ALGEBRAIC HYPERBOLICITY OF THE VERY GENERAL QUINTIC SURFACE IN $\mathbb{P}^3$

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ABSTRACT. We prove that a curve of degree dk on a very general surface of degree $d \geq 5$ in $\mathbb{P}^3$ has geometric genus at least $\frac{dk(d-5)+k}{2}+1$. This gives a substantial improvement on the celebrated genus bounds of Geng Xu. As a corollary, we deduce the algebraic hyperbolicity of a very general quintic surface in $\mathbb{P}^3$, resolving a long-standing conjecture of Demailly. This completely determines which very general hypersurfaces in $\mathbb{P}^3$ are algebraically hyperbolic.

1. INTRODUCTION

In this paper, we give a substantial improvement on Geng Xu's celebrated genus bounds [Xu94]. As a consequence, we prove that a very general quintic surface in $\mathbb{P}^3$ is algebraically hyperbolic, resolving a long-standing conjecture of Demailly. Since quartic K3 surfaces in $\mathbb{P}^3$ have rational curves, they are not hyperbolic. Hence, our results completely determine which very general surfaces in $\mathbb{P}^3$ are algebraically hyperbolic. To prove our results, we introduce a novel twist to the set-up developed by Clemens, Ein, Voisin, Pacienza, and Ran. When $f : C \to X$ is a map from a smooth curve to a very general hypersurface in $\mathbb{P}^3$, we analyze a relation between positivity of the normal bundle to f in X and scrolls in $\mathbb{P}^3$ containing $f(C)$. Our proof, in addition to giving stronger bounds, is significantly simpler than Xu's.

A complex projective variety X is *algebraically hyperbolic* if there exists an $\epsilon > 0$ such that for any curve $C \subset X$ of geometric genus $g(C)$ we have

$$2g(C) - 2 \geq \epsilon \deg(C).$$

In particular, algebraically hyperbolic varieties contain no rational or elliptic curves. Algebraic hyperbolicity is intimately related to complex analytic notions of hyperbolicity. Recall that a variety X is *Brody hyperbolic* if it admits no holomorphic maps from $\mathbb{C}$. By Brody's Theorem [Br78], Brody hyperbolicity on compact manifolds is equivalent to the nondegeneracy of the Kobayashi metric. Brody hyperbolicity is conjectured to control many geometric and arithmetic properties of X. For example, Demailly [De95] proves that Brody hyperbolic varieties are algebraically hyperbolic and conjectures the following.

Conjecture 1.1. *A smooth complex projective variety X is Brody hyperbolic if and only if it is algebraically hyperbolic.*

Hypersurfaces provide a natural testing ground for deep conjectures on hyperbolicity. Hence, we are led to the following question.

Question 1.2. *For which n and d is a very general hypersurface of degree d in $\mathbb{P}^n$ algebraically hyperbolic?*

More generally, one can ask the following.

Question 1.3. *For a very general hypersurface of degree d in $\mathbb{P}^n$, for which pairs (e, g) does there exist a curve of degree e and geometric genus g?*

The study of the hyperbolicity of very general hypersurfaces in $\mathbb{P}^n$ has a long history (see [Co05, De95, DEG00, Mc99, Si04]). For example, Question 1.2 has been resolved in many cases. Results of Voisin [Vo96, Vo98] prove that if $n \geq 4$, then

$$2g - 2 \geq e(d - 2n + 2).$$

Therefore, a very general hypersurface of degree $d \geq 2n - 1$ in $\mathbb{P}^n$ is algebraically hyperbolic for $n \geq 4$. In the case $n = 3$, by the Noether-Lefschetz Theorem, any curve on X is a complete intersection of type (d, k). Geng Xu [Xu94] improved results of Ein [Ei88] and proved that

$$g \geq \frac{dk(d - 5)}{2} + 2,$$

showing that surfaces in $\mathbb{P}^3$ of degree at least 6 are algebraically hyperbolic. Chen [Ch04] studies the case of complements of hypersurfaces. However, despite all this interest, the case of quintics in $\mathbb{P}^3$ has remained open for the past 20 years (see [De95, De18]). Our goal in this paper is to prove the following.

Theorem 1.4. *Let $X \subset \mathbb{P}^3$ be a very general surface of degree $d \geq 5$ and let k be a positive integer. Then the geometric genus of any curve in X of degree dk is at least $\frac{dk(d-5)+k}{2} + 1$.*

In particular, when $d = 5$, the bound on the genus of a curve of degree $5k$ specializes to

$$2g(C) - 2 \geq k = \frac{1}{5}\deg(C).$$

We obtain the following corollary.

Corollary 1.5. *A very general quintic surface in $\mathbb{P}^3$ is algebraically hyperbolic.*

Organization of the paper. In §2, we recall, simplify and further develop the basic setup developed by Ein, Voisin, Pacienza, Clemens and Ran. In §3, we prove Theorem 1.4.

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2. Normal bundles and the universal family

In this section, we recall, simplify and further develop the setup due to [CR04, Ei88, Ei91, Pa03, Pa04, Vo96, Vo98]. The main goal of this section is to prove Corollary 2.6. We always work over the complex numbers $\mathbb{C}$.

Let $S_d = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$. Suppose that a general hypersurface of degree d in $\mathbb{P}^n$ admits a generically injective map from a smooth projective variety of deformation class Y. After an étale base change $U \to S_d$, we can find a generically injective map

$$h : \mathcal{Y} \to \mathcal{X}$$

over U, where $\mathcal{X}$ is the universal hypersurface of degree d in $\mathbb{P}^n$ over U and $\mathcal{Y}$ is a smooth family of pointed varieties over U with members in the deformation class of Y. We choose $\mathcal{Y}$ so that the codimension of $h(\mathcal{Y})$ in $\mathcal{X}$ is $n - 1 - \dim Y$. Let

$$\pi_1 : \mathcal{X} \to U \quad \text{and} \quad \pi_2 : \mathcal{X} \to \mathbb{P}^n$$

denote the two natural projections of the universal hypersurface over U. Let the *vertical tangent sheaf* $T_{\mathcal{X}/\mathbb{P}^n}$ be defined by the natural sequence

$$0 \to T_{\mathcal{X}/\mathbb{P}^n} \to T_{\mathcal{X}} \to \pi_2^* T_{\mathbb{P}^n} \to 0.$$

Since every hypersurface contains a variety of deformation class Y, the family $\mathcal{Y}$ dominates U under $\pi_1 \circ h$. Moreover, without loss of generality, we may assume that $\mathcal{Y}$ is stable under the GL_{n+1} action on $\mathbb{P}^n$, so $\pi_2 \circ h$ dominates $\mathbb{P}^n$ [Pa03, Vo96]. Furthermore, the invariance under GL_{n+1} also implies that the map $T_{\mathcal{Y}} \to h^*(\pi_2^* T_{\mathbb{P}^n})$ is surjective. Let $T_{\mathcal{Y}/\mathbb{P}^n}$ denote the kernel

$$0 \to T_{\mathcal{Y}/\mathbb{P}^n} \to T_{\mathcal{Y}} \to h^*(\pi_2^* T_{\mathbb{P}^n}) \to 0.$$

Let $t \in U$ be a general closed point. Let X_t denote the corresponding fiber of $\mathcal{X}$. Let

$$h_t : Y_t \to X_t$$

denote the corresponding map from the fiber Y_t of $\mathcal{Y}$ over t to X_t. Let

$$i_t : X_t \to \mathcal{X} \quad \text{and} \quad j_t : Y_t \to \mathcal{Y}$$

denote the inclusion of X_t in $\mathcal{X}$ and Y_t in $\mathcal{Y}$, respectively. Observe that $i_t \circ h_t = h \circ j_t$. Let $N_{h/\mathcal{X}}$ denote the normal sheaf to the map $h : \mathcal{Y} \to \mathcal{X}$ and let N_{h_t/X_t} denote the normal sheaf to h_t defined as the cokernels of the following natural sequences

$$0 \to T_{\mathcal{Y}} \to h^* T_{\mathcal{X}} \to N_{h/\mathcal{X}} \to 0$$

$$0 \to T_{Y_t} \to h_t^* T_{X_t} \to N_{h_t/X_t} \to 0.$$

Following Ein, Voisin, Pacienza, Clemens and Ran, it is standard to relate N_{h_t/X_t} to sheaves arising from the family $\mathcal{Y} \to \mathcal{X}$. Unfortunately, the prior work does not explicitly state the theorem we will use. For the reader's convenience, we reprove the main statements, emphasizing the key points from our perspective. We will make repeated use of the following lemma.

Lemma 2.1. *Let $\phi : Z_0 \to Z$ be a morphism of projective varieties. Let*

(1) $$0 \to E \to F \to G \to 0$$

be a short exact sequence of sheaves on Z, where E and F are vector bundles. If $\phi^* E \to \phi^* F$ is generically injective, then the sequence

$$0 \to \phi^* E \to \phi^* F \to \phi^* G \to 0$$

on Z_0 is exact.

Proof. Apply the derived pullback functor to the sequence (1) to obtain the sequence

$$0 \to L^1 \phi^* G \to \phi^* E \to \phi^* F \to \phi^* G \to 0,$$

where $L^1 \phi^* F = 0$ since F is a vector bundle. Since $\phi^* E \to \phi^* F$ is generically injective, $L^1 \phi^* G$ must be torsion. Since E is a vector bundle, $\phi^* E$ is torsion-free and we conclude that $L^1 \phi^* G = 0$. This proves the lemma. $\qquad\square$

We first relate the normal bundles N_{h_t/X_t} and $N_{h/\mathcal{X}}$.

Lemma 2.2. *We have*

$$N_{h_t/X_t} \cong j_t^* N_{h/\mathcal{X}}.$$

Proof. Consider the following diagram.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathcal{O}_{Y_t}^N & \xrightarrow{=} & \mathcal{O}_{Y_t}^N & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & j_t^* T_{\mathcal{Y}} & \longrightarrow & h_t^* i_t^* T_{\mathcal{X}} & \longrightarrow & j_t^* N_{h/\mathcal{X}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \cong \uparrow & & \\
0 & \longrightarrow & T_{Y_t} & \longrightarrow & h_t^* T_{X_t} & \longrightarrow & N_{h_t/X_t} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Since Y_t in $\mathcal{Y}$ and X_t in $\mathcal{X}$ are fibers of fibrations, their normal bundles are trivial of rank $N = \dim S_d$. The first column in the diagram is the definition of the normal bundle $N_{Y_t/\mathcal{Y}}$. The second column in the diagram is the pullback under h_t of the sequence

$$0 \to T_{X_t} \to T_{\mathcal{X}}|_{X_t} \to N_{X_t/\mathcal{X}} \to 0$$

defining $N_{X_t/\mathcal{X}}$. Since h_t is generically injective and $T_{X_t} \to T_{\mathcal{X}}|_{X_t}$ is everywhere injective, by Lemma 2.1 the pullback of the sequence by h_t remains exact. The top row is an isomorphism of sheaves in the natural way. The bottom row is the sequence defining N_{h_t/X_t}. The middle row is the pullback of the sequence defining $N_{h/\mathcal{X}}$ under j_t using the identification $h \circ j_t = i_t \circ h_t$. It is exact by Lemma 2.1 and the fact that Y_t passes through a general point of $\mathcal{Y}$. The required isomorphism $N_{h_t/X_t} \cong j_t^* N_{h/\mathcal{X}}$ follows from the diagram by the Nine Lemma. $\square$

We can also express the normal bundle N_{h_t/X_t} in terms of vertical tangent bundles. Let $\mathcal{K}$ denote the cokernel of the map $T_{\mathcal{Y}/\mathbb{P}^n} \to h^* T_{\mathcal{X}/\mathbb{P}^n}$.

Lemma 2.3. *We have*

$$N_{h_t/X_t} \cong j_t^* \mathcal{K}.$$

Proof. This is another diagram chase. Consider the following diagram.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & j_t^* h^* \pi_2^* T_{\mathbb{P}^n} & \xrightarrow{=} & h_t^* i_t^* \pi_2^* T_{\mathbb{P}^n} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & j_t^* T_{\mathcal{Y}} & \longrightarrow & h_t^* i_t^* T_{\mathcal{X}} & \longrightarrow & j_t^* N_{h/\mathcal{X}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \cong \uparrow & & \\
0 & \longrightarrow & j_t^* T_{\mathcal{Y}/\mathbb{P}^n} & \longrightarrow & h_t^* i_t^* T_{\mathcal{X}/\mathbb{P}^n} & \longrightarrow & j_t^* \mathcal{K} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

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The first two columns come from the definitions of $T_{\mathcal{Y}/\mathbb{P}^n}$ and $T_{\mathcal{X}/\mathbb{P}^n}$, respectively. The restrictions of each to Y_t remain exact by Lemma 2.1. The top isomorphism is also the natural one, since $j_t^* h^* = h_t^* i_t^*$.

The bottom row is exact by Lemma 2.1, and the middle row is exact just as in the proof of Lemma 2.2.

$\square$

Thus, in order to study positivity of N_{h_t/X_t}, we may study positivity of $\mathcal{K}$.

Lemma 2.4 (cf. [Pa04] Section 2)**.** *Let $M_d^{\mathbb{P}^n}$ be the bundle defined by the sequence*

$$0 \to M_d^{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n} \otimes S_d \to \mathcal{O}_{\mathbb{P}^n}(d) \to 0.$$

Then $T_{\mathcal{X}/\mathbb{P}^n} \cong \pi_2^* M_d^{\mathbb{P}^n}$.

Proof. This is yet another diagram chase. Consider the following diagram.

$$
\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
\pi_2^* \mathcal{O}(d) & \xrightarrow{=} & \pi_2^* \mathcal{O}(d) \\
\uparrow & & \uparrow \\
\end{array}
$$

$$
\begin{array}{ccccccc}
0 \to & \mathcal{O} \otimes S_d & \to & \pi_2^* T_{\mathbb{P}^n} \oplus \mathcal{O} \otimes S_d & \to & \pi_2^* T_{\mathbb{P}^n} & \to 0 \\
& \uparrow & & \uparrow & & \cong \uparrow & \\
0 \to & T_{\mathcal{X}/\mathbb{P}^n} & \to & T_{\mathcal{X}} & \to & \pi_2^* T_{\mathbb{P}^n} & \to 0 \\
& \uparrow & & \uparrow & & & \\
& 0 & & 0 & & &
\end{array}
$$

The bottom row is the defining sequence for $T_{\mathcal{X}/\mathbb{P}^n}$. The middle row comes from the natural splitting of $\pi_2^* T_{\mathbb{P}^n} \oplus \mathcal{O} \otimes S_d$.

The second column is the normal bundle sequence for $\mathcal{X} \subset \mathbb{P}^n \times U$. The maps in the first column are naturally induced by the maps in the second and prove the desired result.

$\square$

Lemma 2.5 (cf. [CR04] Section 3)**.** *There exists an integer s so that there is a surjective map*

$$\bigoplus_{i=1}^{s} h^* \pi_2^* M_1^{\mathbb{P}^n} \to \mathcal{K}.$$

Proof. Given a degree $d-1$ polynomial P, we get a natural map $M_1^{\mathbb{P}^n} \to M_d^{\mathbb{P}^n}$, given by multiplication by P. Since polynomials of the form $x_i P$ generate the fiber of $M_d^{\mathbb{P}^n}$ at a point, we see that we can inductively keep adding copies of $M_1^{\mathbb{P}^n}$ to decrease the cokernel of the combined map. $\square$

In Voisin, Pacienza, and Clemens-Ran [Vo98, Pa03, Pa04, CR04], the strategy is to show that if Y_t is not contained in the locus on X_t swept out by lines, then s in Lemma 2.5 is not too large relative to n. Since the first Chern class of $M_1^{\mathbb{P}^n}$ is negative the hyperplane class on $\mathbb{P}^n$, this bounds the negativity of the normal bundle of any subvariety $Y_t \subset X_t$ that is not contained in the locus swept out by lines.

If $n = 3$, we get a map with torsion cokernel if we merely let $s = 1$, so bounding s is not useful in studying curves on surfaces in $\mathbb{P}^3$. Instead, we combine the fact that $M_1^{\mathbb{P}^n} \cong \Omega_{\mathbb{P}^n}(1)$ with our understanding of $\Omega_{\mathbb{P}^n}(1)$ pulled back to Y_t to get better bounds on the degree of N_{h_t/X_t}. The statement we use for our purposes is the following.

Corollary 2.6. *If $n = 3$ and $\mathcal{Y}$ is a family of curves, there is a map $h_t^* i_t^* \pi_2^* \Omega_{\mathbb{P}^3}(1) \to N_{h_t/X_t}$ with torsion cokernel.*

Proof. Pull back the surjective map $\bigoplus_{i=1}^s h^* \pi_2^* M_1^{\mathbb{P}^n} \to \mathcal{K}$ from Lemma 2.5 under j_t. Since $\mathcal{K}$ has rank 1 and $N_{h_t/X_t} \cong j_t^* \mathcal{K}$, the result follows. $\square$

3. Scrolls and degree considerations

In this section, we prove Theorem 1.4. We specialize the discussion in the previous section to the case of curves.

Proposition 3.1. *Let $h : C \to \mathbb{P}^n$ be a generically injective map of degree e from a smooth curve C to $\mathbb{P}^n$. Then line bundle quotients of $h^* \Omega_{\mathbb{P}^n}(1)$ of degree $-m$ give rise to surface scrolls in $\mathbb{P}^n$ of degree at most $e - m$ that contain $h(C)$.*

Proof. The pullback of the Euler sequence on $\mathbb{P}^n$ to C gives rise to the sequence

$$0 \to h^* \Omega_{\mathbb{P}^n}(1) \to \mathcal{O}_C^{n+1} \to \mathcal{O}_C(1) \to 0.$$

Suppose that Q is a degree $-m$ line bundle quotient of $h^* \Omega_{\mathbb{P}^n}(1)$

$$0 \to S \to h^* \Omega_{\mathbb{P}^n}(1) \to Q \to 0$$

with kernel S. Then S is a vector sub-bundle of $h^* \Omega_{\mathbb{P}^n}(1)$ of degree $m - e$. Composing S with the inclusion $h^* \Omega_{\mathbb{P}^n}(1) \to \mathcal{O}_C^{n+1}$ realizes S as a vector sub-bundle of $\mathcal{O}_C^{n+1}$, with quotient Q' of rank 2 and degree $e - m$. By the universal property of projective space, this gives a map from $\psi : \mathbb{P}(Q') \to \mathbb{P}^n$ whose image contains $h(C)$. If the map ψ is generically injective, the scroll has degree $e - m$. If ψ is generically d_0 to one, then the scroll has degree $\frac{e-m}{d_0}$. $\square$

Corollary 3.2. *Let $h : C \to \mathbb{P}^n$ be a generically injective map of degree e from a smooth curve C to $\mathbb{P}^n$. Assume that $h(C)$ does not lie on a surface scroll of degree less than k. Then any rank 1 quotient of $h^* \Omega_{\mathbb{P}^n}(1)$ (not necessarily a line bundle) has degree at least $k - e$.*

Proof. Let Q be a rank 1 quotient, and let L be the line bundle given by Q mod torsion. Then $\deg Q \geq \deg L \geq k - e$ by Proposition 3.1. $\square$

Proof of Theorem 1.4. Let $X \subset \mathbb{P}^3$ be a very general degree d surface. Let $h : C \to X$ be the normalization of a complete intersection curve of type (d, k) in X. Assume that C has genus g. Then by Corollary 2.6, there is a map $\alpha : h^* \Omega_{\mathbb{P}^3}(1) \to N_{h/X}$ with torsion cokernel. Let Q be the image of α. Then since Q injects into $N_{h/X}$, Q is a rank 1 quotient of $h^* \Omega_{\mathbb{P}^3}(1)$ with $\deg Q \leq \deg N_{h/X}$. In order to use Corollary 3.2, we need to understand the smallest possible degree of a scroll containing $h(C)$.

If $k \leq d$, then the degree k hypersurface containing $h(C)$ is the smallest degree surface containing $h(C)$, so any scroll containing $h(C)$ will have degree at least k. If $d > k$, then the only irreducible surface of degree less than k containing $h(C)$ is X. Since $d \geq 5$ and X is very general, X is irreducible and of general type. In particular, X is not a scroll. Thus,

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the complete intersection curve $h(C)$ lies on no surface scrolls of degree less than k. Hence, by Corollary 3.2,

$$\deg N_{h/X} \geq \deg Q \geq k - dk.$$

On the other hand, we know that

$$\deg N_{h/X} = dk(4 - d) + 2g - 2.$$

Rearranging the inequality

$$dk(4 - d) + 2g - 2 \geq k - dk,$$

we obtain

$$2g - 2 \geq dk(d - 5) + k$$

or

$$g \geq \frac{dk(d - 5) + k}{2} + 1$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 3.3. If one had a better bound for the minimal degree of the universal line of a scroll containing $h(C)$, one could get better genus bounds. In particular, if the degree were at least $k + s$, then we would get a genus bound of

$$g \geq \frac{dk(d - 5) + k + s}{2} + 1.$$

One can ask if the cone over $h(C)$ with vertex at the most singular point of $h(C)$ is the scroll containing $h(C)$ with minimal degree of the universal line. If this were true, then the corresponding genus bound would be

$$g \geq \frac{dk(d - 4) - \lfloor \sqrt{dk^2 + 1} \rfloor + 1}{2}.$$

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