm{bd}}(X_0)$ is proper.*

This theorem is proved as Theorem 8. Recall that a map is *proper* if the preimage of a bounded subset is bounded, which makes sense here because the topologies of $\mathcal{T}(X_0)$ and $\mathcal{C}_{\mathrm{bd}}(X_0)$ are defined by families of seminorms.

See Remark 9 for an explanation of why Theorem 1 would fail if $\mathcal{C}_{\mathrm{bd}}(X_0)$ was only endowed with the usual weak* topology, as opposed to the uniform weak* topology.

Following Thurston's original approach, we now consider the rays $\mathbb{R}^+\alpha \subset \mathcal{C}_{\mathrm{bd}}(X_0)$ that are asymptotic to the image $\mathbf{L}\big(\mathcal{T}(X_0)\big)$, namely the set of those bounded geodesic currents $\alpha \in \mathcal{C}_{\mathrm{bd}}(X_0)$ for which there exists a sequence $\big\{[f_n]\big\}_{n\in\mathbb{N}}$ of points of the Teichmüller space and a sequence of positive numbers $\{t_n\}_{n\in\mathbb{N}}$ such that $\alpha = \lim_{n\to\infty} t_n \mathbf{L}\big([f_n]\big)$ and $\lim_{n\to\infty} t_n = 0$. The union of these rays is the *asymptotic cone* of the Liouville embedding $\mathbf{L}$.

Theorem 2 *The asymptotic cone of the Liouville embedding $\mathbf{L}\colon \mathcal{T}(X_0) \to \mathcal{C}_{\mathrm{bd}}(X_0)$ coincides with the subset $\mathcal{ML}_{\mathrm{bd}}(X_0)$ of bounded measured geodesic laminations in X_0, namely with the set of bounded geodesic currents $\alpha \in \mathcal{C}_{\mathrm{bd}}(X_0)$ such that no two geodesics of the support of α in $G(\widetilde{X}_0)$ cross each other in $\widetilde{X}_0$.*

It is not too hard to see that every element of the asymptotic cone of $\mathbf{L}$ is a bounded measured geodesic lamination. It is more difficult to show that every bounded measured geodesic lamination belongs to this cone. For this, we use Thurston's construction of earthquakes [21,33]. A bounded measured geodesic lamination $\alpha \in \mathcal{ML}_{\mathrm{bd}}(X_0)$ defines an *earthquake map* $E^\alpha\colon \mathcal{T}(X_0) \to \mathcal{T}(X_0)$ [10,28,33]. See Remark 29 for comments about the close relationship, when the surface X_0 is noncompact, between the boundedness condition for measured geodesic laminations and the quasiconformal geometry of points of the Teichmüller space $\mathcal{T}(X_0)$.

The following property proves that every bounded measured geodesic lamination belongs to the asymptotic cone of the Liouville embedding. It is also of independent interest as, when the surface X_0 is noncompact, the estimates of [21] or [13, Exp. 8] cannot be used here.

Theorem 3 *Let $\alpha \in \mathcal{ML}_{\mathrm{bd}}(X_0)$ be a bounded measured geodesic lamination in the Riemann surface X_0. Then, for every $[f] \in \mathcal{T}(X_0)$,*

$$\lim_{t\to\infty} \frac{1}{t}\mathbf{L}\big(E^{t\alpha}[f]\big) = \alpha$$

for the uniform weak topology on the space $\mathcal{C}_{\mathrm{bd}}(X_0)$ of bounded geodesic currents.*

The space of rays in the asymptotic cone is the space $\mathcal{PML}_{\mathrm{bd}}(X_0)$ of projective bounded measured geodesic laminations. Theorem 2 enables us to add its elements as boundary points to the Teichmüller space. By analogy with the case of compact surfaces, we call the space $\mathcal{T}(X_0) \cup \mathcal{PML}_{\mathrm{bd}}(X_0)$ the *Thurston bordification* of the Teichmüller space $\mathcal{T}(X_0)$. Note that this bordification is not compact when X_0 is noncompact, as $\mathcal{T}(X_0)$ is not even locally compact in this case. See [18–20,24] for related results.

The article concludes with a result, Proposition 38, which shows that our construction is natural under quasiconformal diffeomorphisms. More precisely, the homeomorphism $\mathcal{T}(X_1) \to \mathcal{T}(X_2)$ induced by a quasiconformal diffeomorphism $X_1 \to X_2$ uniquely extends to a homeomorphism $\mathcal{T}(X_1) \cup \mathcal{PML}_{\mathrm{bd}}(X_1) \to \mathcal{T}(X_2) \cup \mathcal{PML}_{\mathrm{bd}}(X_2)$ between the respective bordifications of the Teichmüller spaces $\mathcal{T}(X_1)$ and $\mathcal{T}(X_2)$. In particular, the quasiconformal mapping class group $\mathbf{MCG}_{\mathrm{qc}}(X_0)$ acts on $\mathcal{T}(X_0) \cup \mathcal{PML}_{\mathrm{bd}}(X_0)$.

This article started as a preprint [30] by the second author alone. The first author, who had been informally involved in the introduction of the uniform weak* topology, later joined to help with the exposition. However, the major technical steps were already fully in [30]. See also [31] for a different approach, in a much more restricted context.

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1 The Teichmüller space of a Riemann surface

Let X_0 be a Riemann surface which is *conformally hyperbolic*. This means that its universal cover $\widetilde{X}_0$ is biholomorphically equivalent to the disk

$$\mathbb{D} = \{z \in \mathbb{C};\ |z| < 1\}.$$

Equivalently, X_0 is not the Riemann sphere $\mathbb{C} \cup \{\infty\}$, the plane $\mathbb{C}$, the punctured plane $\mathbb{C} - \{0\}$, or a torus.

In the disk $\mathbb{D}$, the hyperbolic metric $2|dz|/(1-|z|^2)$ is invariant under the group $\mathbf{H}(\mathbb{D})$ of biholomorphic diffeomorphisms of $\mathbb{D}$. It consequently descends to a hyperbolic metric on X_0 which does not depend on the biholomorphic identification $\widetilde{X}_0 \cong \mathbb{D}$. This is the *Poincaré metric* of the conformally hyperbolic Riemann surface X_0.

All Riemann surfaces in this article will be implicitly assumed to be conformally hyperbolic. We are particularly interested in the case where X_0 is non-compact, and a fundamental example will be that of the unit disk $X_0 = \mathbb{D}$.

Recall that a *quasiconformal diffeomorphism* $f : X_1 \to X_2$ between two Riemann surfaces is an orientation-preserving diffeomorphism such that

$$K(f) = \sup_{z \in X_1} \frac{\left|\frac{\partial f}{\partial z}(z)\right| + \left|\frac{\partial f}{\partial \bar{z}}(z)\right|}{\left|\frac{\partial f}{\partial z}(z)\right| - \left|\frac{\partial f}{\partial \bar{z}}(z)\right|}$$

is finite. Note that the denominator is always positive by the orientation-preserving hypothesis. The number $K(f)$ is the *quasiconformal dilatation* of f.

The *Teichmüller space* $\mathcal{T}(X_0)$ of the Riemann surface X_0 is the space of equivalence classes of all quasiconformal diffeomorphisms $f : X_0 \to X$ from X_0 to another Riemann surface X. Two such quasiconformal maps $f_1 : X_0 \to X_1$ and $f_2 : X_0 \to X_2$ are *equivalent* if there exists a biholomorphic map $g : X_1 \to X_2$ such that $f_2^{-1} \circ g \circ f_1$ is isotopic to the identity by a *bounded isotopy*, namely by an isotopy that moves points of X_0 by a bounded amount for the Poincaré metric of X_0. See [11] for equivalent formulations of this equivalence relation. We denote by $[f] \in \mathcal{T}(X_0)$ the equivalence class of the quasiconformal map $f : X_0 \to X$.

In the fundamental case where X_0 is the unit disk $\mathbb{D}$, the Teichmüller space $\mathcal{T}(\mathbb{D})$ is also known as the *universal Teichmüller space* [3,15].

The Teichmüller space $\mathcal{T}(X_0)$ is endowed with the *Teichmüller distance* defined by

$$d_{\mathrm{T}}\big([f_1], [f_2]\big) = \tfrac{1}{2} \log \inf_g K(g)$$

where the infimum is taken over all quasiconformal maps $g : X_1 \to X_2$ such that $f_2^{-1} \circ g \circ f_1$ is *bounded isotopic* to the identity of X_0 as above, namely isotopic to the identity by an isotopy moving points by a uniformly bounded amount for the Poincaré metric of X_0. Again, see [11] for equivalent formulations.

2 Bounded geodesic currents and the uniform weak* topology

2.1 Geodesic currents

We consider a conformally hyperbolic Riemann surface X_0 of hyperbolic type, with universal cover $\widetilde{X}_0$.

Recall that the group $\mathbf{H}(\mathbb{D})$ of biholomorphic diffeomorphisms of the disk $\mathbb{D}$ consists of all linear fractional maps of the form

$$z \mapsto \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}$$

where $\alpha, \beta \in \mathbb{C}$ are such that $|\alpha|^2 - |\beta|^2 = 1$. In particular, these biholomorphic diffeomorphisms of the open disk $\mathbb{D}$ extend to homeomorphisms of the closed disk $\mathbb{D} \cup \partial\mathbb{D}$.

This enables us to introduce a compactification of the universal cover $\widetilde{X}_0$ by its *circle at infinity* $\partial_\infty \widetilde{X}_0$, intrinsically defined by the property that every biholomorphic diffeomorphism $\widetilde{X}_0 \to \mathbb{D}$ extends to a homeomorphism $\widetilde{X}_0 \cup \partial_\infty \widetilde{X}_0 \to \mathbb{D} \cup \partial\mathbb{D}$.

Each complete hyperbolic geodesic of the disk $\mathbb{D}$ is determined by its two endpoints in $\partial\mathbb{D}$. This identifies the space $G(\mathbb{D})$ of (complete, oriented) geodesics of $\mathbb{D}$ to $\partial\mathbb{D} \times \partial\mathbb{D} - \Delta$, where $\Delta = \{(x,x); x \in \partial\mathbb{D}\}$ is the diagonal of $\partial\mathbb{D} \times \partial\mathbb{D}$.

More generally, let $G(\widetilde{X}_0)$ denote the space of oriented complete geodesics of $\widetilde{X}_0$ for its Poincaré metric. Using a biholomorphic identification $\widetilde{X}_0 \cong \mathbb{D}$, such a geodesic is determined by its endpoints in the circle at infinity $\partial_\infty \widetilde{X}_0$, and this gives a natural identification

$$G(\widetilde{X}_0) = \partial_\infty \widetilde{X}_0 \times \partial_\infty \widetilde{X}_0 - \Delta$$

where $\Delta = \{(x,x); x \in \partial_\infty \widetilde{X}_0\}$ is the diagonal of $\partial_\infty \widetilde{X}_0 \times \partial_\infty \widetilde{X}_0$. In particular, $G(\widetilde{X}_0)$ is homeomorphic to an open annulus.

The fundamental group $\pi_1(X_0)$ acts biholomorphically on the universal cover $\widetilde{X}_0$, and this action therefore respects the Poincaré metric of $\widetilde{X}_0$. As a consequence, $\pi_1(X_0)$ also acts on $G(\widetilde{X}_0)$.

A *geodesic current* in the Riemann surface X_0 is a Radon measure α on $G(\widetilde{X}_0)$ that is invariant under the action of $\pi_1(X_0)$. The Radon property means that the integral $\alpha(K) = \int_K 1\, d\alpha$ is finite and non-negative for every compact subset $K \subset G(\widetilde{X}_0)$.

Most of the geodesic currents considered in this article will be *balanced* (or *unoriented* to use a more topological terminology), in the sense that they are invariant under the involution of $G(\widetilde{X}_0)$ that reverses the orientation of every geodesic.

2.2 Bounded geodesic currents and the uniform weak* topology

As a space of Radon measures on $G(\widetilde{X}_0)$, it would be natural to endow the space $\mathcal{C}(X_0)$ of geodesic currents with the classical *weak* topology* (also called the *vague topology*), defined by the family of semi-norms

$$|\alpha|_\xi = \left| \int_{G(\widetilde{X}_0)} \xi \, d\alpha \right|$$

for $\alpha \in \mathcal{C}(X_0)$, as ξ ranges over all continuous function $\xi : G(\widetilde{X}_0) \to \mathbb{R}$ with compact support.

However, this topology does not quite fit our purposes, because it does not take into account the many symmetries of $\widetilde{X}_0$ provided by the isometric action of the group $\mathbf{H}(\widetilde{X}_0)$ of biholomorphic diffeomorphisms of $\widetilde{X}_0$. It is much better to consider the semi-norms

$$\|\alpha\|_\xi = \sup_{\varphi \in \mathbf{H}(\widetilde{X}_0)} \left| \int_{G(\widetilde{X}_0)} \xi \circ \varphi \, d\alpha \right|$$

as ξ ranges over all continuous function $\xi\colon G(\widetilde{X}_0) \to \mathbb{R}$ with compact support. (We are here using the same letter to denote the biholomorphic map $\varphi\colon \widetilde{X}_0 \to \widetilde{X}_0$, which respects the Poincaré metric of $\widetilde{X}_0$, and its induced homeomorphism $\varphi\colon G(\widetilde{X}_0) \to G(\widetilde{X}_0)$ on the space $G(\widetilde{X}_0)$ of geodesics of $\widetilde{X}_0$.) We will restrict the geodesic currents considered accordingly.

A *bounded geodesic current* is a geodesic current $\alpha \in \mathcal{C}(X_0)$ for which all norms $\|\alpha\|_\xi$ are finite. More precisely, a bounded geodesic current on the Riemann surface X_0 is a Radon measure α on the space $G(\widetilde{X}_0) = \partial_\infty \widetilde{X}_0 \times \partial_\infty \widetilde{X}_0 - \Delta$ of geodesics of $\widetilde{X}_0$ such that:

(1) for every continuous function $\xi\colon G(\widetilde{X}_0) \to \mathbb{R}$ with compact support, the integrals $\left|\int_{G(\widetilde{X}_0)} \xi \circ \varphi \, d\alpha\right|$ are bounded independently of the biholomorphic diffeomorphism $\varphi \in \mathbf{H}(\widetilde{X}_0)$;
(2) α is invariant under the action of the fundamental group $\pi_1(X_0)$ on $G(\widetilde{X}_0)$.

We let $\mathcal{C}_{\mathrm{bd}}(X_0)$ denote the set of bounded geodesic currents in the Riemann surface X_0. The topology defined by the seminorms $\|\alpha\|_\xi$ is the *uniform weak* topology* of $\mathcal{C}_{\mathrm{bd}}(X_0)$.

In particular, a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of bounded geodesic currents $\alpha_n \in \mathcal{C}_{\mathrm{bd}}(X_0)$ converges to α for the uniform weak* topology if and only if

$$\sup_{\varphi\in\mathbf{H}(\widetilde{X}_0)} \left| \int_{G(\widetilde{X}_0)} \xi \circ \varphi \, d\alpha_n - \int_{G(\widetilde{X}_0)} \xi \circ \varphi \, d\alpha \right| \to 0 \text{ as } n \to \infty$$

for every continuous function $\xi\colon G(\widetilde{X}_0) \to \mathbb{R}$ with compact support.

2.3 The weak* and uniform weak* topologies

We collect in this section a few basic properties of the weak* and uniform weak* topologies.

The following easy lemma will enable us to make some of our arguments a little more intuitive, by interpreting continuity properties in terms of sequences.

Lemma 4 *The weak* and uniform weak* topology of* $\mathcal{C}_{\mathrm{bd}}(X_0)$ *are metrizable.*

This property is of course classical for the weak* topology, and we just need to make sure that the argument extends to the uniform weak* topology.

Proof Write $G(\widetilde{X}_0)$ as an increasing union $G(\widetilde{X}_0) = \bigcup_{n=1}^\infty K_n$ of compact subsets K_n, with $K_n \subset K_{n+1}$. Then, for every n, choose a countable family $\mathcal{F}_n$ of continuous functions $\xi\colon G(\widetilde{X}_0) \to \mathbb{R}$ with support contained in K_n, such that the set $\mathcal{F}_n$ is dense in the space of all continuous functions with support in K_n for the metric

$$d(\xi, \xi') = \max_{g\in G(\widetilde{X}_0)} |\xi(g) - \xi'(g)|.$$

For each n, also choose a nonnegative continuous function $\xi^{(n)}\colon G(\widetilde{X}_0) \to [0, \infty[$ with compact support such that $\xi^{(n)}(g) \geqslant 1$ for every $g \in K_n$. Finally, set

$$\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n \cup \{\xi^{(n)}\}.$$

We want to show that, as ξ ranges over all elements of the countable set $\mathcal{F}$, the topology defined by the corresponding family of semi-norms $\| \ \|_\xi$ coincides with the uniform weak* topology (defined by considering all continuous functions $\xi\colon G(\widetilde{X}_0) \to \mathbb{R}$ with compact support).

The uniform weak* topology is defined by the basis consisting of all "balls"

$$\mathcal{B}_{\xi_1,\xi_2,\ldots,\xi_k}(\alpha; r) = \left\{\beta \in \mathcal{C}_{\mathrm{bd}}(X_0);\ \|\alpha - \beta\|_{\xi_i} < r \text{ for all } i = 1, 2, \ldots, k\right\}$$

where $\alpha \in \mathcal{C}_{\mathrm{bd}}(X_0)$, the functions $\xi_i\colon G(\widetilde{X}_0) \to \mathbb{R}$ with $i = 1, 2, \ldots, k$ are continuous with compact support, and $r > 0$.

For such a ball $\mathcal{B}_\xi(\alpha; r)$ associated to a single function ξ, the support of ξ is contained in one of the compact subsets K_n. For an $\varepsilon > 0$ to be specified later, there is by definition of $\mathcal{F}_n$ a function $\xi' \in \mathcal{F}_n$ such that $d(\xi, \xi') < \varepsilon$. As a consequence, remembering that $\xi^{(n)}$ is nonnegative and at least 1 on K_n, we have that $|\xi(g) - \xi'(g)| \leqslant \varepsilon\xi^{(n)}(g)$ for every $g \in G(\widetilde{X}_0)$, and therefore

$$\left| \int_{G(\widetilde{X}_0)} \xi \circ \varphi \, d\alpha - \int_{G(\widetilde{X}_0)} \xi' \circ \varphi \, d\alpha \right| \leqslant \varepsilon \int_{G(\widetilde{X}_0)} \xi^{(n)} \circ \varphi \, d\alpha$$

and

$$\left| \int_{G(\widetilde{X}_0)} \xi \circ \varphi \, d\beta - \int_{G(\widetilde{X}_0)} \xi' \circ \varphi \, d\beta \right| \leqslant \varepsilon \int_{G(\widetilde{X}_0)} \xi^{(n)} \circ \varphi \, d\beta$$

for every $\beta \in \mathcal{C}_{\mathrm{bd}}(X_0)$ and every $\varphi \in \mathbf{H}(\widetilde{X}_0)$. This implies that

$$\|\alpha - \beta\|_\xi \leqslant \|\alpha - \beta\|_{\xi'} + \varepsilon\|\alpha\|_{\xi^{(n)}} + \varepsilon\|\beta\|_{\xi^{(n)}}.$$

If we choose $\varepsilon > 0$ small enough that $\varepsilon\|\alpha\|_{\xi^{(n)}} < \frac{r}{3}$, this enables us to find two functions ξ' and $\xi^{(n)} \in \mathcal{F}$ such that

$$\mathcal{B}_{\xi'}(\alpha; \tfrac{r}{3}) \cap \mathcal{B}_{\xi^{(n)}}(\alpha; \tfrac{r}{3\varepsilon}) \subset \mathcal{B}_\xi(\alpha; r).$$

By taking multiple intersections, it follows that for every ball

$$\mathcal{B}_{\xi_1,\xi_2,\ldots,\xi_k}(\alpha; r) = \mathcal{B}_{\xi_1}(\alpha; r) \cap \mathcal{B}_{\xi_2}(\alpha; r) \cap \cdots \cap \mathcal{B}_{\xi_k}(\alpha; r)$$

there exists $\xi'_1, \xi'_2, \ldots, \xi'_{k'} \in \mathcal{F}$ and $r' > 0$ such that

$$\mathcal{B}_{\xi'_1,\xi'_2,\ldots,\xi'_{k'}}(\alpha; r') \subset \mathcal{B}_{\xi_1,\xi_2,\ldots,\xi_k}(\alpha; r).$$

This shows that the basis consisting of the $\mathcal{B}_{\xi_1',\xi_2',\dots,\xi_{k'}'}(\alpha; r')$ with all $\xi' \in \mathcal{F}$ defines the same topology as the similar basis where all functions with compact support are considered. In other words, the uniform weak* topology $\mathcal{C}_{\mathrm{bd}}(X_0)$ is also the topology defined by the family of seminorms $\| \ \|_\xi$ with $\xi \in \mathcal{F}$.

Since $\mathcal{F}$ is countable, it follows that this topology is metrizable. More precisely, if we list the elements of $\mathcal{F}$ as $\{\xi_i; i = 1, 2, \dots\}$, the uniform weak* topology is the metric topology associated to the metric δ defined by

$$\delta(\alpha, \beta) = \sum_{i=1}^{\infty} 2^{-i} \min\{1, \|\alpha - \beta\|_{\xi_i}\}.$$

The proof that the weak* topology is metrizable is almost identical (and classical). □

Proposition 5 *If the Riemann surface X_0 is compact, the space $\mathcal{C}_{\mathrm{bd}}(X_0)$ of bounded geodesic currents coincide with the space $\mathcal{C}(X_0)$ of all geodesic currents, and the uniform weak* topology coincides with the weak* topology on $\mathcal{C}_{\mathrm{bd}}(X_0)$.*

The two topologies do differ when X_0 is noncompact. For instance, if $g_n \in G(\mathbb{D})$ is a sequence of geodesics of $\mathbb{D}$ that eventually leaves any compact subset of $G(\mathbb{D})$, the Dirac measures $\delta_{g_n} \in \mathcal{C}_{\mathrm{bd}}(\mathbb{D})$ based at g_n provide a sequence of bounded geodesic currents in $\mathcal{C}_{\mathrm{bd}}(\mathbb{D})$ that converges to 0 for the weak* topology but has no limit for the uniform weak* topology. Also, the sum $\sum_{n=1}^{\infty} n\delta_{g_n}$ is a well-defined geodesic current, which is unbounded.

Proof of Proposition 5 We first show that every geodesic current $\alpha \in \mathcal{C}(X_0)$ is bounded.

We want to prove that, for every continuous function $\xi : G(\widetilde{X}_0) \to \mathbb{R}$ with compact support, the semi-norm

$$\|\alpha\|_\xi = \sup_{\varphi \in \mathbf{H}(\widetilde{X}_0)} \left| \int_{G(\widetilde{X}_0)} \xi \circ \varphi \, d\alpha \right| \tag{1}$$

is finite. Because X_0 is compact, there exists a compact subset $K \subset \widetilde{X}_0$ whose image under the action of $\pi_1(X_0)$ covers all of $\widetilde{X}_0$, in the sense that $\widetilde{X}_0 = \bigcup_{\gamma \in \pi_1(X_0)} \gamma(K)$. Pick a base point $x_0 \in K$. Then, for every biholomorphic diffeomorphism $\varphi \in \mathbf{H}(\widetilde{X}_0)$, there exists at least one $\gamma \in \pi_1(X_0)$ such that $\varphi \circ \gamma(x_0) \in K$. Note that $\varphi \circ \gamma$ is also biholomorphic, and that

$$\int_{G(\widetilde{X}_0)} \xi \circ \varphi \circ \gamma \, d\alpha = \int_{G(\widetilde{X}_0)} \xi \circ \varphi \, d\alpha$$

by invariance of the measure α under the action of $\pi_1(X_0)$. Therefore, in the supremum of (1), we can restrict attention to those $\varphi \in \mathbf{H}(\widetilde{X}_0)$ such that $\varphi(x_0) \in K$. Such φ form a compact subset of $\mathbf{H}(\widetilde{X}_0) \cong \mathrm{PSL}_2(\mathbb{R})$, and the supremum is therefore finite. This proves that $\|\alpha\|_\xi < \infty$.

As a conclusion, every geodesic current $\alpha \in \mathcal{C}(X_0)$ is bounded, and therefore $\mathcal{C}(X_0)$ coincides with $\mathcal{C}_{\text{bd}}(X_0)$.

We now prove that the weak* and uniform weak* topologies coincide on $\mathcal{C}(X_0) = \mathcal{C}_{\text{bd}}(X_0)$. By Lemma 4, these topologies are metrizable. Therefore we only need to show that, when X_0 is compact, a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ converges to α for the uniform weak* topology if and only if it converges to α for the weak* topology.

Convergence for the uniform weak* topology clearly implies convergence for the weak* topology. So we can focus on the converse statement.

Suppose that $\alpha_n \in \mathcal{C}_{\text{bd}}(X_0)$ converges to α for the weak* topology. We want to show that, for every continuous function $\xi\colon G(\widetilde{X}_0) \to \mathbb{R}$ with compact support,

$$\|\alpha_n - \alpha\|_\xi = \sup_{\varphi\in\mathbf{H}(\widetilde{X}_0)} \left| \int_{G(\widetilde{X}_0)} \xi \circ \varphi \, d\alpha_n - \int_{G(\widetilde{X}_0)} \xi \circ \varphi \, d\alpha \right| \tag{2}$$

tends to 0 as n tends to ∞.

As before, the compactness of X_0 enables us to restrict attention to those $\varphi \in \mathbf{H}(\widetilde{X}_0)$ such that $\varphi(x_0) \in K$, which form a compact subset of $\mathbf{H}(\widetilde{X}_0)$ (remember that $\mathbf{H}(\widetilde{X}_0)$ is also the set of isometries of the Poincaré metric of $\widetilde{X}_0$). In particular, the supremum of (2) is attained at some $\varphi_n \in \mathbf{H}(\widetilde{X}_0)$, with $\varphi_n(x_0) \in K$ and

$$\|\alpha_n - \alpha\|_\xi = \left| \int_{G(\widetilde{X}_0)} \xi \circ \varphi_n \, d\alpha_n - \int_{G(\widetilde{X}_0)} \xi \circ \varphi_n \, d\alpha \right|.$$

In addition, again by compactness of the set of those $\varphi \in \mathbf{H}(\widetilde{X}_0)$ with $\varphi(x_0) \in K$, we can extract a subsequence $\{\varphi_{n_k}\}_{k\in\mathbb{N}}$ that converges to some $\varphi_\infty \in \mathbf{H}(\widetilde{X}_0)$ uniformly on compact subsets of $\widetilde{X}_0$. In particular,

$$\begin{aligned} \|\alpha_{n_k} - \alpha\|_\xi &= \left| \int_{G(\widetilde{X}_0)} \xi \circ \varphi_{n_k} \, d\alpha_{n_k} - \int_{G(\widetilde{X}_0)} \xi \circ \varphi_{n_k} \, d\alpha \right| \\ &\leqslant \left| \int_{G(\widetilde{X}_0)} \xi \circ \varphi_\infty \, d\alpha_{n_k} - \int_{G(\widetilde{X}_0)} \xi \circ \varphi_\infty \, d\alpha \right| \\ &\quad + \int_{G(\widetilde{X}_0)} |\xi \circ \varphi_{n_k} - \xi \circ \varphi_\infty| \, d\alpha_{n_k} + \int_{G(\widetilde{X}_0)} |\xi \circ \varphi_{n_k} - \xi \circ \varphi_\infty| \, d\alpha \end{aligned} \tag{3}$$

It is now time to use the fact that $\alpha = \lim_{n\to\infty} \alpha_n$ for the weak* topology, which implies that

$$\lim_{k\to\infty} \left| \int_{G(\widetilde{X}_0)} \xi \circ \varphi_\infty \, d\alpha_{n_k} - \int_{G(\widetilde{X}_0)} \xi \circ \varphi_\infty \, d\alpha \right| = 0. \tag{4}$$

Also, pick a nonnegative continuous function $\xi_\infty\colon G(\widetilde{X}_0) \to \mathbb{R}$ with compact support, such that $\xi_\infty \geqslant 1$ on a neighborhood of the support of $\xi \circ \varphi_\infty$. Given $\varepsilon > 0$,

$$|\xi \circ \varphi_{n_k} - \xi \circ \varphi_\infty| \leqslant \varepsilon \xi_\infty$$

for k large enough, since $\varphi_{n_k} \to \varphi_\infty$ as $k \to \infty$ uniformly on compact subsets of $\widetilde{X}_0$ (and therefore uniformly on compact subsets of $G(\widetilde{X}_0)$, if we use the same letter to denote the action of φ_{n_k} on $\widetilde{X}_0$ and on $G(\widetilde{X}_0)$). It follows that

$$\int_{G(\widetilde{X}_0)} |\xi \circ \varphi_{n_k} - \xi \circ \varphi_\infty| \, d\alpha_{n_k} \leqslant \varepsilon \int_{G(\widetilde{X}_0)} \xi_\infty \, d\alpha_{n_k}.$$

Since $\int_{G(\widetilde{X}_0)} \xi_\infty \, d\alpha_{n_k} \to \int_{G(\widetilde{X}_0)} \xi_\infty \, d\alpha_\infty$ as $k \to \infty$ by weak* convergence, we conclude that

$$\lim_{k\to\infty} \int_{G(\widetilde{X}_0)} |\xi \circ \varphi_{n_k} - \xi \circ \varphi_\infty| \, d\alpha_{n_k} = 0. \tag{5}$$

Similarly,

$$\lim_{k\to\infty} \int_{G(\widetilde{X}_0)} |\xi \circ \varphi_{n_k} - \xi \circ \varphi_\infty| \, d\alpha = 0. \tag{6}$$

The combination of the Eqs. (3–6) proves that

$$\lim_{k\to\infty} \|\alpha_{n_k} - \alpha\|_\xi = 0.$$

Therefore, we were able to extract from the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ a subsequence $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$ that converges to α for the uniform weak* topology. If we apply the same process to all subsequences of the original sequence $\{\alpha_n\}_{n\in\mathbb{N}}$, we conclude that this sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ converges to α for the uniform weak* topology.

This completes the proof of Proposition 5. □

Because we will frequently use it, we state as a lemma a well-known property of the weak* topology.

Lemma 6 *Suppose that the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of geodesic currents $\alpha_n \in \mathcal{C}(X_0)$ converges to $\alpha \in \mathcal{C}(X_0)$ for the weak* topology. Then, for every every measurable subset $A \subset G(\widetilde{X}_0)$ whose topological boundary δA has α-mass $\alpha(\delta A)$ equal to* 0*,*

$$\lim_{n\to\infty} \alpha_n(A) = \alpha(A).$$

Proof See for instance [7, chap. IV, §5, n$^{\text{o}}$ 12] or [8] for this classical property of weak* convergence, which holds in a much more general setting. □

The example of Dirac measures shows that the hypothesis that $\alpha(\delta A) = 0$ is really necessary in Lemma 6.

3 The Liouville embedding

3.1 The Liouville geodesic current

We saw that the group $\mathbf{H}(\mathbb{D})$ of biholomorphic diffeomorphisms of $\mathbb{D}$ acts by isometries for the Poincaré metric, and therefore acts on the space $G(\mathbb{D})$ of complete geodesics of $\mathbb{D}$. A computation shows that it respects the *Liouville measure* $L_{\mathbb{D}}$ on $G(\mathbb{D})$ defined by the property that, if we parametrize the unit circle $\partial\mathbb{D} \subset \mathbb{C}$ by $t \mapsto \mathrm{e}^{\mathrm{i}t}$,

$$L_{\mathbb{D}}(A) = \int_A \frac{dt\, ds}{|\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{\mathrm{i}s}|^2}$$

for any Borel subset $A \subset G(\mathbb{D}) = \partial\mathbb{D} \times \partial\mathbb{D} - \Delta$. See for instance Lemma 10 below, and the well-known invariance of crossratios under linear fractional maps.

More generally, if $\widetilde{X}$ is a Riemann surface biholomorphically equivalent to $\mathbb{D}$ by a biholomorphic diffeomorphism $\widetilde{f}\colon \widetilde{X} \to \mathbb{D}$, the induced homeomorphism $\partial_\infty\widetilde{X} \to \partial\mathbb{D}$ provides a homeomorphism from the space $G(\widetilde{X}) = \partial_\infty\widetilde{X} \times \partial_\infty\widetilde{X} - \Delta$ of geodesics of $\widetilde{X}$ to $G(\mathbb{D}) = \partial\mathbb{D} \times \partial\mathbb{D} - \Delta$, which we also denote by $\widetilde{f}$. We can then pull back the Liouville measure $L_{\mathbb{D}}$ to a measure $L_{\widetilde{X}}$ on $G(\widetilde{X})$. The invariance of $L_{\mathbb{D}}$ under the group $\mathbf{H}(\mathbb{D})$ of biholomorphic diffeomorphisms of $\mathbb{D}$ shows that this measure is independent of the choice of the biholomorphic diffeomorphism $\widetilde{f}\colon \widetilde{X} \to \mathbb{D}$. The measure $L_{\widetilde{X}}$ is the *Liouville measure* of the Riemann surface $\widetilde{X} \cong \mathbb{D}$.

Consider an element $[f] \in \mathcal{T}(X_0)$ of the Teichmüller space of the Riemann surface X_0, represented by a quasiconformal diffeomorphism $f\colon X_0 \to X$. Lift f to a quasiconformal diffeomorphism $\widetilde{f}\colon \widetilde{X}_0 \to \widetilde{X}$ between the universal covers. A fundamental property is that this quasiconformal diffeomorphism admits a continuous extension $\widetilde{f}\colon \widetilde{X}_0 \cup \partial_\infty\widetilde{X}_0 \to \widetilde{X} \cup \partial_\infty\widetilde{X}$ (see the Beurling–Ahlfors Theorem 14 below). The restriction of this extension to the circles at infinity induces a homeomorphism from $G(\widetilde{X}_0) = \partial_\infty\widetilde{X}_0 \times \partial_\infty\widetilde{X}_0 - \Delta$ to $G(\widetilde{X}) = \partial_\infty\widetilde{X} \times \partial_\infty\widetilde{X} - \Delta$. We can then pull back the Liouville measure $L_{\widetilde{X}}$ by $\widetilde{f}$ to define a measure L_f on $G(\widetilde{X}_0)$. More precisely, $L_f(A) = L_{\widetilde{X}}\big(\widetilde{f}(A)\big)$ for every measurable subset $A \subset G(\widetilde{X}_0)$, while

$$\int_{G(\widetilde{X}_0)} \xi\, dL_f = \int_{G(\widetilde{X})} \xi \circ \widetilde{f}^{-1}\, dL_{\widetilde{X}}$$

for every continuous function $\xi\colon G(\widetilde{X}_0) \to R$ with compact support.

The action of the fundamental group $\pi_1(X)$ on $\widetilde{X}$ is biholomorphic, and therefore respects the Liouville measure $L_{\widetilde{X}}$ on $G(\widetilde{X})$. Since two lifts $\widetilde{f}\colon \widetilde{X}_0 \to \widetilde{X}$ of f differ only by the action of an element of $\pi_1(X)$, it follows that the measure L_f is independent of the choice of this lift. Also, because $\widetilde{f}$ conjugates the action of $\pi_1(X)$ on $\widetilde{X}$ to the action of $\pi_1(X_0)$ on $\widetilde{X}_0$, the measure L_f is invariant under the action of $\pi_1(X_0)$ on $G(\widetilde{X}_0)$. In other words, L_f is a geodesic current in X_0.

Lemma 7 *The Liouville geodesic current L_f is bounded, and therefore belongs to $\mathcal{C}_{\mathrm{bd}}(X_0)$.*

We postpone the proof of Lemma 7 to Sect. 3.3, where it will be proved as Lemma 16.

If two quasiconformal diffeomorphisms $f_1\colon X_0 \to X_1$ and $f_2\colon X_0 \to X_2$ represent the same element $[f_1] = [f_2]$ in the Teichmüller space $\mathcal{T}(X_0)$, there exists a biholomorphic diffeomorphism $g\colon X_1 \to X_2$ such that $f_2^{-1} \circ g \circ f_1$ is bounded isotopic to the identity in X_0. We can therefore choose lifts $\widetilde{f}_1\colon \widetilde{X}_0 \to \widetilde{X}_1$, $\widetilde{f}_2\colon \widetilde{X}_0 \to \widetilde{X}_2$, $\widetilde{g}\colon \widetilde{X}_1 \to \widetilde{X}_2$ of these diffeomorphisms so that $\widetilde{f}_2^{-1} \circ \widetilde{g} \circ \widetilde{f}_1$ is bounded isotopic to the identity in $\widetilde{X}_0$. A bounded isotopy fixes the boundary at infinity $\partial_\infty \widetilde{X}_0$; indeed, assuming $\widetilde{X}_0 = \mathbb{D}$ without loss of generality, the euclidean distance by which a bounded isotopy moves a point $x \in \mathbb{D}$ tends to 0 as x approaches the boundary circle $\partial\mathbb{D}$. This implies that the restrictions of $\widetilde{f}_2$ and $\widetilde{g} \circ \widetilde{f}_1$ to maps $\partial_\infty \widetilde{X}_0 \to \partial_\infty \widetilde{X}_2$ coincide. As the biholomorphic diffeomorphism $\widetilde{g}$ sends the Liouville measure $L_{\widetilde{X}_1}$ to $L_{\widetilde{X}_2}$, it follows that the measures L_{f_1} and L_{f_2} coincide on $G(\widetilde{X}_0)$.

As a consequence, the Liouville geodesic current $L_f \in \mathcal{C}_{\mathrm{bd}}(X_0)$ depends only on the element $[f]$ of the Teichmüller space $\mathcal{T}(X_0)$ represented by the quasiconformal diffeomorphism $f\colon X_0 \to X$.

The map

$$\mathbf{L}\colon \mathcal{T}(X_0) \to \mathcal{C}_{\mathrm{bd}}(X_0)$$

defined by the property that $\mathbf{L}\big([f]\big) = L_f$ is the *Liouville embedding*.

Theorem 8 *Let X_0 be a conformally hyperbolic Riemann surface, let the Teichmüller space $\mathcal{T}(X_0)$ be equipped with the Teichmüller distance d_{T}, and let the space $\mathcal{C}_{\mathrm{bd}}(X_0)$ of bounded geodesic currents be endowed with the uniform weak* topology defined in Sect. 2. Then, the Liouville embedding $\mathbf{L}\colon \mathcal{T}(X_0) \to \mathcal{C}_{\mathrm{bd}}(X_0)$ is a homeomorphism onto its image, it is a proper map, and its image $\mathbf{L}\big(\mathcal{T}(X_0)\big)$ is closed in $\mathcal{C}_{\mathrm{bd}}(X_0)$.*

Remark 9 The above statement would be false if $\mathcal{C}_{\mathrm{bd}}(X_0)$ was only endowed with the usual weak* topology. Indeed, consider a sequence $\{g_n\}_{n\in\mathbb{N}}$ of geodesics of the disk $\mathbb{D}$ that leaves every compact subset of $G(\mathbb{D})$. For any $[f_0] \in \mathcal{T}(\mathbb{D})$, let $[f_n] = E^1_{g_n}[f_0]$ be obtained from $[f_0]$ by performing an elementary earthquake along g_n (see Sect. 5.2). Then, for every compact subset $K \subset G(\mathbb{D})$, the measure $\mathbf{L}\big([f_n]\big)$ coincides with $\mathbf{L}\big([f_0]\big)$ on K for n sufficiently large. It follows that the sequence $\big\{\mathbf{L}\big([f_n]\big)\big\}_{n\in\mathbb{N}}$ converges to $\mathbf{L}\big([f_0]\big)$ for the weak* topology as n tends to infinity. However, the Teichmüller distance $d_{\mathrm{T}}\big([f_0],[f_n]\big) > 0$ is constant and $[f_n]$ consequently does not converge to $[f_0]$ for the Teichmüller metric on $\mathcal{T}(X_0)$. This shows that the inverse map $\mathbf{L}^{-1}\colon \mathbf{L}\big(\mathcal{T}(X_0)\big) \to \mathcal{T}(X_0)$ is not continuous when its domain is only endowed with the weak* topology, so that the uniform weak* topology is really needed.

The proof of Theorem 8 will take a while. It will be proved in several steps, as Propositions 19, 21, 24 and 25 below. We first introduce a few technical tools to connect the quasiconformal geometry of Riemann surfaces to measures on spaces of geodesics.

3.2 Boxes of geodesics

Let $\widetilde{X}$ be a simply connected conformally hyperbolic Riemann surface, and let $\partial_\infty \widetilde{X}$ be its circle at infinity. Typically, $\widetilde{X}$ will be the universal cover of a conformally hyperbolic Riemann surface X.

The orientation of $\widetilde{X}$ specifies a boundary (counterclockwise) orientation for $\partial_\infty \widetilde{X}$. In particular, two points a, $b \in \partial_\infty \widetilde{X}$ delimit a unique interval $[a, b] \subset \partial_\infty \widetilde{X}$, consisting of those points x such that a, x, b occur in this order for the counterclockwise orientation of $\partial_\infty \widetilde{X}$. Note that $[b, a]$ is different from $[a, b]$, and that $[a, b] \cup [b, a] = \partial_\infty \widetilde{X}$.

Four distinct points a, b, c, $d \in \partial_\infty \widetilde{X}$, occurring counterclockwise in this order, determine two disjoint intervals $[a, b]$, $[c, d] \subset \partial_\infty \widetilde{X}$ and a subset $Q = [a, b] \times [c, d]$ of the space of geodesics $G(\widetilde{X}) = \partial_\infty \widetilde{X} \times \partial_\infty \widetilde{X} - \Delta$. We will refer to such a subset Q as a *box of geodesics of* $\widetilde{X}$, or as a *box in* $G(\widetilde{X})$.

For the disk $\mathbb{D}$ and its Liouville geodesic current $L_{\mathbb{D}} \in \mathcal{C}_{\mathrm{bd}}(\mathbb{D})$, a simple integral computation expresses the Liouville mass of a box of geodesics in terms of the crossratio of the four points of $\partial\mathbb{D}$ determining this box.

Lemma 10 *For a box of geodesics $Q = [a, b] \times [c, d] \subset G(\mathbb{D})$ with a, b, c, $d \in \partial\mathbb{D} \subset \mathbb{C}$,*

$$L_{\mathbb{D}}\big([a, b] \times [c, d]\big) = \iint_Q \frac{ds\, dt}{|\mathrm{e}^{\mathrm{i}s} - \mathrm{e}^{\mathrm{i}t}|^2} = \log \frac{(a-c)(b-d)}{(a-d)(b-c)}.$$

Lemma 11 *Let Q and $Q' \subset G(\widetilde{X})$ be two boxes of geodesics in $\widetilde{X}$. There exists a biholomorphic diffeomorphism $\widetilde{X} \to \widetilde{X}$ sending Q to Q' if and only if they have the same Liouville mass $L_{\widetilde{X}}(Q) = L_{\widetilde{X}}(Q')$.*

Proof Using a biholomorphic diffeomorphism $\widetilde{X} \to \mathbb{D}$, we can assume without loss of generality that $\widetilde{X} = \mathbb{D}$. Then, the biholomorphic diffeomorphisms of $\mathbb{D}$ are the linear fractional maps $z \mapsto \frac{\alpha z+\beta}{\bar\beta z+\bar\alpha}$ where α, $\beta \in \mathbb{C}$ are such that $|\alpha|^2 - |\beta|^2 = 1$. Elementary algebra shows that, given two boxes $Q = [a, b] \times [c, d]$ and $Q' = [a', b'] \times [c', d']$ in $G(\mathbb{D})$, there exists such a linear fractional map sending Q to Q' if and only if the crossratios $\frac{(a-c)(b-d)}{(a-d)(b-c)}$ and $\frac{(a'-c')(b'-d')}{(a'-d')(b'-c')}$ are equal. By Lemma 10, this is equivalent to the property that the Liouville masses $L_{\mathbb{D}}(Q)$ and $L_{\mathbb{D}}(Q')$ are equal. □

For a box of geodesics $Q = [a, b] \times [c, d] \subset G(\widetilde{X})$, its *orthogonal box* is the box $Q^\perp = [b, c] \times [d, a]$.

Note that the definition is not quite as symmetric as one would hope, as $Q^{\perp\perp}$ is different from Q. In fact, $Q^{\perp\perp} = [c, d] \times [a, b]$ consists of all geodesics obtained by reversing the orientation of the geodesics of Q. In particular, $Q^{\perp\perp}$ has the same α-mass as Q for any balanced geodesic current, and the distinction between Q and $Q^{\perp\perp}$ will consequently have little impact in this article since most geodesic currents considered here will be balanced (as defined at the end of Sect. 2.1).

Lemma 12 *Let $L_{\widetilde{X}}$ be the Liouville measure of a simply connected conformally hyperbolic Riemann surface $\widetilde{X}$. For every box of geodesics $Q \subset G(\widetilde{X})$,*

$$\mathrm{e}^{-L_{\widetilde{X}}(Q)} + \mathrm{e}^{-L_{\widetilde{X}}(Q^\perp)} = 1.$$

Proof Using a biholomorphic diffeomorphism $\widetilde{X} \to \mathbb{D}$, we can assume without loss of generality that $\widetilde{X} = X = \mathbb{D}$. Then, for a box $Q = [a, b] \times [c, d] \subset G(\mathbb{D})$, Lemma 10 gives

$$\begin{aligned} \mathrm{e}^{-L_{\mathbb{D}}(Q)} + \mathrm{e}^{-L_{\mathbb{D}}(Q^\perp)} &= \frac{(a-d)(b-c)}{(a-c)(b-d)} + \frac{(b-a)(c-d)}{(b-d)(c-a)} \\ &= \frac{(a-d)(b-c) - (b-a)(c-d)}{(a-c)(b-d)} = 1. \end{aligned}$$

3.3 Quasiconformal and quasisymmetric homeomorphisms

Consider a quasiconformal diffeomorphism $f: X_1 \to X_2$ between conformally hyperbolic Riemann surfaces, and lift it to a map $\widetilde{f}: \widetilde{X}_1 \to \widetilde{X}_2$ between their universal cover. We already mentioned the Beurling–Ahlfors Theorem, which says that $\widetilde{f}$ has a continuous extension $\widetilde{f}: \widetilde{X}_1 \cup \partial_\infty \widetilde{X}_1 \to \widetilde{X}_2 \cup \partial_\infty \widetilde{X}_2$ to the closed disks obtained by adding their circles at infinity to $\widetilde{X}_1$ and $\widetilde{X}_2$. The Beurling–Ahlfors Theorem additionally relates the quasiconformal properties of $\widetilde{f}: \widetilde{X}_1 \to \widetilde{X}_2$ to another regularity property for the boundary extension $\widetilde{f}: \partial_\infty \widetilde{X}_1 \to \partial_\infty \widetilde{X}_2$, as we now explain.

A box $Q \subset G(\widetilde{X}_1)$ is *symmetric* if its Liouville mass $L_{\widetilde{X}_1}(Q)$ is equal to $\log 2$. This property is better explained if we translate it to the disk by a biholomorphic diffeomorphism $\widetilde{X}_0 \to \mathbb{D}$. Indeed, Lemma 11 shows that a box $Q \subset G(\mathbb{D})$ is symmetric if and only if it is the image $\varphi\big([1, \mathrm{i}] \times [-1, -\mathrm{i}]\big)$ under a biholomorphic map $\varphi \in \mathbf{H}(\mathbb{D})$ of the "standard" box $[1, \mathrm{i}] \times [-1, -\mathrm{i}]$ delimited by the points $1, \mathrm{i}, -1, -\mathrm{i} \in \partial\mathbb{D}$. Another characterization is provided by the combination of Lemmas 11 and 12 , which shows that a box Q is symmetric if and only if there is a biholomorphic diffeomorphism of $\widetilde{X}_1$ sending Q to the orthogonal box $Q^\perp$.

A homeomorphism $\widetilde{f}: \partial_\infty \widetilde{X}_1 \to \partial_\infty \widetilde{X}_2$ is *quasisymmetric* if the supremum

$$M(\widetilde{f}) = \sup_{Q \text{ symmetric}} \frac{L_{\widetilde{X}}\big(\widetilde{f}(Q)\big)}{\log 2},$$

as Q ranges over all symmetric boxes $Q \subset G(\widetilde{X}_1)$, is finite. By definition, $M(h)$ is the *quasisymmetric constant* of h.

Note that $M(\widetilde{f}) = 1$ when $\widetilde{f}$ comes from a biholomorphic diffeomorphism $\widetilde{X}_1 \to \widetilde{X}_2$, and that in general $M(\widetilde{f}) \geqslant 1$ by Lemma 12.

Remark 13 The quasisymmetry property is sometimes stated in a different way, by restricting attention to homeomorphisms $f: \mathbb{R} \to \mathbb{R}$ and by requiring that the supremum

$$H(f) = \sup\{\tfrac{|f(x+t) - f(x)|}{|f(x) - f(x-t)|}; x, t \in \mathbb{R}\}$$

be finite; to clarify the terminology, let us say that a homeomorphism $f\colon \mathbb{R} \to \mathbb{R}$ satisfying this property is *weakly quasi-symmetric* (compare [35]). If we identify $\mathbb{R} \cup \{\infty\}$ to $S^1 = \partial\mathbb{D}$ by stereographic projection, a simple algebraic manipulation shows that $\log(1+H(f)) \leqslant M(f)$. As a consequence, if the extension $\mathbb{R}\cup\{\infty\} \to \mathbb{R}\cup\{\infty\}$ of $f\colon \mathbb{R} \to \mathbb{R}$ is quasisymmetric, then f is weakly quasisymmetric. A consequence of the proof [2] of the Beurling–Ahlfors Theorem 14 stated below is that the converse holds, namely that the extension $\mathbb{R}\cup\{\infty\} \to \mathbb{R}\cup\{\infty\}$ of a homeomorphism $f\colon \mathbb{R} \to \mathbb{R}$ is quasisymmetric if and only if f is weakly quasisymmetric. Indeed, that proof only uses the weak quasisymmetry property, whereas the boundary extension of a quasiconformal diffeomorphism is quasisymmetric.

The following fundamental result connects quasiconformal diffeomorphisms between Riemann surfaces and quasisymmetric homeomorphisms between their circles at infinity.

Theorem 14 (Beurling–Ahlfors) *Let $\widetilde{X}_1$ and $\widetilde{X}_2$ be two simply connected conformally hyperbolic Riemann surfaces. Every quasiconformal diffeomorphism $\widetilde{f}\colon \widetilde{X}_1 \to \widetilde{X}_2$ admits a unique extension to a homeomorphism $\widetilde{X}_1 \cup \partial_\infty\widetilde{X}_1 \to \widetilde{X}_2 \cup \partial_\infty\widetilde{X}_2$, whose restriction $\widetilde{f}\colon \partial_\infty\widetilde{X}_1 \to \partial_\infty\widetilde{X}_2$ to the circles at infinity is quasisymmetric. In addition, the quasisymmetric constant $M(\widetilde{f})$ of $\widetilde{f}\colon \partial_\infty\widetilde{X}_1 \to \partial_\infty\widetilde{X}_2$ tends to* 1 *as the quasiconformal dilatation $K(\widetilde{f})$ of $\widetilde{f}\colon \widetilde{X}_1 \to \widetilde{X}_2$ tends to* 1.

Conversely, every quasisymmetric homeomorphism $\widetilde{f}\colon \partial_\infty\widetilde{X}_1 \to \partial_\infty\widetilde{X}_2$ admits a continuous extension $\widetilde{X}_1 \cup \partial_\infty\widetilde{X}_1 \to \widetilde{X}_2 \cup \partial_\infty\widetilde{X}_2$, whose restriction $\widetilde{f}\colon \widetilde{X}_1 \to \widetilde{X}_2$ is a quasiconformal diffeomorphism. In addition, the extension can be chosen so that the quasiconformal dilatation $K(\widetilde{f})$ of $\widetilde{f}\colon \widetilde{X}_1 \to \widetilde{X}_2$ is bounded by a constant $K'(\widetilde{f})$ depending only on the quasisymmetric constant $M(\widetilde{f})$ of $\widetilde{f}\colon \partial_\infty\widetilde{X}_1 \to \partial_\infty\widetilde{X}_2$, and tending to 1 *as $M(\widetilde{f})$ tends to* 1.

Proof See [2], [22, §II.6] or [17, §16], for instance. □

Although the definition of a quasisymmetric homeomorphism $\widetilde{f}\colon \partial_\infty\widetilde{X}_1 \to \partial_\infty\widetilde{X}_2$ involves only symmetric boxes, the quasisymmetry property actually controls the Liouville mass $L_{\widetilde{X}_2}\big(\widetilde{f}(Q)\big)$ for all boxes $Q \subset G(\widetilde{X}_1)$.

Proposition 15 *If a homeomorphism $\widetilde{f}\colon \partial_\infty\widetilde{X}_1 \to \partial_\infty\widetilde{X}_2$ is quasisymmetric, there exists a homeomorphism $\omega\colon [0,\infty[\to [0,\infty[$ depending only on the quasisymmetric constant $M(\widetilde{f})$ such that*

$$L_{\widetilde{X}_2}\big(\widetilde{f}(Q)\big) \leqslant \omega\big(L_{\widetilde{X}_1}(Q)\big)$$

for every box $Q \subset G(\widetilde{X}_1)$.

In addition, the homeomorphism ω can be chosen so that it converges to the identity, uniformly on compact subsets of the open interval $]0,\infty[$, as the quasisymmetric constant $M(\widetilde{f})$ tends to 1.

Proof Although there exists direct proofs of the first half of the statement (see for instance [35]), it is easier to use the full force of the Beurling–Ahlfors Theorem 14.

In addition to its Liouville mass $L_{\widetilde{X}_1}(Q)$, a box $Q = [a, b] \times [c, d]$ in $G(\widetilde{X}_1)$ has a more complex analytic invariant, its *conformal modulus* $\mu_{\widetilde{X}_1}(Q)$. This is defined as the number $\mu = \mu_{\widetilde{X}_1}(Q)$ for which there exists a homeomorphism $\widetilde{X}_1 \cup \partial_\infty \widetilde{X}_1 \to [0, \mu] \times [0, 1]$ that is conformal on $\widetilde{X}$ and sends the corners a, b, c, $d \in \partial_\infty \widetilde{X}$ of Q to the corners $(0, 0)$, $(\mu, 0)$, $(\mu, 1)$, $(0, 1)$ of the rectangle $[0, \mu] \times [0, 1] \subset \mathbb{R}^2$, respectively. These two invariants are classically related by an increasing homeomorphism $\eta\colon\ ]0, \infty[\ \to\ ]0, \infty[$ such that $\mu_{\widetilde{X}_1}(Q) = \eta\big(L_{\widetilde{X}_1}(Q)\big)$; indeed, these two quantities depend continuously on the corners a, b, c, d of Q, they both increase as Q gets larger, they tend to 0 as Q gets arbitrarily small, and they tend to $+\infty$ as Q gets arbitrarily large.

Let $\widetilde{f}\colon \widetilde{X}_1 \to \widetilde{X}_2$ be the quasiconformal extension of $\widetilde{f}\colon \partial_\infty \widetilde{X}_1 \to \partial_\infty \widetilde{X}_2$ provided by Theorem 14. In particular, this quasiconformal extension can be chosen so that its quasiconformal dilatation $K(\widetilde{f})$ is bounded by a constant $K'(\widetilde{f})$ depending only on the quasisymmetric constant $M(\widetilde{f})$, and tending to 1 as $M(\widetilde{f})$ tends to 1. A fundamental consequence of quasiconformality is that

$$\mu_{\widetilde{X}_2}\big(\widetilde{f}(Q)\big) \leqslant K(\widetilde{f})\, \mu_{\widetilde{X}_1}(Q);$$

see for instance [1,22]. Proposition 15 then holds for the homeomorphism ω defined by $\omega(t) = \eta^{-1}\big(K'(\widetilde{f})\eta(t)\big)$. □

An immediate consequence of Proposition 15 is that, if $\widetilde{f}\colon \partial_\infty \widetilde{X}_1 \to \partial_\infty \widetilde{X}_2$ is quasisymmetric, so is its inverse $\widetilde{f}^{-1}\colon \partial_\infty \widetilde{X}_2 \to \partial_\infty \widetilde{X}_1$.

We now have the tools to prove Lemma 7, a task which we had temporarily postponed. We rephrase this statement in the following way.

Lemma 16 *Let $\widetilde{f}\colon \widetilde{X}_1 \to \widetilde{X}_2$ be a quasiconformal diffeomorphism between two simply connected conformally hyperbolic Riemann surfaces. Then, for every continuous function $\xi\colon G(\widetilde{X}_1) \to \mathbb{R}$ with compact support, the supremum*

$$\sup_{\varphi \in \mathbf{H}(\widetilde{X}_1)} \left| \int_{G(\widetilde{X}_1)} \xi \circ \varphi \, dL_{\widetilde{f}} \right|$$

is finite, where the supremum is taken over all biholomorphic diffeomorphisms $\varphi\colon \widetilde{X}_1 \to \widetilde{X}_1$ and where $L_{\widetilde{f}}$ is the pull back under $\widetilde{f}$ of the Liouville measure $L_{\widetilde{X}_2}$ of $\widetilde{X}_2$.

Proof Cover the support of ξ by finitely many boxes $Q_1, Q_2, \ldots, Q_k \subset G(\widetilde{X}_1)$. Then, for every $\varphi \in \mathbf{H}(\widetilde{X}_1)$

$$\begin{aligned} \left| \int_{G(\widetilde{X}_1)} \xi \circ \varphi \, dL_{\widetilde{f}} \right| &\leqslant \Big(\max_{g \in G(\widetilde{X}_1)} |\xi(g)| \Big) \sum_{i=1}^{k} L_{\widetilde{f}}\big(\varphi^{-1}(Q_i)\big) \\ &\leqslant \Big(\max_{g \in G(\widetilde{X}_1)} |\xi(g)| \Big) \sum_{i=1}^{k} L_{\widetilde{X}_2}\big(\widetilde{f} \circ \varphi^{-1}(Q_i)\big). \end{aligned}$$

Since $\widetilde{f}\colon \partial_\infty \widetilde{X}_1 \to \partial_\infty \widetilde{X}_2$ is quasisymmetric, Proposition 15 provides a function ω such that, for each box $Q_i \subset G(\widetilde{X}_1)$,

$$L_{\widetilde{X}_2}\big(\widetilde{f} \circ \varphi^{-1}(Q_i)\big) \leqslant \omega\big(L_{\widetilde{X}_1}(\varphi^{-1} Q_i)\big) = \omega\big(L_{\widetilde{X}_1}(Q_i)\big).$$

This gives the uniform bound requested. □

Theorem 14 provides a correspondence between quasiconformal diffeomorphisms between simply connected Riemann surfaces and quasisymmetric homeomorphisms between their boundaries at infinity. We will need a slight improvement of this correspondence for maps between Riemann surfaces that are not simply connected.

Lift a quasiconformal map $f\colon X_1 \to X_2$ to a quasiconformal diffeomorphism $\widetilde{f}\colon \widetilde{X}_1 \to \widetilde{X}_2$ between universal covers, and consider the quasisymmetric extension $\widetilde{f}\colon \partial_\infty \widetilde{X}_1 \to \partial_\infty \widetilde{X}_2$ provided by the first part of Theorem 14. The quasisymmetry property is invariant under composition with biholomorphic maps of $\widetilde{X}_2$ (as these respect the Liouville measure $L_{\widetilde{X}_2}$). It follows that the quasisymmetric constant $M(\widetilde{f})$ is independent of the choice of the lift $\widetilde{f}\colon \widetilde{X}_1 \to \widetilde{X}_2$. We will refer to $M(\widetilde{f})$ as *the quasisymmetric constant* $M(f)$ of the quasiconformal map $f\colon X_1 \to X_2$.

The first part of Theorem 14 indicates that this quasisymmetric constant $M(f)$ is close to 1 when the quasiconformal dilatation $K(f)$ is close to 1. We will need the following converse statement, which improves the second part of Theorem 14 by ensuring that the quasiconformal extension $\widetilde{f}\colon \widetilde{X}_1 \to \widetilde{X}_2$ comes from a quasiconformal diffeomorphism $f\colon X_1 \to X_2$.

Theorem 17 *Let $f\colon X_1 \to X_2$ be a quasiconformal diffeomorphism between conformally hyperbolic Riemann surfaces, and let $M(f)$ be its quasisymmetric constant. Then, there is another quasiconformal diffeomorphism $f'\colon X_1 \to X_2$ that is bounded isotopic to f and whose quasiconformal dilatation $K(f')$ is bounded by a constant depending only on the quasisymmetric constant $M(f) = M(f')$. In addition, f' can be chosen so that its quasiconformal dilatation $K(f')$ tends to* 1 *as the quasisymmetric constant $M(f)$ tends to* 1.

Proof As usual, lift f to $\widetilde{f}\colon \widetilde{X}_1 \to \widetilde{X}_2$, and consider the quasisymmetric extension $\widetilde{f}\colon \partial_\infty \widetilde{X}_1 \to \partial_\infty \widetilde{X}_2$. A fundamental construction of Douady–Earle [9] provides another continuous extension $\widetilde{f}'\colon \widetilde{X}_1 \cup \partial_\infty \widetilde{X}_1 \to \widetilde{X}_2 \cup \partial_\infty \widetilde{X}_2$ of $\widetilde{f}\colon \partial_\infty \widetilde{X}_1 \to \partial_\infty \widetilde{X}_2$ such that $\widetilde{f}'\colon \widetilde{X}_1 \to \widetilde{X}_2$ is quasiconformal, which has the additional property that it is equivariant with respect to the action of the biholomorphic diffeomorphisms of $\widetilde{X}_1$ and $\widetilde{X}_2$. Namely, for every biholomorphic diffeomorphism $\varphi_1 \in \mathbf{H}(\widetilde{X}_1)$ and $\varphi_2 \in \mathbf{H}(\widetilde{X}_2)$, the Douady–Earle quasiconformal extension of $\varphi_1 \circ \widetilde{f} \circ \varphi_2\colon \partial_\infty \widetilde{X}_1 \to \partial_\infty \widetilde{X}_2$ is $\varphi_1 \circ \widetilde{f}' \circ \varphi_2\colon \widetilde{X}_1 \to \widetilde{X}_2$. In addition, we still have the property that the quasiconformal constant $K(\widetilde{f}')$ of the Douady–Earle extension tends to 1 as the quasisymmetric constant $M(\widetilde{f})$ tends to 1 (although the bound is not as good as for the Beurling–Ahlfors Theorem).

Applying the equivariance property to the (biholomorphic) actions of the fundamental group $\pi_1(X_1) = \pi_1(X_2)$ on $\widetilde{X}_1$ and $\widetilde{X}_2$, it follows that $\widetilde{f}'\colon \widetilde{X}_1 \to \widetilde{X}_2$ descends to a quasiconformal map $f'\colon X_1 \to X_2$. By construction, $K(f') = K(\widetilde{f}')$ tends to 1 as $M(f) = M(\widetilde{f})$ tends to 1.

By construction, the quasisymmetric extensions $\widetilde{f}, \widetilde{f}'\colon \partial_\infty \widetilde{X}_1 \to \partial_\infty \widetilde{X}_2$ of the quasiconformal maps $\widetilde{f}, \widetilde{f}'\colon \widetilde{X}_0 \to \widetilde{X}$ coincide. A result of Earle–McMullen [11] then shows that f and f' are bounded isotopic. □

3.4 The Liouville embedding $\mathbf{L}\colon \mathcal{T}(X_0) \to \mathcal{C}_{\mathrm{bd}}(X_0)$ is injective

We are now ready to begin proving Theorem 8. We begin with the easier part.

Proposition 18 *The Liouville embedding* $\mathbf{L}\colon \mathcal{T}(X_0) \to \mathcal{C}_{\mathrm{bd}}(X_0)$ *is injective.*

Proof Suppose that $\mathbf{L}\big([f_1]\big) = \mathbf{L}\big([f_2]\big)$ for $[f_1], [f_2] \in \mathcal{T}(X_0)$ represented by quasiconformal diffeomorphisms $f_1\colon X_0 \to X_1$, $f_2\colon X_0 \to X_2$. Lift f_1, f_2 to maps $\widetilde{f}_1\colon \widetilde{X}_0 \to \widetilde{X}_1$, $\widetilde{f}_2\colon \widetilde{X}_0 \to \widetilde{X}_2$ between universal covers, and consider the quasisymmetric extensions $\widetilde{f}_1\colon \partial_\infty \widetilde{X}_0 \to \partial_\infty \widetilde{X}_1$, $\widetilde{f}_2\colon \partial_\infty \widetilde{X}_0 \to \partial_\infty \widetilde{X}_2$ provided by Theorem 14.

Since $\mathbf{L}\big([f_1]\big) = \mathbf{L}\big([f_2]\big)$, the homeomorphism $\widetilde{f}_2 \circ \widetilde{f}_1^{-1}\colon \partial_\infty \widetilde{X}_1 \to \partial_\infty \widetilde{X}_2$ sends the Liouville measure $L_{\widetilde{X}_1}$ to $L_{\widetilde{X}_2}$. It follows that the quasisymmetric constant $M(\widetilde{f}_2 \circ \widetilde{f}_1^{-1}) = M(f_2 \circ f_1^{-1})$ is equal to 1. By Theorem 17, it follows that $f_2 \circ f_1^{-1}$ is bounded isotopic to maps $g\colon X_1 \to X_2$ whose quasiconformal dilatation $K(g)$ is arbitrarily close to 1. This proves that the Teichmüller distance $d_{\mathrm{T}}\big([f_1], [f_2]\big)$ is equal to 0, so that $[f_1] = [f_2]$ in $\mathcal{T}(X_0)$ as required. □

3.5 The Liouville embedding $\mathbf{L}\colon \mathcal{T}(X_0) \to \mathcal{C}_{\mathrm{bd}}(X_0)$ is continuous

We now prove a more substantial step in the proof of Theorem 8.

Proposition 19 *The Liouville embedding* $\mathbf{L}\colon \mathcal{T}(X_0) \to \mathcal{C}_{\mathrm{bd}}(X_0)$ *is continuous, for the Teichmüller topology on* $\mathcal{T}(X_0)$ *and the uniform weak* topology on* $\mathcal{C}_{\mathrm{bd}}(X_0)$.

Proof The Teichmüller space is endowed with the topology defined by the Teichmüller metric d_{T}, and the uniform weak* topology on $\mathcal{C}_{\mathrm{bd}}(X_0)$ is metrizable by Lemma 4. It therefore suffices to show that, for every sequence $\big\{[f_n]\big\}_{n\in\mathbb{N}}$ converging to $[f_\infty]$ in $\mathcal{T}(X_0)$, the sequence of Liouville geodesic currents $\mathbf{L}\big([f_n]\big) = L_{f_n}$ converges to $\mathbf{L}\big([f_\infty]\big) = L_{f_\infty}$ in $\mathcal{C}_{\mathrm{bd}}(X_0)$ for the uniform weak* topology. By definition of the uniform weak* topology, this means that

$$\sup_{\varphi \in \mathbf{H}(\widetilde{X}_0)} \left| \int_{G(\widetilde{X}_0)} \xi \circ \varphi \, dL_{f_n} - \int_{G(\widetilde{X}_0)} \xi \circ \varphi \, dL_{f_\infty} \right| \to 0 \text{ as } n \to \infty$$

for every continuous function $\xi\colon G(\widetilde{X}_0) \to \mathbb{R}$ with compact support.

As a first step, we begin by proving a similar statement for boxes of geodesics in $\widetilde{X}_0$.

Lemma 20 *For every box* $Q \subset G(\widetilde{X}_0)$,

$$\sup_{\varphi \in \mathbf{H}(\widetilde{X}_0)} \Big| L_{f_n}(\varphi(Q)) - L_{f_\infty}(\varphi(Q)) \Big| \to 0 \text{ as } n \to \infty.$$

Proof By definition of the Teichmüller topology, the classes $[f_n], [f_\infty] \in \mathcal{T}(X_0)$ can be represented by quasiconformal maps $f_n : X_0 \to X_n$ and $f_\infty : X_0 \to X_\infty$ such that the quasiconformal constant $K(f_n \circ f_\infty^{-1})$ tends to 1 as n tends to ∞.

Lift f_n and f_∞ to quasiconformal maps $\widetilde{f}_n : \widetilde{X}_0 \to \widetilde{X}_n$ and $\widetilde{f}_\infty : \widetilde{X}_0 \to \widetilde{X}_\infty$, respectively, and consider their quasisymmetric extensions $\widetilde{f}_n : \partial_\infty \widetilde{X}_0 \to \partial_\infty \widetilde{X}_n$ and $\widetilde{f}_\infty : \partial_\infty \widetilde{X}_0 \to \partial_\infty \widetilde{X}_\infty$ to the circles at infinity.

A first observation is that, as $\varphi \in \mathbf{H}(\widetilde{X}_0)$ ranges over all biholomorphic diffeomorphisms of $\widetilde{X}_0$, the Liouville mass $L_{\widetilde{X}_0}\big(\varphi(Q)\big)$ is constant by invariance of the Liouville measure $L_{\widetilde{X}_0}$ under the action of $\mathbf{H}(\widetilde{X}_0)$. Applying Proposition 15 to the quasisymmetric maps $\widetilde{f}_\infty$ and $\widetilde{f}_\infty^{-1}$ then shows that $L_{\widetilde{X}_\infty}\big(\widetilde{f}_\infty(\varphi(Q))\big)$ stays in a compact subset of the interval $]0, \infty[$, independent of $\varphi \in \mathbf{H}(\widetilde{X}_0)$.

Since the quasiconformal dilatation $K(\widetilde{f}_n \circ \widetilde{f}_\infty^{-1}) = K(f_n \circ f_\infty^{-1})$ tends to 1, it follows from Theorem 14 that the quasisymmetric constant $M(\widetilde{f}_n \circ \widetilde{f}_\infty^{-1})$ of $\widetilde{f}_n \circ \widetilde{f}_\infty^{-1} : \partial_\infty \widetilde{X}_\infty \to \partial_\infty \widetilde{X}_n$ tends to 1 as $n \to \infty$. By Proposition 15 and using the property that $L_{\widetilde{X}_\infty}\big(\widetilde{f}_\infty(\varphi(Q))\big)$ is bounded away from 0 and ∞, it follows that

$$\limsup_{n\to\infty} \frac{L_{f_n}(\varphi(Q))}{L_{f_\infty}(\varphi(Q))} = \limsup_{n\to\infty} \frac{L_{\widetilde{X}_n}\Big(\widetilde{f}_n \circ \widetilde{f}_\infty^{-1}\big(\widetilde{f}_\infty(\varphi(Q))\big)\Big)}{L_{\widetilde{X}_\infty}\big(\widetilde{f}_\infty(\varphi(Q))\big)} \leqslant 1,$$

and this uniformly in $\varphi \in \mathbf{H}(\widetilde{X}_0)$.

Similarly, since $K(f_n) \leqslant K(f_n \circ f_\infty^{-1})K(f_\infty)$, the maps $f_n : X_0 \to X_n$ are uniformly quasiconformal and, as above, the Liouville masses $L_{f_n}\big(\varphi(Q)\big) = L_{\widetilde{X}_n}\big(\widetilde{f}_n(\varphi(Q))\big)$ stay bounded away from 0 and ∞. Replacing $\widetilde{f}_n \circ \widetilde{f}_\infty^{-1}$ by $\widetilde{f}_\infty \circ \widetilde{f}_n^{-1}$ in the argument above gives that

$$\limsup_{n\to\infty} \frac{L_{f_\infty}(\varphi(Q))}{L_{f_n}(\varphi(Q))} = \limsup_{n\to\infty} \frac{L_{\widetilde{X}_\infty}\Big(\widetilde{f}_\infty \circ \widetilde{f}_n^{-1}\big(\widetilde{f}_n(\varphi(Q))\big)\Big)}{L_{\widetilde{X}_n}\big(\widetilde{f}_n(\varphi(Q))\big)} \leqslant 1,$$

uniformly in $\varphi \in \mathbf{H}(\widetilde{X}_0)$.

Therefore,

$$\lim_{n\to\infty} \frac{L_{f_n}(\varphi(Q))}{L_{f_\infty}(\varphi(Q))} = 1$$

uniformly in $\varphi \in \mathbf{H}(\widetilde{X}_0)$. Since $L_{f_\infty}(\varphi(Q))$ is uniformly bounded away from 0 and ∞, it follows that $L_{f_n}(\varphi(Q))$ tends to $L_{f_\infty}(\varphi(Q))$ as $n \to \infty$, and this uniformly in $\varphi \in \mathbf{H}(\widetilde{X}_0)$. This proves Lemma 20. $\square$

We now return to the proof of Proposition 19. Consider a continuous test function $\xi : G(\widetilde{X}_0) \to \mathbb{R}$ with compact support.

We begin by covering the support of ξ by finitely many boxes $Q_1, Q_2, \dots, Q_m \subset G(\widetilde{X}_0)$.

For a number $\varepsilon_0 > 0$ to be specified later, we then cover the support of ξ by finitely many boxes $Q'_1, Q'_2, \dots, Q'_{m'} \subset G(\widetilde{X}_0)$, contained in the union of the boxes Q_i and small enough that

$$\left|\max_{x\in Q'_i} \xi(x) - \min_{x\in Q'_i} \xi(x)\right| < \varepsilon_0. \tag{7}$$

After subdividing these boxes $Q'_i = [a_i, b_i] \times [c_i, d_i]$, we can arrange that the boxes Q'_i have disjoint interiors. We then approximate ξ by the step function

$$\sigma = \sum_{i=1}^{m'} \xi(x_i^*)\chi_{Q'_i}$$

where x_i^* is an arbitrary point of Q'_i and where $\chi_{Q'_i} : G(\widetilde{X}_0) \to \mathbb{R}$ is the characteristic function of Q'_i. By construction, $|\xi - \sigma| \leqslant \varepsilon_0$ except possibly on the boundary of the boxes Q'_i.

Then, for every $\varphi \in \mathbf{H}(\widetilde{X}_0)$,

$$\begin{aligned}
&\left|\int_{G(\widetilde{X}_0)} (\xi\circ\varphi - \sigma\circ\varphi)\, d(L_{f_n} - L_{f_\infty})\right| \\
&\qquad \leqslant \varepsilon_0 \sum_{i=1}^{m'} \Big(L_{f_n}\big(\varphi^{-1}(Q'_i)\big) + L_{f_\infty}\big(\varphi^{-1}(Q'_i)\big)\Big) \\
&\qquad \leqslant \varepsilon_0 \sum_{j=1}^{m} \Big(L_{f_n}\big(\varphi^{-1}(Q_j)\big) + L_{f_\infty}\big(\varphi^{-1}(Q_j)\big)\Big)
\end{aligned} \tag{8}$$

using the properties that the boundary of a box has Liouville measure 0 and that $\bigcup_{i=1}^{m'} Q'_i$ is contained in $\bigcup_{j=1}^{m} Q_j$.

Similarly, once we have chosen the boxes Q'_i to approximate ξ by a step function, Lemma 20 shows that

$$\begin{aligned}
&\left|\int_{G(\mathbb{D})} (\sigma\circ\varphi)\, d(L_{f_n} - L_{f_\infty})\right| \\
&\qquad = \left|\sum_{i=1}^{m'} \xi\big(\varphi(x_i^*)\big)\Big(L_{f_n}\big(\varphi^{-1}(Q'_i)\big) - L_{f_\infty}\big(\varphi^{-1}(Q'_i)\big)\Big)\right| \\
&\qquad \to 0 \text{ as } n \to \infty,
\end{aligned} \tag{9}$$

and this uniformly in $\varphi \in \mathbf{H}(\widetilde{X}_0)$.

Suppose that we are given $\varepsilon > 0$, and that we have chosen the boxes Q_j to cover the support of ξ. Once this choice is made, Lemma 20 then shows that the term

$$\sum_{j=1}^{m} \Big(L_{f_n}\big(\varphi^{-1}(Q_j)\big) + L_{f_\infty}\big(\varphi^{-1}(Q_j)\big)\Big)$$

occurring on the last line of Eq. (8) is uniformly bounded. We can therefore pick a number $\varepsilon_0 > 0$ so that the contribution of (8) is less than $\varepsilon/2$. After choosing the boxes Q_i' so that (7) holds for this ε_0, the contribution of (9) will be less than $\varepsilon/2$ for n sufficiently large. Combining (8) and (9), we conclude that

$$\left|\int_{G(\widetilde{X}_0)} \xi \circ \varphi \, d(L_{f_\infty} - L_{f_\infty})\right| < \varepsilon$$

for n sufficiently large, and this uniformly in $\varphi \in \mathbf{H}(\widetilde{X}_0)$. This proves the continuity property of Proposition 19. □

3.6 The inverse map $\mathbf{L}^{-1}: \mathbf{L}\big(\mathcal{T}(X_0)\big) \to \mathcal{T}(X_0)$ is continuous

Proposition 21 *The inverse $\mathbf{L}^{-1}\colon \mathbf{L}\big(\mathcal{T}(X_0)\big) \to \mathcal{T}(X_0)$ of the Liouville embedding $\mathbf{L}\colon \mathcal{T}(X_0) \to \mathcal{C}_{\mathrm{bd}}(X_0)$ is continuous, for the Teichmüller topology on $\mathcal{T}(X_0)$ and for the uniform weak* topology on $\mathcal{C}_{\mathrm{bd}}(X_0)$.*

Proof Consider an element $[f_\infty]$ and a sequence $\big\{[f_n]\big\}_{n\in\mathbb{N}}$ of elements of the Teichmüller space $\mathcal{T}(X_0)$ such that the Liouville currents $L_{f_n} \in \mathcal{C}_{\mathrm{bd}}(X_0)$ converge to L_{f_∞} for the uniform weak* topology. We want to show that $[f_n]$ converges to $[f_\infty]$ for the Teichmüller topology of $\mathcal{T}(X_0)$.

As usual, represent the class $[f_n] \in \mathcal{T}(X_0)$ by quasiconformal maps $f_n\colon X_0 \to X_n$, and consider their quasiconformal lifts $\widetilde{f}_n\colon \widetilde{X}_0 \to \widetilde{X}_n$ and quasisymmetric extensions $\widetilde{f}_n\colon \partial_\infty \widetilde{X}_0 \to \partial_\infty \widetilde{X}_n$.

Lemma 22 *The quasisymmetric constants $M(f_n)$ of the quasisymmetric maps $\widetilde{f}_n\colon \partial_\infty \widetilde{X}_0 \to \partial_\infty \widetilde{X}_n$ are uniformly bounded.*

Proof We want to show that, as $Q \subset G(\widetilde{X}_0)$ ranges over all symmetric boxes in $\widetilde{X}_0$, the Liouville masses $L_{f_n}(Q)$ are uniformly bounded, independently of n and Q. For this, choose a symmetric box $Q_0 \subset G(\widetilde{X}_0)$, and a test function $\xi\colon G(\widetilde{X}_0) \to \mathbb{R}$ with compact support such that $\xi \geqslant 1$ over the box Q_0.

By definition of the uniform weak* topology,

$$\int_{G(\widetilde{X}_0)} \xi \circ \varphi \, dL_{f_n} \to \int_{G(\widetilde{X}_0)} \xi \circ \varphi \, dL_{f_\infty} \text{ as } n \to \infty$$

uniformly over all biholomorphic maps $\varphi \in \mathbf{H}(\widetilde{X}_0)$. The limit is uniformly bounded by Lemma 16. It follows that the integrals $\int_{G(\widetilde{X}_0)} \xi \circ \varphi \, dL_{f_n}$ are bounded by a constant C independent of n and $\varphi \in \mathbf{H}(\widetilde{X}_0)$.

Every symmetric box $Q \subset G(\widetilde{X}_0)$ is of the form $\varphi^{-1}(Q_0)$ for some $\varphi \in \mathbf{H}(\widetilde{X}_0)$. Then, since $\xi \geqslant 1$ over Q_0,

$$L_{\widetilde{X}_n}\big(\widetilde{f}_n(Q)\big) = L_{f_n}(Q) = L_{f_n}(\varphi^{-1}(Q_0)) \leqslant \int_{G(\widetilde{X}_0)} \xi \circ \varphi \, dL_{f_n} \leqslant C$$

so that the quasisymmetric constant $M(f_n) = M(\widetilde{f}_n)$ are bounded by $C/\log 2$. □

Lemma 23 *The quasisymmetric constant $M(f_n \circ f_\infty^{-1})$ converges to 1 as n tends to ∞.*

Proof We will use a proof by contradiction. If the property does not hold, there exists an $\varepsilon_0 > 0$ and a subsequence $\{[f_{n_k}]\}_{k\in\mathbb{N}}$ such that $M(f_{n_k} \circ f_\infty^{-1}) > 1 + \varepsilon_0$ for every k. (Recall that the quasisymmetric constant is always greater than or equal to 1). By definition of the quasisymmetric constant, this means that there exists a symmetric box Q'_{n_k} in $\widetilde{X}_\infty$ such that $L_{\widetilde{X}_{n_k}}\big(\widetilde{f}_{n_k} \circ \widetilde{f}_\infty^{-1}(Q'_{n_k})\big) > (1 + \varepsilon_0) \log 2$. We then have a box $Q_{n_k} = \widetilde{f}_\infty^{-1}(Q'_{n_k}) \subset G(\widetilde{X}_0)$ such that $L_{f_\infty}(Q_{n_k}) = \log 2$ and $L_{f_{n_k}}(Q_{n_k}) > (1 + \varepsilon_0) \log 2$.

Fix three points $a_0, b_0, c_0 \in \partial_\infty \widetilde{X}_0$, counterclockwise in this order. Then, there exists a biholomorphic map $\varphi_{n_k} \in \mathbf{H}(\widetilde{X}_0)$ such that the box $\varphi_{n_k}(Q_{n_k})$ is of the form $[a_0, b_0] \times [c_0, d_{n_k}]$ for some point d_{n_k} in the open interval $]c_0, a_0[\subset \partial_\infty \widetilde{X}_0$.

Since $\widetilde{f}_\infty \colon \widetilde{X}_0 \to \widetilde{X}_\infty$ is quasisymmetric and $L_{f_\infty}(Q_{n_k}) = \log 2$, Proposition 15 shows that the Liouville mass $L_{\widetilde{X}_0}(\varphi_{n_k}(Q_{n_k})) = L_{\widetilde{X}_0}(Q_{n_k})$ is bounded between two positive constants. It then follows from Lemma 10 that the point d_{n_k} stays within a compact subset of the interval $]c_0, a_0[$. Refining the subsequence if necessary, we can therefore assume that d_{n_k} converge to some point $d_\infty \in\,]c_0, a_0[$ as k tends to ∞. In other words, the box $\varphi_{n_k}(Q_{n_k})$ converge to the box $Q_\infty = [a_0, b_0] \times [c_0, d_\infty]$ as k tends to ∞.

For an $\varepsilon > 0$ to be specified later, choose intervals $]a'_0, a''_0[$, $]b''_0, b'_0[$, $]c'_0, c''_0[$ and $]d''_\infty, d'_\infty[\subset \partial_\infty \widetilde{X}_0$ respectively containing the points a_0, b_0, c_0, d_∞, and small enough that the following property holds. The box Q_∞ is contained in $Q'_\infty = [a'_0, b'_0] \times [c'_0, d'_\infty]$ and contains $Q''_\infty = [a''_0, b''_0] \times [c''_0, d''_\infty]$. By Lemma 22, the maps $\widetilde{f}_n \colon \widetilde{X}_0 \to \widetilde{X}_n$ are uniformly quasisymmetric. Therefore, noting that the closure of $Q'_\infty - Q''_\infty$ is the union of the four boxes $[a'_0, b'_0] \times [c'_0, c''_0]$, $[a'_0, b'_0] \times [d''_\infty, d'_\infty]$, $[a'_0, a''_0] \times [c'_0, d'_\infty]$ and $[b''_0, b'_0] \times [c'_0, d'_\infty]$, we can use Proposition 15 to choose the intervals $]a'_0, a''_0[$, $]b''_0, b'_0[$, $]c'_0, c''_0[$ and $]d''_\infty, d'_\infty[$ small enough that

$$\begin{aligned} L_{f_n}\big(\varphi(Q'_\infty - Q''_\infty)\big) &< \varepsilon \\ \text{and } L_{f_\infty}\big(\varphi(Q'_\infty - Q''_\infty)\big) &< \varepsilon \end{aligned} \tag{10}$$

for every n and every $\varphi \in \mathbf{H}(\widetilde{X}_0)$.

By construction, Q_∞ is contained in the interior of Q'_∞, and contains Q''_∞ in its interior. Let $\xi \colon G(\widetilde{X}_0) \to [0, 1]$ be a continuous test function that is identically 1 on the box Q''_∞ and 0 outside of Q'_∞. For k large enough, the box $\varphi_{n_k}(Q_{n_k})$ is very close to Q_∞ and therefore $Q''_\infty \subset \varphi_{n_k}(Q_{n_k}) \subset Q'_\infty$. As a consequence, $\chi_{\varphi_{n_k}^{-1}(Q''_\infty)} \leqslant \xi \circ \varphi_{n_k} \leqslant \chi_{\varphi_{n_k}^{-1}(Q'_\infty)}$ and $\chi_{\varphi_{n_k}^{-1}(Q''_\infty)} \leqslant \chi_{Q_{n_k}} \leqslant \chi_{\varphi_{n_k}^{-1}(Q'_\infty)}$ if $\chi_A \colon G(\widetilde{X}_0) \to \{0, 1\}$ denotes the characteristic function of the subset $A \subset G(\widetilde{X}_0)$. It follows that for k sufficiently large

$$\left| \int_{G(\widetilde{X}_0)} \xi \circ \varphi_{n_k}\, dL_{f_{n_k}} - L_{f_{n_k}}(Q_{n_k}) \right| \leqslant L_{f_{n_k}}\big(\varphi_{n_k}^{-1}(Q'_\infty - Q''_\infty)\big) < \varepsilon$$

by (10), and

$$\int_{G(\widetilde{X}_0)} \xi \circ \varphi_{n_k}\, dL_{f_{n_k}} > L_{f_{n_k}}(Q_{n_k}) - \varepsilon \\ > \log 2 + \varepsilon_0 \log 2 - \varepsilon \tag{11}$$

since the boxes Q_{n_k} were chosen so that $L_{f_n}(Q_{n_k}) > (1+\varepsilon_0)\log 2$.

Similarly,

$$\left| \int_{G(\widetilde{X}_0)} \xi \circ \varphi_{n_k}\, dL_{f_\infty} - L_{f_\infty}(Q_{n_k}) \right| \leqslant L_{f_\infty}\big(\varphi_{n_k}^{-1}(Q'_\infty - Q''_\infty)\big) < \varepsilon$$

and

$$\int_{G(\widetilde{X}_0)} \xi \circ \varphi_{n_k}\, dL_{f_\infty} < L_{f_\infty}(Q_{n_k}) + \varepsilon \\ < \log 2 + \varepsilon \tag{12}$$

since $L_{f_\infty}(Q_{n_k}) = \log 2$.

But, if we had chosen $\varepsilon > 0$ small enough that $2\varepsilon < \varepsilon_0 \log 2$, the inequalities (11) and (12) are incompatible with the fact that

$$\int_{G(\widetilde{X}_0)} \xi \circ \varphi_{n_k}\, dL_{f_{n_k}} \to \int_{G(\widetilde{X}_0)} \xi \circ \varphi_{n_k}\, dL_{f_\infty} \text{ as } k \to \infty$$

by uniform weak* convergence of $L_{f_{n_k}}$ to L_{f_∞}. This contradiction proves Lemma 23. □

By the property of Lemma 23, Theorem 17 then shows that $[f_n] \in \mathcal{T}(X_0)$ converges to $[f_\infty]$ for the Teichmüller metric. This completes the proof of Proposition 21. □

3.7 The image $\mathbf{L}\big(\mathcal{T}(X_0)\big)$ of the Liouville embedding is closed

Proposition 24 *The image $\mathbf{L}\big(\mathcal{T}(X_0)\big)$ of the Liouville embedding $\mathbf{L}\colon \mathcal{T}(X_0) \to \mathcal{C}_{\mathrm{bd}}(X_0)$ is closed in the space $\mathcal{C}_{\mathrm{bd}}(X_0)$ of bounded geodesic currents.*

Proof As before, the metrizability property of Lemma 4 enables us to argue in terms of sequences. Let $[f_n] \in \mathcal{T}(X_0)$ be a sequence in the Teichmüller space such that the associated Liouville geodesic currents $\mathbf{L}\big([f_n]\big) = L_{f_n}$ converge to some geodesic current $\alpha_\infty \in \mathcal{C}_{\mathrm{bd}}(X_0)$. We want to show that α_∞ is also in the image $\mathbf{L}\big(\mathcal{T}(X_0)\big)$.

As usual, lift the quasiconformal diffeomorphisms $f_n\colon X_0 \to X_n$ to maps $\widetilde{f}_n\colon \widetilde{X}_0 \to \widetilde{X}_n$ between universal covers, and consider the quasisymmetric extension $\widetilde{f}_n\colon \partial_\infty \widetilde{X}_0 \to \partial_\infty \widetilde{X}_n$. Because the Liouville geodesic currents L_{f_n} converge to α_∞ for the uniform weak* topology and because the limit α_∞ is bounded, the argument that we already used in the proof of Lemma 22 shows that the quasisymmetric constants $M(\widetilde{f}_n)$ are uniformly bounded.

Fix three points a_0, b_0, c_0 in this order in the circle at infinity $\partial_\infty \widetilde{X}_0$. Then, there is a unique biholomorphic map $\widetilde{g}_n \colon \widetilde{X}_n \to \mathbb{D}$ sending $\widetilde{f}_n(a_0)$ to 1, $\widetilde{f}_n(b_0)$ to i and $\widetilde{f}_n(c_0)$ to -1. The maps $\widetilde{g}_n \circ \widetilde{f}_n \colon \partial_\infty \widetilde{X}_0 \to \partial\mathbb{D}$ are uniformly quasisymmetric, and send the three points a_0, b_0, c_0 to the fixed points 1, i, -1. It easily follows that these maps $\widetilde{g}_n \circ \widetilde{f}_n$ are equicontinuous, so that we can extract a subsequence $\widetilde{g}_{n_k} \circ \widetilde{f}_{n_k}$ that converges to a homeomorphism $\widetilde{f}_\infty \colon \partial_\infty \widetilde{X}_0 \to \partial\mathbb{D}$ for the topology of uniform convergence (see for instance [22, §II.5] or [17, §16]).

By uniform quasisymmetry of the $\widetilde{f}_n$, the limit $\widetilde{f}_\infty$ is quasisymmetric. Also, if $\varphi \colon \widetilde{X}_0 \to \widetilde{X}_0$ is the biholomorphic diffeomorphism of $\widetilde{X}_0$ defined by an element $\varphi \in \pi_1(X_0)$ of the fundamental group, $\widetilde{f}_\infty \circ \varphi \circ \widetilde{f}_\infty^{-1} = \lim_{k\to\infty} \widetilde{f}_{n_k} \circ \varphi \circ \widetilde{f}_{n_k}^{-1}$ is a linear fractional map that is the restriction to $\partial\mathbb{D}$ of a biholomorphic diffeomorphism of $\mathbb{D}$. As φ ranges over all elements of $\pi_1(X_0)$, these $\widetilde{f}_\infty \circ \varphi \circ \widetilde{f}_\infty^{-1}$ define a discrete biholomorphic action of $\pi_1(X_0)$ on $\mathbb{D}$, and we can consider the Riemann surface $X_\infty = \mathbb{D}/\pi_1(X_0)$.

The Douady–Earle Extension Theorem [9] (see also our proof of Theorem 17) then provides a quasiconformal extension $\widetilde{f}_\infty \colon \widetilde{X}_0 \to \mathbb{D}$ of $f_\infty \colon \partial_\infty \widetilde{X}_0 \to \partial\mathbb{D}$ that commutes with the actions of $\pi_1(X_0)$ on $\widetilde{X}_0$ and $\mathbb{D}$, and therefore descends to a quasiconformal map $f_\infty \colon X_0 \to X_\infty = \mathbb{D}/\pi_1(X_0)$.

The uniform convergence of $\widetilde{g}_{n_k} \circ \widetilde{f}_{n_k}$ to $\widetilde{f}_\infty$ as $k \to \infty$ does not imply that $[f_{n_k}] \in \mathcal{T}(X_0)$ necessarily converges to $[f_\infty]$ for the Teichmüller topology. However, it is enough to guarantee that the pullback L_{f_∞} of the Liouville measure $L_\mathbb{D}$ by $\widetilde{f}_\infty$ is the weak* limit of the pullback of $L_\mathbb{D}$ by $\widetilde{g}_{n_k} \circ \widetilde{f}_{n_k}$, which also is the pullback $L_{f_{n_k}}$ of $L_{\widetilde{X}_{n_k}}$ by $\widetilde{f}_{n_k}$. Therefore $\alpha_\infty \in \mathcal{C}_{\mathrm{bd}}(X_0)$, which was defined as the uniform weak* limit of the Liouville geodesic currents L_{f_n}, is equal to $L_{f_\infty} = \mathbf{L}\big([f_\infty]\big)$. In particular, α_∞ is in the image of $\mathbf{L}$, as requested. □

3.8 The Liouville embedding is proper

Proposition 25 *The Liouville embedding* $\mathbf{L} \colon \mathcal{T}(X_0) \to \mathcal{C}_{\mathrm{bd}}(X_0)$ *is proper.*

Proof Recall that a map is *proper* if the preimage of a bounded set is bounded. We therefore need to prove the following property: Let B be a subset of $\mathcal{T}(X_0)$ such that

$$\sup_{[f]\in B} \sup_{\varphi\in \mathbf{H}(\widetilde{X}_0)} \left| \int_{G(\widetilde{X}_0)} \xi \circ \varphi \, dL_f \right| \leqslant C(\xi)$$

for every continuous function $\xi \colon G(\widetilde{X}_0) \to \mathbb{R}$ with compact support and for some constant $C(\xi)$ depending on ξ; then B is bounded for the Teichmüller metric of $\mathcal{T}(X_0)$.

For such a subset B, choose a symmetric box $Q_0 \subset G(\widetilde{X}_0)$ and a non-negative function $\xi \colon G(\widetilde{X}_0) \to \mathbb{R}$ with compact support such that $\xi \geqslant 1$ over the box Q_0. Then, as in the proof of Lemma 22, $L_f(Q) \leqslant C(\xi)$ for every symmetric box Q and every $[f] \in B$, and the quasisymmetric constants $M(f)$ are uniformly bounded over B. By Theorem 17, this proves that B is bounded by the Teichmüller metric. □

The combination of Propositions 19, 21, 24 and 25 proves Theorem 8, namely that the Liouville embedding $\mathbf{L}\colon \mathcal{T}(X_0) \to \mathcal{C}_{\mathrm{bd}}(X_0)$ is proper and induces a homeomorphism between $\mathcal{T}(X_0)$ and a closed subset of $\mathcal{C}_{\mathrm{bd}}(X_0)$.

We are going to need a slightly stronger version of this result.

3.9 The projectivization of the Liouville embedding

The group $\mathbb{R}^+$ of positive real numbers acts by multiplication on the space $\mathcal{C}_{\mathrm{bd}}(X_0)$ of bounded geodesic currents. Let $\mathcal{PC}_{\mathrm{bd}}(X_0) = \big(\mathcal{C}_{\mathrm{bd}}(X_0) - \{0\}\big)/\mathbb{R}^+$ be the quotient of $\mathcal{C}_{\mathrm{bd}}(X_0) - \{0\}$ under this action. We endow the space $\mathcal{PC}_{\mathrm{bd}}(X_0)$ with the quotient of the uniform weak* topology of $\mathcal{C}_{\mathrm{bd}}(X_0)$.

The elements of $\mathcal{PC}_{\mathrm{bd}}(X_0)$ are *projective bounded geodesic currents* in the Riemann surface X_0.

Composing the Liouville embedding $\mathbf{L}\colon \mathcal{T}(X_0) \to \mathcal{C}_{\mathrm{bd}}(X_0)$ with the projection $\mathcal{C}_{\mathrm{bd}}(X_0) \to \mathcal{PC}_{\mathrm{bd}}(X_0)$ gives a continuous map $\mathbf{PL}\colon \mathcal{T}(X_0) \to \mathcal{PC}_{\mathrm{bd}}(X_0)$, which we call the *projective Liouville embedding*. The following result shows that this projective Liouville embedding is really an embedding.

Theorem 26 *The map* $\mathbf{PL}\colon \mathcal{T}(X_0) \to \mathcal{PC}_{\mathrm{bd}}(X_0)$ *induces a homeomorphism between the Teichmüller space* $\mathcal{T}(X_0)$ *and a subset of the space* $\mathcal{PC}_{\mathrm{bd}}(X_0)$ *of projective bounded geodesic currents.*

Proof The map $\mathbf{PL}\colon \mathcal{T}(X_0) \to \mathcal{PC}_{\mathrm{bd}}(X_0)$ is injective. Indeed, if $\mathbf{PL}\big([f_1]\big) = \mathbf{PL}\big([f_2]\big)$ in $\mathcal{PC}_{\mathrm{bd}}(X_0)$, the Liouville current $\mathbf{L}\big([f_2]\big) = L_{f_2}$ is equal to $t\mathbf{L}\big([f_1]\big) = tL_{f_1}$ in $\mathcal{C}_{\mathrm{bd}}(X_0)$ for some number $t > 0$. The property of Lemma 12, that

$$\mathrm{e}^{-L_f(Q)} + \mathrm{e}^{-L_f(Q^\perp)} = 1$$

for every $[f] \in \mathcal{T}(X_0)$ and every box $Q \subset G(\widetilde{X}_0)$ with orthogonal box $Q^\perp$, then shows that necessarily $t = 1$. The injectivity of $\mathbf{PL}\colon \mathcal{T}(X_0) \to \mathcal{PC}_{\mathrm{bd}}(X_0)$ then follows from the injectivity of the Liouville embedding $\mathbf{L}\colon \mathcal{T}(X_0) \to \mathcal{C}_{\mathrm{bd}}(X_0)$ (Proposition 18).

The projective Liouville embedding $\mathbf{PL}$ was defined as the composition of two continuous maps, and is consequently continuous. Therefore, we only have to show that its inverse $\mathbf{PL}^{-1}\colon \mathbf{PL}\big(\mathcal{T}(X_0)\big) \to \mathcal{T}(X_0)$ is continuous.

For this, consider a sequence of points $[f_n] \in \mathcal{T}(X_0)$ such that $\lim_{n\to\infty} \mathbf{PL}\big([f_n]\big) = \mathbf{PL}\big([f_\infty]\big)$ in $\mathcal{PC}_{\mathrm{bd}}(X_0)$ for some $[f_\infty] \in \mathcal{T}(X_0)$. We want to show that $\lim_{n\to\infty}[f_n] = [f_\infty]$ in $\mathcal{T}(X_0)$.

By definition of the quotient topology, the property that $\lim_{n\to\infty} \mathbf{PL}\big([f_n]\big) = \mathbf{PL}\big([f_\infty]\big)$ means that there exists a sequence $r_n \in \mathbb{R}^+$ such that $\frac{1}{r_n}\mathbf{L}\big([f_n]\big) = \frac{1}{r_n}L_{f_n}$ converges to $\mathbf{L}\big([f_\infty]\big) = L_{f_\infty}$ in $\mathcal{C}_{\mathrm{bd}}(X_0)$, for the uniform weak* topology. In particular, $\frac{1}{r_n}L_{f_n}$ converges to L_{f_∞} for the (non uniform) weak* topology and, by Lemma 6, it follows that $\frac{1}{r_n}L_{f_n}(Q)$ converges to $L_{f_\infty}(Q)$ for every box $Q \subset G(\widetilde{X}_0)$. Another application of Lemma 12 then shows that necessarily $\lim_{n\to\infty} r_n = 1$.

As a consequence, $\lim_{n\to\infty} \mathbf{L}\big([f_n]\big) = \mathbf{L}\big([f_\infty]\big)$ in $\mathcal{C}_{\mathrm{bd}}(X_0)$. Since the inverse map $\mathbf{L}^{-1}\colon \mathbf{L}\big(\mathcal{T}(X_0)\big) \to \mathcal{T}(X_0)$ is continuous by Proposition 21, if follows that $\lim_{n\to\infty}[f_n] = [f_\infty]$ in $\mathcal{T}(X_0)$ as required. □

4 A boundary for the Teichmüller space

4.1 Measured geodesic laminations

A *measured geodesic lamination* in the Riemann surface X_0 is a geodesic current $\alpha \in \mathcal{C}(X_0)$ such that:

(1) α is balanced, in the sense that it is invariant under the involution $\tau \colon G(\widetilde{X}_0) \to G(\widetilde{X}_0)$ that reverses the orientation of each geodesic $g \in G(\widetilde{X}_0)$;

(2) any two distinct geodesics g, g' of the support $\operatorname{Supp}(\alpha) \subset G(\widetilde{X}_0)$ are disjoint in $\widetilde{X}_0$, unless $g' = \tau(g)$;

By equivariance of α, its support is invariant under the action of $\pi_1(X_0)$ and therefore descends to a *geodesic lamination* λ_α in X_0, namely to a family of disjoint simple complete geodesics (for the Poincaré metric of X_0) whose union forms a closed subset of X_0. Recall that a geodesic is *complete* if it cannot be extended to a longer geodesic, and that it is *simple* if it does not transversely intersect itself.

Beware that, in contrast to the classical case where X_0 is compact, the union of the geodesics of the geodesic lamination λ_α can have nonempty interior in X_0, and that this subset can have several decompositions as a union of pairwise disjoint complete geodesics.

A measured geodesic lamination is *bounded* if it is bounded as a geodesic current, as defined in Sect. 2. Let $\mathcal{ML}_{\mathrm{bd}}(X_0) \subset \mathcal{C}_{\mathrm{bd}}(X_0)$ denote the space of bounded measured geodesic laminations in the Riemann surface X_0.

4.2 The Thurston boundary of $\mathcal{T}(X_0)$

As in Sect. 3.9, consider the projective Liouville embedding $\mathbf{PL} \colon \mathcal{T}(X_0) \to \mathcal{PC}_{\mathrm{bd}}(X_0)$ from the Teichmüller space $\mathcal{T}(X_0)$ to the space $\mathcal{PC}_{\mathrm{bd}}(X_0)$ of projective bounded geodesic currents. We saw in Theorem 26 that $\mathbf{PL}$ induces a homeomorphism from $\mathcal{T}(X_0)$ to its image $\mathbf{PL}\big(\mathcal{T}(X_0)\big) \subset \mathcal{PC}_{\mathrm{bd}}(X_0)$.

By analogy with the case where X_0 is compact, we define the *Thurston boundary* of $\mathcal{T}(X_0)$ as the boundary of this embedding, namely as the set of points of $\mathcal{PC}_{\mathrm{bd}}(X_0)$ that are in the closure of $\mathbf{PL}\big(\mathcal{T}(X_0)\big)$ but are not contained in $\mathbf{PL}\big(\mathcal{T}(X_0)\big)$.

Our next goal is to describe this closure. Note that the space $\mathcal{ML}_{\mathrm{bd}}(X_0)$ of bounded measured geodesic laminations is invariant under the action of $\mathbb{R}^+$ on $\mathcal{C}_{\mathrm{bd}}(X_0)$. It therefore makes sense to consider its image $\mathcal{PML}_{\mathrm{bd}}(X_0) = \big(\mathcal{ML}_{\mathrm{bd}}(X_0) - \{0\}\big)/\mathbb{R}^+$ in $\mathcal{PC}_{\mathrm{bd}}(X_0)$. By definition, the points of $\mathcal{PML}_{\mathrm{bd}}(X_0)$ are *projective bounded measured geodesic laminations* in X_0.

Proposition 27 *The Thurston boundary of the Teichmüller space $\mathcal{T}(X_0)$ is contained in the space $\mathcal{PML}_{\mathrm{bd}}(X_0)$ of projective bounded measured geodesic laminations.*

Proof Let $\alpha \in \mathcal{C}_{\mathrm{bd}}(X_0)$ be a bounded geodesic current whose image $\langle\alpha\rangle \in \mathcal{PC}_{\mathrm{bd}}(X_0)$ is in the Thurston boundary. In particular, $\langle\alpha\rangle$ is in the closure of $\mathbf{PL}\big(\mathcal{T}(X_0)\big)$, and there exists a sequence $[f_n] \in \mathcal{T}(X_0)$ and numbers $t_n > 0$ such that

$$\alpha = \lim_{n\to\infty} \frac{1}{t_n}\mathbf{L}\big([f_n]\big) = \lim_{n\to\infty} \frac{1}{t_n} L_{f_n}.$$

We claim that $t_n \to \infty$ as $n \to \infty$. Indeed, we would otherwise find a subsequence t_{n_k} converging to some $t_\infty \geqslant 0$ as $k \to \infty$. Then, $t_\infty \alpha = \lim_{k\to\infty} L_{f_{n_k}}$ would belong to $\mathbf{L}\big(\mathcal{T}(X_0)\big)$ since this image is closed by Theorem 8. Note that t_∞ cannot be equal to 0, as otherwise $\mathbf{L}\big(\mathcal{T}(X_0)\big)$ would contain the trivial geodesic current $0 \in \mathcal{C}_{\mathrm{bd}}(X_0)$ while Liouville currents clearly are never trivial. But it cannot be different from 0 either, as this would otherwise contradict the fact that $\langle\alpha\rangle$ is not allowed to belong to $\mathbf{PL}\big(\mathcal{T}(X_0)\big)$, by definition of the Thurston boundary.

Now suppose, in search of a contradiction, that α is not a measured geodesic lamination. This means that the support of α contains two geodesics $g, g' \in G(\widetilde{X}_0)$ that cross each other in $\widetilde{X}_0$. We can then find a box $Q \subset G(\widetilde{X}_0)$ containing g in its interior such that the orthogonal box $Q^\perp$ contains g' in its interior (possibly after reversing the orientation of g'). In particular, $\alpha(Q) > 0$ and $\alpha(Q^\perp) > 0$. In addition, by countable additivity of α, we can choose the points of $\partial_\infty \widetilde{X}_0$ delimiting Q so that $\alpha(\partial Q) = \alpha(\partial Q^\perp) = 0$. Then, by weak* convergence (see Lemma 6),

$$\alpha(Q) = \lim_{n\to\infty} \frac{1}{t_n} L_{f_n}(Q) \text{ and } \alpha(Q^\perp) = \lim_{n\to\infty} \frac{1}{t_n} L_{f_n}(Q^\perp),$$

so that

$$\lim_{n\to\infty} L_{f_n}(Q) = \lim_{n\to\infty} L_{f_n}(Q^\perp) = \infty$$

since we established that $t_n \to \infty$ as $n \to \infty$. But this contradicts Lemma 12, and the fact that $\mathrm{e}^{-L_{f_n}(Q)} + \mathrm{e}^{-L_{f_n}(Q^\perp)} = 1$.

Therefore, the support of α is a geodesic lamination, and $\langle\alpha\rangle$ belongs to the space $\mathcal{PML}_{\mathrm{bd}}(X_0)$ of projective bounded measured geodesic laminations. □

We prove the converse of Proposition 27 as Corollary 31 in the next section. The combination of these two statements will show:

Theorem 28 *The Thurston boundary of the Teichmüller space $\mathcal{T}(X_0)$ is exactly equal to the space $\mathcal{PML}_{\mathrm{bd}}(X_0)$ of projective bounded measured geodesic laminations.* □

5 Earthquakes

We will use earthquakes as a tool to show that every projective bounded measured geodesic lamination is contained in the Thurston boundary of $\mathcal{T}(X_0)$. The key technical step is Theorem 30 below, which is of independent interest.

5.1 Earthquakes

Let λ be a geodesic lamination in the Riemann surface X_0, namely a family of disjoint simple complete geodesics in X_0 whose union is closed in X_0. Let $\widetilde{\lambda} \subset G(\widetilde{X}_0)$ consist of those geodesics which project to one of the geodesics of λ. In particular, $\widetilde{\lambda}$ is invariant under the involution $\tau : G(\widetilde{X}_0) \to G(\widetilde{X}_0)$ that acts by reversing the orientation of each geodesic. A simple argument also shows that $\widetilde{\lambda}$ is closed in $G(\widetilde{X}_0)$.

If $[f]$, $[f'] \in \mathcal{T}(X_0)$ are two points of the Teichmüller space of X_0, we say that $[f']$ is obtained from $[f]$ by a *left earthquake along* λ if

$$L_f(Q) \leqslant L_{f'}(Q)$$

for every box of geodesics $Q = [a, b] \times [c, d] \subset G(\widetilde{X}_0)$ such that $\{a, c\} \in \partial_\infty \widetilde{S}$ are the endpoints of one of the geodesics of $\widetilde{\lambda}$.

Thurston [33] shows how to quantify the increase in Liouville masses by a measure on the closed subset $\widetilde{\lambda} \subset G(\widetilde{X}_0)$, namely by a measure α on $G(\widetilde{X}_0)$ whose support is contained in $\widetilde{\lambda}$. In addition, α is invariant under the action of the fundamental group $\pi_1(X_0)$, and consequently is a measured geodesic lamination. A subtler consequence of the fact that f is quasiconformal is that α is bounded; see [12,16,28,29,33].

Thurston also introduced an inverse construction [10,33] which, given a point $[f] \in \mathcal{T}(X_0)$ and a bounded measured geodesic lamination $\alpha \in \mathcal{ML}_{\mathrm{bd}}(X_0)$, produces another element $[f'] \in \mathcal{T}(X_0)$ that is obtained from $[f]$ by a left earthquake along the support λ_α of α, with amplitude determined by the measure α. We then write that $[f'] = E^\alpha[f]$.

Finally, Thurston shows [33] that for any two $[f]$, $[f'] \in \mathcal{T}(X_0)$ there exists a unique $\alpha \in \mathcal{ML}_{\mathrm{bd}}(X_0)$ such that $[f'] = E^\alpha[f]$. See also [21].

Remark 29 We should emphasize the close relationship between the boundedness property for measured geodesic laminations and the quasiconformal geometry underlying the Teichmüller space. Thurston's construction [33] makes sense in the broader context of diffeomorphisms $f: X_0 \to X$ whose lift to universal covers continuously extends to a homeomorphism $\partial_\infty \widetilde{X}_0 \to \partial_\infty \widetilde{X}$. These are not necessarily quasiconformal, so that they do not necessarily define an element $[f] \in \mathcal{T}(X_0)$, but the equivalence relation defining the Teichmüller space makes sense in this more general context. Thurston shows that any two such $f: X_0 \to X$ and $f': X_0 \to X'$ are related by an earthquake, namely that $[f'] = E^\alpha[f]$ for some measured geodesic lamination α which is not necessarily bounded. However, when X_0 is noncompact, there is no easy characterization of which measured geodesic laminations $\alpha \in \mathcal{ML}(X_0)$ occur in this way. The results mentioned above show that, when f is quasiconformal, $E^\alpha[f]$ is well-defined and realized by a quasiconformal diffeomorphism f' precisely when α is bounded.

This distinction is of course irrelevant when X_0 is compact, as every diffeomorphism $f: X_0 \to X$ is then quasiconformal, and every measured geodesic lamination is bounded by Proposition 5.

For a bounded measured geodesic lamination $\alpha \in \mathcal{ML}_{\mathrm{bd}}(X_0)$ and a number $t > 0$, let $t\alpha$ be the bounded measured geodesic lamination obtained by multiplying the measure α by t. The following theorem investigates the behavior of $E^{t\alpha}[f] \in \mathcal{T}(X_0)$ under the Liouville embedding $\mathbf{L}: \mathcal{T}(X_0) \to \mathcal{C}_{\mathrm{bd}}(X_0)$.

Theorem 30 *Let $\alpha \in \mathcal{ML}_{\mathrm{bd}}(X_0)$ be a bounded measured geodesic lamination in the Riemann surface X_0. Then, for every $[f] \in \mathcal{T}(X_0)$,*

$$\lim_{t\to\infty} \frac{1}{t} \mathbf{L}\big(E^{t\alpha}[f]\big) = \alpha$$

for the uniform weak topology on the space $\mathcal{C}_{\mathrm{bd}}(X_0)$ of geodesic currents.*

The proof of Theorem 30 will occupy the rest of this section. However, it has the following immediate corollary, which completes the proof of Theorem 28.

Corollary 31 *The space $\mathcal{PML}_{\mathrm{bd}}(X_0)$ of projective bounded measured geodesic laminations is contained in the Thurston boundary of the Teichmüller space $\mathcal{T}(X_0)$.*

Proof Theorem 30 shows that every projective bounded measured geodesic lamination $\langle\alpha\rangle \in \mathcal{PML}_{\mathrm{bd}}(X_0)$ is in the closure of the image of the projective Liouville embedding $\mathbf{PL}\colon \mathcal{T}(X_0) \to \mathcal{PC}_{\mathrm{bd}}(X_0)$. A Liouville geodesic current has full support in $G(\widetilde{X}_0)$, and a measured geodesic lamination cannot have full support. It follows that $\langle\alpha\rangle \in \mathcal{PML}_{\mathrm{bd}}(X_0)$ does not belong to the image $\mathbf{PL}\big(\mathcal{T}(X_0)\big)$, and therefore is in the Thurston boundary of $\mathcal{T}(X_0)$ by definition of this boundary. □

5.2 Elementary earthquakes

The construction of the earthquake deformations $E^\alpha[f]$ is based on the following special case.

Let $\widetilde{X}_0$ be a simply connected conformally hyperbolic Riemann surface. (We are using a tilde in the notation to remind the reader that the surface is simply connected, and therefore equal to its universal cover.) In particular, $\widetilde{X}_0$ is biholomorphically equivalent to the disk $\mathbb{D}$.

For a geodesic $g \in G(\widetilde{X}_0)$ and a number $t \in \mathbb{R}$, the *elementary earthquake of amplitude t along g* is the homeomorphism $E^t_g\colon \mathcal{T}(\widetilde{X}_0) \to \mathcal{T}(\widetilde{X}_0)$ defined as follows.

Let $[f] \in \mathcal{T}(\widetilde{X}_0)$ be a point in the Teichmüller space of $\widetilde{X}_0$, represented by a quasiconformal diffeomorphism $f\colon \widetilde{X}_0 \to \widetilde{X}_1$. If g_1 is the geodesic of $\widetilde{X}_1$ that is the image of g under the map $f\colon G(\widetilde{X}_0) \to G(\widetilde{X}_1)$ induced by f, and let $\varphi_t\colon \widetilde{X}_1 \to \widetilde{X}_1$ be the hyperbolic isometry that preserves g_1 and acts by translation of $t \in \mathbb{R}$ along g_1 for the orientation of g_1. Then $E^t_g[f] \in \mathcal{T}(\widetilde{X}_0)$ is represented by any quasiconformal extension of the quasisymmetric homeomorphism $E^t_g f\colon \partial_\infty\widetilde{X}_0 \to \partial_\infty\widetilde{X}_1$ that coincides with f on the component of $\partial_\infty\widetilde{X}_0 - \partial g$ that sits to the left of g, and with $\varphi_t \circ f$ on the other component of $\partial_\infty\widetilde{X}_0 - \partial g$. Equivalently, $E^t_g[f]$ is represented by the quasisymmetric homeomorphism $\varphi_t^{-1} \circ E^t_g f\colon \partial_\infty\widetilde{X}_0 \to \partial_\infty\widetilde{X}_1$ that coincides with $\varphi_t^{-1} \circ f$ on the component of $\partial_\infty\widetilde{X}_0 - \partial g$ that sits to the left of g, and with f on the other component of $\partial_\infty\widetilde{X}_0 - \partial g$.

From the fact that φ_t is an isometry of $\widetilde{X}_1$, it easily follows that reversing the orientation of the geodesic g does not change $E^t_g[f] \in \mathcal{T}(\widetilde{X}_0)$.

General earthquakes $E^\alpha\colon \mathcal{T}(\widetilde{X}_0) \to \mathcal{T}(\widetilde{X}_0)$ are constructed from elementary earthquakes as follows.

First consider the case where $\delta \in \mathcal{ML}_{\mathrm{bd}}(\widetilde{X}_0)$ is a Dirac measure with finite support $\{g_1, g_2, \dots, g_k, \bar{g}_1, \bar{g}_2, \dots, \bar{g}_k\} \subset G(\widetilde{X}_0)$, where $\bar{g}_i = \tau(g_i)$ is obtained by reversing the orientation of the geodesic $g_i \in G(\widetilde{X}_0)$. Then, E^δ is defined as

$$E^\delta = E^{d_1}_{g_1} \circ E^{d_2}_{g_2} \circ \cdots \circ E^{d_k}_{g_k}$$

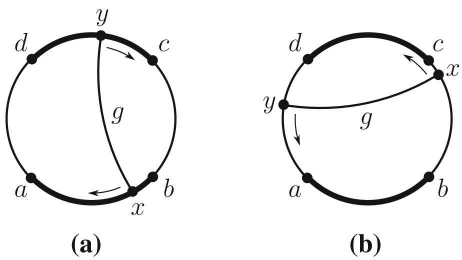


Fig. 1 The arrows indicate the direction in which the endpoints of g can be moved in order to increase $L_{E^t_g[f]}(Q)$ when $t > 0$

where $d_i = \delta(\{g_i\}) = \delta(\{\bar{g}_i\})$. Note that the elementary earthquakes $E^{d_i}_{g_i} : \mathcal{T}(\widetilde{X}_0) \to \mathcal{T}(\widetilde{X}_0)$ commute because the geodesics g_i are disjoint.

In the general case, we approximate the measured geodesic lamination $\alpha \in \mathcal{ML}_{\mathrm{bd}}(\widetilde{X}_0)$ by Dirac measures δ as above, and define

$$E^\alpha[f] = \lim_{\delta \to \alpha} E^\delta[f]$$

for every $[f] \in \mathcal{T}(\widetilde{X}_0)$, where the limit is taken as the Dirac measure δ tends to α for the weak* topology. The boundedness of α is used to show that the limit really exists. See [10,28,33] for details.

When $\widetilde{X}_0$ is the universal cover of a conformally hyperbolic Riemann surface X_0 and when $\alpha \in \mathcal{ML}_{\mathrm{bd}}(X_0) \subset \mathcal{ML}_{\mathrm{bd}}(\widetilde{X}_0)$, the above construction is equivariant with respect to the action of $\pi_1(X_0)$ on $\mathcal{T}(\widetilde{X}_0)$, and the earthquake $E^\alpha : \mathcal{T}(\widetilde{X}_0) \to \mathcal{T}(\widetilde{X}_0)$ therefore descends to a continuous map $E^\alpha : \mathcal{T}(X_0) \to \mathcal{T}(X_0)$.

5.3 Two lemmas on elementary earthquakes

We will make frequent use of the following two lemmas.

Lemma 32 *Let $Q = [a, b] \times [c, d]$ be a box of geodesics in $G(\widetilde{X}_0)$, and let $g \in G(\widetilde{X}_0)$ be a geodesic with endpoints $x, y \in \partial_\infty \widetilde{X}_0 - \{a, b, c, d\}$. Consider the image $E^t_g[f]$ of $[f] \in \mathcal{T}(\widetilde{X}_0)$ under the elementary earthquake of amplitude $t > 0$ along g.*

(0) *If x and y are in the same component of $\partial_\infty \widetilde{X}_0 - \{a, b, c, d\}$, then $L_{E^t_g[f]}(Q) = L_{[f]}(Q)$ is independent of x and y.*
(a) *It $x \in]a, b[$ and $y \in]c, d[$ as in Fig. 1a, $L_{E^t_g[f]}(Q)$ is a decreasing function of x and y for the boundary orientation of $\partial_\infty \widetilde{X}_0$.*
(b) *It $x \in]b, c[$ and $y \in]d, a[$ as in Fig. 1b, $L_{E^t_g[f]}(Q)$ is an increasing function of x and y.*

The statement is expressed in a more pictorial way by Fig. 1.

Proof of Lemma 32(0) If x and y are in the same component of $\partial_\infty \widetilde{X}_0 - \{a, b, c, d\}$, let $[f]$ be represented by a quasisymmetric homeomorphism $f : \partial_\infty \widetilde{X}_0 \to \partial_\infty \widetilde{X}_1$. Then, by definition of the elementary earthquake, $E^t_g[f]$ is represented by a quasisymmetric homeomorphism $E^t_g f$ that coincides with f at the points a, b, c, d. If follows that $E^t_g f(Q) = f(Q)$ in $G(\widetilde{X}_1)$, so that $L_{E^t_g[f]}(Q) = L_{[f]}(Q)$. $\square$

Proof of Lemma 32(a) In this second case (a), we can represent $[f]$ by a quasiconformal diffeomorphism $f\colon \widetilde{X}_0 \to \mathbb{H}$ valued in the upper half-space

$$\mathbb{H} = \{z \in \mathbb{C};\, \mathrm{Im}(z) > 0\}.$$

In addition, we can arrange that $f(y) = \infty$, and set $\alpha = f(a)$, $\beta = f(b)$, $\gamma = f(c)$, $\delta = f(d)$ and $\xi = f(x)$. Note that $\delta < \alpha < \xi < \beta < \gamma$ in $\mathbb{R}$.

Then, by Lemma 10,

$$L_{[f]}(Q) = L_{\mathbb{H}}\big([\alpha, \beta] \times [\gamma, \delta]\big) = \log \frac{(\alpha-\gamma)(\beta-\delta)}{(\alpha-\delta)(\beta-\gamma)}.$$

Also, the hyperbolic isometry of $\mathbb{H}$ that acts by translation of t along the geodesic $\xi\infty$ is the map $z \mapsto \mathrm{e}^t z + \xi - \mathrm{e}^t \xi$. Therefore

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\xi} L_{E^t_g[f]}(Q) &= \frac{\mathrm{d}}{\mathrm{d}\xi} \log \frac{(\alpha - \mathrm{e}^t\gamma - \xi + \mathrm{e}^t\xi)(\mathrm{e}^t\beta + \xi - \mathrm{e}^t\xi - \delta)}{(\alpha-\delta)(\mathrm{e}^t\beta - \mathrm{e}^t\gamma)} \\
&= \frac{-1+\mathrm{e}^t}{\alpha - \mathrm{e}^t\gamma - \xi + \mathrm{e}^t\xi} + \frac{1-\mathrm{e}^t}{\mathrm{e}^t\beta + \xi - \mathrm{e}^t\xi - \delta} \\
&= \frac{1-\mathrm{e}^t}{(\xi-\alpha) + \mathrm{e}^t(\gamma-\xi)} + \frac{1-\mathrm{e}^t}{\mathrm{e}^t(\beta-\xi) + (\xi-\delta)} \quad < 0
\end{aligned}$$

where the inequality comes from the fact that $\delta < \alpha < \xi < \beta < \gamma$ and $t > 0$.

It follows that $L_{E^t_g[f]}(Q)$ is a decreasing function of $\xi = f(x) \in \mathbb{R}$, and therefore of the endpoint $x \in \partial_\infty \widetilde{X}_0$ of the geodesic g.

By symmetry, $L_{E^t_g[f]}(Q)$ is also a decreasing function of the endpoint y. □

Proof of Lemma 32(b) Consider the orthogonal box $Q^\perp$ of Q. Case (a) shows that $L_{E^t_g[f]}(Q^\perp)$ is a decreasing function of the endpoints x and y. The relation between $L_{E^t_g[f]}(Q)$ and $L_{E^t_g[f]}(Q^\perp)$ provided by Lemma 12 then shows that $L_{E^t_g[f]}(Q)$ is an increasing function of x and y. □

Lemma 33 *Let $E^t_{ac}\colon \mathcal{T}(\widetilde{X}_0) \to \mathcal{T}(\widetilde{X}_0)$ be the elementary earthquake associated to the diagonal geodesic ac of the box $Q = [a, b] \times [c, d]$. Then, for every $[f] \in \mathcal{T}(\widetilde{X}_0)$ and every $t > 0$,*

$$t + \log\left(\mathrm{e}^{L_{[f]}(Q)} - 1\right) < L_{E^t_{ac}[f]}(Q) < t + L_{[f]}(Q).$$

Proof Represent the class $[f] \in \mathcal{T}(\widetilde{X}_0)$ by a quasiconformal map $f\colon \widetilde{X}_0 \to \mathbb{H}$ such that $f(a) = 0$, $f(b) = \beta$, $f(c) = \infty$ and $f(d) = -1$. Then, as in the proof of Lemma 32(a) (with $\alpha = \xi = 0$, $\gamma = \eta = \infty$ and $\delta = -1$),

$$L_{E^t_{ac}[f]}(Q) = \log(\mathrm{e}^t\beta + 1).$$

In particular, the case $t = 0$ gives that $\beta = \mathrm{e}^{L_{[f]}(Q)} - 1$.

Then, because $t > 0$,

$$L_{E^t_{ac}[f]}(Q) = t + \log(\beta + e^{-t}) < t + \log(\beta + 1) = t + L_{[f]}(Q)$$

while

$$L_{E^t_{ac}[f]}(Q) = t + \log(\beta + e^{-t}) > t + \log(\beta) = t + \log\left(e^{L_{[f]}(Q)} - 1\right).$$

□

5.4 Simple convergence on boxes

This section is devoted to proving Lemma 35, which is a key technical step in the proof of Theorem 30. As a warm-up, we begin with a simpler statement.

It will be convenient to say that, for a geodesic current $\alpha \in \mathcal{C}_{\mathrm{bd}}(\widetilde{X}_0)$, the box $Q = [a, b] \times [c, d]$ is α-*generic* if the subset of $G(\widetilde{X}_0)$ consisting of those geodesics with one endpoint in $\{a, b, c, d\}$ has α-mass 0. Using the countable additivity of α, there can be at most countably many $x \in \partial_\infty \widetilde{X}_0$ such that the set of geodesics passing through x has positive α-mass. As a consequence, every box can be arbitrarily approximated by an α-generic box.

Lemma 34 *Let $\alpha \in \mathcal{ML}_{\mathrm{bd}}(X_0)$ be a bounded measured geodesic lamination. Then, for every α-generic box $Q \subset G(\widetilde{X}_0)$,*

$$\lim_{t\to+\infty} \frac{1}{t} \mathbf{L}\big(E^{t\alpha}[f]\big)(Q) = \alpha(Q).$$

Proof As usual, let the box Q be described as $Q = [a, b] \times [c, d]$ with a, b, c, $d \in \partial_\infty \widetilde{X}_0$.

We will split the proof into several steps.

STEP 1. $\liminf_{t\to+\infty} \frac{1}{t} \mathbf{L}\big(E^{t\alpha}[f]\big)(Q) \geqslant \alpha(Q)$.

We only need to consider the case where $\alpha(Q) > 0$.

Then, because of the hypothesis that Q is α-generic, there is a strictly smaller box $Q' = [a, b'] \times [c, d']$ such that $a < b' < b$, $c < d' < d$ and $\alpha(Q')$ is arbitrarily close to $\alpha(Q)$. Since $\alpha(Q')$ is close to $\alpha(Q) > 0$ it is different from 0, and Q' meets the support of α. Among the (disjoint) geodesics of the support of α that are contained in Q', let $a''d''$ be the one that is closest to the interval $[d', a] \subset \partial_\infty \widetilde{X}_0$, and let $b''c''$ be the one closest to $[b', c]$, in such a way that $a \leqslant a'' \leqslant b'' \leqslant b'$ and $c \leqslant c'' \leqslant d'' \leqslant d'$. See Fig. 2.

We now consider the box $Q'' = [a'', b] \times [c'', d]$. Our construction is specially designed that the geodesics g of the support of α are of four distinct types with respect to $Q'' = [a'', b] \times [c'', d]$:

(1) g has both endpoints in the closure of the same component of $\partial_\infty \widetilde{X}_0 - \{a'', b, c'', d\}$;
(2) g has one endpoint in $[a'', b]$ and one endpoint in $[b, c'']$;
(3) g has one endpoint in $[c'', d]$ and one endpoint in $[d, a]$;

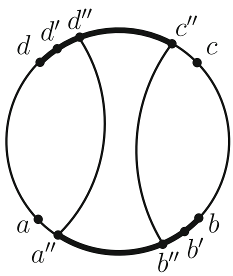


Fig. 2 Step 1 of the proof of Lemma 34

(4) g has one endpoint in $[a'', b]$ and another endpoint in $[c'', d]$.

Indeed, the presence of the geodesics $a''d''$ and $b''c''$ in the support of α excludes all other cases.

We can therefore decompose α as a sum of measured geodesic laminations

$$\alpha = \alpha_o + \alpha_b + \alpha_d + \alpha_{Q''}$$

where

- the support of α_b consists of geodesics of type (2), which encircle the point b;
- the support of α_d consists of geodesics of type (3), which encircle the point d;
- the support of $\alpha_{Q''}$ consists of geodesics of type (4), which are contained in the box Q'' (after a possible orientation reversal);
- the support of α_o consists of geodesics of type (1) (where o stands for "other").

This decomposes the earthquake $E^{t\alpha} : \mathcal{T}(\widetilde{X}_0) \to \mathcal{T}(\widetilde{X}_0)$ as a composition

$$E^{t\alpha} = E^{t\alpha_o} \circ E^{t\alpha_d} \circ E^{t\alpha_b} \circ E^{t\alpha_{Q''}}.$$

For notational convenience, set $[f_1] = E^{t\alpha_{Q''}}[f]$, $[f_2] = E^{t\alpha_b}[f_1]$, $[f_3] = E^{t\alpha_d}[f_2]$ and $[f_4] = E^{t\alpha_o}[f_3] = E^{t\alpha}[f]$.

We begin by estimating $\mathbf{L}\big([f_1]\big)(Q'') = \mathbf{L}\big(E^{t\alpha_{Q''}}[f]\big)(Q'')$.

If we approximate the measured lamination $\alpha_{Q''}$ by a Dirac measure supported on a finite set $\{g_1, g_2, \dots, g_k, \bar{g}_1, \bar{g}_2, \dots, \bar{g}_k\}$ of disjoint geodesics in the support of $\alpha_{Q''}$ and assigning mass $a_i > 0$ to the atom g_i, then by construction $E^{t\alpha''_Q}$ is approximated by the product of elementary earthquakes

$$E^{ta_1}_{g_1} \circ E^{ta_1}_{g_2} \circ \cdots \circ E^{ta_n}_{g_n}.$$

By definition of a'' and c'', the geodesics of the support of $\alpha_{Q''}$ actually have one endpoint in $[a'', b''] \subset [a'', b']$ and one endpoint in $[c'', d''] \subset [c'', d']$. Lemma 32(a) shows that, for each such geodesic g,

$$\mathbf{L}\big(E^u_g[f']\big)(Q) \geqslant \mathbf{L}\big(E^u_{b'd'}[f']\big)(Q)$$

for every $[f'] \in \mathcal{T}(\widetilde{X}_0)$ and every $u > 0$. It follows that

$$\mathbf{L}\big(E^{ta_1}_{g_1} E^{ta_1}_{g_2} \dots E^{ta_n}_{g_n}[f]\big)(Q'') \geqslant \mathbf{L}\big(E^{t(a_1+a_2+\dots+a_n)}_{b'd'}[f]\big)(Q'')$$

and, passing to the limit as we improve the approximation of $\alpha_{Q''}$ by Dirac measures, that

$$\mathbf{L}\big([f_1]\big)(Q'') = \mathbf{L}\big(E^{t\alpha_{Q''}}[f]\big)(Q'') \geqslant \mathbf{L}\big(E^{t\alpha(Q'')}_{b'd'}[f]\big)(Q'')$$

for every $t > 0$.

The box $Q'' = [a'', b] \times [c'', d]$ contains the box $Q''' = [b', b] \times [d', d]$. Lemma 33 then shows that

$$\begin{aligned}\mathbf{L}\big([f_1]\big)(Q'') \geqslant \mathbf{L}\big(E^{t\alpha(Q'')}_{b'd'}[f]\big)(Q'') \geqslant \mathbf{L}\big(E^{t\alpha(Q'')}_{b'd'}[f]\big)(Q''')\\ \geqslant t\alpha(Q'') + \log\big(\mathrm{e}^{L_{[f]}(Q''')} - 1\big).\end{aligned} \tag{13}$$

After this estimate for $\mathbf{L}\big([f_1]\big)(Q'')$, we now consider $[f_2] = E^{t\alpha_b}[f_1]$. By construction, the Liouville current $\mathbf{L}\big([f_2]\big) = \mathbf{L}\big(E^{t\alpha_b}[f_1]\big)$ is the pullback of $\mathbf{L}\big([f_1]\big)$ by a homeomorphism of $G(\widetilde{X}_0)$ that sends $Q'' = [a'', b] \times [c'', d]$ to a larger box $Q''_1 = [a'', b_1] \times [c'', d]$ with $b \leqslant b_1 < c''$. Therefore,

$$\mathbf{L}\big([f_2]\big)(Q'') = \mathbf{L}\big(E^{t\alpha_b}[f_1]\big)(Q'') = \mathbf{L}\big([f_1]\big)(Q''_1) \geqslant \mathbf{L}\big([f_1]\big)(Q'') \tag{14}$$

since Q''_1 contains Q''.

Similarly,

$$\mathbf{L}\big([f_3]\big)(Q'') = \mathbf{L}\big(E^{t\alpha_d}[f_2]\big)(Q'') \geqslant \mathbf{L}\big([f_2]\big)(Q''). \tag{15}$$

Finally, $\mathbf{L}\big([f_4]\big) = \mathbf{L}\big(E^{t\alpha_o}[f_3]\big)$ is the pullback of $\mathbf{L}\big([f_3]\big)$ by a homeomorphism of $G(\widetilde{X}_0)$ that sends Q'' to itself. Therefore

$$\mathbf{L}\big([f_4]\big)(Q'') = \mathbf{L}\big([f_3]\big)(Q''). \tag{16}$$

Combining Eqs. (13–16), we conclude that

$$\mathbf{L}\big(E^{t\alpha}[f]\big)(Q) \geqslant \mathbf{L}\big(E^{t\alpha}[f]\big)(Q'') = \mathbf{L}\big([f_4]\big)(Q'') \geqslant t\alpha(Q'') + \log\big(\mathrm{e}^{L_{[f]}(Q''')} - 1\big). \tag{17}$$

We now use the key property that $b' < b$ and $d' < d$, so that the box $Q''' = [b', b] \times [d', d]$ has nonempty interior and $L_{[f]}(Q''') > 0$. It consequently follows from (17) that

$$\liminf_{t\to+\infty} \frac{1}{t}\mathbf{L}\big(E^{t\alpha}[f]\big)(Q) \geqslant \alpha(Q'').$$

By definition of the box Q'', its mass $\alpha(Q'')$ for the measured lamination α is equal to $\alpha(Q')$. Also, because Q is α-generic, the box $Q' = [a, b'] \times [c, d']$ can be chosen so that $\alpha(Q')$ is arbitrarily close to $\alpha(Q)$. It follows that

$$\liminf_{t\to+\infty} \frac{1}{t}\mathbf{L}\big(E^{t\alpha}[f]\big)(Q) \geqslant \alpha(Q),$$

which completes the proof of this Step 1.

STEP 2. *If* $\alpha(Q) > 0$, then $\limsup_{t\to+\infty} \frac{1}{t}\mathbf{L}\big(E^{t\alpha}[f]\big)(Q) \leqslant \alpha(Q)$.

The property that $\alpha(Q) > 0$ prevents any geodesic of the support of α from having one endpoint in $[b, c]$ and one endpoint in $[d, a]$. As in Step 1, we can therefore break down α as a sum of measured laminations

$$\alpha = \alpha_Q + \alpha_a + \alpha_b + \alpha_c + \alpha_d + \alpha_o$$

where

- each geodesic of the support of α_Q has one endpoint in $[a, b]$ and one endpoint in $[c, d]$, and therefore belongs to $Q = [a, b] \times [c, d]$ after a possible orientation-reversal;
- each geodesic of the support of α_a has one endpoint in $[d, a]$ and one endpoint in $[a, b]$, and therefore encircles a;
- each geodesic of the support of α_b has one endpoint in $[a, b]$ and one endpoint in $[b, c]$, and therefore encircles b;
- each geodesic of the support of α_c has one endpoint in $[b, c]$ and one endpoint in $[c, d]$, and therefore encircles c;
- each geodesic of the support of α_d has one endpoint in $[c, d]$ and one endpoint in $[d, a]$, and therefore encircles d;
- each geodesic of the support of α_o has its two endpoints in the closure of the same component of $\partial_\infty \widetilde{X}_0 - \{a, b, c, d\}$.

Then,

$$E^{t\alpha}[f] = E^{t\alpha_o} \circ E^{t\alpha_a} \circ E^{t\alpha_c} \circ E^{t\alpha_Q} \circ E^{t\alpha_b} \circ E^{t\alpha_d}[f].$$

In order to estimate $\mathbf{L}\big(E^{t\alpha}[f]\big)(Q)$, set $[f_1] = E^{t\alpha_d}[f]$, $[f_2] = E^{t\alpha_b}[f_1]$, $[f_3] = E^{t\alpha_Q}[f_2]$, $[f_4] = E^{t\alpha_c}[f_3]$, $[f_5] = E^{t\alpha_a}[f_4]$ and $[f_6] = E^{t\alpha_o}[f_5] = E^{t\alpha}[f]$.

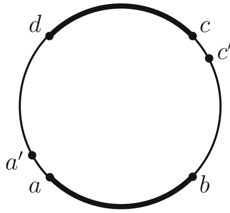
We will proceed backwards in our estimates, beginning with the simpler cases.

By construction of earthquakes, $\mathbf{L}\big(E^{t\alpha}[f]\big) = \mathbf{L}\big([f_6]\big) = \mathbf{L}\big(E^{t\alpha_o}[f_5]\big)$ is the pullback of $\mathbf{L}\big([f_5]\big)$ by a quasi-symmetric homeomorphism of $\partial_\infty \widetilde{X}_0$ which sends the box Q to itself. Therefore,

$$\mathbf{L}\big(E^{t\alpha}[f]\big)(Q) = \mathbf{L}\big([f_6]\big)(Q) = \mathbf{L}\big([f_5]\big)(Q). \tag{18}$$

Again by construction of earthquakes, $\mathbf{L}\big([f_5]\big) = \mathbf{L}\big(E^{t\alpha_a}[f_4]\big)$ is the pullback of $\mathbf{L}\big([f_4]\big)$ by a homeomorphism of $\partial_\infty \widetilde{X}_0$ which fixes the points b, c, d, and which moves the point a in the positive direction of $\partial_\infty \widetilde{X}_0$. As a consequence, this homeomorphism

Fig. 3 Step 2 of the proof of Lemma 34



sends the box $Q = [a, b] \times [c, d]$ to a smaller box $Q_1 = [a_1, b] \times [c, d] \subset Q$ with $a_1 \in [a, b]$, and

$$\mathbf{L}\big([f_5]\big)(Q) = \mathbf{L}\big([f_4]\big)(Q_1) \leqslant \mathbf{L}\big([f_4]\big)(Q). \tag{19}$$

The same argument applied to $\mathbf{L}\big([f_4]\big) = \mathbf{L}\big(E^{t\alpha_c}[f_3]\big)$ shows that

$$\mathbf{L}\big([f_4]\big)(Q) = \mathbf{L}\big([f_3]\big)(Q_2) \leqslant \mathbf{L}\big([f_3]\big)(Q) \tag{20}$$

for some box $Q_2 = [a, b] \times [c_2, d] \subset Q$.

We now use Lemmas 32 and 33 to estimate $\mathbf{L}\big([f_3]\big)(Q) = \mathbf{L}\big(E^{t\alpha_Q}[f_2]\big)(Q)$.

If we approximate the measured lamination α_Q by a Dirac measure based at a finite set $\{g_1, g_2, \dots, g_k, \bar{g}_1, \bar{g}_2, \dots, \bar{g}_k\}$ of disjoint geodesics in Q and assigning mass $a_i > 0$ to the atom g_i, then by construction $E^{t\alpha_Q}$ is approximated by the product of elementary earthquakes

$$E_{g_1}^{ta_1} \circ E_{g_2}^{ta_1} \circ \cdots \circ E_{g_n}^{ta_n}.$$

If ac denotes the diagonal of the box Q, going from a to $c \in \partial_\infty \widetilde{X}_0$, Lemma 32(a) shows that

$$\mathbf{L}\big(E_{g_i}^{ta_i}[f']\big)(Q) \leqslant \mathbf{L}\big(E_{ac}^{ta_i}[f']\big)(Q)$$

for every $[f'] \in \mathcal{T}(\widetilde{X}_0)$. The combination of Lemmas 32 and 33 then shows that

$$\begin{aligned} \mathbf{L}\big(E_{g_1}^{ta_1} E_{g_2}^{ta_1} \dots E_{g_n}^{ta_n}[f_2]\big)(Q) &\leqslant \mathbf{L}\big(E_{ac}^{t(a_1+a_2+\cdots+a_n)}[f_2]\big)(Q) \\ &\leqslant \mathbf{L}\big([f_2]\big)(Q) + t(a_1 + a_2 + \cdots + a_n). \end{aligned}$$

Passing to the limit as we use better and better approximations of α_Q by Dirac measures, we conclude that

$$\mathbf{L}\big([f_3]\big)(Q) = \mathbf{L}\big(E^{t\alpha_Q}[f_2]\big)(Q) \leqslant \mathbf{L}\big([f_2]\big)(Q) + t\alpha(Q). \tag{21}$$

Estimating $\mathbf{L}\big([f_2]\big)(Q) = \mathbf{L}\big(E^{t\alpha_b}[f_1]\big)(Q)$ will require more care. In particular, we need to split the geodesics of the support of α_b into those that have one endpoint near c and those that do not.

Pick a point c' in the open interval $]b, c[$ such that $\alpha\big([a, b] \times \{c'\}\big) = 0$, which can always be done by countable additivity of α. We will later choose c' close enough to c to ensure that $\alpha\big([a, b] \times [c', c]\big)$ is small. See Fig. 3.

Let α'_b be the restriction of α to the box $[a, b]\times[c', c]$, and let α''_b be the restriction of α to $[a, b]\times[b, c']$. In particular, $\alpha_b = \alpha'_b+\alpha''_b$ by the property that $\alpha\big([a, b]\times\{c'\}\big) = 0$.

As in our analysis of $E^{t\alpha_a}[f_4]$ and $E^{t\alpha_c}[f_3]$, the Liouville current $\mathbf{L}\big(E^{t\alpha''_b}[f_1]\big)$ is the pullback of $\mathbf{L}\big([f_1]\big)$ under a homeomorphism of $\partial_\infty \widetilde{X}_0$ which fixes a, c, d and moves b to a point of the interval $[b, c']$. Therefore

$$\mathbf{L}\big(E^{t\alpha''_b}[f_1]\big)(Q) \leqslant \mathbf{L}\big([f_1]\big)(Q''_{c'})$$

where $Q''_{c'} = [a, c'] \times [c, d]$.

Then, as in our analysis of $E^{t\alpha_Q}[f_2]$, the combination of Lemmas 32 and 33 gives that

$$\begin{aligned}\mathbf{L}\big([f_2]\big)(Q) &= \mathbf{L}\big(E^{t\alpha_b}[f_1]\big)(Q) = \mathbf{L}\big(E^{t\alpha'_b}E^{t\alpha''_b}[f_1]\big)(Q)\\ &\leqslant \mathbf{L}\big(E_{ac}^{t\alpha(Q'_{c'})}E^{t\alpha''_b}[f_1]\big)(Q)\\ &\leqslant \mathbf{L}\big(E^{t\alpha''_b}[f_1]\big)(Q) + t\alpha(Q'_{c'})\\ &\leqslant \mathbf{L}\big([f_1]\big)(Q''_{c'}) + t\alpha(Q'_{c'})\end{aligned} \tag{22}$$

where $Q'_{c'} = [a, b] \times [c', c]$ and $Q''_{c'} = [a, c'] \times [c, d]$.

Similarly, to estimate $\mathbf{L}\big([f_1]\big)(Q''_{c'}) = \mathbf{L}\big(E^{t\alpha_d}[f]\big)(Q''_{c'})$, pick a point a' in the open interval $]d, a[$ such that $\alpha\big(\{a'\} \times [c, d]\big) = 0$, and split α_d as $\alpha_d = \alpha'_d + \alpha''_d$, where α'_d and α''_d are the respective restrictions of α_d to $[a', a] \times [c, d]$ and $[d, a'] \times [c, d]$. See Fig. 3.

Then, using the combination of Lemmas 32 and 33 as in our analysis of $[f_2] = E^{t\alpha_b}[f_1]$,

$$\begin{aligned}\mathbf{L}\big([f_1]\big)(Q''_{c'}) &= \mathbf{L}\big(E^{t\alpha_d}[f]\big)(Q''_{c'}) = \mathbf{L}\big(E^{t\alpha'_d}E^{t\alpha''_d}[f]\big)(Q''_{c'})\\ &\leqslant \mathbf{L}\big(E^{t\alpha''_d}[f]\big)(Q''_{c'}) + t\alpha(Q'_{a'})\\ &\leqslant \mathbf{L}\big([f]\big)(Q''_{a'c'}) + t\alpha(Q'_{a'})\end{aligned} \tag{23}$$

where $Q'_{a'} = [a', a] \times [c, d]$ and $Q''_{a'c'} = [a, c'] \times [c, a']$.

Now, if we combine the estimates of (18–23), we get that

$$\mathbf{L}\big(E^{t\alpha}[f]\big)(Q) \leqslant \mathbf{L}\big([f]\big)(Q''_{a'c'}) + t\alpha(Q'_{a'}) + t\alpha(Q'_{c'}) + t\alpha(Q) \tag{24}$$

for the boxes $Q''_{a'c'} = [a, c']\times[c, a']$, $Q'_{a'} = [a', a]\times[c, d]$ and $Q'_{c'} = [a, b]\times[c', c]$.

Passing to the limit as t tends to ∞, this gives

$$\limsup_{t\to+\infty} \frac{1}{t}\mathbf{L}\big(E^{t\alpha}[f]\big)(Q) \leqslant \alpha(Q'_{a'}) + \alpha(Q'_{c'}) + \alpha(Q).$$

This property holds for any choice of points $a' \in]d, a[$ and $c' \in]b, c[$ (with $\alpha\big([a, b] \times \{c'\}\big) = 0$ and $\alpha\big(\{a'\} \times [c, d]\big) = 0$). Letting a' tend to a and c' tend to c, so that $\alpha(Q'_{a'})$ and $\alpha(Q'_{c'})$ respectively converge to $\alpha\big(\{a\} \times [c, d]\big) = 0$ and $\alpha\big([a, b] \times \{c\}\big) = 0$ by our hypothesis that Q is α-generic, we conclude that

$$\limsup_{t\to+\infty} \frac{1}{t} \mathbf{L}\big(E^{t\alpha}[f]\big)(Q) \leqslant \alpha(Q).$$

This concludes the proof of Step 2.

In particular, the combination of Steps 1 and 2 shows that $\lim_{t\to+\infty} \frac{1}{t} \mathbf{L}\big(E^{t\alpha}[f]\big)(Q) = \alpha(Q)$ when $\alpha(Q) > 0$.

We will rely on these first two steps to settle the remaining cases. Recall that $Q^\perp$ denotes the orthogonal box of Q, as defined in Sect. 3.2.

Step 3. *If* $\alpha(Q) = 0$ *and* $\alpha(Q^\perp) > 0$, *then* $\lim_{t\to+\infty} \mathbf{L}\big(E^{t\alpha}[f]\big)(Q) = 0$.

We rely on Lemma 12, which shows that

$$\mathrm{e}^{-\mathbf{L}(E^{t\alpha}[f])(Q)} + \mathrm{e}^{-\mathbf{L}(E^{t\alpha}[f])(Q^\perp)} = 1. \tag{25}$$

Because the box Q is α-generic, so is the orthogonal box $Q^\perp$. We can therefore apply Step 1 to $Q^\perp$, which gives

$$\liminf_{t\to+\infty} \frac{1}{t}\mathbf{L}\big(E^{t\alpha}[f]\big)(Q^\perp) \geqslant \alpha(Q^\perp) > 0$$

and in particular implies that $\mathbf{L}(E^{t\alpha}[f])(Q^\perp) \to +\infty$ as $t \to +\infty$.

We conclude that, as $t \to +\infty$, $\mathrm{e}^{-\mathbf{L}(E^{t\alpha}[f])(Q^\perp)} \to 0$ so that $\mathrm{e}^{-\mathbf{L}(E^{t\alpha}[f])(Q)} \to 1$ by (25), and therefore $\mathbf{L}(E^{t\alpha}[f])(Q) \to 0$.

Step 4. *If* $\alpha(Q) = 0$ *and* $\alpha(Q^\perp) = 0$, *then* $\lim_{t\to+\infty} \frac{1}{t}\mathbf{L}\big(E^{t\alpha}[f]\big)(Q) = 0$.

In the proof of Step 2, the only time we used the hypothesis that $\alpha(Q) > 0$ was to guarantee that the support of α contained no geodesic of the interior of the orthogonal box $Q^\perp$.

In the current setup of Step 4, the hypothesis that $\alpha(Q^\perp) = 0$ implies that the support of α is disjoint from the interior of $Q^\perp$. We can therefore apply the arguments of Step 2 and conclude that

$$\limsup_{t\to+\infty} \frac{1}{t}\mathbf{L}\big(E^{t\alpha}[f]\big)(Q) \leqslant \alpha(Q) = 0$$

as required.

This concludes the proof of Lemma 34, by Steps 1 and 2 when $\alpha(Q) > 0$, and by Steps 3 and 4 when $\alpha(Q) = 0$. □

We will need a more uniform version of Lemma 34. The lemma below will allow us to enhance a weak* convergence to a uniform weak* convergence. Recall that the box $Q = [a, b] \times [c, d]$ is α-generic if the subset of $G(\widetilde{X}_0)$ consisting of those geodesics with one endpoint in $\{a, b, c, d\}$ has α-mass 0.

Lemma 35 *Let $\{\alpha_n\}_n$ be a sequence of bounded measured geodesic laminations converging, as $n \to \infty$, to a measure α on $G(\widetilde{X}_0)$ for the weak* topology. Then, for every sequence $\{t_n\}$ converging to $+\infty$ in $\mathbb{R}$ and for every α-generic box $Q \subset G(\widetilde{X}_0)$,*

$$\lim_{n\to\infty} \frac{1}{t_n}\mathbf{L}\big(E^{t_n\alpha_n}[f]\big)(Q) = \alpha(Q).$$

Note that the α_n are only required to converge to α for the weak* topology, not for the uniform weak* topology. As a consequence, α is clearly a measured geodesic lamination but is not necessarily bounded.

Proof This follows from a careful inspection of the proof of Lemma 34. We repeat the steps of that proof.

STEP 1. $\liminf_{n\to\infty} \frac{1}{t_n}\mathbf{L}\big(E^{t_n\alpha_n}[f]\big)(Q) \geqslant \alpha(Q)$.

As in the proof of Lemma 34, assume $\alpha(Q) > 0$ without loss of generality, and choose a smaller box $Q' = [a, b'] \times [c, d'] \subset Q$ with $a < b' < b$ and $c < d' < d$, and with $\alpha(Q') > 0$ close to $\alpha(Q)$. By countable additivity of α we can arrange that Q' is α-generic and in particular that $\alpha(\partial Q') = 0$.

For n large enough, $\alpha_n(Q') > 0$ by Lemma 6 and our hypothesis that $\alpha(\partial Q') = 0$, and the support of α_n therefore meets Q'. Among the geodesics of the support of α_n that are contained in Q', let $a''_n d''_n$ be the one that is closest to the interval $[d', a] \subset \partial_\infty \widetilde{X}_0$, and let $b''_n c''_n$ be the one closest to $[b', c]$, in such a way that $a \leqslant a''_n \leqslant b''_n \leqslant b'$ and $c \leqslant c''_n \leqslant d''_n \leqslant d'$. Set $Q''_n = [a''_n, b] \times [c''_n, d]$.

The arguments used in Step 1 of the proof of Lemma 34 then show that, as in (16),

$$\mathbf{L}\big(E^{t_n\alpha_n}[f]\big)(Q) \geqslant \mathbf{L}\big(E^{t_n\alpha_n}[f]\big)(Q''_n) \geqslant t_n\alpha_n(Q''_n) + \log\big(\mathrm{e}^{L_{[f]}(Q''')} - 1\big).$$

for the box $Q''' = [b', b] \times [d', d]$.

By definition of the box Q''_n, its mass $\alpha_n(Q''_n)$ for the measured lamination α_n is equal to $\alpha_n(Q')$. Since we arranged that $\alpha(\partial Q') = 0$, Lemma 6 then shows that $\alpha_n(Q''_n) = \alpha_n(Q')$ converges to $\alpha(Q')$ as n tends to infinity. Therefore,

$$\liminf_{n\to\infty} \frac{1}{t_n}\mathbf{L}\big(E^{t_n\alpha_n}[f]\big)(Q) \geqslant \alpha(Q').$$

As Q' can be chosen so that $\alpha(Q')$ is arbitrarily close to $\alpha(Q)$, we conclude that

$$\liminf_{n\to\infty} \frac{1}{t_n}\mathbf{L}\big(E^{t_n\alpha_n}[f]\big)(Q) \geqslant \alpha(Q)$$

as required.

STEP 2. *If* $\alpha(Q) > 0$, *then* $\limsup_{n\to\infty} \frac{1}{t_n}\mathbf{L}\big(E^{t_n\alpha_n}[f]\big)(Q) \leqslant \alpha(Q)$.

As in Step 2 of the proof of Lemma 34, pick a point $c' \in\,]b, c[$ close to c, and a point $a' \in\,]d, a[$ close to a, such that $\alpha\big([a, b] \times \{c'\}\big) = 0$ and $\alpha\big(\{a'\} \times [c, d]\big) = 0$. Then, the same argument as in that Step 2 shows that, for every n,

$$\mathbf{L}\big(E^{t_n\alpha_n}[f]\big)(Q) \leqslant \mathbf{L}\big([f]\big)(Q''_{a'c'}) + t_n\alpha_n(Q'_{a'}) + t_n\alpha_n(Q'_{c'}) + t_n\alpha_n(Q) \quad (26)$$

for the boxes $Q''_{a'c'} = [a, c'] \times [c, a']$, $Q'_{a'} = [a', a] \times [c, d]$ and $Q'_{c'} = [a, b] \times [c', c]$.

By choice of the points a' and c', $\alpha(\partial Q'_{a'}) = \alpha(\partial Q'_{c'}) = 0$. We can therefore apply Lemma 6 when passing to the limit, and conclude that

$$\limsup_{n\to\infty} \frac{1}{t_n} \mathbf{L}\big(E^{t_n\alpha_n}[f]\big)(Q) \leqslant \alpha(Q'_{a'}) + \alpha(Q'_{c'}) + \alpha(Q).$$

Choosing a' and c' so that $\alpha(Q'_{a'})$ and $\alpha(Q'_{c'})$ are arbitrarily small, we conclude that

$$\limsup_{n\to\infty} \frac{1}{t_n} \mathbf{L}\big(E^{t_n\alpha_n}[f]\big)(Q) \leqslant \alpha(Q).$$

STEP 3. *If* $\alpha(Q) = 0$ *and* $\alpha(Q^\perp) > 0$, *then* $\lim_{n\to\infty} \mathbf{L}\big(E^{t_n\alpha_n}[f]\big)(Q) = 0$.

The argument is identical to that used for Step 3 of the proof of Lemma 34.

STEP 4. *If* $\alpha(Q) = 0$ *and* $\alpha(Q^\perp) = 0$, *then* $\lim_{n\to\infty} \frac{1}{t_n}\mathbf{L}\big(E^{t_n\alpha_n}[f]\big)(Q) = 0$.

In the proof of Lemma 34, we used the fact that the support of α is disjoint from the interior of $Q^\perp$ to reduce this step to Step 2. However, although $\alpha(Q^\perp) = 0$, it is here quite possible that $\alpha_n(Q^\perp) > 0$ and that the support of α_n meets the interior of $Q^\perp$.

Let us decompose each α_n as a sum $\alpha_n = \alpha_n^{Q^\perp} + \alpha'_n$ of two measured geodesic laminations $\alpha_n^{Q^\perp}$ and α'_n such that:

- every geodesic of the support of $\alpha_n^{Q^\perp}$ is contained in the orthogonal box $Q^\perp$, after a possible orientation-reversal;
- the support of α'_n is disjoint from the interior of $Q^\perp$.

As in Step 2 of the proof of Lemma 34, pick a point $c' \in \,]b, c[$ close to c, and a point $a' \in \,]d, a[$ close to a, such that $\alpha\big([a, b] \times \{c'\}\big) = 0$ and $\alpha\big(\{a'\} \times [c, d]\big) = 0$. Because the support of α'_n is disjoint from the interior of $Q^\perp$, we can then apply to α'_n this Step 2 of the proof of Lemma 34 and show that, for every n,

$$\mathbf{L}\big(E^{t_n\alpha'_n}[f]\big)(Q) \leqslant \mathbf{L}\big([f]\big)(Q''_{a'c'}) + t_n\alpha'_n(Q'_{a'}) + t_n\alpha'_n(Q'_{c'}) + t_n\alpha'_n(Q) \tag{27}$$

for the boxes $Q''_{a'c'} = [a, c'] \times [c, a']$, $Q'_{a'} = [a', a] \times [c, d]$ and $Q'_{c'} = [a, b] \times [c', c]$. Compare Eq. (24).

Then, by Lemmas 32(b) and 33,

$$\begin{aligned}
\mathbf{L}\big(E^{t_n\alpha_n}[f]\big)(Q) &= \mathbf{L}\big(E^{t_n\alpha_n^{Q^\perp}} E^{t_n\alpha'_n}[f]\big)(Q) \\
&\leqslant \mathbf{L}\big(E_{ac}^{t_n\alpha_n^{Q^\perp}(Q^\perp)} E^{t_n\alpha'_n}[f]\big)(Q) \\
&\leqslant \mathbf{L}\big(E^{t_n\alpha'_n}[f]\big)(Q) + t_n\alpha_n^{Q^\perp}(Q^\perp)
\end{aligned} \tag{28}$$

Combining (27) and (28), we conclude that

$$\begin{aligned}\mathbf{L}\big(E^{t_n\alpha_n}[f]\big)(Q) &\leqslant \mathbf{L}\big([f]\big)(Q''_{a'c'}) + t_n\alpha'_n(Q'_{a'}) + t_n\alpha'_n(Q'_{c'}) + t_n\alpha'_n(Q) + t_n\alpha_n^{Q^\perp}(Q^\perp) \\ &\leqslant \mathbf{L}\big([f]\big)(Q''_{a'c'}) + t_n\alpha_n(Q'_{a'}) + t_n\alpha_n(Q'_{c'}) + t_n\alpha_n(Q) + t_n\alpha_n(Q^\perp).\end{aligned} \tag{29}$$

Because the boxes Q, $Q^\perp$, $Q'_{a'}$, $Q'_{c'}$ are α-generic, $\alpha_n(Q'_{a'}) \to \alpha(Q'_{a'})$, $\alpha_n(Q'_{c'}) \to \alpha(Q'_{c'})$, $\alpha_n(Q) \to \alpha(Q) = 0$ and $\alpha_n(Q^\perp) \to \alpha(Q^\perp) = 0$ as $n \to \infty$. It follows that

$$\limsup_{n\to\infty} \frac{1}{t_n}\mathbf{L}\big(E^{t_n\alpha_n}[f]\big)(Q) \leqslant \alpha(Q'_{a'}) + \alpha(Q'_{c'}).$$

We can make $\alpha(Q'_{a'})$ arbitrarily close to $\alpha\big(\{a\}\times[c,d]\big) = 0$ and $\alpha(Q'_{c'})$ arbitrarily close to $\alpha\big([a,b]\times\{c\}\big) = 0$ by choosing a' sufficiently close to a and c' sufficiently close to c. This proves that

$$\lim_{n\to\infty} \frac{1}{t_n}\mathbf{L}\big(E^{t_n\alpha_n}[f]\big)(Q) = 0.$$

The combination of Steps 1, 2, 3 and 4 completes the proof of Lemma 35. □

5.5 Uniform weak* convergence of earthquake paths

We are now ready to prove Theorem 30, which we restate here as:

Theorem 36 *Let $\alpha \in \mathcal{ML}_{\mathrm{bd}}(X_0)$ be a bounded measured geodesic lamination and let $[f] \in \mathcal{T}(X_0)$ be a point of the Teichmüller space of X_0. Consider the left earthquake $E^{t\alpha}\colon \mathcal{T}(X_0) \to \mathcal{T}(X_0)$ for $t \in \mathbb{R}$, and the Liouville embedding $\mathbf{L}\colon \mathcal{T}(X_0) \to \mathcal{C}_{\mathrm{bd}}(X_0)$ from $\mathcal{T}(X_0)$ to the space $\mathcal{C}_{\mathrm{bd}}(X_0)$ of bounded geodesic currents. Then,*

$$\lim_{t\to\pm\infty} \frac{1}{|t|}\mathbf{L}\big(E^{t\alpha}[f]\big) = \alpha$$

for the uniform weak topology of $\mathcal{C}_{\mathrm{bd}}(X_0)$.*

Proof By symmetry between left and right earthquakes, we can restrict attention to the limit as $t \to +\infty$.

It is easier to use a proof by contradiction. Suppose the property false. Then, because the uniform weak* topology is metrizable (Lemma 4), there exists a sequence of real numbers t_n such that $t_n \to +\infty$ as $n \to \infty$ but such that $\frac{1}{t_n}\mathbf{L}\big(E^{t_n\alpha}[f]\big) = \frac{1}{t_n}L_{E^{t_n\alpha}[f]}$ does not converge to α for the uniform weak* topology. Passing to a subsequence if necessary, this means that there exists a lower bound $\varepsilon > 0$, a test function $\xi\colon G(\widetilde{X}_0) \to \mathbb{R}$ with compact support and a sequence of biholomorphic diffeomorphisms $\varphi_n \in \mathbf{H}(\widetilde{X}_0)$ such that

$$\left| \frac{1}{t_n}\int_{G(\widetilde{X}_0)} \xi\circ\varphi_n \, dL_{E^{t_n\alpha}[f]} - \int_{G(\widetilde{X}_0)} \xi\circ\varphi_n \, d\alpha \right| > \varepsilon \tag{30}$$

for every n.

Let α_n be the push forward of the measure α under the homeomorphism $G(\widetilde{X}_0) \to G(\widetilde{X}_0)$ induced by φ_n. Then α_n is clearly a measured geodesic lamination, and is bounded by definition of this property. Also, by definition of the push forward,

$$\int_{G(\widetilde{X}_0)} \xi \circ \varphi_n \, d\alpha = \int_{G(\widetilde{X}_0)} \xi \, d\alpha_n.$$

Lift the quasiconformal diffeomorphism $f\colon X_0 \to X$ representing $[f] \in \mathcal{T}(X_0)$ to $\widetilde{f}\colon \widetilde{X}_0 \to \widetilde{X}$. Then, in the Teichmüller space $\mathcal{T}(\widetilde{X}_0)$ of the universal cover, diagram chasing in the construction of elementary earthquakes shows that $E^t_g[\widetilde{f} \circ \varphi_n] = E^t_{\varphi_n(g)}[\widetilde{f}]$ for every geodesic $g \in G(\widetilde{X}_0)$ and every $t \in \mathbb{R}$. It follows that $E^{t_n\alpha}[\widetilde{f} \circ \varphi_n] = E^{t_n\alpha_n}[\widetilde{f}]$. As a consequence, the Liouville current $L_{E^{t_n\alpha_n}[f]} = L_{E^{t_n\alpha_n}[\widetilde{f}]}$ is the push forward of $L_{E^{t_n\alpha}[f]} = L_{E^{t_n\alpha}[\widetilde{f}]}$ under the homeomorphism $\varphi_n : G(\widetilde{X}_0) \to G(\widetilde{X}_0)$ induced by $\varphi_n \in \mathbf{H}(\widetilde{X}_0)$. In particular,

$$\int_{G(\widetilde{X}_0)} \xi \circ \varphi_n \, dL_{E^{t_n\alpha}[f]} = \int_{G(\widetilde{X}_0)} \xi \, dL_{E^{t_n\alpha_n}[f]}$$

and we can rewrite (30) as

$$\left| \frac{1}{t_n} \int_{G(\widetilde{X}_0)} \xi \, dL_{E^{t_n\alpha_n} f} - \int_{G(\widetilde{X}_0)} \xi \, d\alpha_n \right| > \varepsilon. \tag{31}$$

For every continuous function $\xi'\colon G(\widetilde{X}_0) \to \mathbb{R}$ with compact support, the associated weak* seminorms

$$|\alpha_n|_{\xi'} = \left| \int_{G(\widetilde{X}_0)} \xi' \, d\alpha_n \right| = \left| \int_{G(\widetilde{X}_0)} \xi' \circ \varphi_n \, d\alpha \right|$$

are uniformly bounded because the measured geodesic lamination $\alpha \in \mathcal{ML}_{\mathrm{bd}}(X_0)$ is bounded. By weak* compactness (see for instance [7, chap. III, §1, n$^{\mathrm{o}}$9]) we can therefore assume, after passing to a subsequence, that α_n converges to some measured geodesic lamination β for the weak* topology (but not necessarily for the uniform weak* topology).

Lemma 35 then states that for every β-generic box Q

$$\lim_{n\to\infty} \frac{1}{t_n} L_{E^{t_n\alpha_n} f}(Q) = \lim_{n\to\infty} \frac{1}{t_n} \mathbf{L}\big(E^{t_n\alpha_n}[f]\big)(Q) = \beta(Q).$$

But this will contradict (31) if we approximate the test function ξ by a β-generic step function, namely by a linear combination of the characteristic functions of a finite family of β-generic boxes.

Therefore, our original assumption cannot hold, and $\frac{1}{|t|}\mathbf{L}\big(E^{t\alpha}[f]\big)$ converges to α for the uniform weak* topology as $t \to +\infty$. $\square$

6 Naturality under quasiconformal diffeomorphisms

We conclude with a remark that our constructions are natural with respect to quasiconformal diffeomorphisms.

Let $f\colon X_1 \to X_2$ be a quasiconformal diffeomorphism between two conformally hyperbolic Riemann surfaces. If we lift f to a quasiconformal diffeomorphism $\widetilde{f}\colon \widetilde{X}_1 \to \widetilde{X}_2$ between universal covers, the quasisymmetric extension $\widetilde{f}\colon \partial_\infty \widetilde{X}_1 \to \partial_\infty \widetilde{X}_2$ induces a homeomorphism $\widetilde{f}\colon G(\widetilde{X}_1) \to G(\widetilde{X}_2)$ and therefore a bijection $F\colon \mathcal{C}(X_1) \to \mathcal{C}(X_2)$ between the corresponding spaces of geodesic currents.

Lemma 37 *The above bijection restricts to a homeomorphism $F\colon \mathcal{C}_{\mathrm{bd}}(X_1) \to \mathcal{C}_{\mathrm{bd}}(X_2)$, when the spaces $\mathcal{C}_{\mathrm{bd}}(X_1)$ and $\mathcal{C}_{\mathrm{bd}}(X_2)$ of bounded geodesic currents are endowed with the uniform weak* topology.*

Proof The main issue to deal with is that the definition of bounded geodesic currents in X_1 and of the uniform weak* topology of $\mathcal{C}_{\mathrm{bd}}(X_1)$ involves the space $\mathbf{H}(\widetilde{X}_1)$ of biholomorphic diffeomorphisms of the universal cover $\widetilde{X}_1$, whereas the corresponding notions in X_2 involve $\mathbf{H}(\widetilde{X}_2)$. Our proof will use an *ad hoc* correspondence between $\mathbf{H}(\widetilde{X}_1)$ and $\mathbf{H}(\widetilde{X}_2)$.

Arbitrarily pick three distinct points $x_1, y_1, z_1 \in \partial_\infty \widetilde{X}_1$, counterclockwise in this order, in the circle at infinity of $\widetilde{X}_1$ and three distinct points $x_2, y_2, z_2 \in \partial_\infty \widetilde{X}_2$, also in counterclockwise order. Then, for every biholomorphic map $\varphi \in \mathbf{H}(\widetilde{X}_2)$, there exists a unique $\rho(\varphi) \in \mathbf{H}(\widetilde{X}_1)$ sending the three points $\widetilde{f}^{-1} \circ \varphi^{-1}(x_2)$, $\widetilde{f}^{-1} \circ \varphi^{-1}(y_2)$, $\widetilde{f}^{-1} \circ \varphi^{-1}(z_2)$ to x_1, y_1, z_1, respectively. This provides a bijection $\rho\colon \mathbf{H}(X_2) \to \mathbf{H}(X_1)$ characterized by the property that for every $\varphi \in \mathbf{H}(\widetilde{X}_2)$ the map $\varphi \circ \widetilde{f} \circ \rho(\varphi)^{-1}$ sends our base points $x_1, y_1, z_1 \in \partial_\infty \widetilde{X}_1$ to the base points $x_2, y_2, z_2 \in \partial_\infty \widetilde{X}_2$, respectively.

We temporarily postpone the proof that F sends $\mathcal{C}_{\mathrm{bd}}(X_1)$ to $\mathcal{C}_{\mathrm{bd}}(X_2)$, as the argument will be a simpler version of our proof that the restriction $F\colon \mathcal{C}_{\mathrm{bd}}(X_1) \to \mathcal{C}_{\mathrm{bd}}(X_2)$ is continuous.

To prove that $F\colon \mathcal{C}_{\mathrm{bd}}(X_1) \to \mathcal{C}_{\mathrm{bd}}(X_2)$ is continuous, consider a sequence of bounded geodesic currents $\alpha_n \in \mathcal{C}_{\mathrm{bd}}(X_1)$ converging to α_∞ as $n \to \infty$, for the uniform weak* topology. We want to show that $F(\alpha_n)$ converges to $F(\alpha_\infty)$ in $\mathcal{C}_{\mathrm{bd}}(X_2)$, namely that

$$\|F(\alpha_n) - F(\alpha_\infty)\|_\xi = \sup_{\varphi \in \mathbf{H}(\widetilde{X}_2)} \left| \int_{G(\widetilde{X}_2)} \xi \circ \varphi \, dF(\alpha_n) - \int_{G(\widetilde{X}_2)} \xi \circ \varphi \, dF(\alpha_\infty) \right| \to 0 \text{ as } n \to \infty \tag{32}$$

for every continuous function $\xi\colon G(\widetilde{X}_2) \to \mathbb{R}$ with compact support. It is easier to use a proof by contradiction.

Suppose that (32) does not hold, in search for a contradiction. Then, passing to a subsequence if necessary, there exists $\delta > 0$ and a sequence of biholomorphic maps $\varphi_n \in \mathbf{H}(\widetilde{X}_2)$ such that

$$\left| \int_{G(\widetilde{X}_2)} \xi \circ \varphi_n \, dF(\alpha_n) - \int_{G(\widetilde{X}_2)} \xi \circ \varphi_n \, dF(\alpha_\infty) \right| > \delta \tag{33}$$

for every n. Then, by definition of the measure $F(\alpha_n)$,

$$\begin{aligned}\int_{G(\widetilde{X}_2)} \xi \circ \varphi_n \, dF(\alpha_n) &= \int_{G(\widetilde{X}_1)} \xi \circ \varphi_n \circ \widetilde{f} \, d\alpha_n \\ &= \int_{G(\widetilde{X}_1)} \xi \circ \left(\varphi_n \circ \widetilde{f} \circ \rho(\varphi_n)^{-1}\right) \circ \rho(\varphi_n) \, d\alpha_n\end{aligned} \tag{34}$$

for the bijection $\rho \colon \mathbf{H}(\widetilde{X}_1) \to \mathbf{H}(\widetilde{X}_2)$ defined above. Similarly,

$$\int_{G(\widetilde{X}_2)} \xi \circ \varphi_n \, dF(\alpha_\infty) = \int_{G(\widetilde{X}_1)} \xi \circ \left(\varphi_n \circ \widetilde{f} \circ \rho(\varphi_n)^{-1}\right) \circ \rho(\varphi_n) \, d\alpha_\infty \tag{35}$$

The functions $\widetilde{f_n} = \varphi_n \circ \widetilde{f} \circ \rho(\varphi_n)^{-1} \colon \partial_\infty \widetilde{X}_1 \to \partial_\infty \widetilde{X}_2$ are uniformly quasisymmetric since $M(\widetilde{f_n}) = M(\widetilde{f})$, and by construction send $x_1, y_1, z_1 \in \partial_\infty \widetilde{X}_1$ to x_2, y_2, $z_2 \in \partial_\infty \widetilde{X}_2$, respectively. By a classical equicontinuity property (see [22, §II.5]), they consequently form a relatively compact family in the space of quasisymmetric homeomorphisms $\partial_\infty \widetilde{X}_1 \to \partial_\infty \widetilde{X}_2$, for the topology of uniform convergence. Passing to a subsequence if necessary, we can therefore assume that the $\widetilde{f_n} \colon \partial_\infty \widetilde{X}_1 \to \partial_\infty \widetilde{X}_2$ uniformly converge to some homeomorphism $\widetilde{f_\infty}$. Then, as $n \to \infty$, the induced homeomorphisms $\widetilde{f_n} \colon G(\widetilde{X}_1) \to G(\widetilde{X}_2)$ converge to $\widetilde{f_\infty} \colon G(\widetilde{X}_1) \to G(\widetilde{X}_2)$ uniformly on compact subsets of $G(\widetilde{X}_1)$.

By Eqs. (34) and (35)

$$\begin{aligned}&\left|\int_{G(\widetilde{X}_2)} \xi \circ \varphi_n \, dF(\alpha_n) - \int_{G(\widetilde{X}_2)} \xi \circ \varphi_n \, dF(\alpha_\infty)\right| \\ &= \left|\int_{G(\widetilde{X}_1)} \xi \circ \widetilde{f_n} \circ \rho(\varphi_n) \, d\alpha_n - \int_{G(\widetilde{X}_1)} \xi \circ \widetilde{f_n} \circ \rho(\varphi_n) \, d\alpha_\infty\right| \\ &\leqslant \left|\int_{G(\widetilde{X}_1)} \xi \circ \widetilde{f_n} \circ \rho(\varphi_n) \, d\alpha_n - \int_{G(\widetilde{X}_1)} \xi \circ \widetilde{f_\infty} \circ \rho(\varphi_n) \, d\alpha_n\right| \\ &\quad + \left|\int_{G(\widetilde{X}_1)} \xi \circ \widetilde{f_\infty} \circ \rho(\varphi_n) \, d\alpha_n - \int_{G(\widetilde{X}_1)} \xi \circ \widetilde{f_\infty} \circ \rho(\varphi_n) \, d\alpha_\infty\right| \\ &\quad + \left|\int_{G(\widetilde{X}_1)} \xi \circ \widetilde{f_\infty} \circ \rho(\varphi_n) \, d\alpha_\infty - \int_{G(\widetilde{X}_1)} \xi \circ \widetilde{f_n} \circ \rho(\varphi_n) \, d\alpha_\infty\right|.\end{aligned} \tag{36}$$

Choose a nonnegative continuous function $\eta \colon G(\widetilde{X}_2) \to \mathbb{R}$ with compact support that is constantly 1 on a neighborhood of the support of ξ. For an arbitrary $\varepsilon > 0$, the fact that $\widetilde{f_n}$ converges to $\widetilde{f_\infty}$ uniformly on compact subsets implies that

$$|\xi \circ \widetilde{f_n} - \xi \circ \widetilde{f_\infty}| \leqslant \varepsilon \eta \circ \widetilde{f_\infty}$$

for n large enough, so that

$$\begin{aligned}\Big|\int_{G(\widetilde{X}_1)} \xi\circ\widetilde{f}_n\circ\rho(\varphi_n)\,d\alpha_n - \int_{G(\widetilde{X}_1)} \xi\circ\widetilde{f}_\infty\circ\rho(\varphi_n)\,d\alpha_n\Big| \\ \leqslant \varepsilon\int_{G(\widetilde{X}_1)} \eta\circ\widetilde{f}_\infty\circ\rho(\varphi_n)\,d\alpha_n \\ \leqslant \varepsilon \sup_{\varphi\in\mathbf{H}(\widetilde{X}_1)}\int_{G(\widetilde{X}_1)} \eta\circ\widetilde{f}_\infty\circ\varphi\,d\alpha_n \\ \leqslant \varepsilon\|\alpha_n\|_{\eta\circ\widetilde{f}_\infty}\end{aligned} \tag{37}$$

for n large enough, where $\| \ \|_{\eta\circ\widetilde{f}_\infty}$ is the (uniform weak*) seminorm on $\mathcal{C}_{\mathrm{bd}}(X_1)$ defined by the function $\eta\circ\widetilde{f}_\infty\colon G(\widetilde{X}_1)\to\mathbb{R}$. Similarly

$$\Big|\int_{G(\widetilde{X}_1)} \xi\circ\widetilde{f}_\infty\circ\rho(\varphi_n)\,d\alpha_\infty - \int_{G(\widetilde{X}_1)} \xi\circ\widetilde{f}_n\circ\rho(\varphi_n)\,d\alpha_\infty\Big| \leqslant \varepsilon\|\alpha_\infty\|_{\eta\circ\widetilde{f}_\infty} \tag{38}$$

for n large enough. Finally,

$$\Big|\int_{G(\widetilde{X}_1)} \xi\circ\widetilde{f}_\infty\circ\rho(\varphi_n)\,d\alpha_n - \int_{G(\widetilde{X}_1)} \xi\circ\widetilde{f}_\infty\circ\rho(\varphi_n)\,d\alpha_\infty\Big| \leqslant \|\alpha_n-\alpha_\infty\|_{\xi\circ\widetilde{f}_\infty} \tag{39}$$

Combining the inequalities of (36–39) we conclude that, for every $\varepsilon>0$,

$$\begin{aligned}\Big|\int_{G(\widetilde{X}_2)} \xi\circ\varphi_n\,dF(\alpha_n) - \int_{G(\widetilde{X}_2)} \xi\circ\varphi_n\,dF(\alpha_\infty)\Big| \leqslant \varepsilon\|\alpha_n\|_{\eta\circ\widetilde{f}_\infty} + \varepsilon\|\alpha_\infty\|_{\eta\circ\widetilde{f}_\infty} \\ +\|\alpha_n-\alpha_\infty\|_{\xi\circ\widetilde{f}_\infty}.\end{aligned} \tag{40}$$

for n large enough.

However, $\|\alpha_n\|_{\eta\circ\widetilde{f}_\infty}\to\|\alpha_\infty\|_{\eta\circ\widetilde{f}_\infty}$ and $\|\alpha_n-\alpha_\infty\|_{\xi\circ\widetilde{f}_\infty}\to 0$ as $n\to\infty$ since $\alpha_n\to\alpha_\infty$ in $\mathcal{C}_{\mathrm{bd}}(X_1)$, so that (40) contradicts (33) for ε small enough.

This contradiction proves (32), and shows that the function $F\colon\mathcal{C}_{\mathrm{bd}}(X_1)\to\mathcal{C}_{\mathrm{bd}}(X_2)$ is continuous.

A symmetric argument shows that the inverse $F^{-1}\colon\mathcal{C}_{\mathrm{bd}}(X_2)\to\mathcal{C}_{\mathrm{bd}}(X_1)$ is continuous, so that $F\colon\mathcal{C}_{\mathrm{bd}}(X_1)\to\mathcal{C}_{\mathrm{bd}}(X_2)$ is a homeomorphism.

We had postponed the proof that our original function $F\colon\mathcal{C}(X_1)\to\mathcal{C}(X_2)$ sends bounded geodesic current to bounded geodesic current. This is a simpler version of the above continuity proof. For a bounded geodesic current $\alpha\in\mathcal{C}_{\mathrm{bd}}(X_2)$, suppose in search of a contradiction that the geodesic current $F(\alpha)\in\mathcal{C}(X_2)$ is not bounded. As in (32) and (33), this means that there exists a continuous function $\xi\colon G(\widetilde{X}_2)\to\mathbb{R}$ with compact support and a sequence of biholomorphic maps $\varphi_n\in\mathbf{H}(\widetilde{X}_2)$ such that

$$\Big|\int_{G(\widetilde{X}_2)} \xi\circ\varphi_n\,dF(\alpha)\Big|\to\infty \text{ as } n\to\infty. \tag{41}$$

Passing to a subsequence if necessary, we can again arrange that the functions $\widetilde{f}_n = \varphi_n \circ \widetilde{f} \circ \rho(\varphi_n)^{-1} \colon G(\widetilde{X}_1) \to G(\widetilde{X}_2)$ converge to some homeomorphism $\widetilde{f}_\infty$, uniformly on compact subsets of $G(\widetilde{X}_1)$. Then, given $\varepsilon > 0$ and a continuous function $\eta \colon G(\widetilde{X}_2) \to \mathbb{R}$ with compact support that is constantly 1 on a neighborhood of the support of ξ,

$$\begin{aligned} \left| \int_{G(\widetilde{X}_2)} \xi \circ \varphi_n \, dF(\alpha) \right| &= \left| \int_{G(\widetilde{X}_2)} \xi \circ \widetilde{f}_n \circ \rho(\varphi_n) \, d\alpha \right| \\ &\leqslant \left| \int_{G(\widetilde{X}_2)} \xi \circ \widetilde{f}_n \circ \rho(\varphi_n) \, d\alpha - \int_{G(\widetilde{X}_2)} \xi \circ \widetilde{f}_\infty \circ \rho(\varphi_n) \, d\alpha \right| \\ &\quad + \left| \int_{G(\widetilde{X}_2)} \xi \circ \widetilde{f}_\infty \circ \rho(\varphi_n) \, d\alpha \right| \\ &\leqslant \varepsilon \|\alpha\|_{\eta \circ \widetilde{f}_\infty} + \|\alpha_n\|_{\xi \circ \widetilde{f}_\infty} \end{aligned} \tag{42}$$

for n large enough, as in (36–40). But this clearly contradicts (41), and therefore concludes our proof that the geodesic current $F(\alpha)$ is bounded.

As a consequence, the bijection $F \colon \mathcal{C}(X_1) \to \mathcal{C}(X_2)$ restricts to a map $F \colon \mathcal{C}_{\mathrm{bd}}(X_1) \to \mathcal{C}_{\mathrm{bd}}(X_2)$, which we already proved is a homeomorphism for the uniform weak* topologies. □

The quasiconformal diffeomorphism $f \colon X_1 \to X_2$ also induces a map $F_{\mathrm{T}} \colon \mathcal{T}(X_1) \to \mathcal{T}(X_2)$ between Teichmüller spaces, by the property that $F_{\mathrm{T}}\big([g]\big) = [g \circ f^{-1}] \in \mathcal{T}(X_2)$ for every $[g] \in \mathcal{T}(X_1)$ represented by a quasiconformal diffeomorphism $g \colon X_1 \to X$. It is immediate from definitions that F_{T} is an isometry for the Teichmüller metrics of $\mathcal{T}(X_1)$ and $\mathcal{T}(X_2)$.

It is also immediate from definitions that this construction is well-behaved with respect to the Liouville embeddings $\mathbf{L}_1 \colon \mathcal{T}(X_1) \to \mathcal{C}_{\mathrm{bd}}(X_1)$ and $\mathbf{L}_2 \colon \mathcal{T}(X_2) \to \mathcal{C}_{\mathrm{bd}}(X_2)$. More precisely, the diagram

$$\begin{tikzcd} \mathcal{C}_{\mathrm{bd}}(X_1) \arrow[r, "F"] & \mathcal{C}_{\mathrm{bd}}(X_2) \\ \mathcal{T}(X_1) \arrow[u, "\mathbf{L}_1"] \arrow[r, "F_{\mathrm{T}}"] & \mathcal{T}(X_2) \arrow[u, "\mathbf{L}_2"'] \end{tikzcd}$$

is commutative.

The following property is then an automatic consequence of the continuity of $F \colon \mathcal{C}_{\mathrm{bd}}(X_1) \to \mathcal{C}_{\mathrm{bd}}(X_2)$.

Proposition 38 *Let $f \colon X_1 \to X_2$ be a quasiconformal diffeomorphism between two conformally hyperbolic Riemann surfaces. Then the isometry $F_{\mathrm{T}} \colon \mathcal{T}(X_1) \to \mathcal{T}(X_2)$ induced by f continuously extends to the Thurston bordifications $\mathcal{T}(X_1) \cup \mathcal{PML}_{\mathrm{bd}}(X_1)$ and $\mathcal{T}(X_2) \cup \mathcal{PML}_{\mathrm{bd}}(X_2)$ of Sect. 4.2.* □

In particular, we can consider the case where $X_1 = X_2$. The *quasiconformal mapping class group* of a conformally hyperbolic Riemann surface X_0 is the group

$$\mathbf{MCG}_{\mathrm{qc}}(X_0) = \{\text{quasiconformal diffeomorphisms } f\colon X_0 \to X_0\}/\sim,$$

where the equivalence relation $\sim$ identifies $f_1, f_2\colon X_0 \to X_0$ when they are isotopic by an isotopy that moves points by a uniformly bounded amount, for the Poincaré metric. We refer to the results of [11] for several equivalent ways of expressing this relation.

A quasiconformal diffeomorphism $g\colon X_0 \to X$ is a quasi-isometry for the Poincaré metrics of X_0 and X. It follows that, if the quasiconformal diffeomorphisms f_1, $f_2\colon X_0 \to X_0$ are isotopic by an isotopy that moves points by a uniformly bounded amount, so are $g\circ f_1^{-1}$ and $g\circ f_2^{-1}\colon X_0 \to X_0$. As a consequence, if $f_1, f_2\colon X_0 \to X_0$ represent the same element of $\mathbf{MCG}_{\mathrm{qc}}(X_0)$, the maps F_1, $F_2\colon \mathcal{T}(X_0) \to \mathcal{T}(X_0)$ respectively induced by f_1 and f_2 coincide. This defines an isometric action of the quasiconformal mapping class group $\mathbf{MCG}_{\mathrm{qc}}(X_0)$ on the Teichmüller space $\mathcal{T}(X_0)$.

Proposition 38 immediately implies the following result.

Corollary 39 *The action of the quasiconformal mapping class group* $\mathbf{MCG}_{\mathrm{qc}}(X_0)$ *on the Teichmüller space* $\mathcal{T}(X_0)$ *continuously extends to the Thurston bordification* $\mathcal{T}(X_0) \cup \mathcal{PML}_{\mathrm{bd}}(X_0)$ *of Sect.* 4.2. □

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