The moduli spaces of sheaves on surfaces, pathologies and Brill-Noether problems

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1 Introduction

Brill-Noether divisors play a central role in the birational geometry of moduli spaces of sheaves on surfaces. In this paper, we survey recent Brill-Noether Theorems for rational surfaces following [CH16] and [CH17b]. The paper grew out of the first author's talk at the Abel Symposium in Svolvær, Norway in August 2017. While this paper is largely a survey, we will also present some new results and examples on surfaces of general type.

Let X be a smooth, complex projective surface and let H be an ample divisor. Let $\mathbf{v} = (\mathrm{rk}, \mathrm{ch}_1, \mathrm{ch}_2)$ be the Chern character of a sheaf on X. Gieseker [Gie77] and Maruyama [Mar78] construct a moduli space $M_{X,H}(\mathbf{v})$ that parameterizes S-equivalence classes of H-Gieseker semistable sheaves on X with Chern character $\mathbf{v}$. The moduli spaces $M_{X,H}(\mathbf{v})$ carry fundamental information on algebro-geometric invariants such as linear systems on X and they play a central role in Donaldson's theory of differentiable structures [Don90], in representation theory [Nak99] and mathematical physics [Wit95].

Rank one stable sheaves are of the form $L \otimes I_Z$, where L is a line bundle on X and I_Z is an ideal sheaf of a zero-dimensional scheme on X. Consequently, when the rank of $\mathbf{v}$ is one, the moduli space $M_{X,H}(\mathbf{v})$ fibers over $\mathrm{Pic}^{\mathrm{ch}_1(\mathbf{v})}(X)$ with fibers isomorphic to a Hilbert scheme of points on X. The Hilbert scheme $X^{[n]}$ of n points on X is a smooth, projective irreducible variety of dimension $2n$ [Fog68]. Hence, the basic geometric invariants of $M_{X,H}(\mathbf{v})$ such as dimension and irreducibility are

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well-understood. When the rank of $\mathbf{v}$ is higher, much less is known. The following questions are open in general.

1. For which Chern characters $\mathbf{v}$ is the moduli space $M_{X,H}(\mathbf{v})$ nonempty?
2. When is $M_{X,H}(\mathbf{v})$ irreducible and of the expected dimension?
3. What are the singularities of $M_{X,H}(\mathbf{v})$? Is $M_{X,H}(\mathbf{v})$ reduced?

The currently known answers to these questions have two flavors. There are results that hold on arbitrary surfaces under numerical restrictions on $\mathbf{v}$. For example, the Bogomolov inequality, which asserts that the discriminant Δ of a stable sheaf has to be nonnegative, imposes strong restrictions on the existence of stable sheaves. When $\Delta(\mathbf{v}) \gg 0$, then theorems of Donaldson [Don90], Li [LiJ93, LiJ94] and O'Grady [O'G96] show that the moduli spaces $M_{X,H}(\mathbf{v})$ behave well. They are nonempty, irreducible, of the expected dimension and generically smooth (see [HuL10]). Then there are results on specific surfaces. The question of when $M_{X,H}(\mathbf{v})$ is nonempty has been answered for surfaces such as K3 surfaces, Abelian surfaces and $\mathbb{P}^2$ (see [DLP85, HuL10, LeP97, Muk84, Yos99, Yos01]). In these cases, the moduli spaces are irreducible and often have more structure. For example, when X is a $K3$ surface, $\mathbf{v}$ is a primitive character and H is sufficiently general, then $M_{X,H}(\mathbf{v})$ is a hyperkähler manifold.

When X is a surface of general type and Δ is positive but small, the moduli space $M_{X,H}(\mathbf{v})$ can exhibit pathological behavior. The moduli spaces can be reducible, nonreduced and can have components of different dimensions (see [Mes97, MS11, MS13a, MS13b]). The pathological behavior is already present in hypersurfaces in $\mathbb{P}^3$. In §3 we will show the following.

Theorem 1. *Given a positive integer k, there exists an integer d_k such that for all $d \geq d_k$, there exists a moduli space $M_{X_d,H}(\mathbf{v}_d)$ with at least k components, where X_d is a very general surface of degree d in $\mathbb{P}^3$, H is the hyperplane class and $\mathbf{v}_d$ is a Chern character of rank 2.*

Consequently, a full understanding of the moduli spaces of sheaves on an arbitrary surface is likely out of reach.

When the moduli space $M_{X,H}(\mathbf{v})$ is irreducible and normal, one can ask for finer topological and birational invariants of $M_{X,H}(\mathbf{v})$.

1. Compute the ample and effective cones of divisors of $M_{X,H}(\mathbf{v})$.
2. Run the minimal model program for $M_{X,H}(\mathbf{v})$ and use wall-crossing to compute topological invariants of $M_{X,H}(\mathbf{v})$.

Brill-Noether divisors provide a large class of natural divisors on $M_{X,H}(\mathbf{v})$ and play a central role in the birational geometry of these moduli spaces (see [ABCH13, CHW17]). Let $\mathbf{w}$ be a Chern character of a sheaf on X such that the Euler characteristic $\chi(\mathbf{v} \otimes \mathbf{w}) = 0$. Note that the condition $\chi(\mathbf{v} \otimes \mathbf{w}) = 0$ makes sense since the Euler characteristic is numerical and can be formally computed using Riemann-Roch. Let $\mathscr{W}$ be a sheaf with Chern character $\mathbf{w}$. Consider the locus

$$D_{\mathscr{W}} := \{\mathscr{V} \in M_{X,H}(\mathbf{v}) | h^1(X, \mathscr{W} \otimes \mathscr{V}) \neq 0\}.$$

When $D_{\mathscr{W}}$ is not the entire moduli space, it is an effective divisor called *the Brill-Noether divisor* associated to $\mathscr{W}$. The Brill-Noether problem asks to determine the invariants $\mathbf{w}$ for which there exists a sheaf $\mathscr{W}$ with Chern character $\mathbf{w}$ such that $D_{\mathscr{W}}$ is an effective divisor. In particular, when $\chi(\mathbf{v}) = 0$, we can take $\mathscr{W} = \mathscr{O}_X$ and ask whether the cohomology of the general sheaf $\mathscr{V} \in M_{X,H}(\mathbf{v})$ vanishes. On certain rational surfaces such as $\mathbb{P}^2$ and Hirzebruch surfaces, it is possible to give a complete classification of Chern characters $\mathbf{v}$ for which the general sheaf $\mathscr{V} \in M_{X,H}(\mathbf{v})$ has no cohomology. In §4, following [CH16] we will recall these results.

Recently, Bridgeland stability conditions have led to significant progress in understanding the birational geometry of $M_{X,H}(\mathbf{v})$ and computing the ample and effective cones of $M_{X,H}(\mathbf{v})$. Using Bridgeland stability, it is possible to construct nef and effective Brill-Noether divisors. This survey will not discuss these developments; we instead refer the reader to [CH15] and [Hui17].

The organization of the paper

In §2, we will recall definitions and results on Gieseker stability and review basic constructions such as elementary modifications and the Serre construction. In §3, we will discuss unexpected behavior of moduli spaces of sheaves on general type surfaces when the discriminant is small. In particular, we will prove Theorem 1. In §4, we will review recent developments on the Brill-Noether Problem for rational surfaces following [CH16] and [CH17b].

2 Preliminaries

In this section, we collect basic definitions and facts on Gieseker semistability and prioritary sheaves.

2.1 Gieseker and μ-stability

We refer the reader to [CH15, HuL10, Hui17] and [LeP97] for more detailed information on Gieseker (semi)stability and moduli spaces of stable sheaves. Let X be a smooth, complex projective surface and let H be an ample divisor on X. Let $\mathbf{v}$ denote a Chern character on X and define the *H-slope* $\mu_H(\mathbf{v})$, the *total slope* $\nu(\mathbf{v})$ and *discriminant* $\Delta(\mathbf{v})$ by the formulae

$$\mu_H(\mathbf{v}) = \frac{c_1(\mathbf{v})\cdot H}{r(\mathbf{v})\cdot H^2}, \quad \nu(\mathbf{v}) = \frac{c_1(\mathbf{v})}{r(\mathbf{v})}, \quad \Delta(\mathbf{v}) = \frac{1}{2}\nu(\mathbf{v})^2 - \frac{\mathrm{ch}_2(\mathbf{v})}{r(\mathbf{v})},$$

respectively. The H-slope, total slope and discriminant of a sheaf $\mathscr{V}$ of positive rank is defined to be the H-slope, total slope and discriminant of its Chern character. The Chern character $(r, \mathrm{ch}_1, \mathrm{ch}_2)$ of a positive rank sheaf can be recovered from (r, ν, Δ). The advantage is that the slope and the discriminant are additive on tensor products

$$\nu(\mathscr{V} \otimes \mathscr{W}) = \nu(\mathscr{V}) + \nu(\mathscr{W})$$
$$\Delta(\mathscr{V} \otimes \mathscr{W}) = \Delta(\mathscr{V}) + \Delta(\mathscr{W}).$$

If L is a line bundle on X, then $\Delta(L) = 0$. Consequently, tensoring a sheaf with a line bundle preserves the discriminant. Set

$$P(\nu) = \chi(\mathscr{O}_X) + \frac{1}{2}\nu \cdot (\nu - K_X).$$

The Riemann-Roch formula in terms of these invariants reads

$$\chi(\mathscr{V}) = r(\mathscr{V})(P(\nu(\mathscr{V})) - \Delta(\mathscr{V})).$$

Definition 1. A torsion-free coherent sheaf $\mathscr{V}$ is called *μ_H-(semi)stable* if for every nonzero subsheaf $\mathscr{W}$ of smaller rank, we have

$$\mu_H(\mathscr{W}) \underset{(-)}{<} \mu_H(\mathscr{V}).$$

The Hilbert polynomial $P_{H,\mathscr{V}}$ and the reduced Hilbert polynomials $p_{H,\mathscr{V}}$ of a pure d-dimensional, coherent sheaf $\mathscr{V}$ with respect to H are defined by

$$P_{H,\mathscr{V}}(m) = \chi(\mathscr{V}(mH)) = a_d \frac{m^d}{d!} + \text{l.o.t}, \qquad p_{H,\mathscr{V}} = \frac{P_{H,\mathscr{V}}}{a_d}.$$

The sheaf $\mathscr{V}$ is *H-Gieseker (semi)stable* if for every proper subsheaf $\mathscr{W}$,

$$p_{H,\mathscr{W}}(m) \underset{(-)}{<} p_{H,\mathscr{V}}(m)$$

for $m \gg 0$.

Expressing the Hilbert polynomial in terms of μ_H and Δ, one obtains the following implications

$$\mu_H\text{-stability} \implies H\text{-Gieseker stability} \implies$$

$$H\text{-Gieseker semistability} \implies \mu_H\text{-semistability}.$$

The reverse implications are false in general. However, when $c_1 \cdot H$ and rH^2 are relatively prime, then μ_H-stability and μ_H-semistability coincide and all 4 concepts agree. When the ample class H is fixed or understood from the context, we will drop it from our notation. We will often refer to Gieseker (semi)stability simply as (semi)stability.

Two sheaves $\mathscr{V}$ and $\mathscr{W}$ are *S-equivalent* with respect to a notion of stability if they have the same Jordan-Hölder factors with respect to that notion of stability. Gieseker [Gie77] and Maruyama [Mar78] prove that there exists a (possibly empty) projective scheme parameterizing S-equivalence classes of H-Gieseker semistable sheaves (see [HuL10, Theorem 4.3.4]).

The Bogomolov inequality

The *Bogomolov inequality* asserts that a μ_H-semistable sheaf $\mathscr{V}$ satisfies $\Delta(\mathscr{V}) \geq 0$ and imposes a strong restriction on the existence of semistable sheaves. Since a line bundle L has $\Delta(L) = 0$, the Bogomolov inequality is sharp. However, the inequalities may be improved for (nonintegral) slopes depending on the surface X. Given a rank r and a total slope ν, let $\Delta^H_{\min,\nu,r}$ denote the minimal discriminant of a μ_H-semistable sheaf with total slope ν and rank at most r. By definition any μ_H-semistable sheaf with total slope ν and rank at most r satisfies the inequality $\Delta \geq \Delta^H_{\min,\nu,r}$. We will refer to such inequalities as *sharp Bogomolov inequalities*.

Remark 1. Determining the sharp Bogomolov inequalities on X is equivalent to classifying Chern characters of μ_H-semistable sheaves on X. Once there exists a μ_H-semistable sheaf $\mathscr{V}$ of rank r, total slope ν, and discriminant $\Delta^H_{\min,\nu,r}$, by performing elementary modifications (explained in detail below) we obtain μ_H-semistable sheaves for all integral Chern characters of rank r, total slope ν and $\Delta > \Delta^H_{\min,\nu,r}$. Similarly, if there exists a μ_H-stable sheaf of rank r, total slope ν and discriminant Δ_0, then there exists a μ_H-stable sheaf for every integral Chern character of rank r, total slope ν and discriminant $\Delta \geq \Delta_0$. Furthermore, the main theorem of [CH17a] shows that the problem of computing the ample cone of $M_{X,H}(\mathbf{v})$ is also intimately tied to sharp Bogomolov inequalities.

The existence of Gieseker semistable sheaves is more subtle. For example, on $\mathbb{P}^2$ with $H = \mathscr{O}_{\mathbb{P}^2}(1)$, $\mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}$ is a Gieseker semistable sheaf with $(r, \mu_H, \Delta) = (2,0,0)$. However, any Gieseker semistable sheaf with $r = 2$, $\mu_H = 0$ and $\Delta > 0$, in fact has $\Delta \geq 1$. There does not exist a Gieseker semistable sheaf with $\Delta = \frac{1}{2}$ since every such sheaf has a section and hence is destabilized by $\mathscr{O}_{\mathbb{P}^2}$. Let I_p denote the ideal sheaf of a point $p \in \mathbb{P}^2$. The sheaf $\mathscr{O}_{\mathbb{P}^2} \oplus I_p$ is a μ_H-semistable sheaf with $\Delta = \frac{1}{2}$.

2.2 Prioritary sheaves

It is often difficult to construct semistable bundles or check that a given bundle is semistable. When K_X is negative, there is a weaker notion which is easier to work with.

Definition 2. Let D be an effective divisor on X. A torsion-free coherent sheaf $\mathscr{V}$ is *D-prioritary* on X if $\mathrm{Ext}^2(\mathscr{V},\mathscr{V}(-D))=0$. Let $\mathscr{P}_{X,D}(\mathbf{v})$ denote the stack of D-prioritary sheaves on X with Chern character $\mathbf{v}$.

If $H\cdot(K_X+D)<0$, then the stack $\mathscr{M}_{X,H}(\mathbf{v})$ of H-Gieseker semistable sheaves is a (possibly empty) open substack of $\mathscr{P}_{X,D}(\mathbf{v})$. If $\mathscr{V}$ is μ_H-semistable, then by Serre duality

$$\mathrm{Ext}^2(\mathscr{V},\mathscr{V}(-D))=\mathrm{Hom}(\mathscr{V},\mathscr{V}(K_X+D))^*=0,$$

where the last equality follows because $\mu_H(\mathscr{V})<\mu_H(\mathscr{V}(K_X+D))$ by assumption. Hence, every μ_H-semistable sheaf is D-prioritary.

This concept is especially useful when X is $\mathbb{P}^2$ and D is the hyperplane class L or X is a birationally ruled surface and D is the fiber class F. We will use the following fundamental theorem of Walter numerous times.

Theorem 2 (Walter [Wal98]). *Let X be a birationally ruled surface with fiber class F and let $\mathbf{v}$ be a Chern character with positive rank. Then the stack $\mathscr{P}_{X,F}(\mathbf{v})$ is irreducible whenever it is nonempty. Moreover, if* $\mathrm{rk}(\mathbf{v})\geq 2$*, then the general element of $\mathscr{P}_{X,F}(\mathbf{v})$ is a vector bundle. In particular, if H is a polarization such that $H\cdot(K_X+F)<0$ and $M_{X,H}(\mathbf{v})$ is nonempty, then $M_{X,H}(\mathbf{v})$ is irreducible.*

Remark 2. Walter's theorem generalizes an earlier theorem of Hirschowitz and Laszlo [HiL93] which asserts that if $\mathbf{v}$ is a positive rank Chern character, then $\mathscr{P}_{\mathbb{P}^2,L}(\mathbf{v})$ is irreducible whenever it is nonempty.

Remark 3. When X is a Hirzebruch surface, K_X+F is anti-effective. Hence, the condition $H\cdot(K_X+F)<0$ holds for every polarization H. On an arbitrary birationally ruled surface, F is nef and $K_X\cdot F<0$ by adjunction. Since ampleness is an open condition and nef divisors are in the closure of the ample cone, there exist polarizations H sufficiently close to F such that $H\cdot(K_X+F)<0$. However, this inequality in general imposes conditions on the polarization H.

Checking a sheaf is prioritary is much easier than checking it is stable. Prioritary sheaves are also easier to construct.

Example 1. Let L be the hyperplane class on $\mathbb{P}^2$ and let a be an integer. Then vector bundles of the form $\mathscr{O}_{\mathbb{P}^2}(a)^m\oplus\mathscr{O}_{\mathbb{P}^2}(a+1)^{r-m}$ are L-prioritary even though they are not μ_L-semistable if $m\neq 0,r$.

Every vector bundle of rank r on a smooth rational curve is a direct sum of line bundles $\bigoplus_{i=1}^r\mathscr{O}_{\mathbb{P}^1}(a_i)$. The vector bundle is called *balanced* if $|a_i-a_j|\leq 1$ for every $1\leq i\leq j\leq r$.

Let D be a smooth curve on X. The condition of being D-prioritary is useful for understanding the restriction of bundles from X to D and especially useful when D is a rational curve. Let $\mathscr{F}_s/S$ be a complete family of D-prioritary sheaves on X which are locally free on D. Recall that a family is *complete* if the Kodaira-Spencer map $\kappa: T_sS\to\mathrm{Ext}^1(\mathscr{F}_s,\mathscr{F}_s)$ is surjective at some point $s\in S$. Then the condition of being D-prioritary implies that the natural map

$$\mathrm{Ext}^1_X(\mathscr{F}_s,\mathscr{F}_s)\to \mathrm{Ext}^1_D(\mathscr{F}_s|_D,\mathscr{F}_s|_D)$$

is surjective. Consequently, we obtain the following.

Proposition 1 ([CH17b], Proposition 2.6). *Let D be a smooth curve on X and let $\mathscr{F}_s/S$ be a complete family of D-prioritary sheaves on X which are locally free on D. Then the restricted family $\mathscr{F}_s|_D/S$ is also a complete family. In particular, if D is a smooth rational curve, then $\mathscr{F}_s|_D$ is balanced for $s\in U$, where U is a nonempty dense open subset of S.*

2.3 Elementary modifications

An *elementary modification* of a torsion-free sheaf $\mathscr{V}$ on X is any sheaf given by an exact sequence

$$0\to\mathscr{V}'\to\mathscr{V}\to\mathscr{O}_p\to 0,$$

where $p\in X$ is a point. Using the defining exact sequence, the following are immediate:

$$\mathrm{rk}(\mathscr{V}')=\mathrm{rk}(\mathscr{V}),\quad c_1(\mathscr{V}')=c_1(\mathscr{V}),\quad \mathrm{ch}_2(\mathscr{V}')=\mathrm{ch}_2(\mathscr{V})-1.$$

In particular,

$$\chi(\mathscr{V}')=\chi(\mathscr{V})-1,\quad \Delta(\mathscr{V}')=\Delta(\mathscr{V})+\frac{1}{r}.$$

Assume $\phi:\mathscr{F}\to\mathscr{V}'$ is an injective sheaf homomorphism. Composing ϕ with the inclusion of $\mathscr{V}'$ into $\mathscr{V}$, we can view $\mathscr{F}$ as a subsheaf of $\mathscr{V}$. Consequently, an elementary modification of a μ_H-(semi)stable sheaf is again μ_H-(semi)stable. As discussed in Remark 1, elementary modifications of Gieseker (semi)stable sheaves do not need to be Gieseker (semi)stable.

For future reference, we observe the following easy lemma.

Lemma 1 ([CH17b], Lemma 2.7). *Let $\mathscr{V}'$ be a general elementary modification of $\mathscr{V}$ at a general point $p\in X$. Then:*

1. *If $\mathscr{V}$ is D-prioritary, then $\mathscr{V}'$ is D-prioritary.*
2. *$H^2(X,\mathscr{V})=H^2(X,\mathscr{V}')$.*
3. *If $h^0(X,\mathscr{V})>0$, then $h^0(X,\mathscr{V}')=h^0(X,\mathscr{V})-1$ and $h^1(X,\mathscr{V}')=h^1(X,\mathscr{V})$. If $h^0(X,\mathscr{V})=0$, then $h^1(X,\mathscr{V}')=h^1(X,\mathscr{V})+1$. In particular, if at most one of h^0 or h^1 is nonzero for $\mathscr{V}$, then at most one of h^0 or h^1 is nonzero for $\mathscr{V}'$.*

2.4 The Serre construction

The Serre construction provides a method for constructing locally free sheaves on any variety, but it takes a particularly simple form on surfaces. Let Z be collection of

n distinct points on X. We say that Z satisfies the *Cayley-Bacharach property* with respect to a line bundle L if any section of L vanishing on a subset $Z' \subset Z$ of $n-1$ points vanishes on Z.

Theorem 3 ([HuL10], Theorem 5.1.1). *There exists a locally free extension $\mathscr{V}$ of the form*

$$0 \to L_1 \to \mathscr{V} \to L_2 \otimes I_Z \to 0$$

if and only if Z satisfies the Cayley-Bacharach property with respect to the line bundle $L_1^{-1} \otimes L_2 \otimes K_X$.

Below we will use the Serre construction to construct vector bundles on hypersurfaces in $\mathbb{P}^3$.

3 Pathological behavior for small Δ

Let $X \subset \mathbb{P}^3$ be a very general surface of degree d and let H denote the hyperplane class. In this section, following ideas of Mestrano and Simpson [MS13b], we use Hilbert schemes of space curves to construct components of moduli spaces $M_X(2,1,n)$ of H-Gieseker semistable sheaves on X with $\mathrm{rk} = 2$, $c_1 = H$ and $c_2 = n$ if $d \gg 0$. As an application, we show that for any number $k > 0$, there is a number d_k such that if $d \geq d_k$, then there are moduli spaces $M_X(2,1,n)$ with at least k irreducible components. We warn the reader that in this section it is more convenient to use $c_2 = n$ instead of $\mathrm{ch}_2 = \frac{d^2}{2} - n$. We will also omit the ample H from our notation since it will always be the hyperplane class.

By the Noether-Lefschetz theorem, $\mathrm{Pic}\, X \cong \mathbb{Z}$, generated by $\mathscr{O}_X(1)$. We have $K_X = \mathscr{O}_X(d-4)$. By the restriction sequence

$$0 \to \mathscr{O}_{\mathbb{P}^3}(k-d) \to \mathscr{O}_{\mathbb{P}^3}(k) \to \mathscr{O}_X(k) \to 0,$$

the line bundles $\mathscr{O}_X(k)$ all have $H^1(\mathscr{O}_X(k)) = 0$, and if $k < d$ then $H^0(\mathscr{O}_X(k)) \cong H^0(\mathscr{O}_{\mathbb{P}^3}(k))$. Thus for $k < d$ the sections of $\mathscr{O}_X(k)$ can be interpreted as surfaces in $\mathbb{P}^3$.

3.1 The construction

Let $\mathscr{H}_{e,g}$ be the Hilbert scheme of curves of degree $e \geq 3$ and genus g in $\mathbb{P}^3$, and let $\mathscr{R} = \mathscr{R}_{e,g} \subset \mathscr{H}_{e,g}$ be an open subset of an irreducible component of $\mathscr{H}_{e,g}$ parameterizing nondegenerate smooth irreducible curves which are transverse to X. Let $C \subset \mathbb{P}^3$ be a general curve parameterized by $\mathscr{R}$. By Riemann-Roch, the Hilbert polynomial of $\mathscr{O}_C$ is

$$\chi(\mathscr{O}_C(m)) = em - g + 1.$$

We further assume that $d \geq 5$ is large enough that the following two properties hold.

1. The following cohomology groups vanish:

$$h^1(\mathscr{O}_C(d-4)) = 0, \quad h^1(I_{C\subset\mathbb{P}^3}(d-4)) = 0, \quad \text{and} \quad h^1(I_{C\subset\mathbb{P}^3}(d-3)) = 0.$$

2. The curve C can be cut out by homogeneous forms of degree $d-3$.

By passing to an open subset of $\mathscr{R}$, we can without loss of generality assume the above properties hold for *every* curve C parameterized by $\mathscr{R}$. These properties guarantee that many other cohomology groups vanish.

Lemma 2. *If $k \geq d-4$, then the sheaves $\mathscr{O}_C(k)$ and $I_{C\subset\mathbb{P}^3}(k)$ have no higher cohomology.*

Proof. Exact sequences of the form

$$0 \to \mathscr{O}_C(d-4) \to \mathscr{O}_C(k) \to \mathscr{O}_Z \to 0$$

for Z zero-dimensional show that $H^1(\mathscr{O}_C(d-4)) = 0$ implies $H^1(\mathscr{O}_C(k)) = 0$. The sequences

$$0 \to I_{C\subset\mathbb{P}^3}(k) \to \mathscr{O}_{\mathbb{P}^3}(k) \to \mathscr{O}_C(k) \to 0$$

then show that $H^2(I_{C\subset\mathbb{P}^3}(k)) = H^3(I_{C\subset\mathbb{P}^3}(k)) = 0$. Recall that a smooth irreducible curve $C \subset \mathbb{P}^3$ is called *k-normal* if $H^1(I_{C\subset\mathbb{P}^3}(k)) = 0$. Since C is $(d-3)$-normal and $\mathscr{O}_C(d-4)$ is nonspecial, we also have that C is k-normal for all $k \geq d-3$ by [ACGH85, Exercise III.D-5].

We put

$$n := n(d,e,g) = h^0(\mathscr{O}_C(d-3)) + 1 = e(d-3) - g + 2.$$

Then $C \cap X$ consists of $de > n$ points. Let $Z \subset C \cap X$ be a collection of n points. We study rank 2 bundles $\mathscr{E}$ on X which fit as extensions

$$0 \to \mathscr{O}_X \to \mathscr{E} \to I_{Z\subset X}(1) \to 0.$$

Proposition 2. *We have* $\mathrm{ext}^1(I_{Z\subset X}(1), \mathscr{O}_X) = 1$. *Let $\mathscr{E}$ be the sheaf given by a non-trivial extension class. Then $\mathscr{E}$ is a μ-stable vector bundle in $M_X(2,1,n)$.*

Let $U_{\mathscr{R}}(n) \subset M_X(2,1,n)$ be the locus parameterizing the sheaves $\mathscr{E}$ which can be constructed by varying C in $\mathscr{R}$ and choosing the scheme $Z \subset C \cap X$ arbitrarily. Then $U_{\mathscr{R}}(n)$ is irreducible and $\dim U_{\mathscr{R}}(n) = \dim \mathscr{R} \geq 4e$.

Proof. First view the curve C and a collection $Z' \subset C$ of $n-1$ points as fixed; we claim that there is a surface $X \subset \mathbb{P}^3$ of degree d which contains Z' but does not contain C. Consider the restriction sequence

$$0 \to I_{C\subset\mathbb{P}^3}(d) \to I_{Z'\subset\mathbb{P}^3}(d) \to I_{Z'\subset C}(d) \to 0.$$

Then

$$\chi(I_{Z'\subset C}(d)) = ed - g + 1 - (n-1) = 3e > 0,$$

so $h^0(I_{Z'\subset C}(d)) > 0$. Also $h^1(I_{C\subset \mathbb{P}^3}(d)) = 0$ by our choice of d, so there is a surface X of degree d which vanishes on Z' and does not contain C.

Let $W \subset \mathbb{P}H^0(\mathscr{O}_{\mathbb{P}^3}(d))$ be the subset of surfaces which do not contain C, and consider the correspondence

$$\Sigma = \{(X,Z') : Z' \subset C \cap X\} \subset W \times \mathrm{Sym}^{n-1} C.$$

Then Σ is irreducible by a standard monodromy argument, and dominates the second factor. Therefore the locus of (X,Z') such that Z' imposes $n-1 = h^0(\mathscr{O}_C(d-3))$ conditions on sections of $\mathscr{O}_C(d-3)$ is a dense open subset. Hence if X is very general and $Z \subset C \cap X$ is any collection of n points, then $h^0(I_{Z\subset C}(d-3)) = 0$ and $h^1(I_{Z\subset C}(d-3)) = 1$. Since $H^0(\mathscr{O}_{\mathbb{P}^3}(d-3)) \to H^0(\mathscr{O}_C(d-3))$ is surjective, we see that Z imposes $n-1$ conditions on surfaces of degree $d-3$. Then by Serre duality we have

$$\mathrm{ext}^1(I_{Z\subset X}(1), \mathscr{O}_X) = \mathrm{ext}^1(\mathscr{O}_X, I_{Z\subset X}(d-3)) =$$
$$h^1(I_{Z\subset X}(d-3)) = h^1(I_{Z\subset \mathbb{P}^3}(d-3)) = 1.$$

The sheaf $\mathscr{E}$ is a vector bundle since the scheme Z satisfies the Cayley-Bacharach property for the line bundle $K_X(1) = \mathscr{O}_X(d-3)$: every section of $\mathscr{O}_X(d-3)$ which vanishes at some $n-1$ points $Z' \subset Z$ vanishes on C and so vanishes at all of Z. The Chern class computation is elementary. The defining sequence for $\mathscr{E}$ shows that no line bundle $\mathscr{O}_X(k)$, $k \geq 1$, admits a nonzero map to $\mathscr{E}$. Since $c_1(\mathscr{E})$ is odd, the μ-stability of $\mathscr{E}$ follows.

Given a bundle $\mathscr{E}$ constructed by this method, observe that every section of $\mathscr{O}_{\mathbb{P}^3}(d-3)$ which vanishes along Z also has to contain the nondegenerate irreducible curve C. This implies that the collection of points Z is not coplanar, and therefore $h^0(\mathscr{E}) = 1$. This unique section allows us to uniquely recover the sheaf $I_Z(1)$ as the cokernel of the inclusion $\mathscr{O}_X \to \mathscr{E}$. Since the ideal of C is generated in degree $d-3$ or less, we can recover C as the common zero locus of all the forms in $H^0(I_Z(d-3))$. Thus there is a quasifinite mapping $U_{\mathscr{R}}(n) \to \mathscr{R}$ given by sending $\mathscr{E}$ to C, and $\dim U_{\mathscr{R}}(n) = \dim \mathscr{R}$. The dimension estimate $\dim \mathscr{R} \geq 4e$ for components of the Hilbert scheme parameterizing smooth curves is well-known. The irreducibility of $U_{\mathscr{R}}(n)$ is immediate from the irreducibility of Σ.

Remark 4. The previous discussion can be modified to allow $\mathscr{R}$ to be either the Hilbert scheme of lines or conics which intersect X transversely. We still have $\mathrm{ext}^1(I_{Z\subset X}(1), \mathscr{O}_X) = 1$, and $\mathscr{E}$ is a μ-stable vector bundle in $M_X(2,1,n)$. Varying C and Z still sweeps out a locus $U_{\mathscr{R}}(n) \subset M_X(2,1,n)$. However, since C is degenerate the dimension estimate for $U_{\mathscr{R}}(n)$ changes as follows:

1. If $\mathscr{R}$ parameterizes lines, we have $n = d-1$. Since $h^0(I_Z(1)) = 2$ we get $h^0(\mathscr{E}) = 3$, and $\mathscr{E}$ no longer determines the line C. Any sheaf $\mathscr{E} \in U_{\mathscr{R}}(n)$ arises from up to a 2-dimensional family of schemes Z, so $\dim U_{\mathscr{R}}(n) \geq 4 - 2 = 2$. Equality holds; see Remark 5.

2. If $\mathscr{R}$ parameterizes conics, we have $n = 2d-4$. Here $h^0(I_Z(1)) = 1$ and $h^0(\mathscr{E}) = 2$, and $\mathscr{E}$ no longer determines the conic C. By the same considerations as above, $\dim U_{\mathscr{R}}(n) \geq 8-1 = 7$, and again equality holds by Remark 5.

3.2 Tangent space

Let $\mathscr{S}_{\mathscr{R}}(n)$ be an irreducible component of $M_X(2,1,n)$ which contains $U_{\mathscr{R}}(n)$. In this section we study the tangent space to $M_X(2,1,n)$ at points of $U_{\mathscr{R}}(n)$ to find an upper bound on the dimension of $\mathscr{S}_{\mathscr{R}}(n)$. This computation generalizes a computation from [MS13b] in the case where $d = 6$ and $\mathscr{R}$ parameterizes twisted cubics.

Let $\mathscr{E} \in U_{\mathscr{R}}(n)$. Since $\mathscr{E}$ is locally free, we have $\mathrm{Ext}^1(\mathscr{E},\mathscr{E}) \cong H^1(\mathscr{E}^* \otimes \mathscr{E})$. Since $\mathscr{E}$ has rank 2, we have $\mathscr{E}^* \cong \mathscr{E}(-1)$. Also, $\mathscr{E}^* \otimes \mathscr{E}$ splits as a direct sum

$$\mathscr{E}^* \otimes \mathscr{E} \cong (\mathscr{E} \otimes \mathscr{E})(-1) \cong (\mathrm{Sym}^2 \mathscr{E} \oplus \bigwedge^2 \mathscr{E})(-1) \cong \mathscr{V} \oplus \mathscr{O}_X$$

where $\mathscr{V} := (\mathrm{Sym}^2 \mathscr{E})(-1)$. Since $h^1(\mathscr{O}_X) = 0$, we find that $h^1(\mathscr{V})$ is the dimension of the tangent space to the moduli space at $\mathscr{E}$.

Applying Sym^2 to the exact sequence defining $\mathscr{E}$ and twisting by $\mathscr{O}_X(-1)$, we see that the bundle $\mathscr{V}$ fits in the exact sequence

$$0 \to \mathscr{E}(-1) \to \mathscr{V} \to I_{2Z\subset X}(1) \to 0.$$

Here, $2Z \subset X$ is the subscheme defined by the square of the ideal $I_{Z\subset X}$ of Z, so it consists of a union of n planar double points and has length $3n$. We write $2C$ for the "rope" in $\mathbb{P}^3$ defined by the square of the ideal $I_{C\subset\mathbb{P}^3}$. Thus a surface contains $2C$ if and only if it is singular at every point of C.

Lemma 3. *We have $H^0(I_{2Z\subset X}(d-3)) \cong H^0(I_{2C\subset\mathbb{P}^3}(d-3))$. That is, every degree $d-3$ form tangent to X at each point of Z is singular along the entire curve C.*

Proof. Let $F \in H^0(I_{2Z\subset X}(d-3)) \subset H^0(\mathscr{O}_{\mathbb{P}^3}(d-3))$ and let Y denote the zero locus of F. Then Y is a surface of degree $d-3$ containing Z, so Y contains C. Since C intersects X transversely and Y is tangent to X at the points of Z, we find that Y is singular at each point of Z. Then the partial derivatives $\partial F/\partial X_i$ each vanish at every point of Z. Since the partials have degree $d-4$, they must then also contain C, and therefore the partials of F vanish identically along C. Therefore

$$F \in H^0(I_{2C\subset\mathbb{P}^3}(d-3)).$$

The opposite containment is obvious.

Slavov computes the Hilbert polynomial of $\mathscr{O}_{2C}$ as follows.

Lemma 4 ([Sla16]). *The Hilbert polynomial of $\mathscr{O}_{2C}$ is*

$$\chi(\mathscr{O}_{2C}(m)) = 3em - 4e - 5g + 5.$$

It is convenient to define

$$\alpha = h^0(I_{2C\subset\mathbb{P}^3}(d-3)) - \chi(I_{2C\subset\mathbb{P}^3}(d-3)),$$

so that in particular $\alpha = 0$ if d is sufficiently large. We now estimate the dimension of the tangent space.

Proposition 3. *The dimension $h^1(\mathscr{V})$ of the tangent space to $M_X(2,1,n)$ at $\mathscr{E} \in U_{\mathscr{R}}(n)$ satisfies the inequalities*

$$4e + 2g + \alpha \le h^1(\mathscr{V}) \le 5e + 2g - 3 + \alpha.$$

Proof. Since $\mathscr{E}$ is stable we have $h^0(\mathscr{V}) = 0$. The Euler characteristic $\chi(\mathscr{V})$ is computed using the exact sequences as follows.

$$\begin{aligned}\chi(\mathscr{V}) &= \chi(\mathscr{E}(-1)) + \chi(I_{2Z\subset X}(1)) \\ &= \chi(\mathscr{O}_X(-1)) + \chi(\mathscr{O}_X) + \chi(\mathscr{O}_X(1)) - 4n \\ &= \binom{d}{3} + 1 + \binom{d-1}{3} + 4 + \binom{d-2}{3} - 4(e(d-3) - g + 2)\end{aligned}$$

We compute $h^2(\mathscr{V})$ by noting $\mathscr{V}$ is self-dual so $h^2(\mathscr{V}) = h^0(\mathscr{V}(d-4))$. Then we have an exact sequence

$$0 \to H^0(\mathscr{E}(d-5)) \to H^0(\mathscr{V}(d-4)) \to H^0(I_{2Z\subset X}(d-3)) \xrightarrow{\delta} H^1(\mathscr{E}(d-5)).$$

Now we compute the cohomology of $\mathscr{E}(d-5)$. We have

$$\begin{aligned}h^0(\mathscr{E}(d-5)) &= h^0(\mathscr{O}_X(d-5)) + h^0(I_{Z\subset X}(d-4)) \\ &= h^0(\mathscr{O}_{\mathbb{P}^3}(d-5)) + h^0(I_{C\subset\mathbb{P}^3}(d-4)) \\ &= \binom{d-2}{3} + h^0(\mathscr{O}_{\mathbb{P}^3}(d-4)) - h^0(\mathscr{O}_C(d-4)) \\ &= \binom{d-2}{3} + \binom{d-1}{3} - (e(d-4) - g + 1).\end{aligned}$$

Note that $\mathscr{E}(d-5)$ is Serre dual to $\mathscr{E}$. So,

$$\begin{aligned}\chi(\mathscr{E}(d-5)) = \chi(\mathscr{E}) &= \chi(\mathscr{O}_X) + \chi(\mathscr{O}_X(1)) - n \\ &= 1 + \binom{d-1}{3} + 4 + \binom{d-2}{3} - (e(d-3) - g + 2).\end{aligned}$$

Also $h^2(\mathscr{E}(d-5)) = h^0(\mathscr{E}) = 1$, and therefore

$$h^1(\mathscr{E}(d-5)) = h^0(\mathscr{E}(d-5)) + h^2(\mathscr{E}(d-5)) - \chi(\mathscr{E}(d-5)) = e - 3.$$

By Lemma 3 and 4 we have

$$\begin{aligned} h^0(I_{2Z\subset X}(d-3)) &= h^0(I_{2C\subset\mathbb{P}^3}(d-3)) \\ &= \alpha + \chi(I_{2C\subset\mathbb{P}^3}(d-3)) \\ &= \alpha + \chi(\mathscr{O}_{\mathbb{P}^3}(d-3)) - \chi(\mathscr{O}_{2C}(d-3)) \\ &= \alpha + \binom{d}{3} - (3e(d-3) - 4e - 5g + 5). \end{aligned}$$

Combining these results, we conclude

$$\begin{aligned} h^1(\mathscr{V}) &= h^2(\mathscr{V}) - \chi(\mathscr{V}) \\ &\leq h^0(\mathscr{E}(d-5)) + h^0(I_{2Z\subset X}(d-3)) - \chi(\mathscr{V}) \\ &= 5e + 2g - 3 + \alpha \end{aligned}$$

with equality if δ is 0 and

$$\begin{aligned} h^1(\mathscr{V}) &= h^2(\mathscr{V}) - \chi(\mathscr{V}) \\ &\geq h^0(\mathscr{E}(d-5)) + h^0(I_{2Z\subset X}(d-3)) - h^1(\mathscr{E}(d-5)) - \chi(\mathscr{V}) \\ &= 4e + 2g + \alpha \end{aligned}$$

with equality if δ is surjective.

Remark 5. As in Remark 4, a similar result holds if $\mathscr{R}$ parameterizes lines or conics transverse to X, but adjustments need to be made since the curve C is degenerate:

1. if $\mathscr{R}$ parameterizes lines then $h^0(\mathscr{E}) = 3$, so $h^1(\mathscr{E}(d-5)) = e-1$ and the bounds become the equality $h^1(\mathscr{V}) = 2 + \alpha$. Furthermore, $\alpha = 0$ so long as $d \geq 3$, so $U_{\mathscr{R}}(n)$ is smooth and dense in $\mathscr{S}_{\mathscr{R}}(n)$.
2. If $\mathscr{R}$ parameterizes conics then $h^0(\mathscr{E}) = 2$, so $h^1(\mathscr{E}(d-5)) = e-2$ and the bounds become the equality $h^1(\mathscr{V}) = 7 + \alpha$. We have $\alpha = 0$ for $d \geq 5$, so $U_{\mathscr{R}}(n)$ is smooth and dense in $\mathscr{S}_{\mathscr{R}}(n)$.

Combining the results in this section yields the following dimension estimates.

Corollary 1. *With the assumptions above, the irreducible component $\mathscr{S}_{\mathscr{R}}(n)$ of $M_X(2,1,n)$ which contains $U_{\mathscr{R}}(n)$ has dimension satisfying*

$$4e \leq \dim \mathscr{S}_{\mathscr{R}}(n) \leq 5e + 2g - 3 + \alpha.$$

It is typically challenging to compute the dimension of $\mathscr{S}_{\mathscr{R}}(n)$ exactly. For example, if $g > 0$ then the expected dimension $4e$ of $U_{\mathscr{R}}(n)$ is strictly smaller than the lower bound $4e + 2g$ on the dimension of the tangent space. It is not clear when the closure of $U_{\mathscr{R}}(n)$ is a component of the moduli space. If $U_{\mathscr{R}}(n)$ is dense in $\mathscr{S}_{\mathscr{R}}(n)$ and of dimension smaller than $4e + 2g$, then $\mathscr{S}_{\mathscr{R}}(n)$ is everywhere nonreduced.

The case when $\mathscr{R}$ parameterizes twisted cubic curves is easy to analyze.

Corollary 2. *Suppose $d \geq 6$ and $\mathscr{R}$ parameterizes twisted cubic curves which are transverse to X. Then the closure of $U_{\mathscr{R}}(n)$ in $M_X(2,1,n)$ is an irreducible component of dimension 12 which is smooth at all points of $U_{\mathscr{R}}(n)$.*

Proof. The inequality $d \geq 5$ is sufficient to ensure that the assumptions on d in this section are satisfied. On the other hand, $d \geq 6$ is needed to give $\alpha = 0$; we have $\alpha = 1$ if $d = 5$. Then by Corollary 1, both $U_{\mathscr{R}}(n)$ and $\mathscr{S}_{\mathscr{R}}(n)$ have dimension 12 and the tangent space at any point of $U_{\mathscr{R}}(n)$ has dimension 12.

3.3 Elementary modifications

In the previous subsection we used an open irreducible subset $\mathscr{R} \subset \mathscr{H}_{e,g}$ to construct a locus $U_{\mathscr{R}}(n)$ in $M_X(2,1,n)$ if $d = \deg X$ is sufficiently large. Here we have $n = n(d,e,g) = e(d-3) - g + 2$. We now use elementary modifications to construct additional loci in $M_X(2,1,s)$ for every $s \geq n$.

Definition 3. Let $s \geq n$. The locus $U_{\mathscr{R}}(s) \subset M_X(2,1,s)$ is the set of all sheaves which can be obtained from sheaves in $U_{\mathscr{R}}(n)$ by a sequence of $s-n$ elementary modifications at distinct points of X.

Given a sheaf $\mathscr{E} \in U_{\mathscr{R}}(n)$, a sheaf in $U_{\mathscr{R}}(s)$ is constructed by choosing $s-n$ points $p_1, \ldots, p_{s-n}$ of X and a hyperplane in the fiber $\mathscr{E}_{p_i}$ for each i. Since $U_{\mathscr{R}}(n)$ is irreducible, it follows that $U_{\mathscr{R}}(s)$ is irreducible of dimension $\dim(\mathscr{R}) + 3(s-n)$. Let $\mathscr{S}_{\mathscr{R}}(s) \subset M_X(2,1,s)$ be an irreducible component containing $U_{\mathscr{R}}(s)$. Our main result in this section bounds the dimension of $\mathscr{S}_{\mathscr{R}}(s)$.

Proposition 4. *We have*

$$4e + 3(s-n) \leq \dim \mathscr{S}_{\mathscr{R}}(s) \leq 5e + 2g - 3 + \alpha + 4(s-n).$$

Proof. The lower bound follows from the previous paragraph. Repeated application of the next lemma and Proposition 3 gives the upper bound.

Lemma 5. *Suppose $\mathscr{E}$ is a stable rank r torsion-free sheaf on a surface X, let $p \in X$ be a point where $\mathscr{E}$ is locally free, and let $\mathscr{E}'$ be an elementary modification of $\mathscr{E}$ at p:*

$$0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{O}_p \to 0. \qquad (1)$$

Then

$$\operatorname{ext}^1(\mathscr{E}', \mathscr{E}') \leq \operatorname{ext}^1(\mathscr{E}, \mathscr{E}) + 2r.$$

Proof. We first apply $\operatorname{Ext}(\mathscr{E}, -)$ to the Sequence (1), and obtain the long exact sequence

$$0 \to \operatorname{Hom}(\mathscr{E}, \mathscr{E}') \to \operatorname{Hom}(\mathscr{E}, \mathscr{E}) \to \operatorname{Hom}(\mathscr{E}, \mathscr{O}_p) \to$$
$$\operatorname{Ext}^1(\mathscr{E}, \mathscr{E}') \to \operatorname{Ext}^1(\mathscr{E}, \mathscr{E}) \to \operatorname{Ext}^1(\mathscr{E}, \mathscr{O}_p).$$

We have $\mathrm{Hom}(\mathscr{E},\mathscr{E}) = \mathbb{C}\cdot \mathrm{id}$ by stability, and the map $\mathrm{Hom}(\mathscr{E},\mathscr{E}) \to \mathrm{Hom}(\mathscr{E},\mathscr{O}_p)$ carries id to the nonzero map $\mathscr{E} \to \mathscr{O}_p$ defining $\mathscr{E}'$. Therefore $\mathrm{Hom}(\mathscr{E},\mathscr{E}) \to \mathrm{Hom}(\mathscr{E},\mathscr{O}_p)$ is injective and $\mathrm{Hom}(\mathscr{E},\mathscr{E}') = 0$. Also

$$\hom(\mathscr{E},\mathscr{O}_p) = r \quad \text{and} \quad \mathrm{ext}^1(\mathscr{E},\mathscr{O}_p) = 0$$

since $\mathscr{E}$ is locally free. Putting this all together,

$$\mathrm{ext}^1(\mathscr{E},\mathscr{E}') = \mathrm{ext}^1(\mathscr{E},\mathscr{E}) + r - 1.$$

Next we apply $\mathrm{Ext}(-,\mathscr{E}')$ to Sequence (1) and get an exact sequence

$$\mathrm{Ext}^1(\mathscr{E},\mathscr{E}') \to \mathrm{Ext}^1(\mathscr{E}',\mathscr{E}') \to \mathrm{Ext}^2(\mathscr{O}_p,\mathscr{E}'),$$

so

$$\mathrm{ext}^1(\mathscr{E}',\mathscr{E}') \le \mathrm{ext}^1(\mathscr{E},\mathscr{E}') + \mathrm{ext}^2(\mathscr{O}_p,\mathscr{E}') = \mathrm{ext}^1(\mathscr{E},\mathscr{E}) + r - 1 + \mathrm{ext}^2(\mathscr{O}_p,\mathscr{E}').$$

Finally $\mathrm{ext}^2(\mathscr{O}_p,\mathscr{E}') = r+1$: by Serre duality, $\mathrm{ext}^2(\mathscr{O}_p,\mathscr{E}') = \hom(\mathscr{E}',\mathscr{O}_p)$. Applying $\mathrm{Ext}(-,\mathscr{O}_p)$ to Sequence (1) we have an exact sequence

$$0 \to \mathrm{Hom}(\mathscr{O}_p,\mathscr{O}_p) \to \mathrm{Hom}(\mathscr{E},\mathscr{O}_p) \to \mathrm{Hom}(\mathscr{E}',\mathscr{O}_p) \to$$
$$\mathrm{Ext}^1(\mathscr{O}_p,\mathscr{O}_p) \to \mathrm{Ext}^1(\mathscr{E},\mathscr{O}_p) = 0.$$

Here $\hom(\mathscr{O}_p,\mathscr{O}_p) = 1$, $\mathrm{ext}^1(\mathscr{O}_p,\mathscr{O}_p) = 2$, and $\hom(\mathscr{E},\mathscr{O}_p) = r$, so $\mathrm{ext}^2(\mathscr{O}_p,\mathscr{E}') = r+1$, completing the proof.

3.4 Comparing components

We now use our dimension estimates on the components $\mathscr{S}_{\mathscr{R}}(s)$ to show that if $d \gg 0$, then there are moduli spaces of sheaves $M_X(2,1,s)$ with as many components as we like.

Separating two loci

First suppose $\mathscr{R} = \mathscr{R}_{e,g} \subset \mathscr{H}_{e,g}$ and $\mathscr{R}' = \mathscr{R}_{e',g'} \subset \mathscr{H}_{e',g'}$ are two open irreducible subsets, where $e < e'$. Then we have

$$n := n(d,e,g) = e(d-3) - g + 2$$
$$n' := n(d,e',g') = e'(d-3) - g' + 2.$$

Therefore for $d \gg 0$, we have $n < n'$. Then for any $s \geq n'$ we can consider the two components $\mathscr{S}_{\mathscr{R}}(s)$ and $\mathscr{S}_{\mathscr{R}'}(s)$ of $M_X(2,1,s)$.

Theorem 4. *With the above notation, if $d \gg 0$, then the components $\mathscr{S}_{\mathscr{R}}(n')$ and $\mathscr{S}_{\mathscr{R}'}(n')$ are distinct.*

Proof. We need only see that $\dim \mathscr{S}_{\mathscr{R}}(n') > \dim \mathscr{S}_{\mathscr{R}'}(n')$. By Proposition 4 and our formulas for n and n', we have

$$\dim \mathscr{S}_{\mathscr{R}}(n') \geq 4e + 3(n'-n) = 3(e'-e)d + C_1$$
$$\dim \mathscr{S}_{\mathscr{R}'}(n') \leq 5e' + 2g' - 3 = C_2$$

where C_i are constants which depend (at most) on e,g,e',g', but not on d. Since $e' > e$, the required inequality follows for $d \gg 0$.

If the surface X is fixed and s increases past n', then the components $S_{\mathscr{R}}(s)$ and $S_{\mathscr{R}'}(s')$ eventually coincide since the moduli space $M_X(2,1,s)$ is irreducible for $s \gg 0$. We now quantify how large we can allow s to be while still guaranteeing that these components are distinct.

Proposition 5. *Suppose $d \gg 0$. Then there is a constant C depending on e,g,e',g' such that if*

$$n' \leq s \leq (4e' - 3e)d + C$$

then the components $\mathscr{S}_{\mathscr{R}}(s)$ and $\mathscr{S}_{\mathscr{R}'}(s)$ are distinct.

Note that $4e' - 3e > e'$ since $e' > e$, while n' grows like $e'd + C$ as d increases. So, the range of numbers s where the components can be separated increases with d.

Proof. Again we use Proposition 4 to estimate

$$\dim \mathscr{S}_{\mathscr{R}}(s) \geq 4e + 3(s-n) = -3ed + 3s + C_3$$
$$\dim \mathscr{S}_{\mathscr{R}'}(s) \leq 5e' + 2g' - 3 + 4(s-n') = -4e'd + 4s + C_4$$

where the C_i are constants depending on e,g,e',g'. Then we will have

$$-3ed + 3s + C_3 > -4e'd + 4s + C_4$$

so long as $s < (4e' - 3e)d + C_5$.

Separating multiple loci

Now suppose we consider a list of k open irreducible sets $\mathscr{R}^i = \mathscr{R}_{e_i,g_i} \subset \mathscr{H}_{e_i,g_i}$, and that the degrees satisfy $e_1 < \cdots < e_k$. Let $n_i = n(d,e_i,g_i)$; then for $d \gg 0$ the largest n_i is n_k. As d increases, the number n_k grows like $e_k d + C$. By Proposition 5, if $4e_{i+1} - 3e_i > e_k$ whenever $1 \leq i < k$ then the component $\mathscr{S}_{\mathscr{R}^{i+1}}(n_k)$ will have smaller dimension than $\mathscr{S}_{\mathscr{R}^i}(n_k)$ for large enough d. Thus we have proved the following result.

Proposition 6. *Suppose $e_1 < \cdots < e_k$ satisfy $4e_{i+1} - 3e_i > e_k$ for $1 \leq i < k$. Then if $d \gg 0$, the components $\mathscr{S}_{\mathscr{R}^i}(n_k)$ are all distinct for $1 \leq i \leq k$.*

This easily implies the following more qualitative theorem.

Theorem 5. *For any integer k, there is a number $d_k \gg 0$ such that if $d \geq d_k$ then a very general surface $X \subset \mathbb{P}^3$ of degree d has some moduli space $M_X(2,1,s)$ with at least k components.*

Proof. By Proposition 6 it is enough to see that there are arbitrarily long sequences of positive integers $e_1 < \cdots < e_k$ such that $4e_{i+1} - 3e_i > e_k$. Such sequences are easy to construct. For a crude example, the sequence $4^k - 2^{k-1} < 4^k - 2^{k-2} < \cdots < 4^k - 2^0$ does the trick since

$$4(4^k - 2^i) - 3(4^k - 2^{i+1}) = 4^k + 2^{i+1} > 4^k - 2^0$$

for $i \geq 0$.

Remark 6. When $\Delta(\mathbf{v})$ is small, we would expect the geometry of $M_{X,H}(\mathbf{v})$ to exhibit the same pathologies as the Hilbert scheme of curves in $\mathbb{P}^3$. It would be interesting to make this precise.

4 Brill-Noether Theorems

In this section, we discuss recent progress in Brill-Noether theory of moduli spaces of sheaves on surfaces. This section is based on [CH16] and [CH17b]. We will first discuss the theory for $\mathbb{P}^2$. We will then discuss the case of Hirzebruch surfaces and del Pezzo surfaces. Finally, we will make some remarks for general surfaces and give a few examples for hypersurfaces in $\mathbb{P}^3$.

Rank 1 sheaves

If $\operatorname{rk} \mathbf{v} = 1$, then any torsion free sheaf with Chern character $\mathbf{v}$ is of the form $L \otimes I_Z$ for a line bundle L and an ideal sheaf of points I_Z. The long exact sequence associated to

$$0 \to L \otimes I_Z \to L \to L \otimes \mathscr{O}_Z \to 0,$$

shows that $H^2(X,L) \cong H^2(X, L \otimes I_Z)$. Furthermore, if Z is a general set of n points, the map $H^0(X,L) \to H^0(X, L \otimes \mathscr{O}_Z)$ has maximal rank. Consequently, if L has no higher cohomology, then $L \otimes I_Z$ has no higher cohomology as long as Z is a general set of points on X with $|Z| \leq h^0(X,L)$. Furthermore, when $|Z| \geq h^0(X,L)$, then $L \otimes I_Z$ has no global sections. We conclude that for a general set of points Z on X, $L \otimes I_Z$ has at most one nonzero cohomology group if and only if one of the following holds

1. The line bundle L has no higher cohomology, or
2. We have $h^2(X,L)=0$ and $|Z| \geq h^0(X,L)$, or
3. We have $h^0(X,L)=|Z|$ and $h^1(X,L)=0$.

From now on, we will always assume that $\operatorname{rk} \mathbf{v} \geq 2$.

The projective plane

Let L denote the hyperplane class on $\mathbb{P}^2$. Göttsche and Hirschowitz [GHi94] show that the general sheaf in $M_{\mathbb{P}^2,L}(\mathbf{v})$ has at most one nonzero cohomology group.

Theorem 6 (Göttsche-Hirschowitz [GHi94]). *Let* $\mathbf{v}$ *be a stable Chern character with* $\operatorname{rk}(\mathbf{v}) \geq 2$. *Then the general sheaf* $\mathscr{V} \in M_{\mathbb{P}^2,L}(\mathbf{v})$ *has at most one nonzero cohomology group.*

In particular, if $\chi(\mathbf{v})<0$, then the general stable sheaf $\mathscr{V}$ has $h^1(\mathscr{V}) = -\chi(\mathbf{v})$. If $\chi(\mathbf{v}) \geq 0$ and $\mu_H(\mathbf{v}) \geq 0$, then $h^0(\mathscr{V}) = \chi(\mathbf{v})$. If $\chi(\mathbf{v}) \geq 0$ and $\mu_H(\mathbf{v})<0$, then $h^2(\mathscr{V}) = \chi(\mathbf{v})$. Hence, the Göttsche-Hirschowitz Theorem computes the cohomology of a general stable sheaf on $\mathbb{P}^2$. We will give two simple proofs of the theorem to illustrate the techniques.

Proof (Proof Sketch 1). First, by Serre duality, we may assume that $\mu(\mathbf{v}) \geq -\frac{3}{2}$. If the Serre dual sheaf has only one nonzero cohomology group, so does the original sheaf. We can apply Serre duality because the general sheaf of rank at least 2 in $M_{\mathbb{P}^2,L}(\mathbf{v})$ is a vector bundle. This fails when $\operatorname{rk}(\mathbf{v})=1$. For example, $\chi(I_p(-3))=0$, but $h^1(\mathbb{P}^2, I_p(-3)) = h^2(\mathbb{P}^2, I_p(-3)) = 1$ for any ideal sheaf of a point $p \in \mathbb{P}^2$.

The general stable sheaf $\mathscr{V}$ on $\mathbb{P}^2$ admits a Gaeta resolution of the form

$$0 \to \mathscr{O}_{\mathbb{P}^2}(a-2)^k \to \mathscr{O}_{\mathbb{P}^2}(a-1)^l \oplus \mathscr{O}_{\mathbb{P}^2}(a)^m \to \mathscr{V} \to 0, \quad \text{or}$$

$$0 \to \mathscr{O}_{\mathbb{P}^2}(a-2)^k \oplus \mathscr{O}_{\mathbb{P}^2}(a-1)^l \to \mathscr{O}_{\mathbb{P}^2}(a)^m \to \mathscr{V} \to 0,$$

where a is the largest integer such that $\chi(\mathscr{V}(-a)) \geq 0$ but $\chi(\mathscr{V}(-a-1))<0$,

$$m = \chi(\mathscr{V}(-a)), \quad k = -\chi(\mathscr{V}(-a-1) \quad \text{and} \quad l = |\operatorname{rk}(\mathscr{V})+k-m|.$$

The sign of $\operatorname{rk}(\mathscr{V})+k-m$ determines which of the two resolutions $\mathscr{V}$ admits (see [Gae51] for ideal sheaves of general points).

If $a \geq 0$, then $\mathscr{V}$ clearly has no higher cohomology. Since $\mu(\mathscr{V}) \geq -\frac{3}{2}$,

$$\mu(\mathscr{V}^*(-3)) \leq -\frac{3}{2}.$$

By Serre duality and stability, $h^2(\mathbb{P}^2, \mathscr{V}) = h^0(\mathbb{P}^2, \mathscr{V}^*(-3)) = 0$. When $a<0$, then $\mathscr{V}$ clearly has no global sections. Since $h^2(\mathbb{P}^2, \mathscr{V})=0$, we conclude that the only nonzero cohomology group can be $H^1(\mathbb{P}^2, \mathscr{V})$.

Proof (Proof Sketch 2). Alternatively, we can prove a slightly more general theorem. By Serre duality, we may assume that $\mu(\mathbf{v}) \geq -\frac{3}{2}$. By the division algorithm, we can write $\mu(\mathbf{v}) = a + \frac{m}{r}$, where a is an integer and $0 \leq m < r$. Then

$$\mathscr{V} = \mathscr{O}_{\mathbb{P}^2}(a)^{r-m} \oplus \mathscr{O}_{\mathbb{P}^2}(a+1)^m,$$

is an L-prioritary sheaf with slope $\mu(\mathbf{v})$. Since $\mu(\mathbf{v}) \geq -\frac{3}{2}$, $a \geq -2$. Consequently, $\mathscr{V}$ has no higher cohomology. A simple computation shows that $\Delta(\mathscr{V}) \leq 0$. By Lemma 1, taking general elementary modifications of $\mathscr{V}$, we obtain L-prioritary sheaves with at most one nonzero cohomology group for every integral Chern character $\mathbf{v}$ with $\mathrm{rk}(\mathbf{v}) \geq 2$ and $\Delta(\mathbf{v}) \geq 0$. Since the stack of prioritary sheaves is irreducible and vanishing of cohomology is an open condition, we conclude that the general sheaf in the corresponding stacks also have at most one nonzero cohomology group. In particular, if $\mathbf{v}$ is a stable Chern character, the Gieseker semistable sheaves form an open subset of $\mathscr{P}_L(\mathbf{v})$ and the general semistable sheaf has at most one nonzero cohomology group.

We obtain the following corollary of the proof.

Corollary 3. *Let* $\mathbf{v}$ *be a Chern character such that* $\mathrm{rk}(\mathbf{v}) \geq 2$ *and* $\Delta(\mathbf{v}) \geq 0$. *Then the general prioritary sheaf* $\mathscr{V} \in \mathscr{P}_L(\mathbf{v})$ *has at most one nonzero cohomology group.*

Both of these strategies can be used to obtain Brill-Noether theorems on other surfaces. The weak Brill-Noether theorem has many applications. One application is the classification of globally generated vector bundles. Define a Chern character $\mathbf{v}$ to be a *globally generated Chern character* if the general prioritary sheaf with character $\mathbf{v}$ is globally generated. One needs to exercise some caution with this notion because being globally generated is not an open condition.

Example 2. The vector bundle $\mathscr{V}$ defined as the cokernel of the natural map

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-d) \to \mathscr{O}_{\mathbb{P}^2} \otimes H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(d)) \to \mathscr{V} \to 0$$

is semistable and globally generated [LeP97]. However, when $d \geq 3$, the general member of the moduli space is not globally generated. This is easiest to see when $d \geq 4$. In that case, $\chi(\mathbf{v}) < \mathrm{rk}(\mathbf{v})$. The general sheaf has only a $\chi(\mathbf{v})$-dimensional space of sections, so has no chance of being globally generated. When $d = 3$, $\chi(\mathbf{v}) = \mathrm{rk}(\mathbf{v})$ and the moduli space is positive dimensional. The general sheaf has only 9 sections which fail to generate the sheaf along a curve.

However, if the higher cohomology of the sheaves vanishes, then the condition of being globally generated is an open condition

Theorem 7 ([BGJ16], [CH17b]). *Let* $\mathbf{v}$ *be an integral Chern character on* $\mathbb{P}^2$ *such that* $\mathrm{rk}(\mathbf{v}) \geq 2$, $\Delta(\mathbf{v}) \geq 0$. *Then the Chern character* $\mathbf{v}$ *is globally generated if and only if* $\mu(\mathbf{v}) \geq 0$ *and one of the following holds:*

1. *We have* $\mu(\mathbf{v}) > 0$ *and* $\chi(\mathbf{v}(-1)) \geq 0$.
2. *We have* $\mu(\mathbf{v}) > 0$, $\chi(\mathbf{v}(-1)) < 0$, *and* $\chi(\mathbf{v}) \geq \mathrm{rk}(\mathbf{v}) + 2$.

3. We have $\mu(\mathbf{v}) > 0$, $\chi(\mathbf{v}(-1)) < 0$, *and* $\chi(\mathbf{v}) \geq \mathrm{rk}(\mathbf{v}) + 1$ *and*

$$\mathbf{v} = (\mathrm{rk}\,\mathbf{v} + 1)\,\mathrm{ch}(\mathscr{O}_{\mathbb{P}^2}) - \mathrm{ch}(\mathscr{O}_{\mathbb{P}^2}(-2)).$$

4. We have $\mu(\mathbf{v}) = 0$ *and* $\mathbf{v} = \mathrm{rk}(\mathbf{v})\,\mathrm{ch}(\mathscr{O}_{\mathbb{P}^2})$.

Proof. If $\mathscr{V}$ is globally generated, then its determinant is also globally generated. We therefore have $\mu(\mathscr{V}) \geq 0$. If $\mu(\mathscr{V}) = 0$, then by Riemann-Roch $\chi(\mathscr{V}) \leq \mathrm{rk}(\mathscr{V})$ with equality if and only if $\Delta(\mathscr{V}) = 0$. Since a globally generated bundle $\mathscr{V}$ needs to have at least $\mathrm{rk}(\mathscr{V})$ independent sections and for the general sheaf there is only one nonzero cohomology group, we conclude that $\mu(\mathscr{V}) = \Delta(\mathscr{V}) = 0$ and $\mathbf{v} = \mathrm{rk}(\mathbf{v})\,\mathrm{ch}(\mathscr{O}_{\mathbb{P}^2})$.

If $\chi(\mathbf{v}(-1)) \geq 0$, then the general sheaf in $\mathscr{P}_{\mathbb{P}^2,L}(\mathbf{v})$ has a Gaeta resolution with $a \geq 1$. Then the general sheaf is clearly a quotient of a globally generated bundle. If $\chi(\mathbf{v}(-1)) < 0$ and $\chi(\mathbf{v}) \geq \mathrm{rk}(\mathbf{v}) + 2$, then the general sheaf in $\mathscr{P}_{\mathbb{P}^2,L}(\mathbf{v})$ has a Gaeta resolution of the form

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-2)^k \oplus \mathscr{O}_{\mathbb{P}^2}(-1)^l \to \mathscr{O}_{\mathbb{P}^2}^m \to \mathscr{V} \to 0, \text{ or}$$

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-2)^k \to \mathscr{O}_{\mathbb{P}^2}(-1)^l \oplus \mathscr{O}_{\mathbb{P}^2}^m \to \mathscr{V} \to 0.$$

In the first case, $\mathscr{V}$ is the quotient of a globally generated vector bundle, hence globally generated. The most interesting case is the second case. By the assumption that $\chi(\mathscr{V}) \geq \mathrm{rk}(\mathscr{V}) + 2$, we have that $m \geq \mathrm{rk}(\mathscr{V}) + 2$. Therefore, $k \geq l + 2$. To show that $\mathscr{V}$ is globally generated, it suffices to show that $H^1(\mathbb{P}^2, \mathscr{V} \otimes I_p) = 0$ for every point $p \in \mathbb{P}^2$. By the long exact sequence of cohomology, it suffices to show that the map

$$\phi : H^1(\mathbb{P}^2, I_p(-2))^k \to H^1(\mathbb{P}^2, I_p(-1))^l$$

is surjective. Consider the sequence

$$0 \to M \to \mathscr{O}_{\mathbb{P}^2}(-2)^k \to \mathscr{O}_{\mathbb{P}^2}(-1)^l \to 0.$$

Since the map is general, it is surjective and M is a vector bundle. Clearly M does not have any cohomology. Tensoring the standard exact sequence

$$0 \to I_p \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{O}_p \to 0$$

with M, we see that $H^2(\mathbb{P}^2, I_p \otimes M) = 0$. Consequently, the map ϕ is surjective and $\mathscr{V}$ is globally generated. Finally, if $\chi(\mathscr{V}) = \mathrm{rk}(\mathscr{V}) + 1$ and $\mathscr{V}$ is globally generated, then there is a surjective map $\mathscr{O}_{\mathbb{P}^2}^{r+1} \to \mathscr{V}$. The kernel of this map is a line bundle $\mathscr{O}_{\mathbb{P}^2}(-d)$. If $d = 1$, then $\chi(\mathscr{V}(-1)) = 0$. If $d \geq 3$, then $\chi(\mathscr{V}) < r$ and it is not possible for the general prioritary sheaf with Chern character $\mathbf{v}$ to be globally generated. The only remaining possibility is for $d = 2$. In that case, $\chi(\mathscr{V}) = r + 1$ and this is the Gaeta resolution of the general sheaf. This concludes the classification of globally generated Chern characters on $\mathbb{P}^2$.

The following problem remains open.

Problem 1. Classify the Chern characters $\mathbf{v}$ on $\mathbb{P}^2$ such that the general prioritary sheaf of character $\mathbf{v}$ is ample.

Note that if $\mathscr{V}$ is a vector bundle such that $\mathscr{V}(-1)$ is globally generated, then $\mathscr{V}$ is ample. Thus the classification of globally generated Chern characters gives a sufficient condition for the general bundle to be ample. In particular, if $\mathrm{rk}(\mathbf{v}) \geq 2$, $\mu(\mathbf{v}) \geq 1$, $\Delta(\mathbf{v}) \geq 0$ and $\chi(\mathbf{v}(-1)) \geq \mathrm{rk}(\mathbf{v})+2$, then the general prioritary sheaf with Chern character $\mathbf{v}$ is ample. However, an ample vector bundle on $\mathbb{P}^2$ does not have to have any sections. For example, Gieseker [Gie71] proves that a general vector bundle with resolution

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-d)^2 \to \mathscr{O}_{\mathbb{P}^2}(-1)^4 \to \mathscr{V} \to 0$$

is ample for $d \gg 0$. It is easy to see that we need $d \geq 7$. However, we do not know whether $d = 7$ is sufficient. In general, an ample bundle must satisfy $\mu(\mathbf{v}) \geq 1$ and $\frac{\mu^2}{2} > \frac{\Delta}{r+1}$. It would be interesting to determine conditions under which the converse also holds.

Hirzebruch surfaces

Following [CH16] and [CH17b], we now explain how to obtain analogues of Corollary 3 and Theorem 7 for Hirzebruch surfaces.

Let e be a nonnegative integer and let $\mathbb{F}_e$ denote the Hirzebruch surface $\mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(e))$. We refer the reader to [Cos06a] or [Har77] for a detailed description of Hirzebruch surfaces. The surface $\mathbb{F}_e$ admits a natural projection π to $\mathbb{P}^1$. Let F denote the class of a fiber of π. The surface contains a section E with self-intersection $-e$. When $e \geq 1$, this section is unique. The Picard group $\mathrm{Pic}(\mathbb{F}_e) = \mathbb{Z}E \oplus \mathbb{Z}F$ and the intersection product is given by

$$E^2 = -e, \quad E \cdot F = 1, \quad F^2 = 0.$$

Express the total slope of a Chern character $\mathbf{v}$ by

$$\nu(\mathbf{v}) = \frac{k}{r}E + \frac{l}{r}F.$$

Let $\mathscr{V} = \mathscr{O}_{\mathbb{F}_e}(-E-(e+1)F)^a \oplus \mathscr{O}_{\mathbb{F}_e}(-F)^b \oplus \mathscr{O}_{\mathbb{F}_e}^c$. Then a simple calculation shows that $\Delta(\mathscr{V}) \leq 0$ and $\mathscr{V}$ is both F-prioritary and E-prioritary. Furthermore, every slope in the quadrilateral in the $(\frac{k}{r}, \frac{l}{r})$-plane with vertices

$$(-1,-e-1), \quad (0,0), \quad (0,-1), \quad (-1,-e-1)$$

can be expressed as the slope of a bundle $\mathscr{V}$ or $\mathscr{V}^*(-E-(e+1)F)$. Furthermore, these bundles have no higher cohomology. Translates of this quadrilateral by classes of nef line bundles, covers the region defined by the inequalities $\nu(\mathbf{v}) \cdot F \geq -1$ and

$\nu(\mathbf{v}) \cdot E \geq -1$. Using elementary modifications and Lemma 1, one concludes the following.

Theorem 8 ([CH17b], Theorem 3.1). *Let* $\mathbf{v}$ *be an integral Chern character on* $\mathbb{F}_e$ *with positive rank* r *and* $\Delta \geq 0$. *Then the stack* $\mathscr{P}_{\mathbb{F}_e,F}(\mathbf{v})$ *of* F*-prioritary sheaves is nonempty and irreducible. Let* $\mathscr{V} \in \mathscr{P}_{\mathbb{F}_e,F}(\mathbf{v})$ *be a general sheaf.*

1. *If* $\nu(\mathbf{v}) \cdot F \geq -1$, *then* $h^2(\mathbb{F}_e, \mathscr{V}) = 0$. *If* $\nu(\mathbf{v}) \cdot F \leq -1$, *then* $h^0(\mathbb{F}_e, \mathscr{V}) = 0$. *In particular, if* $\nu(\mathbf{v}) \cdot F = -1$, *then both* h^0 *and* h^2 *vanish and* $h^1(\mathbb{F}_e, \mathscr{V}) = -\chi(\mathbf{v})$.
2. *If* $\nu(\mathbf{v}) \cdot F > -1$ *and* $\nu(\mathbf{v}) \cdot E \geq -1$, *then* $\mathscr{V}$ *has at most one nonzero cohomology group. Thus if* $\chi(\mathbf{v}) \geq 0$, *then* $h^0(\mathbb{F}_e, \mathscr{V}) = \chi(\mathbf{v})$, *and if* $\chi(\mathbf{v}) \leq 0$, *then* $h^1(\mathbb{F}_e, \mathscr{V}) = -\chi(\mathbf{v})$.
3. *If* $\nu(\mathbf{v}) \cdot F > -1$ *and* $\nu(\mathbf{v}) \cdot E < -1$, *then* $H^0(\mathbb{F}_e, \mathscr{V}) = H^0(\mathbb{F}_e, \mathscr{V}(-E))$, *hence the Betti numbers of* $\mathscr{V}$ *are inductively determined using the previous two parts.*
4. *If* $\nu(\mathbf{v}) \cdot F < -1$ *and* $\mathrm{rk}(\mathbf{v}) \geq 2$, *then Serre duality determines the Betti numbers of* $\mathscr{V}$.

We call a Chern character $\mathbf{v}$ *nonspecial* if there exists an F-prioritary sheaf $\mathscr{V}$ with Chern character $\mathbf{v}$ such that $\mathscr{V}$ has at most one nonzero cohomology group. In particular, we obtain a classification of nonspecial Chern characters on $\mathbb{F}_e$.

Corollary 4 ([CH17b], Corollary 3.9). *Let* $\mathbf{v}$ *be an integral Chern character on* $\mathbb{F}_e$ *with positive rank and* $\Delta(\mathbf{v}) \geq 0$, *and suppose* $\nu(\mathbf{v}) \cdot F \geq -1$. *Then* $\mathbf{v}$ *is nonspecial if and only if one of the following holds.*

1. *We have* $\nu(\mathbf{v}) \cdot F = -1$.
2. *We have* $\nu(\mathbf{v}) \cdot F > -1$ *and* $\nu(\mathbf{v}) \cdot E \geq -1$.
3. *If* $\nu(\mathbf{v}) \cdot F > -1$ *and* $\nu(\mathbf{v}) \cdot E < -1$, *let* m *be the smallest positive integer such that either* $\nu(\mathbf{v}(-mE)) \cdot F \leq -1$ *or* $\nu(\mathbf{v}(-mE)) \cdot E \geq -1$.
   a. *If* $\nu(\mathbf{v}(-mE)) \cdot F \leq -1$, *then* $\mathbf{v}$ *is nonspecial.*
   b. *If* $\nu(\mathbf{v}(-mE)) \cdot F > -1$, *then* $\mathbf{v}$ *is nonspecial if and only if* $\chi(\mathbf{v}(-mE)) \leq 0$.

The following corollary when $\chi(\mathbf{v}) \geq 0$ is easier to remember.

Corollary 5 ([CH17b], Corollary 3.10). *Let* $\mathbf{v}$ *be a positive rank Chern character on* $\mathbb{F}_e$ *such that* $\Delta(\mathbf{v}) \geq 0$, $\chi(\mathbf{v}) \geq 0$ *and* $F \cdot \nu(\mathbf{v}) \geq -1$. *Then* $\mathbf{v}$ *is nonspecial if and only if* $F \cdot \nu(\mathbf{v}) = -1$ *or* $E \cdot \nu(\mathbf{v}) \geq -1$.

As in the case of $\mathbb{P}^2$, we may use the Brill-Noether theorems to characterize the globally generated Chern characters. Let $\mathscr{V}$ a general prioritary sheaf in $\mathscr{P}_{\mathbb{F}_e,F}(\mathbf{v})$ with $\Delta(\mathscr{V}) \geq 0$. If $\mathscr{V}$ is globally generated, then its determinant has to be globally generated and nef. If in addition $\nu(\mathscr{V}) \cdot F = 0$, then the restriction of $\mathscr{V}$ to every fiber must be trivial. Hence, $\mathscr{V}$ must be a pullback from $\mathbb{P}^1$. Since $\mathscr{V}$ is F-prioritary and globally generated, we conclude that $\mathscr{V} = \mathscr{O}_{\mathbb{F}_e}(aF)^m \oplus \mathscr{O}_{\mathbb{F}_e}((a+1)F)^{r-m}$ for some $a \geq 0$ and $m \geq 0$. We may now assume that $\nu(\mathscr{V}) \cdot F > 0$. Since $\mathscr{V}$ is general, the restriction of $\mathscr{V}$ to every fiber will be globally generated. If $\chi(\mathscr{V}(-F)) \geq 0$, then the exact sequence

$$0 \to \mathscr{V}(-F) \to \mathscr{V} \to \mathscr{V}|_F \to 0$$

allows us to lift the section of $\mathscr{V}|_F$ to sections of $\mathscr{V}$ on $\mathbb{F}_e$ since by the cohomology computations $H^1(\mathbb{F}_e, \mathscr{V}(-F)) = 0$. If $\chi(\mathscr{V}(-F)) < 0$, then as in the case of $\mathbb{P}^2$, we need to resort to a Gaeta-type resolution.

Theorem 9 ([CH17b], Theorem 4.1). *Let* $\mathbf{v}$ *be an integral Chern character on* $\mathbb{F}_e$ *of positive rank and assume that*

$$\Delta(\mathbf{v}) \geq \frac{1}{4} \text{ if } e = 0, \quad \Delta(\mathbf{v}) \geq \frac{1}{8} \text{ if } e = 1, \quad \Delta(\mathbf{v}) \geq 0 \text{ if } e \geq 2.$$

Then the general sheaf $\mathscr{V} \in \mathscr{P}_{\mathbb{F}_e,F}(\mathbf{v})$ *admits a Gaeta-type resolution*

$$0 \to L(-E-(e+1)F)^a \to L(-E-eF)^b \oplus L(-F)^c \oplus L^d \to \mathscr{V} \to 0, \qquad (2)$$

for some line bundle L *and nonnegative integers* a,b,c,d.

First, assume that there exists an exact sequence of the form (2). Then the exponents in the exact sequence (2) can be formally calculated using the Euler characteristic:

$$a = -\chi(\mathscr{V}(-L-E-F)), \quad b = -\chi(\mathscr{V}(-L-E)),$$
$$c = -\chi(\mathscr{V}(-L-F)), \quad d = \chi(\mathscr{V}(-L)).$$

We now treat a,b,c,d as functions of L defined by these Euler characteristics. Assume that we can find a line bundle L such that a,b,c,d are all nonnegative. Then we can define $\mathscr{V}$ by the sequence (2). Since $\mathrm{Hom}(L(-E-(e+1)F)^a, L(-E-eF)^b \oplus L(-F)^c \oplus L^d)$ is globally generated, the cokernel of a general homomorphism defines a torsion free sheaf $\mathscr{V}$. An easy check shows that the general sheaf given by such a resolution is F-prioritary and such sheaves provide a complete family of F-prioritary sheaves. It then follows that the general F-prioritary sheaf has such a resolution since the stack of F-prioritary sheaves is irreducible. Finally, using the inequalities on Δ, one shows that one can always find a line bundle L that makes the exponents a,b,c,d nonnegative (see [CH17b]).

If $\chi(\mathscr{V}(-F)) < 0$ and $\mathscr{V}$ is globally generated, we consider

$$0 \to \mathscr{M} \to \mathscr{O}_{\mathbb{F}_e}^{\chi(\mathscr{V})} \to \mathscr{V} \to 0,$$

where $\mathscr{M}$ is a vector bundle with character $\mathbf{v}$. Then $\mathscr{M}$ has no cohomology, $h^1(\mathbb{F}_e, \mathscr{M}(-F)) = 0$ and $\mathscr{M}^*$ is globally generated. Conversely, if we can construct such a vector bundle $\mathscr{M}$, we obtain a globally generated F-prioritary vector bundle $\mathscr{V}$ with Chern character $\mathbf{v}$. As in the case of $\mathbb{P}^2$, one constructs $\mathscr{M}$ and check that $\mathscr{M}^*$ is globally generated directly from the Gaeta-type resolution provided by Theorem 9. We obtain the following classification of globally generated Chern characters on $\mathbb{F}_e$.

Theorem 10 ([CH17b], Theorem 5.1). *Let* $\mathbf{v}$ *be a Chern character on* $\mathbb{F}_e$, $e \geq 1$ *such that* $\mathrm{rk}(\mathbf{v}) \geq 2$, $\Delta(\mathbf{v}) \geq 0$ *and* $\nu(\mathbf{v})$ *is nef. Then* $\mathbf{v}$ *is globally generated if and only if one of the following holds:*

1. *We have* $\nu(\mathbf{v}) \cdot F = 0$ *and* $\mathbf{v} = \mathrm{ch}(\pi^*(\mathscr{O}_{\mathbb{P}^1}(a)^m \oplus \mathscr{O}_{\mathbb{P}^1}(a+1)^{r-m}))$ *for some* $a \geq 0$.
2. *We have* $\nu(\mathbf{v}) \cdot F > 0$ *and* $\chi(\mathbf{v}(-F)) \geq 0$.
3. *We have* $\nu(\mathbf{v}) \cdot F > 0$, $\chi(\mathbf{v}(-F)) < 0$ *and* $\chi(\mathbf{v}) \geq r+2$.
4. *We have* $e = 1$, $\nu(\mathbf{v}) \cdot F > 0$, $\chi(\mathbf{v}(-F)) < 0$, $\chi(\mathbf{v}) \geq r+1$ *and*

$$\mathbf{v} = (\mathrm{rk}(\mathbf{v})+1)\,\mathrm{ch}(\mathscr{O}_{\mathbb{F}_1}) - \mathrm{ch}(\mathscr{O}_{\mathbb{F}_1}(-2E-2F)).$$

Since $\mathbb{P}^1 \times \mathbb{P}^1$ admits two fibration structures, the theorem has to account for both fibrations.

Theorem 11 ([CH17b], Theorem 5.2). *Let* $\mathbf{v}$ *be a Chern character on* $\mathbb{P}^1 \times \mathbb{P}^1$ *such that* $\mathrm{rk}(\mathbf{v}) \geq 2$, $\Delta(\mathbf{v}) \geq 0$ *and* $\nu(\mathbf{v})$ *is nef. Let* F_1 *and* F_2 *be the classes of the two rulings. The Chern character* $\mathbf{v}$ *is globally generated if and only if one of the following holds*

1. *We have* $\nu(\mathbf{v}) \cdot F_i = 0$ *for some* $i \in \{1,2\}$ *and*

$$\mathbf{v} = \mathrm{ch}(\mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(aF_i)^m \oplus \mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^1}((a+1)F_i)^{r-m})$$

   for some $a \geq 0$.
2. *We have* $\nu(\mathbf{v}) \cdot F_i > 0$ *for* $i \in \{1,2\}$ *and* $\chi(\mathbf{v}(-F_i)) \geq 0$ *for some* $i \in \{1,2\}$.
3. *We have* $\nu(\mathbf{v}) \cdot F_i > 0$ *and* $\chi(\mathbf{v}(-F_i)) \leq 0$ *for* $i \in \{1,2\}$ *and* $\chi(\mathbf{v}) \geq \mathrm{rk}(\mathbf{v}) + 2$.

As in the case of $\mathbb{P}^2$, it would be very interesting to classify the Chern characters of ample stable (or F-prioritary) bundles on $\mathbb{F}_e$.

Del Pezzo surfaces and more general rational surfaces

Let X be the blowup of $\mathbb{P}^2$ at r points $p_1, \dots, p_k$. If $k \leq 8$ and the points are in general position, then X is a del Pezzo surface. We refer the reader to [Bea83, Cos06b, Har77] for more detailed information on del Pezzo surfaces. Let L denote the pullback of the hyperplane class from $\mathbb{P}^2$ and let $E_1, \dots, E_r$ denote the exceptional divisors lying over $p_1, \dots, p_k$. Then $\mathrm{Pic}(X) \cong \mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_k$ and the intersection form is given by

$$L^2 = 1, \quad L \cdot E_i = 0, \quad E_i \cdot E_j = -\delta_{i,j},$$

where $\delta_{i,j}$ denotes the Krönecker delta function. When X is a del Pezzo surface, the (-1)-curves and $-K_X$ generate the effective cone of curves on X. Furthermore, the cohomology of line bundles is completely known. When X is a more general blowup, even the cohomology of line bundles is not known. Consequently, it is unrealistic to expect a full computation of the cohomology of all higher rank sheaves.

Let $\mathbf{v}$ be a Chern character of rank r and let the total slope be $\nu(\mathbf{v}) = \delta L - \sum_{i=1}^{k} \alpha_i E_i$. Then we have that

$$\delta = d + \frac{q}{r}, \quad \alpha_i = a_i + \frac{q_i}{r}$$

for some integers d, q, a_i and q_i with $0 \leq q < r$ and $0 \leq q_i < r$. Set

$$\gamma(\mathbf{v}) = \frac{q^2}{2r^2} - \frac{q}{2r} + \sum_{i=1}^{k} \left(\frac{q_i}{2r} - \frac{q_i^2}{2r^2} \right).$$

Theorem 12 ([CH16], Theorem 4.5). *Let X be the blowup of $\mathbb{P}^2$ at k distinct points. Let $\mathbf{v}$ be a positive rank Chern character on X with total slope*

$$\nu(\mathbf{v}) = \delta L - \alpha_1 E_1 \cdots - \alpha_k E_k$$

with $\delta \geq 0$ and $\alpha_i \geq 0$. Suppose that the line bundle

$$\lfloor \delta \rfloor L - \lceil \alpha_1 \rceil E_1 \cdots - \lceil \alpha_k \rceil E_k$$

does not have higher cohomology. Assume that $\Delta(\mathbf{v}) \geq \gamma(\mathbf{v})$. Then $\mathscr{P}_{X,L-E_1}(\mathbf{v})$ is nonempty and the general sheaf in $\mathscr{P}_{X,L-E_1}(\mathbf{v})$ has at most one nonzero cohomology group.

To prove the theorem it suffices to consider the direct sum of line bundles $\mathscr{V} = L_1 \oplus \cdots \oplus L_k$, where each line bundle L_j has the form

$$n_j L - \sum_{i=1}^{k} m_{j,i} E_i$$

with $n_j \in \{\lfloor \delta \rfloor, \lceil \delta \rceil\}$ and $m_{j,i} \in \{\lfloor \alpha_i \rfloor, \lceil \alpha_i \rceil\}$. By choosing L_j appropriately, we can arrange that $\nu(\mathscr{V}) = \nu(\mathbf{v})$. Then each L_j is a line bundle with no higher cohomology. It is easy to check that $\mathscr{V}$ is $(L-E_1)$-prioritary and has $\Delta(\mathscr{V}) = \gamma(\mathbf{v})$. The theorem follows by taking elementary modifications of $\mathscr{V}$.

It is possible to choose the direct sum of line bundles more carefully to obtain sharper bounds when k is small. We refer the reader to [CH16, §5] when X is a del Pezzo surface of large degree.

Other surfaces

Determining the cohomology of the general stable sheaf and classifying the Chern characters $\mathbf{v}$ such that the general stable sheaf with Chern character $\mathbf{v}$ is globally generated or stable are important problems on any surface. The theory is most developed for K3 surfaces thanks to the work of Leyenson, Markman, Mukai, O'Grady, Yoshioka and many others. We refer the reader to [O'G97, Ley12, Mrk01] for further details and references. We close this section with a few general remarks on Brill-Noether statements on general surfaces. First, an asymptotic weak Brill-Noether statement holds on any smooth projective surface.

Proposition 7. *Let X be a smooth projective surface and let H be an ample divisor. Let $\mathbf{v}$ be a Chern character with $\Delta(\mathbf{v}) \gg 0$. Let $\mathscr{V} \in M_{X,H}(\mathbf{v})$ be a general sheaf. Then the only nonzero cohomology group of $\mathscr{V}$ can be $H^1(X,\mathscr{V})$.*

Proof. Let $\mathbf{v}^D$ denote the Serre dual Chern character of $\mathbf{v}$. Observe that $\Delta(\mathbf{v}) = \Delta(\mathbf{v}^D)$. Assume that $\Delta(\mathbf{v}) \geq \delta$, where δ is the O'Grady bound that guarantees that both $M_{X,H}(\mathbf{v})$ and $M_{X,H}(\mathbf{v}^D)$ are irreducible and the general member is a stable bundle. If the general sheaf $\mathscr{V} \in M_{X,H}(\mathbf{v})$ has only H^1, we are done. If $\mathscr{V}$ has any global sections, replace $\mathscr{V}$ by $h^0(X,\mathscr{V})$ general elementary modifications $\mathscr{V}_1$. Then $\mathscr{V}_1$ is a slope stable sheaf, has no H^0, and has discriminant $\Delta(\mathscr{V}_1) = \Delta(\mathscr{V}) + \frac{h^0(X,\mathscr{V})}{\mathrm{rk}(\mathscr{V})} > \delta$. Hence, the moduli space containing $\mathscr{V}_1$ is irreducible. We can find a locally free slope-stable sheaf $\mathscr{V}_2$ with no H^0 since vanishing of H^0 is an open condition. If $\mathscr{V}_2$ has only H^1, we are done. Otherwise, replace $\mathscr{V}_2$ by its Serre dual $\mathscr{V}_3$. Apply $h^0(X,\mathscr{V}_3) = h^2(X,\mathscr{V}_2)$ general elementary modifications to $\mathscr{V}_3$. The resulting sheaf $\mathscr{V}_4$ has vanishing H^0 and H^2 and is slope stable. A general deformation $\mathscr{V}_5$ of $\mathscr{V}_4$ is locally free and also has vanishing H^0 and H^2. The Serre dual of $\mathscr{V}_5$ gives the desired bundle. For $\mathbf{v}$ with $\Delta(\mathbf{v}) \geq \Delta(\mathscr{V}_5)$, the only nonzero cohomology group of a general sheaf in $M_{X,H}(\mathbf{v})$ can be H^1.

Since the moduli spaces for small Δ are not necessarily irreducible, even when there is a stable sheaf with at most one nonzero cohomology group, there may still be entire components of the moduli space where more than one cohomology group is nonzero. One can already find such examples on Enriques surfaces. The following example is due to Nuer and Yoshioka.

Example 3 ([NY17], §10). Nuer and Yoshioka show that there is a component of the moduli space of rank 2 sheaves on an Enriques surface X whose general element is given by an extension of the form

$$0 \to I_Z \to \mathscr{V} \to \mathscr{O}_X(K_X) \to 0,$$

where Z is a zero-dimensional scheme of length 2. Observe that $h^1(X,\mathscr{V}) = h^2(X,\mathscr{V}) = 1$.

On surfaces of general type, even very ample line bundles can have nonzero higher cohomology. Consequently, we would not expect the higher cohomology of higher rank sheaves to vanish either.

Example 4. Let X be a very general hypersurface of degree $d \geq 5$ in $\mathbb{P}^3$. Let Z be $d-1$ collinear points on X. Then a general extensions of the form

$$0 \to \mathscr{O}_X \to \mathscr{V} \to I_Z(1) \to 0$$

is slope stable. By §3, such extensions form a component of the moduli space $M_X((2,1,d-1))$ of rank 2 sheaves with c_1 equal to the hyperplane class and $c_2 = d-1$. The long exact sequence of cohomology associated to the defining sequence of $\mathscr{V}$ and Serre duality show that

$$h^0(X,\mathscr{V}) = 3, \quad h^2(X,\mathscr{V}) = \binom{d-1}{3} + \binom{d-2}{3} - d + 3.$$

The line bundle $\mathscr{O}_X(m)$ has no higher cohomology for $m \geq d-3$. More generally, consider extensions of the form

$$0 \to \mathscr{O}_X(m) \to \mathscr{V}(m) \to I_Z(m+1) \to 0.$$

Since $h^2(X, I_Z(m)) \neq 0$ for $m < d-4$, we conclude that $\mathscr{V}$ has nonvanishing h^0 and h^2 for $0 \leq m < d-4$. On the other hand, the higher cohomology of $\mathscr{V}(m)$ vanishes for $m \geq d-4$. This is immediate for $m \geq d-3$ by the long exact sequence of cohomology. The only case to discuss is $m = d-4$. We have

$$0 \to H^1(X,\mathscr{V}(d-4)) \to H^1(X,I_Z(d-3)) \to$$

$$H^2(X,\mathscr{O}_X(d-4)) \to H^2(X,V(d-4)) \to 0.$$

Moreover, $H^1(X,I_Z(d-3)) \cong H^2(X,\mathscr{O}_X(d-4)) \cong \mathbb{C}$. The Serre dual $\mathscr{W}$ of $\mathscr{V}(d-4)$ fits in the exact sequence

$$0 \to \mathscr{O}_X(-1) \to \mathscr{W} \to I_Z \to 0,$$

hence $h^0(X,\mathscr{W}) = h^2(X,\mathscr{V}(d-4)) = 0$. We conclude that the higher cohomology of $\mathscr{V}(m)$ vanishes for $m \geq d-4$. Observe that $\mathscr{V}$ is μ-stable and has $\Delta(\mathscr{V}) = \frac{3}{8}d - \frac{1}{2}$. Let $\mathbf{v}$ be a Chern character of rank 2 on X with $\mu(\mathbf{v}) = \frac{2m+1}{2}$ for $m \geq d-4$. If $\Delta(\mathbf{v}) \geq \frac{3}{8}d - \frac{1}{2}$, there exist stable sheaves with Chern character $\mathbf{v}$ that have at most one nonzero cohomology group.

Remark 7. It would be interesting to explore the following questions further.

1. Let X be a projective surface such that $\mathrm{Pic}(X) = \mathbb{Z}H$ for an ample divisor H. Assume that $H^1(X,mH) = H^2(X,mH) = 0$ for $m \geq m_0$. Assume that $\nu(\mathbf{v}) = \mu H$ for $\mu > m_0$. Does there exist a sheaf $\mathscr{V} \in M_{X,H}(\mathbf{v})$ with at most one nonzero cohomology group? What additional assumptions are necessary for surfaces of higher Picard rank?
2. Assume that the general sheaf in $M_{X,H}(\mathbf{v})$ has no higher cohomology and $\nu(\mathbf{v})$ is ample. Let $\mathscr{V} \in M_{X,H}(\mathbf{v})$ be a general sheaf. What additional assumptions guarantee that if $h^0(X,\mathscr{V}) \geq r+2$, then $\mathscr{V}$ is globally generated?

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