

PROPAGATION OF SMALLNESS IN ELLIPTIC PERIODIC HOMOGENIZATION*

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Abstract. The paper is mainly concerned with an approximate three-ball inequality for solutions in elliptic periodic homogenization. We consider a family of second order operators $\mathcal{L}_\epsilon$ in divergence form with rapidly oscillating and periodic coefficients. It is the first time such an approximate three-ball inequality for homogenization theory is obtained. It implies an approximate quantitative propagation of smallness. The proof relies on a representation of the solution by the Poisson kernel and the Lagrange interpolation technique. Another full propagation of smallness result is also shown.

Key words. homogenization, three-ball theorem, approximate propagation of smallness, Poisson kernel

AMS subject classifications. 35B27, 35J61, 35C15

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1. Introduction. Quantitative propagation of smallness plays an important role in the quantitative study of solutions of elliptic and parabolic equations. It can be stated as follows: a solution u of a second order PDE $Lu = 0$ on a domain X can be made arbitrarily small on any given compact subset of X by making it sufficiently small on an arbitrary given subdomain Y . The quantitative propagation of smallness has found many important applications, such as the stability estimates in inverse problems for the Cauchy problem [ARRV] and the Hausdorff measure estimates of nodal sets of eigenfunctions [Lin], [Lo]. Hadamard’s three-circle or three-ball theorem is the simplest quantitative statement for propagation of smallness. The Hadamard three-ball theorem (or three-ball inequality) for a function u is the inequality

$$(1.1) \quad \|u\|_{r_2} \leq C \|u\|_{r_1}^\alpha \|u\|_R^{1-\alpha},$$

where $0 < r_1 < r_2 < R$ and $\|\cdot\|_r$ is the L^2 or L^∞ norm on the ball centered at the origin with radius r , $\alpha \in (0, 1)$ and C is a constant depending on r_1 , r_2 , and R . We will call the inequality (1.1) the standard three-ball inequality. For holomorphic functions, the three-ball inequality (1.1) is a consequence of the convexity of the logarithm of the modulus of holomorphic functions and the maximum principle. The strategy of using logarithmic convexity of the modulus has played a central role in obtaining three-ball theorems (or three-ball inequalities).

For solutions of second order elliptic equations

$$(1.2) \quad Lu = -D_j(a_{ij}(x)D_i u) + b_i D_i u + c(x)u = 0$$

(the summation convention is used throughout the paper), the logarithmic convexity of the L^2 -norm of solutions has been applied in different ways to obtain the three-ball

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inequality by Agmon [Ag] (for L^2 -norm) and Landis [L] (for L^∞ -norm). For harmonic functions in higher dimensions, Korevaar and Meyers [KM] obtained the three-ball inequality in (1.1) with sharp coefficient $C = 1$ using the logarithmic convexity of the L^2 norm and the expansion on eigenfunctions of the Laplace–Beltrami operator on the sphere. Brummelhuis [B] further pushed this technique to prove three-ball inequalities for solutions to second order elliptic equations.

The frequency function and Carleman estimates are tools often used to obtain a three-ball inequality. The frequency function is given as a quotient involving L^2 -norms of solutions. The monotonicity of the frequency function implies the convexity of the logarithm of L^2 -norm of solutions (see, e.g., [Al], [GL], [Z]). Carleman estimates are weighted integral inequalities with suitable weight functions satisfying some convexity properties. The three-ball inequality is obtained by applying the Carleman estimates to the product of the solution and an appropriate cut-off function and then by choosing the Carleman parameter appropriately (see, e.g., [DF], [JL], [K]). Recently, a proof of the three-ball inequality using the Poisson kernel for harmonic functions was developed by Guadie and Malinnikova [GM].

The theory of homogenization identifies the average, macroscopic behavior of a phenomenon that is subject to microscopic effects. Homogenization, including stochastic homogenization and numerical homogenization, has shown important impacts in applications, e.g., material sciences and elasticity theory. See, e.g., [ALi], [Ba], [FM], to just name a few. In order to obtain the standard three-ball inequality (1.1), the Lipschitz continuity of the leading coefficient $a_{ij}(x)$ is required in the usual proof. This is consistent with the fact that there exist operators with Hölder continuous coefficients a_{ij} with nontrivial solutions vanishing on open sets, which is precluded by (1.1) (see, e.g., [M]). All the aforementioned techniques result in taking the derivative of the leading coefficient $a_{ij}(x)$. The situation for the three-ball inequality for elliptic homogenization changes drastically since the derivative of the leading coefficient $a_{ij}(\frac{x}{\epsilon})$ leads to a “bad” term $\frac{1}{\epsilon}$. In this paper, we intend to study the propagation of smallness for solutions in elliptic periodic homogenization. To achieve this, the main goal of this paper is to address a version of the three-ball inequality for elliptic periodic homogenization.

We consider a family of elliptic operators in divergence form with rapidly oscillating periodic coefficients

$$(1.3) \quad \mathcal{L}_\epsilon u_\epsilon = -\operatorname{div}\left(A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon\right) = 0 \quad \text{in } \Omega,$$

where $\epsilon > 0$, Ω is a bounded domain in $\mathbb{R}^d$, and $A(y) = (a_{ij}(y))$ is a symmetric $d \times d$ matrix-valued function in $\mathbb{R}^d$ for $d \geq 2$. Assume that $A(y)$ satisfies the following assumptions:

(i) Ellipticity: For some $0 < \lambda < 1$ and all $y, \xi \in \mathbb{R}^d$, it holds that

$$(1.4) \quad \lambda|\xi|^2 \leq \langle A(y)\xi, \xi \rangle \leq \frac{1}{\lambda}|\xi|^2.$$

(ii) Periodicity:

$$(1.5) \quad A(y + \mathbf{z}) = A(y) \quad \text{for } y \in \mathbb{R}^d \text{ and } \mathbf{z} \in \mathbb{Z}^d.$$

(iii) Hölder continuity: There exist constants $\tau > 0$ and $0 < \mu < 1$ such that

$$(1.6) \quad |A(x) - A(y)| \leq \tau|x - y|^\mu$$

for any $x, y \in \mathbb{R}^d$.

We are able to establish the following approximate three-ball inequality in ellipsoids. The definition of ellipsoids E_r depending on the coefficients $A(y)$ is given in section 2.

THEOREM 1. *Let u_ϵ be a solution of (1.3) in $\mathbb{B}_{10}$. For $0 < r_1 < r_2 < \frac{R}{4} < 1$,*

$$(1.7) \quad \sup_{E_{r_2}} |u_\epsilon| \leq C \left\{ \frac{r_2}{R} \left(\sup_{E_{r_1}} |u_\epsilon| \right)^\alpha \left(\sup_{E_R} |u_\epsilon| \right)^{1-\alpha} + \frac{R^2}{r_1^2} [\epsilon \ln(\epsilon^{-1} + 2)]^\alpha \sup_{E_R} |u_\epsilon| \right\},$$

where $\alpha = \ln \frac{R}{2r_2} / \ln \frac{R}{r_1}$ and C depends on λ and (τ, μ) .

Let $\mathbb{B}_r(0)$ be the ball centered at the origin with radius r . We usually write it as $\mathbb{B}_r$ if the context is understood. Since balls are more convenient in applications than ellipsoids, a direct consequence of Theorem 1 is the following approximate three-ball inequality in balls.

COROLLARY 1. *Let u_ϵ be a solution of (1.3) in $\mathbb{B}_{10}$. For $0 < R_1 < R_2 < \frac{\lambda R_3}{4} < \frac{\lambda}{4}$,*

$$(1.8) \quad \|u_\epsilon\|_{L^\infty(\mathbb{B}_{R_2})} \leq C \left\{ \frac{R_2}{R_3} \|u_\epsilon\|_{L^\infty(\mathbb{B}_{R_1})}^\beta \|u_\epsilon\|_{L^\infty(\mathbb{B}_{R_3})}^{1-\beta} + \frac{R_3^2}{R_1^2} [\epsilon \ln(\epsilon^{-1} + 2)]^\beta \|u_\epsilon\|_{L^\infty(\mathbb{B}_{R_3})} \right\},$$

where $\beta = \ln \frac{\lambda R_3}{R_2} / \ln \frac{R_3}{R_1}$ and C depends on λ and (τ, μ) .

Let us address some important issues about the difference between the standard three-ball inequality (1.1) and inequality (1.8).

Remark 1. The three-ball inequality (1.8) in Corollary 1 is different from the standard three-ball inequality in (1.1). The inequality (1.1) implies the weak unique continuation property, which states that the solution vanishes globally if it vanishes in an open set. However, the estimate (1.8) does not. As $\epsilon \rightarrow 0$, the inequality (1.8) converges to the standard three-ball inequality in (1.1). Compared with the Lipschitz regularity needed to obtain the inequality (1.1), only Hölder continuity is imposed to obtain the inequality (1.8).

The three-ball inequality with consecutive balls having radii of a fixed ratio is very useful for applications. If we choose R_1, R_2, R_3 in an appropriate scale, e.g., let $R_1 = r$, $R_2 = 2r$, and $R_3 = \frac{9r}{\lambda}$, we have the following corollary.

COROLLARY 2. *Let u_ϵ be a solution of (1.3) in $\mathbb{B}_{10}$. For $r \leq \frac{\lambda}{9}$,*

$$(1.9) \quad \|u_\epsilon\|_{L^\infty(\mathbb{B}_{2r})} \leq C \left\{ \|u_\epsilon\|_{L^\infty(\mathbb{B}_r)}^\beta \|u_\epsilon\|_{L^\infty(\mathbb{B}_{\frac{9r}{\lambda}})}^{1-\beta} + [\epsilon \ln(\epsilon^{-1} + 2)]^\beta \|u_\epsilon\|_{L^\infty(\mathbb{B}_{\frac{9r}{\lambda}})} \right\},$$

where $\beta = \ln \frac{9}{4} / \ln \frac{9}{\lambda}$ and C depends on λ and (τ, μ) .

The eigenvalue problem in elliptic homogenization is of importance in the study of homogenization. Since the eigenvalue can be arbitrarily large, it is interesting to know what role the eigenvalue plays in the propagation of smallness. One can consider the eigenvalue type equation

$$(1.10) \quad -\operatorname{div} \left(A \left(\frac{x}{\epsilon} \right) \nabla u_\epsilon \right) = \Lambda_{\epsilon,k} u_\epsilon \quad \text{in } \mathbb{B}_{10}.$$

Assume that $\Lambda_{\epsilon,k}$ are positive constants and $\Lambda_{\epsilon,k} \rightarrow \infty$ as $k \rightarrow \infty$. The following corollary holds.

COROLLARY 3. Let u_ϵ be a solution of (1.10) in $\mathbb{B}_{10}$. For $0 < R_1 < R_2 < \frac{\lambda R_3}{4} < \frac{\lambda}{4}$,

$$(1.11) \quad \|u_\epsilon\|_{L^\infty(\mathbb{B}_{R_2})} \leq C e^{2R_3 \sqrt{\Lambda_{\epsilon,k}}} \left\{ \frac{R_2}{R_3} \|u_\epsilon\|_{L^\infty(\mathbb{B}_{R_1})}^\beta \|u_\epsilon\|_{L^\infty(\mathbb{B}_{R_3})}^{1-\beta} + \frac{R_3^2}{R_1^2} [\epsilon \ln(\epsilon^{-1} + 2)]^\beta \|u_\epsilon\|_{L^\infty(\mathbb{B}_{R_3})} \right\},$$

where $\beta = \ln \frac{\lambda R_3}{2R_2} / \ln \frac{R_3}{R_1}$ and C depends on λ and (τ, μ) .

Thanks to the three-ball inequality (1.9), we can show an approximate quantitative propagation of smallness result for solutions of (1.3).

COROLLARY 4. Let u_ϵ be a solution of (1.3) in $\mathbb{B}_{10}$. Assume that $\|u_\epsilon\|_{L^\infty(\mathbb{B}_r)} \leq \delta$ for some small $r < \frac{\lambda}{9}$ and $\|u_\epsilon\|_{L^\infty(\mathbb{B}_{10})} \leq 1$. Then, with $\beta = \ln \frac{9}{4} / \ln \frac{9}{\lambda}$ and C depending on λ and (τ, μ) ,

$$(1.12) \quad \|u_\epsilon\|_{L^\infty(\mathbb{B}_9)} \leq C \delta^{\beta^m} + mC [\epsilon \ln(\epsilon^{-1} + 2)]^{\beta^m}$$

for some positive integer m depending only on r , for $\epsilon \ln(\epsilon^{-1} + 2) \leq 1$.

From the inequality (1.12), we learn that u_ϵ in any compact domain of $\mathbb{B}_{10}$ can be made small if δ is sufficiently small and ϵ is sufficiently small, with an explicit dependence of δ and ϵ .

Recently, an interesting doubling inequality was shown by Lin and Shen [LS] under a stronger regularity of $A(y)$. Assume that $A(y)$ satisfies the following Lipschitz continuous condition.

(iv) Lipschitz continuity: There exists a constant $\tau > 0$ such that

$$(1.13) \quad |A(x) - A(y)| \leq \tau |x - y|$$

for any $x, y \in \mathbb{R}^d$.

Let $\mathcal{f}_{\mathbb{B}_r}$ be the average integral in $\mathbb{B}_r$. Under the assumptions of (1.4), (1.5), and (1.13) for $A(y)$, the doubling inequality results in [LS] can be formulated as follows.

THEOREM A. Let u_ϵ be a solution of (1.3) in $\mathbb{B}_{4/\sqrt{\lambda}}$. Assume that

$$(1.14) \quad \mathcal{f}_{\mathbb{B}_{2/\sqrt{\lambda}}} u_\epsilon^2 dx \leq N \mathcal{f}_{\mathbb{B}_{\sqrt{\lambda}}} u_\epsilon^2 dx$$

for some $N > 1$. Then

$$(1.15) \quad \mathcal{f}_{\mathbb{B}_r} u_\epsilon^2 dx \leq C(N) \mathcal{f}_{\mathbb{B}_{\frac{r}{2}}} u_\epsilon^2 dx$$

for any $0 < r < 1$, where $C(N)$ depends on N, λ, τ, d .

Note that $C(N)$ can be chosen increasing with respect to N . By standard arguments, we can replace the average L^2 norm by L^∞ norm in Theorem A. Thanks to Theorem A, we are able to show the following full propagation of smallness result.

COROLLARY 5. Assume that $\|u_\epsilon\|_{L^\infty(\mathbb{B}_{2/\sqrt{\lambda}})} \leq 1$. Then, given any constants $\eta_0 > 0$, $0 < r_0 < \frac{\sqrt{\lambda}}{2}$, there exists a positive constant $\delta_0 = \delta_0(\lambda, d, \tau, r_0, \eta_0)$ such that if $\|u_\epsilon\|_{L^\infty(\mathbb{B}_{r_0})} \leq \delta_0$, then $\|u_\epsilon\|_{L^\infty(\mathbb{B}_{\sqrt{\lambda}})} \leq \eta_0$.

Note that the dependence of δ_0 is not explicit in the corollary.

Remark 2. A direct consequence of Theorem A is the following doubling inequality with translation (Theorem 3.3 in [LS]):

$$(1.16) \quad \oint_{\mathbb{B}_r(x_0)} u_\epsilon^2 dx \leq C(N) \oint_{\mathbb{B}_{\frac{r}{2}}(x_0)} u_\epsilon^2 dx$$

for $0 < r < \frac{3}{4}$ and $|x_0| \leq \frac{\sqrt{\lambda}}{2}$. We can also replace the average L^2 norm by L^∞ norm in (1.16). Assuming that $\|u_\epsilon\|_{L^\infty(\mathbb{B}_{\sqrt{\lambda}})} \geq 1$ and $\|u_\epsilon\|_{L^\infty(\mathbb{B}_{2/\sqrt{\lambda}})} \leq M$, a uniform in ϵ vanishing order estimate for u_ϵ can be shown in $\mathbb{B}_{\sqrt{\lambda}/2}$ by iteration of (1.16) in L^∞ norm. However, such a vanishing order estimate is implicit since it depends on $C(M)$. See, e.g., [K], [Z], for some literature on the explicit vanishing order estimates for solutions in (1.2).

To study the propagation of smallness in the half-space, one needs to establish the three half-balls inequality. We can extend the three-ball inequality (1.8) to elliptic homogenization with Dirichlet or Neumann boundary conditions. We consider elliptic homogenization with the Dirichlet or Neumann boundary conditions in half balls

$$(1.17) \quad \begin{cases} -\operatorname{div}(A(\frac{x}{\epsilon})\nabla u_\epsilon) = 0 & \text{in } \mathbb{B}_{10}^+ = \mathbb{B}_{10} \cap \{x_d \geq 0\}, \\ u_\epsilon = 0 \text{ or } \frac{\partial u_\epsilon}{\partial \nu} = 0 & \text{on } \partial\mathbb{B}_{10}^+ \cap \{x_d = 0\}, \end{cases}$$

where

$$(1.18) \quad A(y) = \begin{pmatrix} a_{11}(y) & \cdots & a_{1(d-1)}(y) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{(d-1)1}(y) & \cdots & a_{(d-1)(d-1)}(y) & 0 \\ 0 & \cdots & 0 & a_{dd}(y) \end{pmatrix},$$

the conormal derivative $\frac{\partial u_\epsilon}{\partial \nu_\epsilon}$ is defined as

$$\frac{\partial u_\epsilon}{\partial \nu_\epsilon} = a_{ij} \frac{\partial u_\epsilon}{\partial x_j} n_i,$$

and $n = (n_1, \dots, n_d)$ is a unit outer normal. We are able to show the approximate three-ball inequality for solutions of (1.17) in half balls.

THEOREM 2. *Let u_ϵ be a solution of (1.17) with $A(y)$ satisfying (1.18) in $\mathbb{B}_{10}^+$. For $0 < R_1 < R_2 < \frac{\lambda R_3}{4} < \frac{\lambda}{4}$,*

$$(1.19) \quad \|u_\epsilon\|_{L^\infty(\mathbb{B}_{R_2}^+)} \leq C \left\{ \frac{R_2}{R_3} \|u_\epsilon\|_{L^\infty(\mathbb{B}_{R_1}^+)}^\beta \|u_\epsilon\|_{L^\infty(\mathbb{B}_{R_3}^+)}^{1-\beta} + \frac{R_3^2}{R_1^2} [\epsilon \ln(\epsilon^{-1} + 2)]^\beta \|u_\epsilon\|_{L^\infty(\mathbb{B}_{R_3}^+)} \right\},$$

where $\beta = \ln \frac{\lambda R_3}{2R_2} / \ln \frac{R_3}{R_1}$ and C depends on λ and (τ, μ) .

Similarly, we are able to consider an approximate three-ball inequality for eigenvalue type equations in half balls with the Dirichlet or Neumann boundary conditions

$$(1.20) \quad \begin{cases} -\operatorname{div}(A(\frac{x}{\epsilon})\nabla u_{\epsilon,k}) = \Lambda_{\epsilon,k} u_{\epsilon,k} & \text{in } \mathbb{B}_{10}^+ = \mathbb{B}_{10} \cap \{x_d \geq 0\}, \\ u_{\epsilon,k} = 0 \text{ or } \frac{\partial u_{\epsilon,k}}{\partial \nu} = 0 & \text{on } \partial\mathbb{B}_{10}^+ \cap \{x_d = 0\}, \end{cases}$$

where $A(y)$ is in the form of (1.18). The constants $\Lambda_{\epsilon,k}$ are assumed to be positive and $\Lambda_{\epsilon,k} \rightarrow \infty$ as $k \rightarrow \infty$. We can show the following corollary.

COROLLARY 6. *Let $u_{\epsilon,k}$ be a solution of (1.20) with $A(y)$ satisfying (1.18) in $\mathbb{B}_{10}^+$. For $0 < R_1 < R_2 < \frac{\lambda R_3}{4} < \frac{\lambda}{4}$,*

$$(1.21) \quad \|u_{\epsilon,k}\|_{L^\infty(\mathbb{B}_{R_2}^+)} \leq C e^{2R_3 \sqrt{\Lambda_{\epsilon,k}}} \left\{ \frac{R_2}{R_3} \|u_{\epsilon,k}\|_{L^\infty(\mathbb{B}_{R_1}^+)}^\beta \|u_{\epsilon,k}\|_{L^\infty(\mathbb{B}_{R_3}^+)}^{1-\beta} + \frac{R_3^2}{R_1^2} [\epsilon \ln(\epsilon^{-1} + 2)]^\beta \|u_{\epsilon,k}\|_{L^\infty(\mathbb{B}_{R_3}^+)} \right\},$$

where $\beta = \ln \lambda R_3 2R_2 / \ln \frac{R_3}{R_1}$ and C depends on λ and (τ, μ) .

One can also consider the asymptotical behavior of eigenvalue $\Lambda_{\epsilon,k}$ in terms of ϵ and k in the model as (1.20), which is a difficult issue. Related results in this direction can be found in, e.g., [KLS1], [Zh].

The paper is organized as follows. Section 2 is devoted to some basic material on elliptic periodic homogenization. In section 3, we prove Theorems 1 and 2 and the corollaries. The letter C denotes a positive constant that does not depend on ϵ or u_ϵ . It may vary from line to line.

Added remark. After the paper was posted in arXiv and submitted for publication, an improvement in the error bound for the three-ball inequality in Theorem 1 (under certain extra conditions) was discovered by Armstrong, Kuusi, and Smart in [AKS].

2. Preliminaries. This section introduces some background results on elliptic periodic homogenization. Most of the material can be found in books, e.g., [BLP], [S]. Let $\chi(\mathbf{y}) = (\chi_1(\mathbf{y}), \dots, \chi_d(\mathbf{y})) \in H^1(\mathbb{T}^d; \mathbb{R}^d)$ denote the corrector of $\mathcal{L}_\epsilon$, where $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$. It is known that $\chi(\mathbf{y})$ is the unique 1-periodic solution in $H^1(\mathbb{T}^d)$ satisfying

$$(2.1) \quad \begin{cases} \mathcal{L}_1(\chi_j) + \mathcal{L}_1(\mathbf{y}_j) = 0 & \text{in } \mathbb{R}^d, \\ \int_{\mathbb{T}^d} \chi_j d\mathbf{y} = 0 & \text{for } j = 1, \dots, d. \end{cases}$$

Using the De Giorgi–Nash estimates, $\chi_j(\mathbf{y})$ is Hölder continuous. If $A(\mathbf{y})$ is Hölder continuous, then $\nabla \chi$ is bounded. See, e.g., [S]. Assume that Ω is a bounded $C^{2,\eta}$ domain for some $\eta \in (0, 1)$ in $\mathbb{R}^d$ in the rest of the paper. If $\{u_\epsilon\}$ is bounded in $H^1(\Omega)$, then any sequence $\{u_{\epsilon_l}\}$ with $\epsilon_l \rightarrow 0$ contains a subsequence that converges weakly in $H^1(\Omega)$. By the homogenization theory, one may conclude that as $\epsilon \rightarrow 0$,

$$(2.2) \quad \begin{cases} A(\frac{x}{\epsilon}) \nabla u_\epsilon \rightharpoonup \hat{A} \nabla u & \text{weakly in } L^2(\Omega), \\ u_\epsilon \rightharpoonup u & \text{weakly in } L^2(\Omega), \\ u_\epsilon \rightarrow u & \text{strongly in } L^2(\Omega), \end{cases}$$

where $\hat{A} = (\hat{a}_{ij})$ and

$$(2.3) \quad \hat{A} = \hat{a}_{ij} = \int_{\mathbb{T}^d} \left[a_{ij} + a_{ik} \frac{\partial \chi_j}{\partial \mathbf{y}_k} \right] d\mathbf{y}.$$

It is also true that the constant matrix $\hat{a}_{ij}$ is symmetric and satisfies (1.4) with the same parameter λ . Thus, the homogenized equation for $\mathcal{L}_\epsilon u_\epsilon = 0$ is given by

$$(2.4) \quad \mathcal{L}_0 u = -\operatorname{div}(\hat{A} \nabla u) = 0.$$

Since $\widehat{A}$ is symmetric and positive definite, there exists a $d \times d$ matrix S such that $S\widehat{A}S^T = I_{d \times d}$. Note that $\widehat{A}^{-1} = S^T S$ and

$$(2.5) \quad \langle \widehat{A}^{-1}x, x \rangle = |Sx|^2.$$

We introduce a family of ellipsoids as

$$(2.6) \quad E_r(\widehat{A}) = \{x \in \mathbb{R}^n : \langle \widehat{A}^{-1}x, x \rangle \leq r^2\}.$$

By the assumption that $\widehat{A}$ satisfies (1.4) as well, we can show that

$$(2.7) \quad \mathbb{B}_{\sqrt{\lambda}r}(0) \subset E_r(\widehat{A}) \subset B_{\frac{r}{\sqrt{\lambda}}}(0).$$

We will write $E_r(\widehat{A})$ as E_r if the context is understood.

The Dirichlet corrector is used to control the influence of the boundary data. The Dirichlet corrector $\Psi_{\epsilon,k}$ for the operator $\mathcal{L}_\epsilon$ in Ω is defined as

$$\begin{cases} \mathcal{L}_\epsilon(\Psi_{\epsilon,k}) = 0 & \text{in } \Omega, \\ \Psi_{\epsilon,k} = P_k & \text{on } \partial\Omega, \end{cases}$$

where $P_k = x_k$ for $1 \leq k \leq d$. It was shown by Avellaneda and Lin in [AL] that

$$(2.8) \quad \|\nabla \Psi_{\epsilon,k}\|_{L^\infty(\Omega)} \leq C,$$

where C depends on λ , (τ, μ) , and Ω . In the discussion in the next section, we need to know how the constant C depends on r in the ellipsoid E_r . By a rescaling argument, we can actually show that

$$(2.9) \quad \|\nabla \Psi_{\epsilon,k}\|_{L^\infty(E_r)} \leq C$$

with C depends only on λ and (τ, μ) . For $x \in E_r$, let $y = \frac{x}{r}$. Then $y \in E_1$. We perform a rescaling as

$$(2.10) \quad \Psi_{\epsilon,k}(x) = r\tilde{\Psi}_{\epsilon,k}\left(\frac{x}{r}\right) = r\tilde{\Psi}_{\epsilon,k}(y),$$

where $\tilde{\Psi}_{\epsilon,k}$ is the Dirichlet corrector for the operator $\mathcal{L}_\epsilon$ in E_1 . We can see that $\Psi_{\epsilon,k}(x)$ is the Dirichlet corrector for the operator $\mathcal{L}_\epsilon$ in E_r , since

$$\begin{cases} -\operatorname{div}(A(\frac{x}{\epsilon r})\nabla \Psi_{\epsilon,k}(x)) = -\operatorname{div}(A(y)\nabla \tilde{\Psi}_{\epsilon,k}(y)) = 0 & \text{in } E_r, \\ \Psi_{\epsilon,k} = r\tilde{\Psi}_{\epsilon,k}(\frac{x}{r}) = r \cdot \frac{x_k}{r} = x_k & \text{on } \partial E_r. \end{cases}$$

It follows from (2.8) that

$$\|\nabla_y \tilde{\Psi}_{\epsilon,k}\|_{L^\infty(E_1)} \leq C$$

with C depending only on λ and (τ, μ) . Thus, it implies that

$$(2.11) \quad \|\nabla_x \Psi_{\epsilon,k}\|_{L^\infty(E_r)} = \|\nabla_y \tilde{\Psi}_{\epsilon,k}\|_{L^\infty(E_1)} \leq C$$

with the same constant C in the last inequality. Therefore, we arrive at the estimates (2.9).

The asymptotic expansion of the Poisson kernel is well studied in homogenization theory. Let $P_\epsilon(x, y)$ denote the Poisson kernel for $\mathcal{L}_\epsilon$ in a bounded $C^{2,\eta}$ domain Ω . It was shown by Kenig, Lin, and Shen in [KLS2] that we can write

$$(2.12) \quad P_\epsilon(x, y) = P_0(x, y)w_\epsilon(y) + R_\epsilon(x, y),$$

where $P_0(x, y)$ is the Poisson kernel for the homogenized operator $\mathcal{L}_0$,

$$(2.13) \quad w_\epsilon(y) = (\hat{a}_{ij}n_in_j)^{-1} \frac{\partial \Psi_{\epsilon,k}}{\partial n} n_k \cdot a_{ij} \left(\frac{y}{\epsilon} \right) n_in_j,$$

$n = (n_1, \dots, n_n)$ is a unit outer normal on $\partial\Omega$, and

$$(2.14) \quad |R_\epsilon(x, y)| \leq \frac{C\epsilon \ln |\epsilon^{-1}|x - y| + 2|}{|x - y|^d} \quad \text{for any } x \in \Omega \text{ and } y \in \partial\Omega.$$

The constant C in (2.14) depends on λ , (τ, μ) , and Ω . From (1.4), (2.11), and (2.13), it follows that

$$(2.15) \quad \|w_\epsilon(y)\|_{L^\infty(E_r)} \leq C$$

with C independent of r . In the following discussions, we also need to know what the dependence of the constant C in (2.14) on the ellipsoid E_r is. Note that for any $x \in \Omega$ and $y \in \partial\Omega$,

$$(2.16) \quad P_\epsilon(x, y) = -\frac{\partial G_\epsilon(x, y)}{\partial n(y)} a_{ij} \left(\frac{y}{\epsilon} \right) n_i(y) n_j(y),$$

and

$$(2.17) \quad P_0(x, y) = -\frac{\partial G(x, y)}{\partial n(y)} \hat{a}_{ij}(y) n_i(y) n_j(y),$$

where $G_\epsilon(x, y)$ and $G(x, y)$ are the Green functions for the operator $\mathcal{L}_\epsilon$ and $\mathcal{L}_0$, respectively. It follows from (2.12) that

$$(2.18) \quad \begin{aligned} R_\epsilon(x, y) = & -\frac{\partial G_\epsilon(x, y)}{\partial n(y)} a_{ij} \left(\frac{y}{\epsilon} \right) n_i(y) n_j(y) \\ & + \frac{\partial G(x, y)}{\partial n(y)} \frac{\partial \Psi_{\epsilon,k}}{\partial n_k(y)} n_k(y) a_{ij} \left(\frac{y}{\epsilon} \right) n_i(y) n_j(y). \end{aligned}$$

Thus, the estimate (2.14) for $R_\epsilon(x, y)$ is shown in [KLS2] as

$$(2.19) \quad \begin{aligned} |R_\epsilon(x, y)| & \leq C \left| \frac{\partial G_\epsilon(x, y)}{\partial y_i} - \frac{\partial \Psi_{\epsilon,k}}{\partial y_i} \frac{\partial G(x, y)}{\partial y_j} \right| \\ & \leq C \frac{\epsilon \ln |\epsilon^{-1}|x - y| + 2|}{|x - y|^d}. \end{aligned}$$

To see how the C in the last inequality depends on r in E_r , we use the rescaling argument again. Let $x' = \frac{x}{r}$ and $y' = \frac{y}{r}$, where $x, y \in E_r$. Then $x', y' \in E_1$. We do a rescaling as

$$(2.20) \quad G_\epsilon(x, y) = r^{2-d} \tilde{G}_\epsilon(x', y'),$$

where $\tilde{G}_\epsilon(x', y')$ is the Green function for the operator $\mathcal{L}_\epsilon$ in E_1 . Since

$$\begin{cases} -\operatorname{div}(a_{ij}(\frac{x}{\epsilon r}) \nabla G_\epsilon(x, y)) = -r^{-d} \operatorname{div}(a_{ij}(\frac{x'}{\epsilon}) \nabla \tilde{G}_\epsilon(y')) \\ \quad = r^{-d} \delta(x' - y') = \delta(x - y) & \text{in } E_r, \\ G_\epsilon(x, y) = r^{2-d} \tilde{G}_\epsilon(x', y') = 0 & \text{on } \partial E_r, \end{cases}$$

it implies that $G_\epsilon(x, y)$ is the Green function for the operator $\mathcal{L}_\epsilon$ in E_r . From (2.19), we have

$$\begin{aligned} \left| \frac{\partial G_\epsilon(x, y)}{\partial y_i} - \frac{\partial \Psi_{\epsilon, k}}{\partial y_i} \frac{\partial G(x, y)}{\partial y_j} \right| &= r^{1-d} \left| \frac{\partial G_\epsilon(x', y')}{\partial y'_i} - \frac{\partial \Psi_{\epsilon, k}}{\partial y'_i} \frac{\partial G(x', y')}{\partial y'_j} \right| \\ &\leq \frac{Cr^{1-d} \epsilon \ln[\epsilon^{-1} |x' - y'| + 2]}{|x' - y'|^d} \\ (2.21) \quad &= \frac{Cr \epsilon \ln[\epsilon^{-1} \frac{|x-y|}{r} + 2]}{|x - y|^d}, \end{aligned}$$

where C in the last inequality depends only on $\lambda, (\tau, \mu)$ since $x', y' \in E_1$. This implies that

$$(2.22) \quad |R_\epsilon(x, y)| \leq \frac{Cr \epsilon \ln[\epsilon^{-1} \frac{|x-y|}{r} + 2]}{|x - y|^d}$$

for $x \in E_r$ and $y \in \partial E_r$ and C is independent of r .

3. Approximate three-ball inequality. This section is devoted to the proof of three-ball inequality (1.8) in Theorem 1. We adapt the method of using the Poisson kernel in [GM], for harmonic functions, to elliptic homogenization. We want to have an explicit form for the Poisson kernel of the homogenized operator $\mathcal{L}_0$. We transform the operator $\mathcal{L}_0$ into the classical Laplacian operator. Let $u(x) = w(Sx)$. Then

$$(3.1) \quad 0 = \operatorname{div}(\hat{A} \nabla u) = \operatorname{div}(S \hat{A} S^T \nabla w) = \Delta w.$$

We can represent the harmonic function $w(x)$ by Poisson kernel as

$$(3.2) \quad w(x) = \int_{S_r} \gamma_d \frac{r^2 - |x|^2}{r|x - y|^d} w(y) dy,$$

where S_r is the $d - 1$ dimensional sphere centered at the origin with radius r and $\gamma_d = 2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$ is the surface area of S_1 with $d \geq 2$. By the relation of $u(x) = w(Sx)$, we can transform (3.2) to the solution u for homogenized equation (2.4). It follows that

$$(3.3) \quad u(x) = \int_{\partial E_r} P_0(x, y) u(y) dy,$$

where

$$(3.4) \quad P_0(x, y) = \gamma_d |S| \frac{r^2 - |Sx|^2}{r|Sx - Sy|^d}.$$

Following [GM], we are going to apply the Lagrange interpolation method to obtain the three-ball inequality (1.8). Intuitively, the main idea of the Lagrange interpolation method is to use well-behaved polynomials to approximate a given function

and efficiently estimate the error term. Let us briefly review the standard Lagrange interpolation method in numerical analysis [DB]. Set

$$(3.5) \quad \Phi_m(z) = (z - u_1)(z - u_2) \cdots (z - u_m)$$

for $z, u_j \in \mathcal{C}$ with $j = 1, \dots, m$. Let $\mathcal{D}$ be a simply connected open domain in the complex plane $\mathcal{C}$ that contains the nodes $\tilde{u}, u_1, \dots, u_m$. Assume that f is an analytic function without poles in the closure of $\mathcal{D}$. By well-known calculations, it holds that

$$(3.6) \quad \frac{1}{z - \tilde{u}} = \sum_{j=1}^m \frac{\Phi_{j-1}(\tilde{u})}{\Phi_j(z)} + \frac{\Phi_m(\tilde{u})}{(z - \tilde{u})\Phi_m(z)}.$$

Multiplying the last identity by $\frac{1}{2\pi i} f(z)$ and integrating along the boundary of $\mathcal{D}$ leads to

$$(3.7) \quad \frac{1}{2\pi i} \int_{\partial \mathcal{D}} \frac{f(z)}{z - \tilde{u}} dz = \sum_{j=1}^m \frac{\Phi_{j-1}(\tilde{u})}{2\pi i} \int_{\partial \mathcal{D}} \frac{f(z)}{\Phi_j(z)} dz + (R_m f)(\tilde{u}),$$

where

$$(R_m f)(\tilde{u}) = \frac{1}{2\pi i} \int_{\partial \mathcal{D}} \frac{\Phi_m(\tilde{u})f(z)}{(z - \tilde{u})\Phi_m(z)} dz.$$

By the residue theorem, we obtain that

$$(3.8) \quad \begin{aligned} (R_m f)(\tilde{u}) &= \sum_{j=1}^m \frac{\Phi_m(\tilde{u})}{(u_j - \tilde{u})\Phi'_m(u_j)} f(u_j) + f(\tilde{u}) \\ &= - \sum_{j=1}^m \prod_{i \neq j}^m \frac{\tilde{u} - u_i}{u_j - u_i} f(u_j) + f(\tilde{u}), \end{aligned}$$

where $(R_m f)(\tilde{u})$ is called the interpolation error. Based on the identity (3.8), the idea is to approximate $f(\tilde{u})$ by a linear combination of the polynomials

$$- \sum_{j=1}^m \prod_{i \neq j}^m \frac{\tilde{u} - u_i}{u_j - u_i} f(u_j)$$

and then control the error term $(R_m f)(\tilde{u})$ efficiently. See chapter 4 in [DB] for more information.

With these preliminaries, we are able to present the proof of Theorem 1.

Proof of Theorem 1. In order to obtain the approximate three-ball theorem for the solution in (1.3) in elliptic periodic homogenization, we consider the Lagrange interpolation for $f(t) = P_0(tx_0 \frac{r_1}{r_2}, y)$, where $0 < r_1 < r_2 < \frac{R}{4} < 1$. We fix a point x_0 such that $\sqrt{\langle \hat{A}^{-1}x_0, x_0 \rangle} = |Sx_0| \leq r_2$. We approximate $P_0(x_0, y)$ by a linear combination of the form $\sum_{i=1}^m c_i P_0(x_i, y)$ with $|Sx_i| \leq r_1$. Then we need to estimate the sum of the absolute values of the coefficients c_i in the linear combination and the error $(R_m P_0)(x_0, y)$ of the approximation.

We choose points $x_i = t_i x_0 \frac{r_1}{r_2}$ on the segment $[0, x_0 \frac{r_1}{r_2}]$ with $t_i \in (0, 1)$. We select $u_i = t_i$ in the definition of Φ_m in (3.5) and $\tilde{u} = \frac{r_2}{r_1}$. Define

$$(3.9) \quad c_j = \prod_{i \neq j}^m \frac{r_2 r_1^{-1} - t_i}{t_j - t_i}.$$

Since $0 < t_i < 1$, direct calculations show that

$$(3.10) \quad |c_j| \leq \frac{(r_2 r_1^{-1})^m}{|\Phi'_m(t_j)|}.$$

To further estimate $|c_j|$, we choose t_i to be the Chebyshev nodes, i.e., $t_i = \cos(\frac{(2i-1)\pi}{2m})$. Then we can write

$$\Phi_m(t) = 2^{1-m} T_m(t),$$

where T_m is the Chebyshev polynomial of the first kind. We also know that

$$(3.11) \quad \Phi'_m(t) = m 2^{1-m} U_{m-1}(t),$$

where U_{m-1} is the Chebyshev polynomial of the second kind. See, e.g., section 3.2.3 in [DB]. At each t_i , we have

$$U_{m-1}(t_i) = U_{m-1}\left(\cos \frac{(2i-1)\pi}{2m}\right) = \frac{\sin \frac{(2i-1)\pi}{2}}{\sin \frac{(2i-1)\pi}{2m}} = \frac{(-1)^{i-1}}{\sin \frac{(2i-1)\pi}{2m}}.$$

By (3.11), it follows that

$$|\Phi'_m(t_i)| \geq m 2^{1-m}.$$

Therefore, we can show that

$$(3.12) \quad |c_j| \leq (2m)^{-1} \left(\frac{2r_2}{r_1}\right)^m.$$

To estimate the error of the approximation $(R_m P_0)(x_0, y)$, we do an analytic extension of the function $f(t) = P_0(tx_0 \frac{r_1}{r_2}, y)$ to the disc of radius $\frac{R}{2r_1}$ centered at the origin in the complex plane $\mathcal{C}$. Note that $P_0(x, y)$ is independent of ϵ . By the explicit form of $P_0(x, y)$ in (3.4) for ∂E_R , we have

$$(3.13) \quad f(z) = \gamma_d |S| \frac{R^2 - |Sx_0|^2 r_1^2 r_2^{-2} z^2}{R |r_1 r_2^{-1} z Sx_0 - Sy|^d}.$$

If R is fixed, then

$$(3.14) \quad |f(z)| \leq CR^{-(d-1)}$$

in the disc. Hence, $f(z)$ is bounded. By (3.5), we can also see that

$$(3.15) \quad |\Phi_m(z)| \geq \left(\left(\frac{R}{2r_1}\right) - 1\right)^m \quad \text{on the circle } |z| = \frac{R}{2r_1}$$

and

$$(3.16) \quad \left|\Phi_m\left(\frac{r_2}{r_1}\right)\right| \leq \left(\frac{r_2}{r_1}\right)^m.$$

By the identity (3.8), we estimate the interpolation error as follows:

$$\begin{aligned}
|(R_m P_0)(x_0, y)| &= \left| P_0(x_0, y) - \sum_{i=1}^m c_i P_0(x_i, y) \right| \\
&= \left| f\left(\frac{r_2}{r_1}\right) - \sum_{i=1}^m c_i f(t_i) \right| \\
&= \left| \frac{1}{2\pi i} \int_{|z|=\frac{R}{2r_1}} \frac{\Phi_m(r_2 r_1^{-1}) f(z)}{(z - r_2 r_1^{-1}) \Phi_m(z)} dz \right| \\
&\leq C \frac{2^m r_2^m}{R^{d-2} (R - 2r_1)^m (R - 2r_2)} \\
(3.17) \quad &\leq C \frac{2^m r_2^m}{R^{d+m-1}},
\end{aligned}$$

where we have used estimates (3.15), (3.16) and the assumption that $0 < r_1 < r_2 < \frac{R}{4} < R$ in the last inequality, and the constant C in the last inequality does not depend on m . Note that (3.12) yields

$$(3.18) \quad \sum_{j=1}^m |c_j| \leq 2^{-1} \left(\frac{2r_2}{r_1} \right)^m.$$

With these estimates for the Poisson kernel, we are going to estimate the supremum u_ϵ in ellipsoids. We can write the solutions of (1.3) in E_R as

$$(3.19) \quad u_\epsilon(x) = \int_{\partial E_R} P_\epsilon(x, y) u_\epsilon(y) dy.$$

Thanks to (2.12), we can split the last integral as

$$\begin{aligned}
&\left| \int_{\partial E_R} P_\epsilon(x, y) u_\epsilon(y) dy \right| \\
&\leq \left| \int_{\partial E_R} \sum_{i=1}^m c_i P_\epsilon(x_i, y) u_\epsilon(y) dy \right| \\
&\quad + \left| \int_{\partial E_R} (P_\epsilon(x, y) - \sum_{i=1}^m c_i P_\epsilon(x_i, y)) u_\epsilon(y) dy \right| \\
&\leq \sum_{i=1}^m |c_i| |u_\epsilon(x_i)| + \int_{\partial E_R} \left| P_0(x, y) - \sum_{i=1}^m c_i P_0(x_i, y) \right| |u_\epsilon(y)| dy \\
(3.20) \quad &+ \int_{\partial E_R} \left| R_\epsilon(x, y) - \sum_{i=1}^m c_i R_\epsilon(x_i, y) \right| |u_\epsilon(y)| dy.
\end{aligned}$$

Let $x = x_0$. By the relations of $x_i = t_i x_0 \frac{r_1}{r_2}$ and $|Sx_0| \leq r_2$, then all $x_i \in E_{r_1}$. Taking into account the estimates (2.15), (2.22), (3.17), and (3.18) gives that

$$(3.21) \quad |u_\epsilon(x_0)| \leq \left(\frac{2r_2}{r_1} \right)^m \sup_{E_{r_1}} |u_\epsilon| + C \frac{2^m r_2^m}{R^m} \sup_{E_R} |u_\epsilon| + C \epsilon \ln(\epsilon^{-1} + 2) \left(\frac{2r_2}{r_1} \right)^m \sup_{E_R} |u_\epsilon|,$$

where those C do not depend on m , r_1 , r_2 , or R . Since x_0 is an arbitrary point in E_{r_2} , it follows that

$$(3.22) \quad \sup_{E_{r_2}} |u_\epsilon| \leq C \left\{ \left(\frac{2r_2}{r_1} \right)^m \sup_{E_{r_1}} |u_\epsilon| + \left(\frac{2r_2}{R} \right)^m \sup_{E_R} |u_\epsilon| + \epsilon \ln(\epsilon^{-1} + 2) \left(\frac{2r_2}{r_1} \right)^m \sup_{E_R} |u_\epsilon| \right\}.$$

Next we aim to minimize the summation of the terms in the right-hand side of (3.22) by choosing the integer value m . For ease of notation, let

$$(3.23) \quad \sup_{E_{r_1}} |u_\epsilon| = \delta, \quad \sup_{E_R} |u_\epsilon| = M.$$

First, we choose a value of m such that

$$(3.24) \quad \left(\frac{2r_2}{r_1} \right)^m \delta = \frac{(2r_2)^m}{R^m} M.$$

Solving the equality gives that

$$m = \frac{\ln M/\delta}{\ln R/r_1}.$$

We define an integer value

$$m_0 = \left\lfloor \frac{\ln M/\delta}{\ln R/r_1} \right\rfloor + 1,$$

where $\lfloor \cdot \rfloor$ denotes its integer part. We split the discussion of the minimization into two cases.

Case 1: The case $\epsilon \ln(\epsilon^{-1} + 2) \left(\frac{2r_2}{r_1} \right)^{m_0} \leq \left(\frac{2r_2}{R} \right)^{m_0}$. In this case, let $m = m_0$ in the estimate (3.22). The third term can be incorporated into the second term in the right-hand side of (3.22). It follows that

$$(3.25) \quad \begin{aligned} \sup_{E_{r_2}} |u_\epsilon| &\leq C \left\{ \left(\frac{2r_2}{r_1} \right)^{m_0} \delta + \left(\frac{2r_2}{R} \right)^{m_0} M \right\} \\ &\leq C \left(\frac{2r_2}{R} \right) M^{1-\alpha} \delta^\alpha, \end{aligned}$$

where

$$(3.26) \quad \alpha = \frac{\ln \frac{R}{2r_2}}{\ln \frac{R}{r_1}}.$$

Case 2: The case $\epsilon \ln(\epsilon^{-1} + 2) \left(\frac{2r_2}{r_1} \right)^{m_0} > \left(\frac{2r_2}{R} \right)^{m_0}$. In this case, from the definition of m_0 , we see that

$$(3.27) \quad \epsilon \ln(\epsilon^{-1} + 2) > \frac{\delta r_1}{MR}.$$

That is,

$$(3.28) \quad \sup_{E_{r_1}} |u_\epsilon| \leq \epsilon \ln(\epsilon^{-1} + 2) \frac{R}{r_1} \sup_{E_R} |u_\epsilon|.$$

In order to obtain the minimum for the terms in the right-hand sides of (3.22), we choose the value of $\hat{m}$ such that

$$(3.29) \quad \epsilon \ln(\epsilon^{-1} + 2) \left(\frac{2r_2}{r_1} \right)^{\hat{m}} = \left(\frac{2r_2}{R} \right)^{\hat{m}}.$$

Solving the equality yields that

$$(3.30) \quad \hat{m} = \frac{\ln(\epsilon \ln(\epsilon^{-1} + 2))}{\ln \frac{r_1}{R}}.$$

We define the integer value as

$$(3.31) \quad m_1 = \left\lfloor \frac{\ln(\epsilon \ln(\epsilon^{-1} + 2))}{\ln \frac{r_1}{R}} \right\rfloor + 1.$$

Substituting $m = m_1$ in the inequality (3.22) again, we can incorporate the second term into the third term in the right-hand side of (3.22). Taking (3.28) into account, we obtain that

$$\begin{aligned} \sup_{E_{r_2}} |u_\epsilon| &\leq C \left\{ \left(\frac{2r_2}{r_1} \right)^{m_1} \epsilon \ln(\epsilon^{-1} + 2) \frac{R}{r_1} M + \epsilon \ln(\epsilon^{-1} + 2) \left(\frac{2r_2}{r_1} \right)^{m_1} M \right\} \\ &\leq C \frac{2r_2 R}{r_1^2} \exp \left\{ \frac{\ln \frac{2r_2}{r_1} \ln[\epsilon \ln(\epsilon^{-1} + 2)]}{\ln \frac{r_1}{R}} \right\} \epsilon \ln(\epsilon^{-1} + 2) M \\ &\leq C \frac{R^2}{r_1^2} [\epsilon \ln(\epsilon^{-1} + 2)]^{\frac{\ln \frac{R}{2r_2}}{\ln \frac{R}{r_1}}} M \\ (3.32) \quad &= C \frac{R^2}{r_1^2} [\epsilon \ln(\epsilon^{-1} + 2)]^\alpha M \end{aligned}$$

with

$$(3.33) \quad \alpha = \frac{\ln \frac{R}{2r_2}}{\ln \frac{R}{r_1}}.$$

Combining the estimates in (3.25) and (3.32) in these two cases, we arrive at

$$(3.34) \quad \sup_{E_{r_2}} |u_\epsilon| \leq C \left\{ \frac{r_2}{R} \left(\sup_{E_{r_1}} |u_\epsilon| \right)^\alpha \left(\sup_{E_R} |u_\epsilon| \right)^{1-\alpha} + \frac{R^2}{r_1^2} [\epsilon \ln(\epsilon^{-1} + 2)]^\alpha \sup_{E_R} |u_\epsilon| \right\}$$

in three different sizes of ellipsoids. By the assumption of r_1 , r_2 , and R , we also see that $0 < \alpha < 1$. This completes the proof of Theorem 1. $\square$

Now we are ready to show the proof of Corollary 1.

Proof of Corollary 1. We make use of the approximate three-ball inequality in ellipsoids in (3.34). We choose $0 < r_1 \leq \frac{r_1}{\lambda} < r_2 < \frac{R}{4} < \frac{R}{4\lambda}$ for $R < \sqrt{\lambda}$. By the relation of ellipsoids and balls in (2.7), we have

$$(3.35) \quad \sup_{\mathbb{B}_{\sqrt{\lambda}r_2}} |u_\epsilon| \leq C \left\{ \frac{r_2}{R} \left(\sup_{\mathbb{B}_{r_1/\sqrt{\lambda}}} |u_\epsilon| \right)^\alpha \left(\sup_{\mathbb{B}_{R/\sqrt{\lambda}}} |u_\epsilon| \right)^{1-\alpha} + \frac{R^2}{r_1^2} [\epsilon \ln(\epsilon^{-1} + 2)]^\alpha \sup_{\mathbb{B}_{R/\sqrt{\lambda}}} |u_\epsilon| \right\}.$$

Let $R_1 = r_1/\sqrt{\lambda}$, $R_2 = \sqrt{\lambda}r_2$, and $R_3 = R/\sqrt{\lambda}$, Then $0 < R_1 < R_2 < \frac{R_3}{4} < \frac{1}{4}$. The last inequality implies that

$$(3.36) \quad \sup_{\mathbb{B}_{R_2}} |u_\epsilon| \leq C \left\{ \frac{R_2}{R_3} \left(\sup_{\mathbb{B}_{R_1}} |u_\epsilon| \right)^\beta \left(\sup_{\mathbb{B}_{R_3}} |u_\epsilon| \right)^{1-\beta} + \frac{R_3^2}{R_1^2} [\epsilon \ln(\epsilon^{-1} + 2)]^\beta \sup_{\mathbb{B}_{R_3}} |u_\epsilon| \right\},$$

where

$$(3.37) \quad \beta = \frac{\ln \frac{\lambda R_3}{2R_2}}{\ln \frac{R_3}{R_1}}$$

is derived from the α value in (3.26) and C depends only on λ and (τ, μ) . It is easy to see that $\beta < 1$. By the value of λ , it is possible that $\beta < 0$. However, the second term in the right-hand side of (3.36) implies that such inequality holds trivially when ϵ is sufficiently small. To have $\beta > 0$, we need to have $R_2 < \frac{\lambda R_3}{2}$. Together with $R_2 < \frac{R_3}{4} < \frac{1}{4}$, we choose $0 < R_1 < R_2 < \frac{\lambda R_3}{4} < \frac{1}{4}$. Thus, the inequality (3.36) holds with $0 < \beta < 1$. This completes the proof of Corollary. $\square$

From Corollary 1, we can give the proof of Corollary 3.

Proof of Corollary 3. We introduce a new function $v_\epsilon(x, t)$ as

$$v_\epsilon(x, t) = e^{\sqrt{\Lambda_{\epsilon,k}}t} u_\epsilon(x) \quad \text{in } \mathbb{B}_{10} \times (-10, 10).$$

From eigenvalue type equation (1.10), the new function $v_\epsilon(x, t)$ solves the equation

$$(3.38) \quad \mathcal{L}_\epsilon v_\epsilon - \partial_t^2 v_\epsilon = 0 \quad \text{in } \mathbb{B}_{10} \times (-10, 10).$$

This new homogenization equation (3.38) has the coefficient matrix

$$\begin{pmatrix} (a_{ij})_{d \times d} & 0 \\ 0 & 1 \end{pmatrix},$$

which also satisfies conditions (1.4), (1.5), and (1.6). Let $\tilde{\mathbb{B}}_r$ be the ball with radius r in $\mathbb{B}_{10} \times (-10, 10)$. We may write $\tilde{\mathbb{B}}_r$ as $\mathbb{B}_r \times (-r, r)$. From Corollary 1, it holds that

$$(3.39) \quad \|v_\epsilon\|_{L^\infty(\tilde{\mathbb{B}}_{R_2})} \leq C \left\{ \frac{R_2}{R_3} \|v_\epsilon\|_{L^\infty(\tilde{\mathbb{B}}_{R_1})}^\beta \|v_\epsilon\|_{L^\infty(\tilde{\mathbb{B}}_{R_3})}^{1-\beta} + \frac{R_3^2}{R_1^2} [\epsilon \ln(\epsilon^{-1} + 2)]^\beta \|v_\epsilon\|_{L^\infty(\tilde{\mathbb{B}}_{R_3})} \right\}$$

for R_1, R_2, R_3, β given in Corollary 1. By the definition of $v_\epsilon(x, t)$ and $\tilde{\mathbb{B}}_r$, we obtain that

$$(3.40) \quad e^{-\sqrt{\Lambda_{\epsilon,k}}R_2} \|u_\epsilon\|_{L^\infty(\mathbb{B}_{R_2})} \leq C \left\{ \frac{R_2}{R_3} e^{\sqrt{\Lambda_{\epsilon,k}}R_3} \|u_\epsilon\|_{L^\infty(\mathbb{B}_{R_1})}^\beta \|u_\epsilon\|_{L^\infty(\mathbb{B}_{R_3})}^{1-\beta} + \frac{R_3^2}{R_1^2} e^{\sqrt{\Lambda_{\epsilon,k}}R_3} [\epsilon \ln(\epsilon^{-1} + 2)]^\beta \|u_\epsilon\|_{L^\infty(\mathbb{B}_{R_3})} \right\}.$$

Thus, the corollary follows. $\square$

Let us show another consequence of the three-ball inequality in Corollary 1, that is, the approximate propagation of smallness estimates.

Proof of Corollary 4. As a consequence of De Giorgi–Nash estimates, the solution u_ϵ in (1.3) is a continuous function. Hence there exists $\bar{x} \in \mathbb{B}_9$ such that $u_\epsilon(\bar{x}) = \sup_{\mathbb{B}_9} |u_\epsilon(x)|$. We select a sequence of balls with radius $2r$ centered at $x_1 = 0, \dots, x_m$ so that $x_{i+1} \in \mathbb{B}_r(x_i)$, $\mathbb{B}_r(x_{i+1}) \subset \mathbb{B}_{2r}(x_i)$ and $\bar{x} \in \mathbb{B}_{2r}(x_m)$. The number m depends on r . By the way we choose x_{i+1} , it holds that

$$(3.41) \quad \|u_\epsilon\|_{L^\infty(\mathbb{B}_r(x_{i+1}))} \leq \|u_\epsilon\|_{L^\infty(\mathbb{B}_{2r}(x_i))}.$$

Applying the three-ball inequality (1.9) for $x_1 = 0$ and using that $\|u_\epsilon\|_{L^\infty(\mathbb{B}_{10})} \leq 1$, we have

$$(3.42) \quad \|u_\epsilon\|_{L^\infty(\mathbb{B}_{2r}(x_1))} \leq C\{\delta^\beta + [\epsilon \ln(\epsilon^{-1} + 2)]^\beta\}$$

with $\beta = \frac{\ln \frac{9}{4}}{\ln \frac{9}{\lambda}}$. Then we choose $x_2 \in \mathbb{B}_{2r}(x_1)$ such that $\mathbb{B}_r(x_2) \subset \mathbb{B}_{2r}(x_1)$. Thus,

$$(3.43) \quad \|u_\epsilon\|_{L^\infty(\mathbb{B}_r(x_2))} \leq \|u_\epsilon\|_{L^\infty(\mathbb{B}_{2r}(x_1))}.$$

Applying the three-ball inequality (1.9) for balls centered at x_2 and using (3.42) and (3.43), we have

$$(3.44) \quad \begin{aligned} \|u_\epsilon\|_{L^\infty(\mathbb{B}_{2r}(x_2))} &\leq C\{C\delta^\beta + C[\epsilon \ln(\epsilon^{-1} + 2)]^\beta\}^\beta + C[\epsilon \ln(\epsilon^{-1} + 2)]^\beta \\ &= C^{\beta+1}\delta^{\beta^2} + C^{\beta+1}[\epsilon \ln(\epsilon^{-1} + 2)]^{\beta^2} + C[\epsilon \ln(\epsilon^{-1} + 2)]^\beta. \end{aligned}$$

Iterating this argument with three-ball inequality at points $x_3, \dots$, up to x_m and using (3.41) and the fact that $\|u_\epsilon\|_{L^\infty(\mathbb{B}_{10})} \leq 1$, we obtain that

$$\|u_\epsilon\|_{L^\infty(\mathbb{B}_{2r}(x_m))} \leq C^{\sum_{i=1}^m \beta^{m-i}} \delta^{\beta^m} + \sum_{j=1}^m C^{\sum_{i=1}^j \beta^{i-1}} [\epsilon \ln(\epsilon^{-1} + 2)]^{\beta^j}.$$

Since $0 < \beta < 1$, we have

$$(3.45) \quad \|u_\epsilon\|_{L^\infty(\mathbb{B}_{2r}(x_m))} \leq C^{\frac{1}{1-\beta}} \delta^{\beta^m} + mC^{\frac{1}{1-\beta}} [\epsilon \ln(\epsilon^{-1} + 2)]^{\beta^m}.$$

If r is fixed, then m is a fixed number. Thus,

$$(3.46) \quad \|u_\epsilon\|_{L^\infty(\mathbb{B}_9)} \leq C^{\frac{1}{1-\beta}} \delta^{\beta^m} + mC^{\frac{1}{1-\beta}} [\epsilon \ln(\epsilon^{-1} + 2)]^{\beta^m}$$

for $\epsilon \ln(\epsilon^{-1} + 2) \leq 1$. $\square$

Compared with the previous approximate propagation of smallness result, we are able to show a full propagation of smallness result with the aid of the doubling inequality in Theorem A.

Proof of Corollary 5. Let $r_k = 2^k r_0$. Choose k_0 such that $2^{k_0} r_0 = \sqrt{\lambda}$. Then $k_0 \approx \log_2(\frac{\sqrt{\lambda}}{r_0})$. Let $M_k = \|u_\epsilon\|_{L^\infty(\mathbb{B}_{r_k})}$ so that $M_0 = \|u_\epsilon\|_{L^\infty(\mathbb{B}_{r_0})}$ and $M_{k_0} = \|u_\epsilon\|_{L^\infty(\mathbb{B}_{\sqrt{\lambda}})}$. Choosing

$$(3.47) \quad N = \frac{\|u_\epsilon\|_{L^\infty(\mathbb{B}_{2/\sqrt{\lambda}})}}{\|u_\epsilon\|_{L^\infty(\mathbb{B}_{\sqrt{\lambda}})}},$$

from Theorem A, we have

$$(3.48) \quad M_0 = \frac{M_0}{M_1} \cdot \frac{M_1}{M_2} \cdot \frac{M_2}{M_3} \cdot \frac{M_{k_0-1}}{M_{k_0}} \cdot M_{k_0} \geq \left(\frac{1}{C(N)} \right)^{k_0} M_{k_0}.$$

It also follows that

$$(3.49) \quad M_{k_0} \leq M_0 C(N)^{k_0}.$$

From the N value in (3.47), the boundedness assumption of u_ϵ , and the definition of M_{k_0} , it follows that $N \leq \frac{1}{M_{k_0}}$. By (3.49) and monotonicity of $C(N)$, we arrive at

$$(3.50) \quad M_{k_0} \leq M_0 C\left(\frac{1}{M_{k_0}}\right)^{k_0}.$$

Supposing that $M_{k_0} > \eta_0$ and $M_0 = \|u_\epsilon\|_{L^\infty(\mathbb{B}_{r_0})} \leq \delta_0$, from (3.50),

$$(3.51) \quad M_{k_0} \leq \delta_0 C\left(\frac{1}{M_{k_0}}\right)^{k_0} \leq \delta_0 C\left(\frac{1}{\eta_0}\right)^{k_0}.$$

If we choose δ_0 such that $\delta_0 C\left(\frac{1}{\eta_0}\right)^{k_0} \leq \frac{\eta_0}{2}$, this gives that $M_{k_0} \leq \frac{\eta_0}{2}$, which contradicts the assumption $M_{k_0} > \eta_0$. Hence, if $\delta_0 \leq \frac{\eta_0}{2} (C(\frac{1}{\eta_0}))^{-k_0}$, we must have $M_{k_0} \leq \eta_0$ as desired. This completes the corollary. Notice that the dependence of δ_0 is not explicit. $\square$

As a consequence of Corollary 1, we can give the proof of Theorem 2.

Proof of Theorem 2. To use the conclusion in Corollary 1, we extend (1.17) in half balls to balls. We do an even extension for the metric $A(y)$ across the half space $\{x|x_d = 0\}$. Then we define a new coefficient matrix as

$$\tilde{A}\left(\frac{x}{\epsilon}\right) = \begin{cases} A\left(\frac{x'}{\epsilon}, \frac{x_d}{\epsilon}\right) & \text{for } x_d \geq 0, \\ A\left(\frac{x'}{\epsilon}, -\frac{x_d}{\epsilon}\right) & \text{for } x_d < 0, \end{cases}$$

where $x' = (x_1, \dots, x_{d-1})$. We can see that $\tilde{A}(y)$ still satisfies the ellipticity condition (1.4) and 1-periodic condition (1.5). To verify that $\tilde{A}(y)$ also satisfies the Hölder continuity (1.6), we just need to check the Hölder continuity for $\tilde{A}(y)$ in the y_d direction. Let $y_d^1 > 0$ and $y_d^2 < 0$. By the condition (1.6) for $A(y)$, we have

$$(3.52) \quad \begin{aligned} |\tilde{A}(y', y_d^1) - \tilde{A}(y', y_d^2)| &= |A(y', y_d^1) - A(y', 0) + A(y', 0) - A(y', -y_d^2)| \\ &\leq \tau |y_d^1|^\mu + \tau |y_d^2|^\mu \\ &\leq 2\tau |y_d^1 - y_d^2|^\mu. \end{aligned}$$

This implies Hölder continuity as (1.6) holds for $\tilde{A}(y)$. For the Dirichlet boundary conditions, we do an odd extension for $u_\epsilon(x)$ across the half space $\{x|x_d = 0\}$. Then we have a new function $v_\epsilon(x)$ defined as follows:

$$v_\epsilon(x) = \begin{cases} u_\epsilon(x', x_d) & \text{for } x_d \geq 0, \\ -u_\epsilon(x', -x_d) & \text{for } x_d < 0. \end{cases}$$

Since the solution $u_\epsilon(x)$ in (1.17) for the Dirichlet boundary conditions is Lipschitz continuous by the homogenization theory, e.g., [AL], the new $v_\epsilon(x)$ is Lipschitz continuous and is in $H^1(\mathbb{B}_{10})$. We claim that $v_\epsilon(x)$ satisfies the elliptic homogenization

$$(3.53) \quad -\operatorname{div}\left(\tilde{A}\left(\frac{x}{\epsilon}\right)\nabla v_\epsilon\right) = 0 \quad \text{in } \mathbb{B}_{10}.$$

To verify that $v_\epsilon(x)$ is a weak solution of (3.53), we need to check that the following holds:

$$(3.54) \quad \int_{\mathbb{B}_{10}} \tilde{A}\left(\frac{x}{\epsilon}\right) \nabla v_\epsilon \cdot \nabla \phi(x) dx = 0 \quad \text{for any } \phi(x) \in C_0^\infty(\mathbb{B}_{10}).$$

Recall that the weak solution $u_\epsilon(x)$ of (1.17) for the Dirichlet boundary condition is given as

$$(3.55) \quad \int_{\mathbb{B}_{10}^+} A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon \cdot \nabla \phi(x) dx = \int_{\partial \mathbb{B}_{10}^+ \cap \{x_d=0\}} A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon \cdot n \phi(x) d\sigma$$

for any $\phi(x) \in C_0^\infty(\mathbb{B}_{10}^+)$. By the assumption of $A(y)$ in (1.18), (3.55) is equivalent to

$$(3.56) \quad \int_{\mathbb{B}_{10}^+} \sum_{i,j=1}^{d-1} a_{ij} \frac{\partial u_\epsilon}{\partial x_j} \frac{\partial \phi}{\partial x_i} dx + \int_{\mathbb{B}_{10}^+} a_{dd} \frac{\partial u_\epsilon}{\partial x_d} \frac{\partial \phi}{\partial x_d} dx = - \int_{\partial \mathbb{B}_{10}^+ \cap \{x_d=0\}} a_{dd} D_d u_\epsilon \phi d\sigma.$$

In order to show (3.54), we split the integral in the left-hand side of (3.54) in $\mathbb{B}_{10}$ into the integrations in $\mathbb{B}_{10}^+$ and $\mathbb{B}_{10}^-$, where $\mathbb{B}_{10}^- = \mathbb{B}_{10} \cap \{x_d < 0\}$. For any $\phi(x) \in C_0^\infty(\mathbb{B}_{10})$, by the definition of $\tilde{A}(\frac{x}{\epsilon})$ and $v_\epsilon(x)$, we have

$$(3.57) \quad \begin{aligned} \int_{\mathbb{B}_{10}} \tilde{A}\left(\frac{x}{\epsilon}\right) \nabla v_\epsilon \cdot \nabla \phi(x) dx &= \int_{\mathbb{B}_{10}^+} \tilde{A}\left(\frac{x}{\epsilon}\right) \nabla v_\epsilon \cdot \nabla \phi(x) dx + \int_{\mathbb{B}_{10}^-} \tilde{A}\left(\frac{x}{\epsilon}\right) \nabla v_\epsilon \cdot \nabla \phi(x) dx \\ &= \int_{\mathbb{B}_{10}^+} A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon \cdot \nabla \phi(x) dx + \int_{\mathbb{B}_{10}^-} \tilde{A}\left(\frac{x}{\epsilon}\right) \nabla v_\epsilon \cdot \nabla \phi(x) dx. \end{aligned}$$

Applying the integration by parts and change of variables to transform the integration from $\mathbb{B}_{10}^-$ to $\mathbb{B}_{10}^+$, we can show that

$$(3.58) \quad \begin{aligned} \int_{\mathbb{B}_{10}^-} \tilde{A}\left(\frac{x}{\epsilon}\right) \nabla v_\epsilon \cdot \nabla \phi(x) dx &= - \int_{\mathbb{B}_{10}^-} \sum_{i,j=1}^{d-1} a_{ij}\left(\frac{x'}{\epsilon}, \frac{-x_d}{\epsilon}\right) \frac{\partial u_\epsilon(x', -x_d)}{\partial x_j} \frac{\partial \phi}{\partial x_i} dx \\ &\quad - \int_{\mathbb{B}_{10}^-} a_{dd}\left(\frac{x'}{\epsilon}, \frac{-x_d}{\epsilon}\right) \frac{\partial u_\epsilon(x', -x_d)}{\partial x_d} \frac{\partial \phi}{\partial x_d} dx \\ &= - \int_{\mathbb{B}_{10}^+} \sum_{i,j=1}^{d-1} a_{ij}\left(\frac{x'}{\epsilon}, \frac{x_d}{\epsilon}\right) \frac{\partial u_\epsilon}{\partial x_j} \frac{\partial \phi(x', -x_d)}{\partial x_i} dx \\ &\quad - \int_{\mathbb{B}_{10}^+} a_{dd}\left(\frac{x'}{\epsilon}, \frac{x_d}{\epsilon}\right) \frac{\partial u_\epsilon}{\partial x_d} \frac{\partial \phi(x', -x_d)}{\partial x_d} dx \\ &= \int_{\partial \mathbb{B}_{10}^+ \cap \{x_d=0\}} a_{dd} D_d u_\epsilon \phi d\sigma, \end{aligned}$$

where we have used (3.56). Then (3.54) is verified by combining the estimates (3.56), (3.57), and (3.58). Thus, $v_\epsilon(x)$ is the solution of (3.53).

Thanks to Corollary 1, we have the approximate three-ball inequality for v_ϵ . For $0 < R_1 < R_2 < \frac{\lambda R_3}{4} < \frac{\lambda}{4}$, we obtain that

$$(3.59) \quad \|v_\epsilon\|_{L^\infty(\mathbb{B}_{R_2})} \leq C \left\{ \frac{R_2}{R_3} \|v_\epsilon\|_{L^\infty(\mathbb{B}_{R_1})}^\beta \|v_\epsilon\|_{L^\infty(\mathbb{B}_{R_3})}^{1-\beta} + \frac{R_3^2}{R_1^2} [\epsilon \ln(\epsilon^{-1} + 2)]^\beta \|v_\epsilon\|_{L^\infty(\mathbb{B}_{R_3})} \right\},$$

where $\beta = \ln \frac{\lambda R_3}{2R_2} / \ln \frac{R_3}{R_1}$ and C depends on λ and (τ, μ) . Since v_ϵ is an odd extension of u_ϵ , the estimates (1.19) for the Dirichlet boundary conditions in half balls follow from (3.59).

For the Neumann boundary conditions, we do an even extension. We define a new function $v_\epsilon(x)$ as follows:

$$v_\epsilon(x) = \begin{cases} u_\epsilon(x', x_d) & \text{for } x_d \geq 0, \\ u_\epsilon(x', -x_d) & \text{for } x_d < 0. \end{cases}$$

The solution $u_\epsilon(x)$ in (1.17) for the Neumann boundary conditions is Lipschitz continuous, e.g., [KLS1], [AS]. Hence, $v_\epsilon(x)$ is Lipschitz continuous and is in $H^1(\mathbb{B}_{10})$. We also claim that $v_\epsilon(x)$ is the solution for

$$(3.60) \quad -\operatorname{div}\left(\tilde{A}\left(\frac{x}{\epsilon}\right)\nabla v_\epsilon\right) = 0 \quad \text{in } \mathbb{B}_{10}.$$

Thus, we need to show that

$$(3.61) \quad \int_{\mathbb{B}_{10}} \tilde{A}\left(\frac{x}{\epsilon}\right)\nabla v_\epsilon \cdot \nabla \phi(x) dx = 0 \quad \text{for any } \phi(x) \in C_0^\infty(\mathbb{B}_{10}).$$

The weak solution $u_\epsilon(x)$ of (1.17) for the Neumann boundary condition is given as

$$(3.62) \quad \int_{\mathbb{B}_{10}^+} A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon \cdot \nabla \phi(x) dx = 0$$

for any $\phi(x) \in C_0^\infty(\mathbb{B}_{10}^+)$. By the definition of $A(y)$, it also holds that

$$(3.63) \quad \int_{\mathbb{B}_{10}^+} \sum_{i,j=1}^{d-1} a_{ij} \frac{\partial u_\epsilon}{\partial x_j} \frac{\partial \phi}{\partial x_i} dx + \int_{\mathbb{B}_{10}^+} a_{dd} \frac{\partial u_\epsilon}{\partial x_d} \frac{\partial \phi}{\partial x_d} dx = 0.$$

We split the integral in the left-hand side of (3.61) in $\mathbb{B}_{10}$ into the integrations in $\mathbb{B}_{10}^+$ and $\mathbb{B}_{10}^-$ as

$$(3.64) \quad \begin{aligned} \int_{\mathbb{B}_{10}} \tilde{A}\left(\frac{x}{\epsilon}\right)\nabla v_\epsilon \cdot \nabla \phi(x) dx &= \int_{\mathbb{B}_{10}^+} \tilde{A}\left(\frac{x}{\epsilon}\right)\nabla v_\epsilon \cdot \nabla \phi(x) dx + \int_{\mathbb{B}_{10}^-} \tilde{A}\left(\frac{x}{\epsilon}\right)\nabla v_\epsilon \cdot \nabla \phi(x) dx \\ &= \int_{\mathbb{B}_{10}^+} A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon \cdot \nabla \phi(x) dx + \int_{\mathbb{B}_{10}^-} \tilde{A}\left(\frac{x}{\epsilon}\right)\nabla v_\epsilon \cdot \nabla \phi(x) dx. \end{aligned}$$

By (3.63), the definition of $\tilde{A}(\frac{x}{\epsilon})$, $v_\epsilon(x)$, and the change of variables, we can show that

$$(3.65) \quad \begin{aligned} \int_{\mathbb{B}_{10}^-} \tilde{A}\left(\frac{x}{\epsilon}\right)\nabla v_\epsilon \cdot \nabla \phi(x) dx &= \int_{\mathbb{B}_{10}^-} \sum_{i,j=1}^{d-1} a_{ij}\left(\frac{x'}{\epsilon}, \frac{-x_d}{\epsilon}\right) \frac{\partial u_\epsilon(x', -x_d)}{\partial x_j} \frac{\partial \phi}{\partial x_i} dx \\ &\quad + \int_{\mathbb{B}_{10}^-} a_{dd}\left(\frac{x'}{\epsilon}, \frac{-x_d}{\epsilon}\right) \frac{\partial u_\epsilon(x', -x_d)}{\partial x_d} \frac{\partial \phi}{\partial x_d} dx \\ &= \int_{\mathbb{B}_{10}^+} \sum_{i,j=1}^{d-1} a_{ij}\left(\frac{x'}{\epsilon}, \frac{x_d}{\epsilon}\right) \frac{\partial u_\epsilon}{\partial x_j} \frac{\partial \phi(x', -x_d)}{\partial x_i} dx \\ &\quad + \int_{\mathbb{B}_{10}^+} a_{dd}\left(\frac{x'}{\epsilon}, \frac{x_d}{\epsilon}\right) \frac{\partial u_\epsilon}{\partial x_d} \frac{\partial \phi(x', -x_d)}{\partial x_d} dx \\ &= 0 \end{aligned}$$

for any $\phi(x) \in C_0^\infty(\mathbb{B}_{10})$. Therefore, the combination of (3.63), (3.64), and (3.65) verifies the claim and $v_\epsilon(x)$ is the solution in (3.60). Following the same strategy as the case of the Dirichlet boundary conditions, the estimate (1.19) is arrived at for equations with the Neumann boundary conditions. $\square$

Following the ideas in Corollary 3 and Theorem 2, we can easily show the proof of Corollary 6. For completeness of presentation, we present the main ideas in the argument.

Proof of Corollary 6. We consider a new function $v_\epsilon(x, t)$ as

$$(3.66) \quad v_\epsilon(x, t) = e^{\sqrt{\Lambda_{\epsilon, k}} t} u_\epsilon(x) \quad \text{in } \mathbb{B}_{10}^+ \times (-10, 10).$$

Then the new function $v_\epsilon(x, t)$ satisfies the homogenization equation

$$(3.67) \quad \begin{cases} -\operatorname{div}(A(\frac{x}{\epsilon}) \nabla v_\epsilon) - \partial_t^2 v_\epsilon = 0 & \text{in } \mathbb{B}_{10}^+ \times (-10, 10), \\ v_\epsilon = 0 \text{ or } \frac{\partial v_\epsilon}{\partial \nu} = 0 & \text{on } \{\partial \mathbb{B}_{10}^+ \cap \{x_d = 0\}\} \times (-10, 10). \end{cases}$$

We denote the new coefficient matrix in $\mathbb{B}_{10}^+ \times (-10, 10)$ as

$$\bar{A}\left(\frac{x}{\epsilon}, t\right) = \begin{pmatrix} (a_{ij}(\frac{x}{\epsilon}))_{d \times d} & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, the equation in (3.67) can be written as

$$(3.68) \quad -\operatorname{div}\left(\bar{A}\left(\frac{x}{\epsilon}, t\right) \nabla v_\epsilon\right) = 0 \quad \text{in } \mathbb{B}_{10}^+ \times (-10, 10).$$

We do an even extension for the matrix $\bar{A}(\frac{x}{\epsilon}, t)$ across the half space $\{(x, t) | x_d = 0\}$ and write the new coefficient matrix as $\tilde{A}(\frac{x}{\epsilon}, t)$. This coefficient matrix $\tilde{A}(\frac{x}{\epsilon}, t)$ still satisfies the conditions (1.4), (1.5), and (1.6). As in the proof of Theorem 2, we do an odd extension for v_ϵ in (3.67) for the Dirichlet boundary conditions across the half space $\{(x, t) | x_d = 0\}$ and write it as $\tilde{v}_\epsilon(x, t)$. We identify the half ball with radius r in the cylinder $\mathbb{B}_{10}^+ \times (-10, 10)$ as $\mathbb{B}_r^+ \times (-r, r)$. Recall that $\mathbb{B}_r \times (-r, r)$ is written as $\tilde{\mathbb{B}}_r$. We can check as in the proof of Theorem 2 that $\tilde{v}_\epsilon(x, t)$ satisfies the equation

$$(3.69) \quad -\operatorname{div}\left(\tilde{A}\left(\frac{x}{\epsilon}, t\right) \nabla \tilde{v}_\epsilon\right) = 0 \quad \text{in } \tilde{\mathbb{B}}_{10}.$$

Thanks to Corollary 1, the solutions $\tilde{v}_\epsilon$ in (3.69) satisfy the three-ball inequality

$$(3.70) \quad \|\tilde{v}_\epsilon\|_{L^\infty(\tilde{\mathbb{B}}_{R_2})} \leq C \left\{ \frac{R_2}{R_3} \|\tilde{v}_\epsilon\|_{L^\infty(\tilde{\mathbb{B}}_{R_1})}^\beta \|\tilde{v}_\epsilon\|_{L^\infty(\tilde{\mathbb{B}}_{R_3})}^{1-\beta} + \frac{R_3^2}{R_1^2} [\epsilon \ln(\epsilon^{-1} + 2)]^\beta \|\tilde{v}_\epsilon\|_{L^\infty(\tilde{\mathbb{B}}_{R_3})} \right\}$$

for $0 < R_1 < R_2 < \frac{\lambda R_3}{4} < \frac{\lambda}{4}$ with $\beta = \ln \frac{\lambda R_3}{2 R_2} / \ln \frac{R_3}{R_1}$. Since we have done an odd extension for $\tilde{v}_\epsilon$, the following inequality holds for $v_\epsilon(x, t)$:

$$(3.71) \quad \begin{aligned} \|v_\epsilon\|_{L^\infty(\mathbb{B}_{R_2}^+ \times (-R_2, R_2))} &\leq \left\{ \frac{R_2}{R_3} \|v_\epsilon\|_{L^\infty(\mathbb{B}_{R_1}^+ \times (-R_1, R_1))}^\beta \|v_\epsilon\|_{L^\infty(\mathbb{B}_{R_3}^+ \times (-R_3, R_3))}^{1-\beta} \right. \\ &\quad \left. + \frac{R_3^2}{R_1^2} [\epsilon \ln(\epsilon^{-1} + 2)]^\beta \|v_\epsilon\|_{L^\infty(\mathbb{B}_{R_3}^+ \times (-R_3, R_3))} \right\}. \end{aligned}$$

By the definition of $v_\epsilon(x, t)$ in (3.66), we further obtain that

$$(3.72) \quad \begin{aligned} \|u_\epsilon\|_{L^\infty(\mathbb{B}_{R_2}^+)} &\leq C e^{2R_3\sqrt{\Lambda_{\epsilon,k}}} \left\{ \frac{R_2}{R_3} \|u_\epsilon\|_{L^\infty(\mathbb{B}_{R_1}^+)}^\beta \|u_\epsilon\|_{L^\infty(\mathbb{B}_{R_3}^+)}^{1-\beta} \right. \\ &\quad \left. + \frac{R_3^2}{R_1^2} [\epsilon \ln(\epsilon^{-1} + 2)]^\beta \|u_\epsilon\|_{L^\infty(\mathbb{B}_{R_3}^+)} \right\}. \end{aligned}$$

Thus, (1.21) is arrived at for solutions of (1.20) with the Dirichlet boundary conditions. For (3.67) with the Neumann boundary conditions, we do an even extension for $v_\epsilon(x, t)$. As in the proof of Theorem 2, the approximate three-ball inequality (1.21) can be obtained. $\square$

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REFERENCES

- [Ag] S. AGMON, *Unicité et convexité dans les problèmes différentiels*, in Séminaire de Mathématiques Supérieures 13 (Été, 1965), Les Presses de l'Université de Montréal, Montreal, Quebec, 1966.
- [ALi] B. ALALI AND R. LIPTON, *Multiscale dynamics of heterogeneous media in the peridynamic formulation*, J. Elasticity, 106 (2012), pp. 71–103.
- [ARRV] G. ALESSANDRINI, L. RONDÌ, E. ROSSET, AND S. VESSELLA, *The stability for the Cauchy problem for elliptic equations*, Inverse Problems, 25 (2009), 123004.
- [Al] F. J. ALMGREN, *Dirichlet's problem for multiple valued functions and the regularity of mass minimizing integral currents*, in Minimal Submanifolds and Geodesics, M. Obata, ed., North Holland, Amsterdam, 1979, pp. 1–6.
- [AKS] S. ARMSTRONG, T. KUUSI, AND C. SMART, *Large-Scale Analyticity and Unique Continuation for Periodic Elliptic Equations*, arXiv:2005.01199, 2020.
- [AS] S. N. ARMSTRONG AND S. SHEN, *Lipschitz estimates in almost-periodic homogenization*, Comm. Pure Appl. Math., 69 (2016), pp. 1882–1923.
- [AL] M. AVELLANEDA AND F.-H. LIN, *Compactness methods in the theory of homogenization*, Comm. Pure Appl. Math., 40 (1987), pp. 803–847.
- [Ba] I. BABUŠKA, *Homogenization and its application: Mathematical and computational problems*, in Numerical Solution of Partial Differential Equations, III Synspade, B. Hubbard, ed., Academic Press, New York, 1976, pp. 89–116.
- [BLP] A. BENSOUSSAN, J.-L. LIONS, AND G. PAPANICOLAOU, *Asymptotic Analysis for Periodic Structures*, Stud. Math. Appl. 5, North-Holland, Amsterdam, 1978.
- [B] R. BRUMMELHUIS, *Three-spheres theorem for second order elliptic equations*, J. Anal. Math., 65 (1995), pp. 179–206.
- [DB] G. DAHLQUIST AND A. BJÖRCK, *Numerical Methods in Scientific Computing*, SIAM, Philadelphia, 2008.
- [DF] H. DONNELLY AND C. FEFFERMAN, *Nodal sets of eigenfunctions on Riemannian manifolds*, Invent. Math., 93 (1988), pp. 161–183.
- [FM] G. FRANCFORT AND F. MURAT, *Homogenisation and optimal bounds in linear elasticity*, Arch. Ration. Mech. Anal., 94 (1986), pp. 307–334.
- [GL] N. GAROFALO AND F.-H. LIN, *Monotonicity properties of variational integrals, A_p weights and unique continuation*, Indiana Univ. Math. J., 35 (1986), pp. 245–268.
- [GM] M. GUADIE AND E. MALINNIKOVA, *On three balls theorem for discrete harmonic functions*, Comput. Methods Funct. Theory, 14 (2014), pp. 721–734.
- [JL] D. JERISON AND G. LEBEAU, *Nodal sets of sums of eigenfunctions*, in Harmonic Analysis and Partial Differential Equations, Chicago Lectures in Math., University of Chicago Press, Chicago, IL, 1999, pp. 223–239.
- [K] C. KENIG, *Some recent applications of unique continuation*, in Recent Developments in Nonlinear Partial Differential Equations, Contemp. Math. 439, AMS, Providence, RI, 2007, pp. 25–56.

- [KLS] C. KENIG, F.-H. LIN, AND Z. SHEN, *Homogenization of elliptic systems with Neumann boundary conditions*, J. Amer. Math. Soc., 26 (2013), pp. 901–937.
- [KLS1] C. KENIG, F. LIN, AND Z. SHEN, *Estimates of eigenvalues and eigenfunctions in periodic homogenization*, J. Eur. Math. Soc. (JEMS), 15 (2013), pp. 1901–1925.
- [KLS2] C. KENIG, F.-H. LIN, AND Z. SHEN, *Periodic homogenization of Green and Neumann functions*, Comm. Pure Appl. Math., 64 (2014), pp. 1–44.
- [KM] J. KOREVAAR AND J. L. H. MEYERS, *Logarithmic convexity for supremum norms of harmonic functions*, Bull. Lond. Math. Soc., 26 (1994), pp. 353–362.
- [L] E. M. LANDIS, *Some problems of the qualitative theory of second order elliptic equations*, Russian Math. Surveys, 18 (1963), pp. 1–62.
- [Lin] F.-H. LIN, *Nodal sets of solutions of elliptic equations of elliptic and parabolic equations*, Comm. Pure Appl. Math., 44 (1991), 287–308.
- [LS] F.-H. LIN AND Z. SHEN, *Nodal sets and doubling conditions in elliptic homogenization*, Acta Math. Sin. (Engl. Ser.), 35 (2019), pp. 815–831.
- [Lo] A. LOGUNOV, *Nodal sets of Laplace eigenfunctions: Polynomial upper estimates of the Hausdorff measure*, Ann. of Math., 187 (2018), pp. 221–239.
- [M] K. MILLER, *Non-unique continuation for uniformly parabolic and elliptic equations in self-adjoint divergence form with Hölder-continuous coefficients*, Arch. Ration. Mech. Anal., 54 (1974), pp. 105–117.
- [S] Z. SHEN, *Periodic Homogenization of Elliptic Systems*, Oper. Theory Adv. Appl. 269, Birkhäuser/Springer, Cham, 2018.
- [Z] J. ZHU, *Quantitative uniqueness of elliptic equations*, Amer. J. Math., 138 (2016), pp. 733–762.
- [Zh] J. ZHUGE, *First-Order Expansions for Eigenvalues and Eigenfunctions in Periodic Homogenization*, arXiv:1804.10739, 2018.