

Tangle decompositions of alternating link complements

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Abstract Decomposing knots and links into tangles is a useful technique for understanding their properties. The notion of prime tangles was introduced by Kirby and Lickorish; Lickorish proved that by summing prime tangles one obtains a prime link. In a similar spirit, summing two prime alternating tangles will produce a prime alternating link if summed correctly with respect to the alternating property. Given a prime alternating link, we seek to understand whether it can be decomposed into two prime tangles, each of which is alternating. We refine results of Menasco and Thistlethwaite to show that if such a decomposition exists, either it is visible in an alternating link diagram or the link is of a particular form, which we call a pseudo-Montesinos link.

1. Overview

We review some definitions and give an outline of the paper.

Let L be a non-split prime alternating link in S^3 . A properly imbedded surface in the complement of L is *essential* if it is incompressible, boundary incompressible, and non-boundary parallel in $S^3 - L$.

A *Conway sphere* for L is an essential 4-punctured sphere properly imbedded in the complement of L with meridional boundary components. We say that a Conway sphere splits L into two *2-tangles* (i.e., two 3-balls, each of which contains two strands of L). A 3-ball may also additionally contain components of L disjoint from F . Since L is prime by hypotheses, the tangles on each side are *prime*. A 2-tangle in which the two strands are boundary parallel is called a *rational* tangle.

The notion of a prime tangle was suggested by Kirby and Lickorish in [3]. Lickorish shows in [5] that by summing prime tangles one obtains a prime link. Similarly, summing two prime alternating tangles produces a prime alternating link if summed correctly with respect to the alternating property. Here, we look at the converse: given a prime alternating link diagram, can one determine whether it is a sum of two prime alternating tangles?

Menasco proved in [6] that a Conway sphere is realized in a reduced alternating diagram as either visible (represented by a *PPPP* curve in standard position) or hidden (two *PSPS* curves). He also proved that two *PSPS* curves in a prime alternating diagram represent a 4-punctured sphere that is essential [7]. It is known how to

determine whether a given *PPPP* curve (visible case) represents a 4-punctured sphere that is essential: see Menasco and Thistlethwaite [9], and Thistlethwaite [10]. All these facts together give a purely diagrammatic algorithm to determine whether the link has a decomposition into two prime tangles, though possibly not alternating.

We revisit the case of a “hidden” Conway sphere. We prove that such a sphere forces the presence of another visible Conway sphere right next to it, in the same diagram, unless the link is a pseudo-Montesinos link (Proposition 3.2). Pseudo-Montesinos links will be defined in Section 3 and are a subset of arborescent links [1]. This was not known before: rather, Thistlethwaite noted that a visible Conway sphere is always visible in an alternating diagram (i.e., if visible in one alternating diagram, then also visible in other alternating diagrams of the same link), and therefore a hidden one is always hidden [10]. We also prove that a pseudo-Montesinos link has no visible Conway spheres in any alternating diagram (Proposition 3.3 with corollaries). Our two results together yield that a decomposition into two prime alternating tangles can always be detected by looking at an alternating diagram (Theorem 3.1). In the final section, we also show that for a closed braid L , detecting prime alternating tangle decompositions takes an even easier form. Indeed, once the decomposing sphere is essential, it must be positioned in a special way with respect to D (Proposition 4.1).

Our proofs mainly use two techniques. The first one is inherited from Menasco’s work: we use standard position, which helps to translate topology of a surface into combinatorics of curves and diagrams. The second component is a careful topological analysis of certain isotopies and strong isotopies of surfaces embedded in 3-manifolds.

The visibility of a prime alternating tangle decomposition (and of the genus-2 surfaces that yield an essential 4-punctured sphere after meridional compressions) aligns with well-known results of Menasco, who noted that some of the basic topological properties of alternating link complements can be seen directly in reduced alternating link diagrams [6]. Among them is the property of a link being non-split (and respectively, the presence of an essential genus-0 surface in the link complement) and the property of being prime (and the presence of an essential genus-1 surface).

In Section 2, we recall results of Menasco and Thistlethwaite on surfaces in alternating link complements. In Section 3, we state and prove our main theorem. In Section 4, we closely examine tangle decompositions of alternating braids.

2. Conway spheres and standard position

In this preliminary section, we recall Menasco’s techniques and a key lemma, as well some of the results of Thistlethwaite on rational tangles.

Let D be a reduced alternating diagram of the link L . Let Q be the projection sphere, where D lies except for perturbations at crossings. We review the notion of a surface in standard position (see [6, Section 2]).

The link L lies on a union of two spheres in S^3 , S_+^2 , and S_-^2 , which agree with Q except in a bubble around each crossing. At the bubbles, S_+^2 and S_-^2 go over the top and bottom hemispheres, respectively. We will denote by B_+ and B_- the parts of S^3 lying above and below Q , respectively.

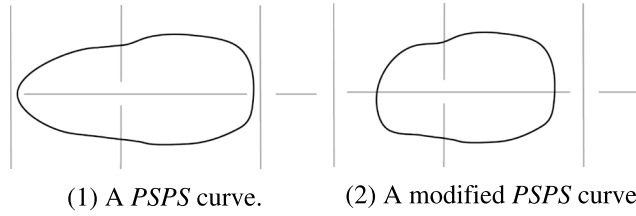


Figure 1. A curve of $F \cap S_{\pm}^2$ and its modification.

Let F be an essential surface properly embedded in the complement of L such that every boundary component of F is meridional. If F is closed, we meridionally compress it until no further meridional compression is possible. Then $F \cap S_{\pm}^2$ consists of simple closed curves bounding disks of $B_{\pm} \cap F$. Following Menasco's technique, we encode such a curve of intersection C by a word consisting of the letters P and S . The letter P means C intersects a strand of a link not at a crossing—i.e., F has a meridional boundary component there. The letter S means C intersects a crossing—i.e., F passes between two strands of a crossing and is shaped as a saddle there. Figure 1(1) depicts an example of a PPS curve, projected on Q from $F \cap S_{+}^2$ (a fragment of a link diagram is pictured in grey). There are multiple ways to put a surface F in standard position. If it is done so that the total number of S 's, P 's, and the curves of $F \cap S_{\pm}^2$ is minimized, then F is in *complexity minimizing standard position*.

The following observations follow from the techniques of Menasco [6]. For details, see [2].

A segment of D from a crossing to an adjacent crossing will be called *an edge*.

LEMMA 2.1

Suppose F is a closed essential 4-punctured sphere in the complement $S^3 - L$ of a prime alternating non-split link L (i.e., F is a Conway sphere for L). Then F can be placed in standard position relative to $S_{+}^2 \cup S_{-}^2$ so that

- (1) F intersects S_{+}^2 in either a single PPPP curve or in two PPS curves.
- (2) No curve passes through a saddle and then crosses an edge of D adjacent to the saddle.
- (3) No curve crosses an edge of D twice consecutively.

REMARK 2.2

Menasco's techniques found application in the work of Thistlethwaite on rational tangles [10], in which he describes precisely what an alternating diagram of a rational tangle looks like, as follows:

Start with a diagram of a 2-string tangle that has no crossings. Then surround this diagram by annuli, each annulus containing four arcs of the link diagram joining distinct boundary components of the annulus, and connected to the arcs in the neighboring annuli or the described 2-string tangle. Each annulus contains a single crossing between

two of the arcs. See the comments after Corollary 3.2 in [10], as well as [4, Figure 2] for an illustration. In particular, one can determine whether an alternating tangle diagram represents a rational tangle just by looking at the diagram.

3. Decomposition into two prime alternating tangles

A *Montesinos link* is a link obtained by taking a cyclic sum of a finite number of rational 2-tangles (see top of Figure 2 for an example; inside each circle insert a diagram of a rational tangle). We will often refer to these four tangles as *sub-tangles* since together they may form larger tangles that we consider. Given a Montesinos link with four rational sub-tangles $T_i, i = 1, 2, 3, 4$, we construct a *pseudo-Montesinos link* as follows: delete four strands, one each connecting T_1 to T_2 , T_2 to T_3 , T_3 to T_4 , and T_4 to T_1 . Then, following the pattern shown in the bottom of Figure 2, replace these strands with four strands, two between T_1 and T_3 and two between T_2 and T_4 . If the resulting

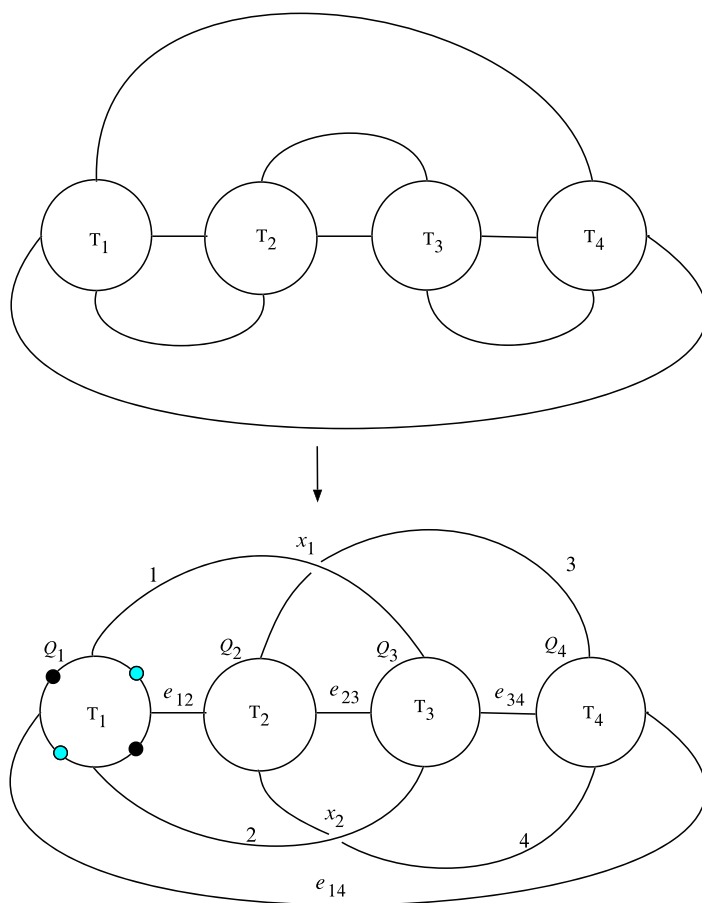


Figure 2. (Color online) Constructing a pseudo-Montesinos link.

diagram is reduced and alternating and each rational sub-tangle has at least one crossing, we say that it is a *standard diagram of an alternating pseudo-Montesinos link*. Figure 3(2) is an example of standard diagram of an alternating pseudo-Montesinos link.

In the terminology of Thistlethwaite [10], a *visible* 4-punctured sphere in an alternating diagram D is one that appears in the plane of the diagram (after isotopy to standard position) represented by a $PPPP$ curve. A *hidden* Conway sphere is one that is represented by two $PSPS$ curves. We extend this to call any 2-tangle T in L *visible* if a $PPPP$ curve c intersects four arcs of D such that all of T lies on one side of c , and the complement of T (denote it by T^c) lies entirely on the other side of c .

Our main theorem is the following.

THEOREM 3.1

Suppose L is a prime alternating non-split link, D is a reduced alternating diagram for L , and there is an essential Conway sphere embedded in $S^3 - L$. Then a prime tangle decomposition of L is visible in D if and only if D is not a standard diagram of an alternating pseudo-Montesinos link. Further, if D is a standard diagram of an alternating pseudo-Montesinos link, then no prime tangle decomposition for L is visible in any reduced alternating diagram for L .

The rest of the section is devoted to the proof of Theorem 3.1 via a sequence of propositions.

PROPOSITION 3.2

Suppose L is a prime alternating non-split link, D is a reduced alternating diagram for L , and there is an essential Conway sphere embedded in $S^3 - L$. Then either a prime tangle decomposition of L is visible in D , or D is a standard diagram of an alternating pseudo-Montesinos link.

Proof

Let Z be an embedded 4-punctured sphere splitting D into two prime 2-tangles R and Q . Choose one of the two spheres S_-^2 and S_+^2 , say S_+^2 , and consider its intersections with Z . By Lemma 1, we can place Z in standard position so that either Z intersects S_+^2 in a single $PPPP$ curve or in exactly two $PSPS$ curves.

If Z intersects S_+^2 in a single $PPPP$, then the tangle decomposition is visible in the diagram and we are done.

Assume Z intersects S_+^2 in two $PSPS$ curves. The two curves naturally divide the diagram into four 2-string sub-tangles (see Figure 3(1)) along simple closed curves that intersect the link in four points. One of these sub-tangles is labeled T in the figure, with the simple closed curve c (depicted by the dotted line) on its boundary. Note that all four sub-tangles are visible; by standard position and Lemma 2.1(2), note that each sub-tangle contains at least one crossing.

There are two possibilities:

1. Each of these sub-tangles is a rational tangle.

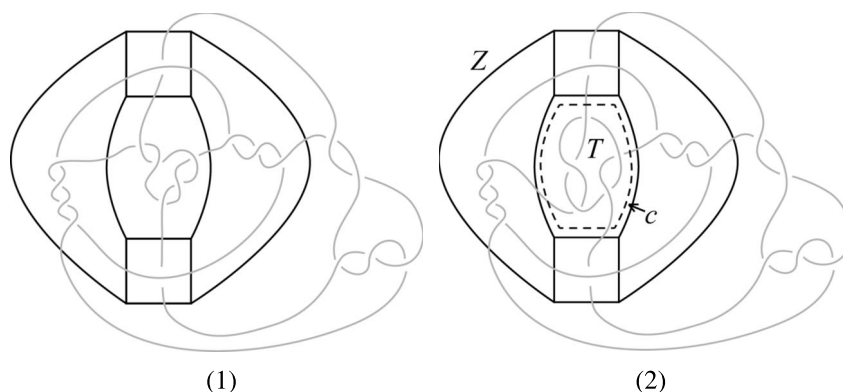


Figure 3. Two *PPSP* curves in a link diagram.

2. At least one of the sub-tangles is prime.

We examine each sub-tangle. By Remark 2.2, we can determine whether each of the sub-tangles is rational just by looking at the link diagram. If possibility 1 holds, then D is a standard diagram of an alternating pseudo-Montesinos link and we are done. Otherwise at least one of the sub-tangles is prime.

Assume at least one of the sub-tangles T is prime, as in Figure 3(1). We will show that the complementary tangle T^c is also prime. This proves that the curve c (which is a *PPPP* curve) that we see on the diagram D describes a decomposition of L into two prime tangles.

Capping the curve c with disks on both sides of the projection sphere forms a 4-punctured 2-sphere W . We call W the 2-sphere associated to c . W splits the knot into the two 2-tangles, T and T^c . Since W and Z can be assumed disjoint, one of the two tangles defined by Z therefore also lies completely inside T^c . Assume the tangle Q lies completely inside T^c .

We claim that T^c cannot be a rational tangle. Assume to the contrary that T^c is rational. Then the two arcs in T^c are parallel through disks E_1 and E_2 to the 4-punctured sphere W . We consider how these disks intersect the 4-punctured sphere Z . There are two cases: Z intersects both arcs of the tangle T^c (in two points each), or Z intersects one arc of T^c (in four points) and is disjoint from the other.

In either case, we use the fact that Z is incompressible in the complement of the knot to remove simple closed curves of intersection between $E_1 \cup E_2$ and Z . An outermost (in $E_1 \cup E_2$) arc of intersection with Z can then be doubled to yield a compressing disk for Z in the complement of the link. Since Z is incompressible in the link complement, this is a contradiction. Hence, T^c is not rational.

A similar argument shows T^c cannot contain any essential twice-punctured sphere, so T^c is a prime tangle, as required. $\square$

Note that every standard diagram of an alternating pseudo-Montesinos link gives rise to two *PPSP* curves, as in Figure 3(1). By [7, Theorem 2], the resulting sphere is

essential—i.e., every standard diagram of an alternating pseudo-Montesinos link has a hidden Conway sphere. We now consider the possibility of a visible Conway sphere.

Suppose D is a standard diagram of an alternating pseudo-Montesinos link L . Let c be a $PPPP$ curve in D , and let W be the 4-punctured 2-sphere associated to c . A *strong isotopy* of $W \text{ rel } D$ is an isotopy of W that induces a planar isotopy of c . We say that c is a *flying curve* for D if there exists another $PPPP$ curve b disjoint from c and the annulus between c and b contains a single crossing of the diagram. Then there is a flype of the diagram, which changes which strands have the single crossing between c and b , and which turns the tangle inside c upside down. We call this *flying the diagram along c* ; this preserves the alternating property (see [8, Figure 1] for more details; our “flying curve” is the boundary of the tangle S_A). Notice that if the disk bounded by c that is also contained in b contains zero or one crossing, flying along c leaves the diagram unchanged. The following results use the labeling from Figure 2; in particular, Q_i is the closed punctured curve bounding the tangle T_i in D .

PROPOSITION 3.3

Suppose D is a standard diagram of an alternating pseudo-Montesinos link L . Let c be a $PPPP$ curve in D , and let W be the 4-punctured 2-sphere associated to c . Then there is a strong isotopy of $W \text{ rel } D$ such that c is isotoped to a $PPPP$ curve c' which is either parallel to Q_i for some i or which lies completely inside T_i for some i , or c' bounds a disk which is disjoint from all the T_i 's and which contains zero or one crossing. In the last case, note that flying along c' does not change the diagram.

Assuming this proposition, we can prove the following.

COROLLARY 3.4

Suppose D is a standard diagram of an alternating pseudo-Montesinos link L . Let c be a $PPPP$ curve in D , and let W be the 4-punctured 2-sphere associated to c . Then W is not a Conway sphere for L . If c is a flying curve for D , flying along c yields another standard alternating pseudo-Montesinos diagram for L .

Proof

By Proposition 3.3, there is strong isotopy of $W \text{ rel } D$ such that c is isotoped to a $PPPP$ curve c' which is parallel to Q_i for some i , or which lies completely inside T_i for some i , or bounds a disk containing a single crossing completely exterior to $\cup T_i$. In all cases, c bounds a rational tangle. If c is a flying curve for D , flying along c preserves the rationality of T_i , hence preserving the standard alternating pseudo-Montesinos diagram structure of D . $\square$

COROLLARY 3.5

Suppose D is a standard diagram of an alternating pseudo-Montesinos link L . Then no reduced alternating diagram for L contains a $PPPP$ curve c corresponding to a Conway sphere.

Proof

Since any two reduced alternating diagrams are related by a sequence of flypes [8], Corollary 3.5 follows from Corollary 3.4. $\square$

Theorem 3.1 follows immediately from Propositions 3.2 and Corollary 3.5.

The remainder of this section is devoted to the proof of Proposition 3.3.

Proof

Minimize the number of points of intersection $c \cap \bigcup Q_i$ up to strong isotopy of $W \text{ rel } D$.

Case 1: $c \cap \bigcup Q_i = \emptyset$.

- Subcase a: $c \subset T_i$ for some i ; then we are done.
- Subcase b: $c \cap \bigcup T_i = \emptyset$.

Proof for Subcase b:

Up to relabeling, one of the following must hold:

- i. c does not separate the T_i 's.
- ii. c separates T_1 from T_2, T_3, T_4 .
- iii. c separates the T_i 's in pairs.

i. Then c either contains a single crossing or it bounds a 2-stranded tangle with no crossings. In both cases, flying over c leaves D unchanged.

ii. Then c is parallel to Q_1 and we are done.

iii. In both cases (c separates T_1 and T_2 from the rest, or T_1 and T_3 from the rest), examination of Figure 2 shows c must intersect D in at least six points, a contradiction.

Case 2: $c \cap \bigcup Q_i \neq \emptyset$. Assume c intersects Q_1 .

The points of intersection between c and Q_1 divide c into subarcs $\beta_1, \beta_2, \dots, \beta_{2m}$ with $\beta_{2j} \subset T_1$.

Since we have minimized the number of points of intersection in $c \cap \bigcup Q_i$ up to strong isotopy of W , $\beta_i \cap D \neq \emptyset$ for each i . Therefore, there are at most four such subarcs of c .

If $\beta_j \subset T_1$ intersects D in only a single point for some j , we can (strongly) isotop β_j out of T_1 , contradicting minimality. Therefore, there are at most two such subarcs of c , β_1 and β_2 , with $\beta_2 \subset T_1$, and each subarc intersects D in exactly two points.

The endpoints of β_2 must separate the intersection points of D with Q_1 into two pairs. We now apply the above arguments to β_1 , which shares these two endpoints with β_2 . It is useful to refer back to Figure 2, where the endpoints of β_2 (and β_1) are marked on Q_1 , as either the pair of black points or the pair of blue points (shown in color online). We note that β_1 cannot intersect any other T_i because any subarc of c contained in a T_i must intersect D in at least two points, and the sections of β_1 disjoint from all T_i 's would also have to intersect D at least once, a contradiction. Hence, β_1 must be a subarc disjoint from $T_2 \cup T_3 \cup T_4$, with both endpoints on Q_1 , intersecting L in exactly two points. By inspection, we see that β_1 can be strongly isotoped into T_1 , contradicting minimality. Hence, $c \cap \bigcup Q_i = \emptyset$. $\square$

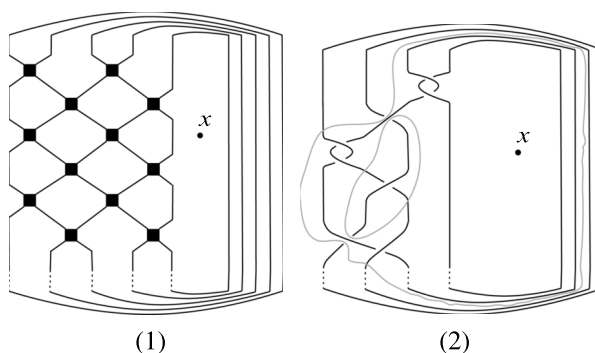


Figure 4. Diagrams of alternating braids.

4. Tangle decompositions of alternating braids

Let L be an alternating, prime, non-split closed n -braid with a reduced alternating braid diagram D , $n \geq 3$. For a diagram D , see Figure 4(1), where every square represents either a twist or a 2-tangle with no crossings. By a *twist*, we mean either a single crossing, or a connected sequence of bigon regions of D that is not a part of another such sequence. The point x in the figure represents the braid axis.

As before, let D lie on the projection plane Q . A curve C on Q will be called *special* with respect to D if the following holds:

- (1) C intersects D transversally in exactly four points. Denote them x_1, x_2, x_3, x_4 .
- (2) C intersects every edge of D at most once.
- (3) C is *monotone*—i.e., it can be isotoped (where the intersection points of C with the link may slide along a link strand until they reach a crossing, but not further) so that a ray from x always intersects C in a single point. Figure 5(1) shows an example of a monotone curve, and Figure 5(2) shows an example of a curve that is not monotone. If there is a choice whether to put x inside or outside of C , while x stays in the same region of D , we always assume x is outside of C . This is illustrated in Figure 5(3), where the curve is not monotone.

Note that the last condition above also implies that, up to isotopy, C winds exactly once around x .

Recall that by Lemma 2.1(1), a Conway sphere F in standard position results in either one *PPPP* or two *PSPS* curves in $F \cap S_+^2$ (similarly, in $F \cap S_-^2$). Modify a *PSPS* curve as follows. For every saddle it passes, push the curve from the saddle at a crossing into one of the edges adjacent to this crossing, as in Figure 1(2). Call the new curve a *modified PSPS curve*. This modification does not correspond to an isotopy of F .

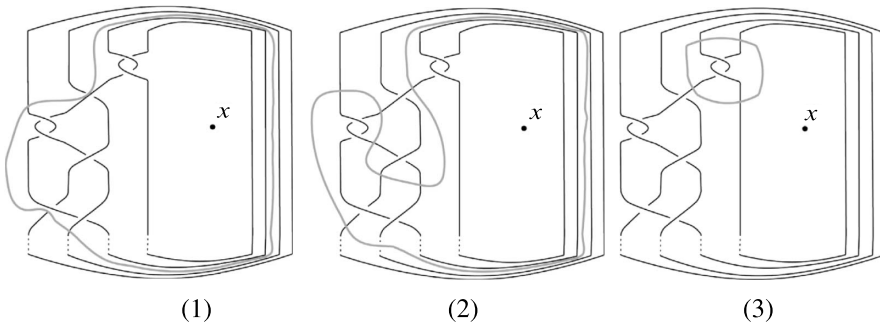


Figure 5. In gray, a curve that is monotone (left) and two curves that are not (center, right).

THEOREM 4.1

Suppose D is a reduced alternating diagram of a prime alternating non-split n -braid L . Assume $S^3 - L$ contains a Conway sphere F . Then either

- D admits a special curve that is a *PPPP* or modified *PSPS* curve for F , or
- D is a diagram of the 3-braid $\sigma_1 \sigma_2^{-n_1} \sigma_1^{n_2} \sigma_2^{-1} \sigma_1^{n_3} \sigma_2^{-n_4}$, where n_1, n_2, n_3, n_4 are positive integers.

Proof

Note that for $n = 1$, we obtain an unknot that contains no essential surfaces. For $n = 2$, we obtain a $(2, n)$ -torus link, where an existence of a *PSPS* curve contradicts Lemma 2.1(2), and a *PPPP* curve bounds a rational tangle. Hence, there are no Conway spheres, and the theorem holds vacuously. We therefore need to prove the theorem for $n \geq 3$.

We claim that for F , any *PPPP* or modified *PSPS* curve C_1 coming from F either is special or bounds a 2-tangle whose diagram contains exactly one twist.

The curve C_1 cuts D into two 2-tangles and travels around x not more than once since it does not have self-intersections. In addition, C_1 cannot intersect an edge of D more than once by Lemma 2.1(2, 3).

If a *PSPS* curve enters a bigon through a saddle at a crossing (with S), it must exit the bigon through an edge (with P). But this contradicts Lemma 2.1(2). Hence, if C_1 is a modified *PSPS* curve, it cannot intersect a bigon. If C_1 is an actual *PPPP* curve, it can be isotoped so that it does not cross any bigons.

Therefore, we can depict C_1 on the diagram from Figure 4(1) so that if it passes through a black square, then the square represents a tangle with no crossings, and C_1 does not intersect L in that square. The intersections of C_1 with L can be grouped in consecutive pairs (a pair of intersections corresponds to C_1 entering/exiting a region of D), each pair arranged on a vertical, horizontal, or diagonal line through a region of D . There are four intersections of C_1 with D , which can be denoted by x_1, x_2, x_3, x_4 . C_1 is a closed curve, and once a region where it starts is chosen, the fourth intersection, x_4 , must allow the curve to return to the same region.

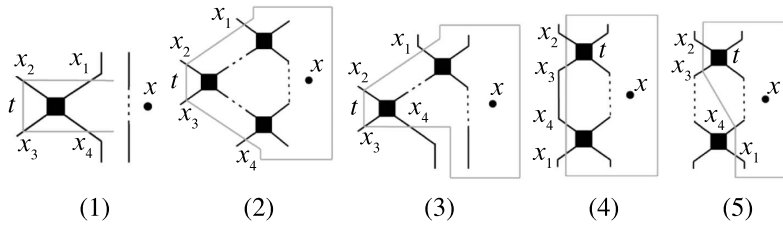


Figure 6. Patterns of intersection.

Figure 6 shows all possible patterns once the starting region is chosen, and D is placed on S^2 . In the figure, C_1 is depicted in gray, and dotted lines mean that more twisting may occur there. Between two consecutive points of intersection, x_i and x_{i+1} (up to cyclic order), the curve may pass through some black squares that have no crossings in them, without intersecting the link. To see that these are all possible patterns, note that C_1 must bound at least one twist but is allowed only four intersections with L . Call the twist t : it is represented by a black square in the figure. Hence, a part of C_1 is necessarily a vertical segment between intersections with two strands of L coming out of t . Suppose the intersections are on the left of t , and denote them by x_2, x_3 (for another side, the argument is similar). Then C_1 must close up in some way on the right from t . Since there are just four intersections of L with C_1 , and t already has four strands, C_1 cannot proceed to the left before x_2 and after x_3 . Figure 6 then demonstrates all ways for C_1 to close up, up to a symmetry/reflection, where the fact that other segments of C_1 must be horizontal, vertical, or diagonal, is used.

Situation (1) includes the two possible ways of closing up C_1 : either making a full circle around x , or through the shortest segment on the picture that connects its free ends. Figure 4(2) shows an actual example of a braid and two *PSPS* curves that yield modified curves of types (1) and (3) from Figure 6.

In each of the depicted situations, either C_1 bounds just one twist of D , or it is monotone and therefore special. This concludes the proof of our claim.

Now consider the *PPPP* or *PSPS* curve C'_1 represented by C_1 . Assume C_1 is not special. Then by the claim, it bounds just one twist. If a *PPPP* curve bounds just one twist, F is compressible. Hence, only a *PSPS* curve can bound a twist. The second *PSPS* curve coming from F , denoted by C'_2 , hits saddles at the same two crossings as C'_1 . By the claim above, the modified *PSPS* curve C_2 resulting from C'_2 either bounds a twist or is special itself.

If C_2 is special, we are done. Otherwise, each of C_1 and C_2 bounds a twist. Then there are six twists in the diagram: two enclosed by C'_1 and C'_2 , two enclosed by the two modified *PSPS* curves in S^2_- , and two one-crossing twists outside of all four of these modified *PSPS* curves. This is depicted in Figure 7, where C_1, C_2 are in gray. Hence, L is a 3-braid. Each of the enclosed twists has just one crossing because we isotoped F so that a *PSPS* curve does not go through a crossing of a bigon of D . Therefore, L is the braid $\sigma_1 \sigma_2^{-n_1} \sigma_1^{n_2} \sigma_2^{-1} \sigma_1^{n_3} \sigma_2^{-n_4}$. $\square$

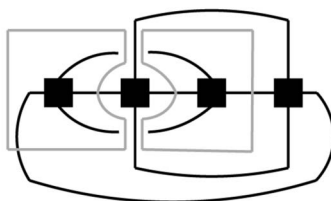


Figure 7. Two modified *PSPS* curves are depicted in gray, and a link diagram is schematically depicted in black, with squares representing twists.

EXAMPLE 4.2

Suppose that every black square in Figure 4(1) represents a tangle with at least one crossing. One can immediately see that such a diagram contains no special curves, and is not a diagram of a 3-braid. Therefore, by Theorem 4.1, $S^3 - L$ contains no Conway sphere.

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