

Eccentric-orbit extreme-mass-ratio-inspiral radiation. II. 1PN correction to leading-logarithm and subleading-logarithm flux sequences and the entire perturbative 4PN flux

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In a recent paper we showed that for eccentric-orbit extreme-mass-ratio inspirals the analytic forms of the leading-logarithm energy and angular momentum post-Newtonian (PN) flux terms (radiated to infinity) can, to arbitrary PN order, be determined by sums over the Fourier spectrum of the Newtonian quadrupole moment. We further showed that an essential part of the eccentricity dependence of the related subleading-logarithm PN sequences, at lowest order in the symmetric mass ratio ν , stems as well from the Newtonian quadrupole moment. Once that part is factored out, the remaining eccentricity dependence is more easily determined by black hole perturbation theory. In this paper we show how the sequences that are the 1PN corrections to the entire leading-logarithm series, namely terms that appear at PN orders $x^{3k+1} \log^k(x)$ and $x^{3k+5/2} \log^k(x)$ (for PN compactness parameter x and integers $k \geq 0$), at lowest order in ν , are determined by the Fourier spectra of the Newtonian mass octupole, Newtonian current quadrupole, and 1PN part of the mass quadrupole moments. We also develop a conjectured (but plausible) form for 1PN correction to the leading logs at second order in ν . Further, in analogy to the first paper, we show that these same source multipole moments also yield nontrivial parts of the 1PN correction to the subleading-logarithm series, and that the remaining eccentricity dependence (at lowest order in ν) can then more easily be determined using black hole perturbation theory. We use this method to determine the entire analytic eccentricity dependence of the perturbative (i.e., lowest order in ν) 4PN nonlog terms, $\mathcal{R}_4(e_i)$ and $\mathcal{Z}_4(e_i)$, for energy and angular momentum respectively.

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I. INTRODUCTION

With development proceeding on the Laser Interferometer Space Antenna (LISA) gravitational wave mission [1,2], the need for accurate theoretical models of eccentric extreme-mass-ratio inspirals (EMRIs) has continued to grow [3–6]. In previous work [7–9] complementary approaches from post-Newtonian (PN) theory and black hole perturbation theory (BHPT) were combined to generate new information on the orbit-averaged energy and angular momentum fluxes radiated to infinity in (nonspinning) eccentric-orbit systems. In a recent one of these papers [8] (hereafter Paper I), we showed that the Fourier amplitudes of the Newtonian mass quadrupole moment, and the function $g(n, e_i)$ [10,11] proportional to their complex square, determine the functional dependence in the quasi-Keplerian (time) eccentricity e_i [12] of the entire leading-logarithm sequence (i.e., to arbitrary PN order) of these fluxes. The functional dependence in eccentricity of each such flux term, relative to the circular orbit limit, is commonly referred to as an enhancement function. It is apparent that the quadrupole moment, as embodied by $g(n, e_i)$, enters into each new type of radiative process in the PN expansion: it provides the leading (quadrupole) order

emission; it determines the leading tail emission at 1.5PN order [13]; and it provides the first appearance of a log term at 3PN order [14]. Paper I showed the continuation of this pattern for arbitrarily higher PN order terms in the fluxes. A similar conclusion holds for terms in the conservative sector, such as the orbital energy of the system [15].

We then went further in Paper I to show that additional sums over the quadrupole spectrum determine essential parts of the eccentricity dependence of the subleading-logarithm series, which are terms associated with leading logs at the same PN order but with one less power of $\log(x)$, where x is a PN compactness parameter. Specifically, we define $x = ((m_1 + m_2)\Omega_\varphi)^{2/3}$ [11], where m_1 and m_2 are the primary and secondary masses and Ω_φ is the mean frequency of azimuthal motion. A subleading-logarithm term can be thought of alternatively as the 3PN correction to the leading-logarithm term of the same power of $\log(x)$ (or henceforth referred to as the corresponding term in the 3PN log series).

At lowest order in ν , these quadrupole-dependent parts can be reexpressed in terms of the Darwin [16,17] definition of eccentricity e . Each entire subleading-log term is

then taken to have an assumed form for its expansion in powers of e^2 , with the quadrupole-dependent part being built in. This quadrupole portion subsumes all of the transcendental number coefficients. The remaining unknown structure in each flux term is found to be either a closed form expression (at integer PN orders) or an infinite series (at half-integer PN orders) with rational number coefficients that can then in principle be determined more easily by BHPT calculations.

Explicitly, as mentioned in Paper I, the established methods of computing PN expansions within BHPT, whether numeric analytic [9] or all analytic [18], yield eccentricity functions as simple Taylor series to finite orders in e . In the former case involving numeric fits to highly accurate flux data then converted to analytic form, the integer-relation algorithm is much more likely to succeed if the dimensionality of the search vector is held to a minimum. The reduction of dimensionality is made possible by understanding and removing the quadrupole-dependent, transcendental-number-dependent part of the subleading log terms. Additionally, the BHPT approach does not easily present separable pieces involving functions of e like $\log(1 - e^2)$. Instead, the knowledge from PN theory allows us to resum the finite eccentricity expansions into those compact forms. Most importantly, for integer-order subleading logs, the separable quadrupole portion contains the only part of the analytic eccentricity dependence that is not closed in form. It is multiple orders of magnitude faster to obtain the expansion of this piece through the Fourier methods of Paper I than through BHPT alone. This process is described in detail in Paper I, and a variant is used in this paper to find new sequences in the PN structure of the fluxes. In either event, once the terms are determined as functions of e , then (at least at lowest order in ν) the fluxes can be transformed back to e_t . We showed this procedure in action by extracting the entire analytic dependence of the 6PN log energy and angular momentum terms, $\mathcal{R}_{6L}(e_t)$ and $\mathcal{Z}_{6L}(e_t)$, to arbitrary powers of e_t^2 .

Thus, one conclusion of Paper I is that two *diagonal strips* in the high-order PN structure of the fluxes [i.e., the leading logarithms at PN orders $x^{3k} \log^k(x)$ and $x^{3k+3/2} \log^k(x)$ for integers $k \geq 0$] are determined by the Fourier spectrum of the Newtonian quadrupole moment. See Fig. 1 for a graphical depiction of these leading log strips in the PN structure. The second main conclusion is that two additional diagonal strips, the subleading logs at PN orders $x^{3k} \log^{k-1}(x)$ and $x^{3k+3/2} \log^{k-1}(x)$ for integers $k \geq 1$, are also partly determined at lowest order in ν by the quadrupole spectrum, with the remaining eccentricity dependence having a closed form (integer order) or infinite series (half-integer order) and being more easily determined by BHPT. The leading log and 3PN log sequences are represented in the figure as (solid and dashed) red and green lines, respectively. The question then arises is it possible to

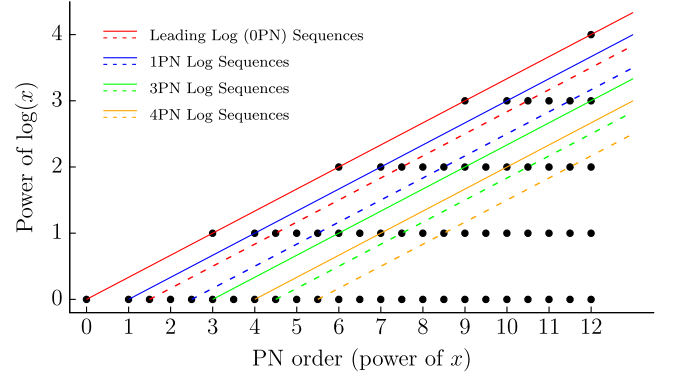


FIG. 1. Schematic depiction of the presence of terms (as black filled circles) in the high PN order relative fluxes for successively higher powers of compactness x (horizontal axis) and higher powers of $\log(x)$ (vertical axis). The Peters-Mathews [10] flux is symbolized by the left-most point at the origin of the plot. This representation of the PN structure of the fluxes allows a graphical explanation of the various “log” sequences that are the focus of this paper and Paper I. The red lines show the leading-log sequences, both integer order (solid) and half-integer order (dashed) detailed previously in Paper I. The 3PN log sequences (previously called subleading logs), also the subject of Paper I, are shown as green lines, both integer order and half-integer order. The blue lines represent the 1PN log sequences and the orange lines denote the 4PN log sequences, all of which are the focus of this paper.

determine additional entire diagonal strips in the PN structure of the fluxes with only limited additional knowledge of low-order source multipole moments? As we show in this paper, the answer is yes if we focus on the 1PN corrections to the leading- and subleading-logarithm sequences.

The first term in the leading-log series is the Newtonian quadrupole flux, i.e., the Peters-Mathews [10,19] term $\mathcal{R}_0(e_t)$ itself. The enhancement function in this case arises from simply summing the Newtonian quadrupole moment spectrum $g(n, e_t)$ over all harmonics n in the eccentric motion. The next order term $\mathcal{R}_1(e_t)$ is the 1PN correction to the gravitational wave flux, which has been known since Wagoner and Will [20] (see also [11,21]). In this case determining the enhancement function requires the Fourier spectra of the Newtonian mass octupole, the Newtonian current quadrupole, and the 1PN-corrected mass quadrupole moments (hereafter called the 1PN multipoles). The $\mathcal{R}_1(e_t)$ flux is the first term in one of the two new diagonal sequences of 1PN-corrected leading logarithms, which we will refer to as a *1PN log series* (Fig. 1, solid blue line). This sequence has PN orders $x^{3k+1} \log^k(x)$ for $k \geq 0$. The other (half-integer) 1PN log series (dashed blue line in the figure) begins with the 2.5PN tail at $x^{5/2}$ and has PN orders $x^{3k+5/2} \log^k(x)$ for $k \geq 0$. A principal result of this paper is to show that it is merely the spectra of the three 1PN multipoles that are required to determine these two 1PN log

series in their entirety to arbitrary PN order (at lowest order in ν).

Calculation of the Newtonian mass octupole and current quadrupole for eccentric bound motion is fairly straightforward and can be found in the original, earlier papers [12,20], the review by Blanchet [11], or extrapolated from techniques reviewed in Paper I. Calculation of the 1PN correction to the mass quadrupole is more involved. At 1PN order, the determination of the mass quadrupole must account for relativistic orbital precession, which means that the spectrum cannot be represented as a single Fourier series but instead requires a double Fourier sum over harmonics of the two different frequencies, Ω_r (radial libration) and Ω_φ [12,14,22–24]. Once these spectra are computed for given orbital parameters, their sums weighted by powers of n over all harmonics combine to give terms in the 1PN log series. One key difference though between the 1PN log series and the leading logs themselves is that the former now have contributions beyond lowest order in the mass ratio ν . Because the multipole moment analysis in this paper makes no *a priori* assumptions on the mass ratio, we are able to extract the likely forms for these $\mathcal{O}(\nu)$ corrections, though without (presently) second-order BHPT to assist in verification. At lowest order in ν , the analysis found in this paper provided a theoretical underpinning for several previously known closed-form flux terms [7–9].

With the 1PN log series thus understood, we then find that the same set of 1PN multipoles again appear in the 1PN correction to the subleading logarithms. These sequences will be referred to as *4PN log series*, since for a given power of $\log(x)$ each term in this series occurs at order x^4 relative to the corresponding leading log term. In other words, the 4PN log sequences are two diagonal strips in the high PN order flux structure that appear at orders $x^{3k+1} \log^{k-1}(x)$ and $x^{3k+5/2} \log^{k-1}(x)$ for $k \geq 1$ (solid and dashed orange lines in the figure). The first sequence begins with the 4PN nonlog flux and the second with the 5.5PN nonlog term. In direct analogy to our findings in Paper I, the set of 1PN multipoles provides essential separable portions of the terms in the 4PN log series, which include all transcendental coefficients, leaving only rational series which at lowest order in ν can then be calculated (more) easily with BHPT. For the first (integer-order) sequence (solid orange line), the remaining parts can be factored into closed forms with rational coefficients, and it is possible to determine their entire analytic eccentricity dependence in this manner. For the second (half-integer-order) sequence (dashed orange line), the remaining eccentricity dependence is an infinite power series with rational coefficients, and BHPT can be used to determine coefficients to some depth in the e^2 expansion. We illustrate this procedure in detail by obtaining the 4PN nonlog energy and angular momentum fluxes at lowest order in the mass ratio.

The layout of this paper is as follows. In Sec. II we review the PN expansion for radiated energy and angular momentum, with an illustration of the terms that will be computed in this analysis. There we also derive the Fourier expansion for each of the 1PN multipole moments, and in Sec. III we detail their previously known contributions to the energy and angular momentum flux expansions. Section IV shows how these source multipole spectra contribute to the 1PN log series, with explicit general formulas which generate all members of those sequences. We proceed in Sec. V to derive the 4PN tail flux using these same Fourier spectra in order to check various results and to aid our extraction of the full 4PN log series fluxes at lowest order in ν . Then, in Sec. VI we illustrate how the various 1PN Fourier summations manifest specifically in the 4PN flux (and more generally in higher-order terms in the 4PN log series) and combine these observations with BHPT flux calculations from [9] to compute $\mathcal{R}_4(e_t)$ and $\mathcal{Z}_4(e_t)$ in compact form. This result is quite timely, as it will provide a valuable check for the PN community as they close in on a full description of the orbital mechanics and radiative losses at 4PN. That section also gives our analysis of the 5.5PN nonlog energy flux, showing the procedure carries over to half-integer order 4PN log terms. We conclude in Sec. VII with discussion of potentially extending this process to the 2PN-corrected logarithm series (i.e., the 2PN log sequence and 5PN log sequence).

In this paper we use units such that $c = G = 1$. As in Paper I, for any pair of functions with names distinguished by a tilde [e.g., $g(n, e_t)$ and $\tilde{g}(n, e_t)$], the “plain” quantity will relate to the energy flux while the “tilde” version will correspond to its angular momentum counterpart (see e.g., [25] also).

II. ECCENTRIC-ORBIT PN FLUX EXPANSION AND FOURIER DECOMPOSITION OF 1PN MULTIPOLES

In this section we lay out the parts of the PN expansion of the orbit-averaged fluxes that are of interest in this paper and review the calculation of the Fourier spectra of the 1PN multipoles. The focus is on eccentric EMRIs with the binary consisting of two nonspinning bodies of mass m_1 (primary) and mass m_2 (secondary). We are primarily concerned with $m_2 \ll m_1$ but keep the symmetric mass ratio $\nu = m_1 m_2 / (m_1 + m_2)^2$ as a variable.

A. PN flux expansions

In the modified harmonic gauge [11,14,21,25], the flux expansions are parametrized by the aforementioned ν and compactness parameter x , as well as the quasi-Keplerian time eccentricity e_t (also reviewed below). The expansion of the energy flux at infinity has the following form [7–9,11,26–28]:

$$\begin{aligned} \left\langle \frac{dE}{dt} \right\rangle = & \frac{32}{5} \nu^2 x^5 \left[\mathcal{R}_0 + x \mathcal{R}_1 + x^{3/2} \mathcal{R}_{3/2} + x^2 \mathcal{R}_2 + x^{5/2} \mathcal{R}_{5/2} + x^3 (\mathcal{R}_3 + \log(x) \mathcal{R}_{3L}) + x^{7/2} \mathcal{R}_{7/2} \right. \\ & + x^4 (\mathcal{R}_4 + \log(x) \mathcal{R}_{4L}) + x^{9/2} (\mathcal{R}_{9/2} + \log(x) \mathcal{R}_{9/2L}) + x^5 (\mathcal{R}_5 + \log(x) \mathcal{R}_{5L}) + x^{11/2} (\mathcal{R}_{11/2} + \log(x) \mathcal{R}_{11/2L}) \\ & \left. + x^6 (\mathcal{R}_6 + \log(x) \mathcal{R}_{6L} + \log^2(x) \mathcal{R}_{6L^2}) + x^{13/2} (\mathcal{R}_{13/2} + \log(x) \mathcal{R}_{13/2L}) + \dots \right]. \end{aligned} \quad (2.1)$$

In this expression the Newtonian circular-orbit energy flux has been factored out. Each quantity $\mathcal{R}_i = \mathcal{R}_i(e_t, \nu)$ is a function of eccentricity and mass ratio that helps determine the flux radiated at PN order i . The scripts denoting PN order track both the power of x and the presence of powers of $\log(x)$. The dependence of each term on e_t and ν differs notationally from Paper I, where the flux terms were considered only at lowest order in ν and thus taken to be functions of e_t alone. In this paper, while we retain both parameters, we will be interested occasionally in just the lowest order in ν limit. In those circumstances we revert to writing explicitly $\mathcal{R}_i(e_t)$ or $\mathcal{R}_i(e_t, 0)$. With x as the compactness parameter, each flux function is known to diverge as $e_t \rightarrow 1$ (see however [9] for an alternative).

In both Paper I and this paper we are concerned with diagonal strips in the high order PN structure where for each unit increase in power of $\log(x)$ there is an increase of three powers of x . As mentioned, the first example of such strips were the two leading-logarithm series, with (integer) orders $x^{3k} \log^k(x)$ and (half-integer) orders $x^{3k+3/2} \log^k(x)$ (for $k \geq 0$), which were given by Eq. (2.2) in Paper I. That work also dealt with what were there called the subleading-logarithm sequences, which here we refer to as the 3PN log sequences, with (integer) orders $x^{3k} \log^{k-1}(x)$ and (half-integer) orders $x^{3k+3/2} \log^{k-1}(x)$ (for $k \geq 1$).

In this paper our attention is initially on a pair of diagonal sequences that can be considered the 1PN correction to the two leading-log series and which form the following subset of the flux terms in (2.1):

$$\begin{aligned} \left\langle \frac{dE}{dt} \right\rangle^{\text{1L}} = & \frac{32}{5} \nu^2 x^5 [x \mathcal{R}_1 + x^{5/2} \mathcal{R}_{5/2} + x^4 \log(x) \mathcal{R}_{4L} + x^{11/2} \log(x) \mathcal{R}_{11/2L} \\ & + x^7 \log^2(x) \mathcal{R}_{7L^2} + x^{17/2} \log^2(x) \mathcal{R}_{17/2L^2} + x^{10} \log^3(x) \mathcal{R}_{10L^3} + \dots]. \end{aligned} \quad (2.2)$$

These 1PN log series, with integer PN order $x^{3k+1} \log^k(x)$ and half-integer PN order $x^{3k+5/2} \log^k(x)$ (for $k \geq 0$), are evident. Later in this paper we focus on yet another pair of diagonal sequences, the 4PN log series, which make up another subset of the flux terms in (2.1)

$$\begin{aligned} \left\langle \frac{dE}{dt} \right\rangle^{4L} = & \frac{32}{5} \nu^2 x^5 [x^4 \mathcal{R}_4 + x^{11/2} \mathcal{R}_{11/2} + x^7 \log(x) \mathcal{R}_{7L} + x^{17/2} \log(x) \mathcal{R}_{17/2L} \\ & + x^{10} \log^2(x) \mathcal{R}_{10L^2} + x^{23/2} \log^2(x) \mathcal{R}_{23/2L^2} + x^{13} \log^3(x) \mathcal{R}_{13L^3} + \dots]. \end{aligned} \quad (2.3)$$

The average loss of angular momentum is an expansion similar to (2.1) but with a different Newtonian circular-orbit factor and with new flux (enhancement) functions that are referred to by $\mathcal{Z}_i(e_t, \nu)$ instead of $\mathcal{R}_i(e_t, \nu)$. The analogous 1PN and 4PN log series in angular momentum are

$$\begin{aligned} \left\langle \frac{dL}{dt} \right\rangle^{\text{1L}} = & \frac{32}{5} \nu^2 (m_1 + m_2) x^{7/2} [x \mathcal{Z}_1 + x^{5/2} \mathcal{Z}_{5/2} + x^4 \log(x) \mathcal{Z}_{4L} + x^{11/2} \log(x) \mathcal{Z}_{11/2L} \\ & + x^7 \log^2(x) \mathcal{Z}_{7L^2} + x^{17/2} \log^2(x) \mathcal{Z}_{17/2L^2} + x^{10} \log^3(x) \mathcal{Z}_{10L^3} + \dots], \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} \left\langle \frac{dL}{dt} \right\rangle^{4L} = & \frac{32}{5} \nu^2 (m_1 + m_2) x^{7/2} [x^4 \mathcal{Z}_4 + x^{11/2} \mathcal{Z}_{11/2} + x^7 \log(x) \mathcal{Z}_{7L} + x^{17/2} \log(x) \mathcal{Z}_{17/2L} \\ & + x^{10} \log^2(x) \mathcal{Z}_{10L^2} + x^{23/2} \log^2(x) \mathcal{Z}_{23/2L^2} + x^{13} \log^3(x) \mathcal{Z}_{13L^3} + \dots]. \end{aligned} \quad (2.5)$$

B. The 1PN equations of motion

The 1PN source multipoles include the 1PN correction to the mass quadrupole moment. Its computation requires the 1PN correction to the equations of motion, i.e., treatment of the two-body motion as a precessing ellipse. The other two 1PN multipoles, the mass octupole and current quadrupole, need only be computed to Newtonian order.

We take the total mass to be $M = m_1 + m_2$ and assume the motion occurs in the x, y plane. Coordinates $r = r(t)$ and $\varphi = \varphi(t)$ represent the separation distance and the azimuthal angle, respectively. We introduce then the well-known quasi-Keplerian parametrization of the motion [12,29–31] involving three anomalies, $u(t)$, $l(t)$, $V(u)$, three eccentricities, e_t , e_φ , e_r , the two frequencies, Ω_r , Ω_φ , and the semimajor axis, a . In this description, $u(t)$ is the eccentric anomaly, $l(t)$ is the mean anomaly, $V(u)$ is the true anomaly, e_φ is the azimuthal eccentricity, e_r is the radial eccentricity, and e_t is the aforementioned time eccentricity. At the 1PN level these quantities can be related by the following equations:

$$\begin{aligned} r &= a(1 - e_r \cos u), & \varphi &= \left(\frac{\Omega_\varphi}{\Omega_r}\right)V(u), \\ l &= \Omega_r(t - t_P) = \frac{2\pi}{T_r}(t - t_P) = u - e_t \sin u, \\ \frac{du}{dt} &= \frac{\Omega_r}{1 - e_t \cos u}, & \beta_\varphi &= \frac{1 - (1 - e_\varphi^2)^{1/2}}{e_\varphi}, \\ V(u) &= u + 2 \arctan\left(\frac{\beta_\varphi \sin u}{1 - \beta_\varphi \cos u}\right), \end{aligned} \quad (2.6)$$

where t_P is the time of last periastron crossing and $V(u)$ is written in a form that preserves continuity across $u = 2\pi$. A more detailed description of these equations is given in [11,12,31].

Our goal is to obtain all quantities in terms of u , e_t , and $x = (M\Omega_\varphi)^{2/3}$ prior to transformation to the frequency domain. As part of this process, e_r and e_φ must be expressed in terms of e_t to 1PN order. We find

$$e_r = e_t \left[1 + \left(4 - \frac{3}{2}\nu\right)x + \dots \right], \quad (2.7)$$

$$e_\varphi = e_t [1 + (4 - \nu)x + \dots]. \quad (2.8)$$

The semimajor axis can be expressed simply in terms of the (dimensionless) energy ε [11] and ε can itself be PN expanded. Through 1PN order these are found to be

$$a = \frac{M}{\varepsilon} \left(1 + \frac{\varepsilon}{4}(-7 + \nu) \right), \quad (2.9)$$

$$\varepsilon = x + \left(\frac{3 + 5e_t^2}{4(-1 + e_t^2)} - \frac{\nu}{12} \right) x^2, \quad (2.10)$$

from which we obtain the 1PN expansion of a ,

$$a = \frac{M}{x} \left(1 - \frac{(1 - 3e_t^2)}{1 - e_t^2} x + \frac{\nu}{3} x \right). \quad (2.11)$$

Similarly, the radial frequency Ω_r can be PN expanded in straightforward fashion, simultaneously providing the expansion for the frequency ratio $K = \Omega_\varphi/\Omega_r$. We obtain

$$\Omega_r = \frac{x^{3/2}}{M} \left(1 - \frac{3x}{1 - e_t^2} \right), \quad K = 1 + \frac{3x}{1 - e_t^2}. \quad (2.12)$$

The motion in the coordinate φ combines a mean advance at the rate Ω_φ and a periodic motion at the frequency Ω_r . In the Fourier expansion of gravitational wave source terms this produces a bi-periodic expansion. Defining the 1PN difference in the mean angular advance as $kl = (K - 1)l$ and starting with $\varphi = KV$, we separate the advance of $\varphi(t)$ into parts as follows:

$$\varphi(t) = kl + l + K(V(u) - l). \quad (2.13)$$

With this done, all of the previous relations can be combined to give the coordinate positions and velocities in terms of x , u , and e_t to the desired order. Because of the particular manifestation of velocity in the 1PN mass quadrupole, only the lowest order in x is required for the coordinate velocity components. We obtain

$$\begin{aligned} \frac{r}{M} &= \frac{1 - e_t \cos u}{x} - \frac{1 - 3e_t^2 + 3e_t \cos u - e_t^3 \cos u}{1 - e_t^2} \\ &\quad + \frac{1}{6}(2 + 7e_t \cos u)\nu, \end{aligned} \quad (2.14)$$

$$\begin{aligned} \varphi &= kl + u + 2 \arctan\left(\frac{e_t \sin u}{1 + \sqrt{1 - e_t^2} - e_t \cos u}\right) \\ &\quad + \left[\frac{4e_t \sin u}{\sqrt{1 - e_t^2}(1 - e_t \cos u)} + \frac{3e_t \sin u}{1 - e_t^2} \right. \\ &\quad \left. + \left(\frac{6}{1 - e_t^2}\right) \arctan\left(\frac{e_t \sin u}{1 + \sqrt{1 - e_t^2} - e_t \cos u}\right) \right] x \\ &\quad - \frac{e_t \sin u}{\sqrt{1 - e_t^2}(1 - e_t \cos u)} x\nu, \end{aligned} \quad (2.15)$$

$$\begin{aligned} \frac{dr}{dt} &= \frac{e_t \sin u}{1 - e_t \cos u} x^{1/2}, & M \frac{d\varphi}{dt} &= \frac{\sqrt{1 - e_t^2}}{(1 - e_t \cos u)^2} x^{3/2}, \\ v^2 &= \left(\frac{1 + e_t \cos u}{1 - e_t \cos u} \right) x. \end{aligned} \quad (2.16)$$

C. Review of calculating the Newtonian mass octupole and current quadrupole moments

1. Fourier decomposition

We review here the calculation of the Fourier series of the mass octupole and current quadrupole. For more details see [14,21,25] and the review [11]. The calculation is also a straightforward extension of our review of the mass quadrupole Fourier calculation presented in Paper I.

The symmetric tracefree (STF) Newtonian mass octupole tensor is defined as

$$I_{ijk} = Q_{ijk} - \delta_{ij}Q_{aak}/5 - \delta_{jk}Q_{ibb}/5 - \delta_{ik}Q_{cjc}/5, \\ Q_{ijk} = \sum_{\alpha} m_{\alpha} x_i^{\alpha} x_j^{\alpha} x_k^{\alpha}. \quad (2.17)$$

The nonzero components are given by

$$I_{xxx} = \frac{\mu}{20} \sqrt{1-4\nu} r^3 (3 \cos \varphi + 5 \cos 3\varphi), \\ I_{xxy} = I_{xyx} = I_{yxx} = \frac{\mu}{20} \sqrt{1-4\nu} r^3 (\sin \varphi + 5 \sin 3\varphi), \\ I_{xyy} = I_{yyx} = I_{yxy} = \frac{\mu}{20} \sqrt{1-4\nu} r^3 (\cos \varphi - 5 \cos 3\varphi), \\ I_{yyy} = \frac{\mu}{20} \sqrt{1-4\nu} r^3 (3 \sin \varphi - 5 \sin 3\varphi), \\ I_{xzz} = I_{zxx} = I_{zzx} = -\frac{\mu}{5} \sqrt{1-4\nu} r^3 \cos \varphi, \\ I_{yzz} = I_{zyz} = I_{zzx} = -\frac{\mu}{5} \sqrt{1-4\nu} r^3 \sin \varphi, \quad (2.18)$$

where μ is the reduced mass.

Similarly, the STF form of the current quadrupole is given by

$$J_{ij} = \frac{1}{2} \sum_{\alpha} m_{\alpha} [x_i(\vec{x} \times \vec{v})_j + x_j(\vec{x} \times \vec{v})_i], \\ J_{xz} = J_{zx} = \frac{1}{2} \mu \sqrt{1-4\nu} \cos(\varphi) r^3 \frac{d\varphi}{dt}, \\ J_{yz} = J_{zy} = \frac{1}{2} \mu \sqrt{1-4\nu} \sin(\varphi) r^3 \frac{d\varphi}{dt}. \quad (2.19)$$

The transformation is now made from (r, φ) to variables (x, ν, e_t, u) using the relations of the previous section. Without listing every component, we find for example

$$I_{xxx} = \frac{M^3 \mu \sqrt{1-4\nu}}{x^3} \left[(e_t - \cos u)^3 - \frac{3}{5} (e_t - \cos u)(1 - e_t \cos u)^2 \right], \quad (2.20)$$

$$J_{xz} = -\frac{M^2 \mu \sqrt{1-4\nu}}{x^{3/2}} \left(\frac{\sqrt{1-e_t^2}}{2} \right) (e_t - \cos u), \quad (2.21)$$

with obvious extension to the other tensor components. There is no difference between the time eccentricity and the Keplerian eccentricity at Newtonian order, but we use the notation e_t uniformly to prepare for more general expansions. This also allows us to reserve the symbol e for the relativistic Darwin eccentricity.

In each multipole component, the scale and dimension are carried by the initial prefactor. Since we are concerned with the dimensionless eccentricity enhancement functions that will appear in the fluxes, we remove these factors now and define

$$\hat{I}_{ijk} = \frac{x^3}{M^3 \mu \sqrt{1-4\nu}} I_{ijk}, \quad \hat{J}_{ij} = \frac{x^{3/2}}{M^2 \mu \sqrt{1-4\nu}} J_{ij}. \quad (2.22)$$

It is then these scaled multipole moment tensors that we represent with Fourier series

$$\hat{I}_{ijk} = \sum_{n=-\infty}^{\infty} \hat{I}_{ijk(n)} e^{inl}, \quad \hat{J}_{ij} = \frac{1}{2\pi} \int_0^{2\pi} \hat{J}_{ij(n)} e^{-inl} dl, \quad (2.23)$$

with a similar expression for $\hat{J}_{ij}$. As mentioned in Paper I, the Fourier components are most easily evaluated as integrals over u . For instance,

$$\hat{I}_{ijk(n)} = \frac{1}{2\pi} \int_0^{2\pi} \hat{I}_{ijk} e^{-in(u-e_t \sin u)} (1 - e_t \cos u) du.$$

Then, a closed-form expression can be obtained through multiple applications of the Bessel integral formula

$$J_p(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ipu+x \sin u} du. \quad (2.24)$$

We find the following expressions for the mass octupole moment components:

$$\begin{aligned} \hat{I}_{xxx(n)} &= -\frac{3(3-4e_t^2+e_t^4)}{e_t^3 n^2} J_n(ne_t) \\ &\quad + \frac{3(10-6e_t^2+5(1-e_t^2)^2 n^2)}{5e_t^2 n^3} J'_n(ne_t), \\ \hat{I}_{xxy(n)} &= -\frac{3i\sqrt{1-e_t^2}(2(5-e_t^2)+5(1-e_t^2)^2 n^2)}{5e_t^3 n^3} J_n(ne_t) \\ &\quad + \frac{i\sqrt{1-e_t^2}(9-5e_t^2)}{e_t^2 n^2} J'_n(ne_t), \\ \hat{I}_{xyy(n)} &= \frac{9-13e_t^2+4e_t^4}{e_t^3 n^2} J_n(ne_t) \\ &\quad - \frac{3(10-8e_t^2+(1-e_t^2)^2 n^2)}{5e_t^2 n^3} J'_n(ne_t), \end{aligned}$$

$$\begin{aligned}
\hat{I}_{yyy(n)} &= \frac{3i\sqrt{1-e_t^2}(10-4e_t^2+5(1-e_t^2)^2n^2)}{5e_t^3n^3}J_n(ne_t) \\
&\quad - \frac{3i\sqrt{1-e_t^2}(3-2e_t^2)}{e_t^2n^2}J'_n(ne_t), \\
\hat{I}_{xzz(n)} &= \frac{1-e_t^2}{e_tn^2}J_n(ne_t) - \frac{6}{5n^3}J'_n(ne_t), \\
\hat{I}_{yzz(n)} &= \frac{6i\sqrt{1-e_t^2}}{5e_tn^3}J_n(ne_t) - \frac{i\sqrt{1-e_t^2}}{n^2}J'_n(ne_t), \quad (2.25)
\end{aligned}$$

and the following for the current quadrupole moment components:

$$\begin{aligned}
\hat{J}_{xz(n)} &= -\frac{1}{2n}\sqrt{1-e_t^2}J'_n(ne_t), \\
\hat{J}_{yz(n)} &= -\frac{i}{2e_tn}(1-e_t^2)J_n(ne_t). \quad (2.26)
\end{aligned}$$

We successively applied the well-known Bessel function identities

$$\begin{aligned}
J_{n+1}(ne_t) &= -\frac{nJ'_n(ne_t)}{n} + \frac{J_n(ne_t)}{e_t}, \\
J_{n-1}(ne_t) &= -J_{n+1}(ne_t) + \frac{2J_n(ne_t)}{e_t}, \quad (2.27)
\end{aligned}$$

in order to simplify the above expressions for the components of the multipoles (see also [8,32]).

2. Partial flux functions

To derive the 1PN log series, the Fourier amplitudes of the two multipoles given above are not used directly but rather go into forming a pair of (flux) spectral functions. This is similar to the derivation of the Newtonian (Peters-Mathews [10,19]) energy flux term $f(e_t)$, which was called $\mathcal{R}_0(e_t)$ in Paper I. In that case a quadrupole Fourier spectrum $g(n, e_t) = (1/16)n^6|_{(n)}\hat{I}_{ij}|^2$ is derived from the complex square of the Newtonian quadrupole Fourier amplitudes. The function $g(n, e_t)$ was derived by Peters and Mathews [10] (with a correction to their printed expression pointed out by [13]). The power spectrum then produces $\mathcal{R}_0(e_t)$ as the direct sum over the harmonics

$$\begin{aligned}
h(n, e_t) &= \frac{5}{3024}n^8|\hat{I}_{ijk}|^2 = \frac{n^2(1-e_t^2)}{504e_t^6}[12(10-5e_t^2+e_t^4)+5n^2(78-153e_t^2+91e_t^4-16e_t^6)+30n^4(1-e_t^2)^4]J_n(ne_t)^2 \\
&\quad + \frac{n^2}{504e_t^4}[12(10-15e_t^2+6e_t^4)+5(78-183e_t^2+142e_t^4-37e_t^6)n^2+30(1-e_t^2)^4n^4]J'_n(ne_t)^2 \\
&\quad - \frac{5n^3}{14e_t^5}[(2-3e_t^2+e_t^4)(2-e_t^2+(1-e_t^2)^2n^2)]J_n(ne_t)J'_n(ne_t), \quad (2.33)
\end{aligned}$$

$$\mathcal{R}_0(e_t) = \sum_{n=0}^{\infty} g(n, e_t) = \frac{1}{(1-e_t^2)^{7/2}} \left(1 + \frac{73}{24}e_t^2 + \frac{37}{96}e_t^4 \right). \quad (2.28)$$

In Paper I we showed that $g(n, e_t)$ [and its angular momentum counterpart $\tilde{g}(n, e_t)$] could generate the entire leading log series through sums of $g(n, e_t)$ over different powers of n . Here we show that spectral functions similar to $g(n, e_t)$ are formed from complex squares of the mass octupole (MO) and current quadrupole (CQ) Fourier amplitudes. Then, later in the paper, these spectral functions are shown to generate part of, but not all of, the various 1PN log series terms.

The Fourier amplitudes of $\hat{I}_{ijk}$ and $\hat{J}_{ij}$ each contribute to both the energy flux and the angular momentum flux. Calculation of all four pieces follows in close analogy to that of the mass quadrupole as reviewed in Paper I. The corresponding lowest-order energy and angular momentum fluxes are written as [11,33–35]

$$\left\langle \frac{dE}{dt} \right\rangle_1^{\text{MO}} = \frac{1}{189} \langle \ddot{I}_{ijk} \ddot{I}_{ijk} \rangle, \quad (2.29)$$

$$\left\langle \frac{dL}{dt} \right\rangle_1^{\text{MO}} = \frac{1}{63} \epsilon_{ijl} \hat{L}_i \langle \ddot{I}_{jab} \ddot{I}_{lab} \rangle, \quad (2.30)$$

$$\left\langle \frac{dE}{dt} \right\rangle_1^{\text{CQ}} = \frac{16}{45} \langle \ddot{J}_{ij} \ddot{J}_{ij} \rangle, \quad (2.31)$$

$$\left\langle \frac{dL}{dt} \right\rangle_1^{\text{CQ}} = \frac{32}{45} \epsilon_{ijl} \hat{L}_i \langle \ddot{J}_{ja} \ddot{J}_{la} \rangle, \quad (2.32)$$

where angled brackets denote the time average over an orbital period, the subscript 1 indicates these are contributions to the 1PN fluxes, and $\hat{L}_i$ is the unit vector in the direction of the angular momentum vector (which we take to be in the z direction). To compute these 1PN fluxes the mass octupole and current quadrupole moments need only be calculated at Newtonian order.

Inserting the Fourier expansions, integrating, and pulling out the Newtonian circular orbit limit and added power of x for a 1PN term [see (2.1)], we obtain the following functions as analogs of $g(n, e_t)$ and $\tilde{g}(n, e_t)$:

$$\begin{aligned}\tilde{h}(n, e_t) = & -\frac{5i}{1008} n^7 \epsilon_{ijl} \hat{L}_i \hat{J}_{jab} \hat{I}_{lab}^* = \frac{5(1-e_t^2)^{3/2} n^2}{168 e_t^6} [15e_t^6 n^2 - 36(2+n^2) - 2e_t^4(4+33n^2) + e_t^2(48+87n^2)] J_n(ne_t)^2 \\ & + \frac{5\sqrt{1-e_t^2} n^2}{168 e_t^4} [-8(3-2e_t^2)^2 + 3(1-e_t^2)^2(-12+7e_t^2)n^2] J'_n(ne_t)^2 \\ & + \frac{n\sqrt{1-e_t^2}}{126 e_t^5} [36(5-5e_t^2+e_t^4) + 5(1-e_t^2)(-3+2e_t^2)(-39+19e_t^2)n^2 + 45(1-e_t^2)^4 n^4] J_n(ne_t) J'_n(ne_t),\end{aligned}\quad (2.34)$$

$$k(n, e_t) = \frac{1}{9} n^6 |\hat{J}_{ij}|^2 = \frac{1}{18 e_t^2} (1-e_t^2)^2 n^4 J_n(ne_t)^2 + \frac{1}{18} (1-e_t^2) n^4 J'_n(ne_t)^2, \quad (2.35)$$

$$\tilde{k}(n, e_t) = -\frac{2i}{9} n^5 \epsilon_{ijl} \hat{L}_i \hat{J}_{ja} \hat{J}_{la}^* = \frac{(1-e_t^2)^{3/2}}{9 e_t} n^3 J_n(ne_t) J'_n(ne_t). \quad (2.36)$$

Contributions can then be found to the full 1PN energy and angular momentum fluxes, for example, by summing each of these expressions over n . To focus on one particular example, the mass octupole contribution to the energy flux is found by calculating

$$\begin{aligned}\mathcal{R}_1^{\text{MO}}(e_t) &= (1-4\nu) \sum_{n=0}^{\infty} h(n, e_t) \\ &= \frac{1-4\nu}{(1-e_t^2)^{9/2}} \left(\frac{1367}{1008} + \frac{18509e_t^2}{2016} \right. \\ &\quad \left. + \frac{2395e_t^4}{384} + \frac{1697e_t^6}{5376} \right).\end{aligned}\quad (2.37)$$

Additional explicit expansions for component sums like this one are given in Appendix A. We note that this term in the flux became a simple closed form expression once the specific eccentricity singular factor was pulled out. This particular eccentricity singular factor bears an extra power of $(1-e_t^2)^{-1}$ over that found in $\mathcal{R}_0$. Clearly, this mass octupole contribution to the energy flux is not the entirety of the 1PN flux, as can be seen by examining equation (356b) of [11].

D. The 1PN mass quadrupole

The next step is to find the 1PN correction to the mass quadrupole. Fourier decomposition at 1PN order presents a considerable increase in difficulty. The motion no longer closes, which implies that the simple Fourier series, as found in the expansion of the mass octupole and current quadrupole, must be replaced by a double Fourier sum over harmonics of the two frequencies, Ω_r and Ω_ϕ . This Fourier structure, first identified by [22–24], was laid out for use with hereditary contributions to the flux by Arun *et al.* in [14].

We follow some of the procedure and notation found in Loutrel and Yunes [32], who provided a detailed derivation of the 1PN expansion as part of their work. The expression for the components of the mass quadrupole tensor at 1PN order is

$$\begin{aligned}I_{ij} = & \mu \left[\left(1 + v^2 \left(\frac{29}{42} - \frac{29\nu}{14} \right) - \frac{M}{r} \left(\frac{5}{7} - \frac{8\nu}{7} \right) \right) x_{\langle i} x_{j \rangle} \right. \\ & \left. + \left(\frac{11}{21} - \frac{11\nu}{7} \right) r^2 v_{\langle i} v_{j \rangle} - \left(\frac{4}{7} - \frac{12\nu}{7} \right) r r' x_{\langle i} v_{j \rangle} \right],\end{aligned}\quad (2.38)$$

where bracketed indices denote STF projection [14]. Given the 1PN equations of motion, this tensor is converted from polar coordinates to the parameters x , ν , e_t , u , and k through 1PN order. At the same time a factor $\mu M^2/x^2$ is pulled out of I_{ij} to provide a dimensionless quadrupole moment tensor

$$\hat{I}_{ij} = \frac{x^2}{\mu M^2} I_{ij}, \quad (2.39)$$

similar to what we did with I_{ijk} and J_{ij} .

To obtain the Fourier expansion, the u dependence of $\hat{I}_{ij}$ is expressed in terms of complex exponentials and the result is collected over powers of e^{ikl} . The coefficient of each power of e^{ikl} is singly periodic in t , meaning that each can themselves be expressed as a simple Fourier series. The entire tensor can then be written as

$$\hat{I}_{ij}(t) = \sum_{n=-\infty}^{\infty} \sum_{p=-2,0,2} \hat{I}_{ij}^{(n+p)} e^{i(n+p)l}, \quad (2.40)$$

where the k dependence has introduced a magnetic-type separation of components due to 1PN differences in Ω_ϕ and

Ω_r . The goal is then to determine the Fourier coefficients ${}_{(n,p)}\hat{I}_{ij}$.

Proceeding further, we find that the (magnetic) term for each p can be written as the product of a single function (e.g., one of the components) with a constant matrix. Explicitly,

$$\hat{I}_{ij} = \hat{I}_{xx} M_{ij}, \quad (2.41)$$

where

$$M_{ij}^{(2)} = \begin{bmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_{ij}^{(0)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

$$M_{ij}^{(-2)} = \begin{bmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (2.42)$$

It is most convenient to separate each ${}_{(n)}\hat{I}_{ij}$ on powers of x and ν before making the Fourier transformation. To facilitate the process, we introduce a superscript notation, $\hat{I}_{ij}^{ab}$, where a represents the order in x (0 or 1) and b the

order in ν (also 0 or 1). Then each Fourier coefficient will formally separate into

$$\hat{I}_{ij} = \hat{I}_{ij}^{00} + x(\hat{I}_{ij}^{10} + \nu \hat{I}_{ij}^{11}). \quad (2.43)$$

Since we will just need to compute the $\hat{I}_{xx}^{ab}$ functions, as the full tensors will be determined from these functions via multiplication by ${}_{(p)}M_{ij}$, we can drop the lower ij indices, leaving $\hat{I}^{ab}$, to simplify the notation.

Starting at lowest order, the Fourier components are found to be

$$\hat{I}_{(n,\pm 2)}^{00} = \frac{1}{2e_t^2 n^2} (e_t^2 - 2 \pm 2n(1 - e_t^2)^{3/2}) J_n(ne_t) + \frac{\sqrt{1 - e_t^2}}{e_t n^2} (1 \mp n\sqrt{1 - e_t^2}) J'_n(ne_t),$$

$$\hat{I}_{(n,0)}^{00} = -\frac{J_n(ne_t)}{3n^2}, \quad (2.44)$$

which are precisely the terms needed to generate $g(n, e_t)$ and reproduce the Peters-Mathews flux. Then we jump to next order in both x and ν , and find that these coefficients can also be expressed cleanly in terms of Bessel functions

$$\hat{I}_{(n,\pm 2)}^{11} = -\frac{1}{84e_t^2 n^2} (134 + 17e_t^2 \mp 146n\sqrt{1 - e_t^2} \mp 22ne_t^2\sqrt{1 - e_t^2} + 12(1 - e_t^2)^2 n^2) J_n(ne_t) + \frac{1}{42e_t(1 - e_t^2)n^2} (\pm(67 - 25e_t^2)\sqrt{1 - e_t^2} \pm 6n^2(1 - e_t^2)^{5/2} - n(73 - 65e_t^2 - 8e_t^4)) J'_n(ne_t),$$

$$\hat{I}_{(n,0)}^{11} = \frac{17J_n(ne_t)}{126n^2} - \frac{e_t J'_n(ne_t)}{21n}. \quad (2.45)$$

Finally, we arrive at the portion that is first order in x and zeroth order in ν . Here some difficulty arises, as the integrals for ${}_{(n,\pm 2)}\hat{I}^{10}$ have terms that apparently cannot be expressed in closed form. We find

$$\hat{I}_{(n,\pm 2)}^{10} = \frac{1}{84e_t^2(1 - e_t^2)n^3} [\mp 756(-2 + e_t^2) + 4n^3(1 - e_t^2)^3 - 3n(-74 + 19e_t^4 + \sqrt{1 - e_t^2}(756 - 420e_t^2) + 111e_t^2) \pm 2n^2(1 - e_t^2)(378 - \sqrt{1 - e_t^2}(113 - 22e_t^2) - 378e_t^2)] J_n(ne_t) - \frac{1}{42e_t(1 - e_t^2)n^3} [-756\sqrt{1 - e_t^2} \mp 3n(-378 + 37(1 - e_t^2)^{3/2} + 273e_t^2) - n^2(1 - e_t^2)(-113 + 23e_t^2 + 378\sqrt{1 - e_t^2}) \mp 2n^3(1 - e_t^2)^{5/2}] J'_n(ne_t) - \frac{3i}{16(1 - e_t^2)\pi} \int_0^{2\pi} e^{-in(u - e_t \sin u)} (1 - e_t \cos u) [\pm 3e_t^2 \mp 4e_t \cos u \mp (-2 + e_t^2) \cos 2u - 4i\sqrt{1 - e_t^2}(e_t - \cos u) \sin u] \times \arctan\left(\frac{e_t \sin u}{1 + \sqrt{1 - e_t^2} - e_t \cos u}\right) du,$$

$$\hat{I}_{(n,0)}^{10} = -\frac{(75 - 19e_t^2)J_n(ne_t)}{42(1 - e_t^2)n^2} + \frac{26e_t n J'_n(ne_t)}{21n^2}. \quad (2.46)$$

Note that these results directly reveal the crossing relations, ${}_{(n,p)}I^* = {}_{(-n,-p)}I$.

When computing eccentricity enhancement functions, these unevaluated integrals must be expanded in e_t before proceeding. One might presume that this precludes the possibility of eventually finding closed form expressions in the fluxes, but surprisingly this is not the case. Instead, in the case of certain flux terms, once the appropriate eccentricity singular factor is pulled out, we find that parts of the expansion of ${}_{(n,p)}\hat{I}^{10}$ and of ${}_{(n,p)}\hat{I}^{00}$ conspire perfectly to cancel all coefficients beyond certain orders in e_t^2 in the remaining power series.

E. Discussion

The mass octupole and current quadrupole power spectra $h(n, e_t)$, $\tilde{h}(n, e_t)$, $k(n, e_t)$, $\tilde{k}(n, e_t)$, along with the Fourier decomposition of the 1PN-corrected mass quadrupole (MQ), will be shown to generate the entire 1PN log series. In order to more clearly explain the calculation of each flux term, we introduce the notation $\mathcal{R}_i = \mathcal{R}_i^{\text{MQ}} + \mathcal{R}_i^{\text{MO}} + \mathcal{R}_i^{\text{CQ}}$ (with a similar form for $\mathcal{Z}$) to represent the contributions from the (1PN) mass quadrupole, mass octupole, and current quadrupole, respectively. For the latter two (Newtonian) multipole moments, this categorization will be sufficient, as the spectral functions presented in Sec. II C 2 compactly express the entirety of those contributions to the 1PN logarithms. Note also that in $\mathcal{R}_i^{\text{MO}}$ and $\mathcal{R}_i^{\text{CQ}}$ the $\mathcal{O}(\nu)$ contributions will be immediately accessible through the ν -dependent prefactors in (2.22).

Unfortunately, the 1PN mass quadrupole contribution is not encoded in a single spectral function and so instead we work directly with the Fourier components introduced in the last section. The dependence on the two orders in ν is more subtle also. In what follows we are led to separate the relative flux $\mathcal{R}_i^{\text{MQ}}$ into five terms

$$\begin{aligned} \mathcal{R}_i^{\text{MQ}} &= \mathcal{R}_i^{\text{MQ0}} + \nu \mathcal{R}_i^{\text{MQ1}} \\ &= \mathcal{R}_i^{\text{MQ01}} + \mathcal{R}_i^{\text{MQ02}} + \mathcal{R}_i^{\text{MQ03}} + \nu(\mathcal{R}_i^{\text{MQ11}} + \mathcal{R}_i^{\text{MQ12}}), \end{aligned} \quad (2.47)$$

distinguished by the a, b superscripts in $\mathcal{R}_i^{\text{MQ}ab}$. The a represents the relative order in ν and $b \in \{1, 2, 3\}$ represents a particular “type” of summation over different parts in the decomposition of $\hat{I}_{ij}$ (see the next section for explicit examples). This notation for separating the relative flux functions carries over to the corresponding absolute flux, e.g.,

$$\left\langle \frac{dE}{dt} \right\rangle_1^{\text{MQ03}} \quad (2.48)$$

represents the mass quadrupole contribution of the “3-type” summation to the full 1PN (subscript) flux at lowest order in ν (0 superscript).

III. RECOVERING THE 1PN AND 2.5PN RELATIVE FLUXES: FIRST ELEMENTS IN THE 1PN LOG SEQUENCES

Using the frequency-domain tools developed above, this section demonstrates the recovery of the previously known first elements in the 1PN log sequences—namely the instantaneous 1PN fluxes $\mathcal{R}_1(e_t)$ and $\mathcal{Z}_1(e_t)$ and the hereditary 2.5PN tail fluxes $\mathcal{R}_{5/2}(e_t)$ and $\mathcal{Z}_{5/2}(e_t)$.

A. The full mass octupole and current quadrupole relative flux contributions

The contributions from the spectra of the two Newtonian-order moments are intuitive in form and mirror the way $g(n, e_t)$ contributed to the leading logarithms (see the discussion in Paper I). We examine first the 1PN fluxes. These enhancement functions have been known from PN analysis for some time and, since they are entirely instantaneous in nature, are easily calculated through time domain methods [20,36]. Here we give for the first time (as far as we know) their calculation via frequency domain analysis. The mass octupole and current quadrupole contributions to energy flux are trivial in this approach, and are simply given by sums over $h(n, e_t)$ and $k(n, e_t)$

$$\begin{aligned} \mathcal{R}_1^{\text{MO}} &= (1 - 4\nu) \sum_{n=1}^{\infty} h(n, e_t), \\ \mathcal{R}_1^{\text{CQ}} &= (1 - 4\nu) \sum_{n=1}^{\infty} k(n, e_t). \end{aligned} \quad (3.1)$$

Similarly the angular momentum terms are found by substituting the use of $\tilde{h}(n, e_t)$ and $\tilde{k}(n, e_t)$

$$\begin{aligned} \mathcal{Z}_1^{\text{MO}} &= (1 - 4\nu) \sum_{n=1}^{\infty} \tilde{h}(n, e_t), \\ \mathcal{Z}_1^{\text{CQ}} &= (1 - 4\nu) \sum_{n=1}^{\infty} \tilde{k}(n, e_t). \end{aligned} \quad (3.2)$$

The 2.5PN tail functions require a bit more work [14,32] as these hereditary terms do not lend themselves to a time domain approach. The results, though, follow exactly what one would expect from Newtonian-order moments based on the analysis found in Paper I [see that paper for a review of the construction of the 1.5PN tail flux from an analogous sum over $g(n, e_t)$]. We find

$$\begin{aligned} \mathcal{R}_{5/2}^{\text{MO}} &= 2\pi(1 - 4\nu) \sum_{n=1}^{\infty} n h(n, e_t), \\ \mathcal{R}_{5/2}^{\text{CQ}} &= 2\pi(1 - 4\nu) \sum_{n=1}^{\infty} n k(n, e_t), \end{aligned} \quad (3.3)$$

$$\begin{aligned}\mathcal{Z}_{5/2}^{\text{MQ}} &= 2\pi(1-4\nu) \sum_{n=1}^{\infty} n \tilde{h}(n, e_t), \\ \mathcal{Z}_{5/2}^{CQ} &= 2\pi(1-4\nu) \sum_{n=1}^{\infty} n \tilde{k}(n, e_t).\end{aligned}\quad (3.4)$$

B. Relative flux contributions from the mass quadrupole at lowest order in ν

As discussed in Sec. II D, the contribution to the fluxes from the mass quadrupole, calculated through 1PN order, is more involved and is best split into parts. A significant part of the split involves considering the two orders in ν separately. This subsection focuses only on the flux terms at lowest order in ν , which in the previously defined notation means

$$\left\langle \frac{dE}{dt} \right\rangle^{\text{MQ}} = \frac{1}{5} \langle \ddot{I}_{ij} \ddot{I}_{ij} \rangle_1 = \frac{1}{5} \left\langle \sum_{n,m=-\infty}^{\infty} \sum_{p,s=-2}^2 (\Omega_r)^6 (i(n+pk))^3 (i(m+sk))^3 I_{ij}^{(n,p)} I_{ij}^{(m,s)} e^{i(n+m+(p+s)k)l} \right\rangle_1, \quad (3.6)$$

which is here converted in the second equality from the time domain to the frequency domain. All of the terms on the right-hand side must be expanded and retained through 1PN order including the quadrupole moment I_{ij} , the frequency, the polynomial terms, and the exponential factor. We recall the 1PN expansions for Ω_r and k , given by

$$\begin{aligned}k &= \frac{3x}{1-e_t^2} + \mathcal{O}(x^2) \\ \Omega_r &= \frac{x^{3/2}}{M} \left(1 - \frac{3x}{1-e_t^2} \right) + \mathcal{O}(x^{7/2}) \\ &= \Omega_\varphi \left(1 - \frac{3x}{1-e_t^2} \right) + \mathcal{O}(x^{7/2}).\end{aligned}\quad (3.7)$$

Expanding some of the terms and retaining factors linear in k and x , we can write this summation as

$$\begin{aligned}\left\langle \frac{dE}{dt} \right\rangle^{\text{MQ}} &= -\frac{1}{5} \sum_{n,m=-\infty}^{\infty} \sum_{p,s=-2}^2 (\Omega_r)^6 \\ &\quad \times (m^3 n^3 + 3m^2 n^2 (ns + mp)k) \\ &\quad \times I_{ij}^{(n,p)} I_{ij}^{(m,s)} \langle e^{i(n+m+(p+s)k)l} \rangle.\end{aligned}\quad (3.8)$$

Next we expand the time average. The deficit in the frequency ratio, k , is a small quantity, so the integrand in the integral for the time average can be expanded about $k = 0$

$$\begin{aligned}\mathcal{R}_i^{\text{MQ0}} &= \mathcal{R}_i^{\text{MQ01}} + \mathcal{R}_i^{\text{MQ02}} + \mathcal{R}_i^{\text{MQ03}}, \\ \mathcal{Z}_i^{\text{MQ0}} &= \mathcal{Z}_i^{\text{MQ01}} + \mathcal{Z}_i^{\text{MQ02}} + \mathcal{Z}_i^{\text{MQ03}}.\end{aligned}\quad (3.5)$$

(The next subsection will handle the next order in ν terms.) These expressions add up the three summation types and i refers to 1 or 5/2 order. Subject to this split, we show in this subsection the contributions from the 1PN-corrected mass quadrupole to all of the 1PN and 2.5PN relative flux terms.

1. Contributions to the 1PN relative energy flux from the mass quadrupole at lowest order in ν

The 1PN mass quadrupole energy flux follows from retaining 1PN corrections to the well-known quadrupole formula

$$\begin{aligned}\langle e^{i(n+m+(p+s)k)l} \rangle &= \int_0^{2\pi} \frac{e^{i(n+m+(p+s)k)l}}{2\pi} dl \\ &\simeq \int_0^{2\pi} \frac{e^{i(n+m)l}}{2\pi} (1 + i(p+s)kl) dl.\end{aligned}\quad (3.9)$$

This leads to two cases. If $m \neq -n$, the lowest order term ($k = 0$) vanishes, leaving

$$\int_0^{2\pi} \frac{e^{i(n+m)l}}{2\pi} (i(p+s)kl) dl = \frac{p+s}{n+m} k. \quad (3.10)$$

On the other hand, when $m = -n$ we find

$$\int_0^{2\pi} \frac{1 + i(p+s)kl}{2\pi} dl = 1 + i\pi(p+s)k. \quad (3.11)$$

However, it turns out that when these averages are inserted in the full sums in (3.8) the linear in k parts vanish in both cases. To see this, consider the matrices ${}_{(p)}M_{ij}$. Direct calculation shows that the sum ${}_{(p)}M_{ij(s)}M_{ij}$ vanishes whenever $p + s \neq 0$. Therefore,

$${}_{(p)}M_{ij(s)}M_{ij} = 0, \quad (3.12)$$

which is precisely the form of the terms produced when the two linear-in- k terms above are inserted in (3.8). This identity turns out to have strong consequences on the calculation of 1PN log series fluxes (see Appendix B for details). Here the result is that the 1PN time average reduces to the simple Kronecker delta, $\delta_{m,-n}$, leaving

$$\left\langle \frac{dE}{dt} \right\rangle^{\text{MQ}} = \frac{1}{5} \sum_{n=-\infty}^{\infty} \sum_{p,s=-2}^2 (\Omega_r)^6 (n^6 + 3n^5(p-s)k) I_{ij}^{(n,p)} I_{ij}^{(-n,s)}. \quad (3.13)$$

The other consequence of (3.12) is that only elements in the double sum with $s = -p$ will survive, so that

$$\left\langle \frac{dE}{dt} \right\rangle^{\text{MQ}} = \frac{1}{5} \sum_{n=-\infty}^{\infty} \sum_{p=-2}^2 (\Omega_r)^6 (n^6 + 6n^5 pk) I_{ij}^{(n,p)} I_{ij}^{(-n,-p)}. \quad (3.14)$$

We then make a PN expansion of (3.14), combining expansions for the moments, the frequency, and the polynomial factor. Once the Newtonian order flux is discarded, the remainder is the 1PN mass quadrupole flux, which we split into three sums

$$\begin{aligned} \left\langle \frac{dE}{dt} \right\rangle_1^{\text{MQ01}} &= \frac{x}{5} \sum_{n=-\infty}^{\infty} \sum_{p=-2}^2 (\Omega_\varphi)^6 n^6 \left[I_{ij}^{00} I_{ij}^{10} + I_{ij}^{10} I_{ij}^{00} \right]_{(n,p)(-n,-p)}, \\ \left\langle \frac{dE}{dt} \right\rangle_1^{\text{MQ02}} &= -\frac{1}{5} \sum_{n=-\infty}^{\infty} \sum_{p=-2}^2 \left(\frac{18x}{1-e_t^2} \right) (\Omega_\varphi)^6 (n^6) I_{ij}^{00} I_{ij}^{00} \Big|_{(n,p)(-n,-p)}, \\ \left\langle \frac{dE}{dt} \right\rangle_1^{\text{MQ03}} &= \frac{1}{5} \sum_{n=-\infty}^{\infty} \sum_{p=-2}^2 (\Omega_\varphi)^6 (6n^5 pk) I_{ij}^{00} I_{ij}^{00} \Big|_{(n,p)(-n,-p)}. \end{aligned} \quad (3.15)$$

In each case, negative n terms duplicate positive n terms (see Appendix B). Applying the crossing relations and pulling out the Newtonian circular-orbit factor of $(32/5)\nu^2 x^5$, we arrive at the following relative flux contributions:

$$\begin{aligned} \mathcal{R}_1^{\text{MQ01}} &= \frac{1}{16} \sum_{n=1}^{\infty} n^6 \left[\hat{I}_{ij}^{00} \hat{I}_{ij}^{10*} + \hat{I}_{ij}^{10} \hat{I}_{ij}^{00*} \right]_{(n)}, \\ \mathcal{R}_1^{\text{MQ02}} &= -\frac{9}{8(1-e_t^2)} \sum_{n=1}^{\infty} n^6 |\hat{I}_{ij}^{00}|^2_{(n)}, \\ \mathcal{R}_1^{\text{MQ03}} &= \frac{9}{8(1-e_t^2)} \sum_{n=1}^{\infty} \sum_{p=-2}^2 n^5 p |\hat{I}_{ij}^{00}|^2_{(n,p)}, \end{aligned} \quad (3.16)$$

where we define $\hat{I}_{ij}^{00} = \hat{I}_{ij}^{00} \Big|_{(n,-2)} + \hat{I}_{ij}^{00} \Big|_{(n,0)} + \hat{I}_{ij}^{00} \Big|_{(n,2)}$.

2. Contributions to the 1PN relative angular momentum flux from the mass quadrupole at lowest order in ν

Similarly, the angular momentum flux is given by the 1PN correction to the formula

$$\begin{aligned} \left\langle \frac{dL}{dt} \right\rangle_1^{\text{MQ}} &= \frac{2}{5} \epsilon_{ijl} \hat{L}_i \langle \ddot{I}_{ja} \ddot{I}_{la} \rangle, \\ &= \frac{2}{5} \epsilon_{3jl} \left\langle \sum_{n,m=-\infty}^{\infty} \sum_{p,s=-2}^2 (\Omega_r)^5 \right. \\ &\quad \times (i(n+pk))^2 (i(m+sk))^3 \\ &\quad \times I_{ja}^{(n,p)} I_{la}^{(m,s)} e^{i(n+m+(p+s)k)t} \Big\rangle_1 \hat{z}, \end{aligned} \quad (3.17)$$

where as mentioned earlier $\hat{L}_i = \hat{z}$ for Kepler motion in the x, y plane. This sum simplifies in almost the same manner as the energy flux. There is a key identity involving $(p+s)$ in the angular momentum summations that is analogous to the one in the energy flux. We find

$$\sum_{p,s} (p+s) \epsilon_{3jl} I_{ja}^{(n,p)} I_{la}^{(m,s)} = 0. \quad (3.18)$$

The angular momentum also has the identity $\epsilon_{3jl} I_{ja}^{(n,0)} I_{la}^{(m,0)} = 0$, so that only $\epsilon_{3jl} M_{ja} M_{la} = \pm 4i$ survives. Inserting $\delta_{m,-n}$ for $(\pm 2) (\mp 2)$

the time average and taking $s \rightarrow -s$ as above, the expression reduces to

$$\begin{aligned} \left\langle \frac{dL}{dt} \right\rangle_1^{\text{MQ}} &= -\frac{2i}{5} \epsilon_{3jl} \sum_{n=-\infty}^{\infty} \sum_{p=-2,2} (\Omega_r)^5 (n^5 + 5n^4 pk) \\ &\quad \times I_{ja}^{(n,p)} I_{la}^{(-n,-p)} \hat{z}. \end{aligned} \quad (3.19)$$

As expected, we are left with three sums, all similar in form to their energy counterparts. We apply the crossing relation and simplify to obtain the following flux contributions:

$$\begin{aligned} \mathcal{Z}_1^{\text{MQ01}} &= -\frac{i}{8} \epsilon_{3jl} \sum_{n=-\infty}^{\infty} n^5 \left[\hat{I}_{ja}^{00} \hat{I}_{la}^{10*} + \hat{I}_{ja}^{10} \hat{I}_{la}^{00*} \right] \hat{z}_{(n)}, \\ \mathcal{Z}_1^{\text{MQ02}} &= \frac{15i}{8(1-e_t^2)} \epsilon_{3jl} \sum_{n=1}^{\infty} n^5 \hat{I}_{ja}^{00} \hat{I}_{la}^{00*} \Big|_{(n)}, \\ \mathcal{Z}_1^{\text{MQ03}} &= -\frac{15i}{8(1-e_t^2)} \epsilon_{3jl} \sum_{n=1}^{\infty} \sum_{p=-2,2} n^4 p \hat{I}_{ja}^{00} \hat{I}_{la}^{00*} \hat{z}_{(n,p)}. \end{aligned} \quad (3.20)$$

As stated previously, the biperiodicity of the 1PN mass quadrupole introduces three separate sums in the calculation of the 1PN flux. It turns out that these three sums characterize the entirety of both 1PN logarithm series at lowest order in ν . As we will see next, transition to the next highest 1PN logarithm flux (at 2.5PN order) will involve an increase in the power of n in the sums, along with multiplication by a different leading coefficient.

3. Contributions to the 2.5PN relative energy flux from the mass quadrupole at lowest order in ν

In the time domain, the mass quadrupole part of the energy tail flux [14,32] is given by

$$\mathcal{P}_{\infty}^{\text{MQtail}} = \frac{4\mathcal{M}}{5} \ddot{I}_{ij}(t) \int_0^{\infty} I_{ij}^{(5)}(t-\tau) \left[\log\left(\frac{\tau}{2r_0}\right) + \frac{11}{12} \right] d\tau, \quad (3.21)$$

where $\mathcal{M}$ is the Arnowitt-Deser-Misner (ADM) mass $\mathcal{M} = M(1 - \nu x/2 + \mathcal{O}(x^2))$. This expression gives the time-dependent flux, which will subsequently be time averaged over an orbital libration. It represents a nonlinear interaction between the mass quadrupole and ADM mass monopole of the system. However, because we are currently working at lowest order in ν , $\mathcal{M}$ can be replaced with M .

We insert the biperiodic Fourier expansion (2.40) for the quadrupole moment, replace time derivatives, and take the time average to find

$$\begin{aligned} \left\langle \frac{dE}{dt} \right\rangle_{5/2}^{\text{MQ}} &= \frac{4M}{5} \sum_{n=-\infty}^{\infty} \sum_{p,s} (\Omega_r)^8 (-n^8 + n^7(5s-3p)k) \\ &\times I_{ij}^{(n,p)} I_{ij}^{(-n,s)} \int_0^{\infty} e^{i(n-sk)\Omega_r\tau} \left[\log\left(\frac{\tau}{2r_0}\right) + \frac{11}{12} \right] d\tau. \end{aligned} \quad (3.22)$$

The only significant difference between this summation and that at 1PN order is the last integral term, which can be rewritten slightly to aid subsequent evaluation

$$\begin{aligned} &\int_0^{\infty} e^{i(n-sk)\Omega_r\tau} \left[\log\left(\frac{\tau}{2r_0}\right) + \frac{11}{12} \right] d\tau \\ &= \int_0^{\infty} e^{i(n-sk)\Omega_r\tau} \log\left(\frac{\tau}{2r_0 e^{-(11/12)}}\right) d\tau. \end{aligned} \quad (3.23)$$

This expression is regularized by rotating the mean motion into the complex plane. We refer the reader to [14,32,37] as well as Paper I (Sec. IV C and Appendix A) for details. The result is

$$-\frac{i}{(n-sk)\Omega_r} \left[\frac{\pi i}{2} \text{sign}(-n) + \log(2\Omega_r |n-sk| r_0) + \gamma_E - \frac{11}{12} \right] \quad (3.24)$$

$$\begin{aligned} &\approx -\frac{1}{n\Omega_r} \left[\frac{\pi}{2} \text{sign}(n) + i \left(\log(2\Omega_{\varphi} |n| r_0) + \gamma_E - \frac{11}{12} \right) \right] \\ &- \frac{sk}{n^2\Omega_r} \left[\frac{\pi}{2} \text{sign}(n) + i \left(\log(2\Omega_{\varphi} |n| r_0) - \frac{n}{s} + \gamma_E - \frac{23}{12} \right) \right], \end{aligned} \quad (3.25)$$

where the second line is an expansion to first order in k .

Appendix B shows that the imaginary portion will identically vanish in sums over positive and negative n , thus allowing those terms to be eliminated. Using the remaining factor, taking $s \rightarrow -s$, and then setting $s = p$, as in our earlier derivation, leads to

$$\begin{aligned} \left\langle \frac{dE}{dt} \right\rangle_{5/2}^{\text{MQ}} &= \frac{4M}{5} \sum_{n=-\infty}^{\infty} \sum_p (\Omega_r)^7 (n^7 + 8n^6 pk) I_{ij}^{(n,p)} I_{ij}^{(-n,-p)} \\ &\times \left[\frac{\pi}{2} \text{sign}(n) - \left(\frac{pk}{n} \right) \frac{\pi}{2} \text{sign}(n) \right]. \end{aligned} \quad (3.26)$$

As in the 1PN case, this result splits into three well-defined sums, which can be written as

$$\begin{aligned} \mathcal{R}_{5/2}^{\text{MQ01}} &= \frac{\pi}{8} \sum_{n=1}^{\infty} n^7 [\hat{I}_{ij}^{00} \hat{I}_{ij}^{10*} + \hat{I}_{ij}^{10} \hat{I}_{ij}^{00*}], \\ \mathcal{R}_{5/2}^{\text{MQ02}} &= -\frac{21\pi}{8(1-e_t^2)} \sum_{n=1}^{\infty} n^7 |\hat{I}_{ij}^{00}|^2, \\ \mathcal{R}_{5/2}^{\text{MQ03}} &= \frac{21\pi}{8(1-e_t^2)} \sum_{n=1}^{\infty} \sum_p n^6 p |\hat{I}_{ij}^{00}|^2. \end{aligned} \quad (3.27)$$

Summed together and normalized, these terms will recover the enhancement function $\alpha(e_t)$ defined by Arun *et al.* [14,32].

4. Contributions to the 2.5PN relative angular momentum flux from the mass quadrupole at lowest order in ν

Similarly, the (time-dependent) angular momentum tail flux [32] is given by

$$\begin{aligned} \mathcal{G}_{\infty}^{\text{MQtail}} &= \frac{4\mathcal{M}}{5} \epsilon_{3jl} \left\{ \ddot{I}_{ja}(t) \int_0^{\infty} I_{la}^{(5)}(t-\tau) \left[\log\left(\frac{\tau}{2r_0}\right) + \frac{11}{12} \right] d\tau \right. \\ &\quad \left. + \ddot{I}_{la}(t) \int_0^{\infty} I_{ja}^{(4)}(t-\tau) \left[\log\left(\frac{\tau}{2r_0}\right) + \frac{11}{12} \right] d\tau \right\} \hat{z}. \end{aligned} \quad (3.28)$$

By inserting the Fourier series and performing the same simplifications as in the energy case, we arrive at

$$\begin{aligned} \mathcal{Z}_{5/2}^{\text{MQ01}} &= -\frac{\pi i}{4} \epsilon_{3jl} \sum_{n=1}^{\infty} n^6 [\hat{I}_{ja}^{00} \hat{I}_{la}^{10*} + \hat{I}_{ja}^{10} \hat{I}_{la}^{00*}] \hat{z}, \\ \mathcal{Z}_{5/2}^{\text{MQ02}} &= \frac{9\pi i}{2(1-e_t^2)} \epsilon_{3jl} \sum_{n=1}^{\infty} n^6 \hat{I}_{ja}^{00} \hat{I}_{la}^{00*} \hat{z}, \\ \mathcal{Z}_{5/2}^{\text{MQ03}} &= -\frac{9\pi i}{2(1-e_t^2)} \epsilon_{3jl} \sum_{n=1}^{\infty} \sum_{p=-2,2} (n^5 p) \hat{I}_{ja}^{00} \hat{I}_{la}^{00*} \hat{z}. \end{aligned} \quad (3.29)$$

Despite the factor of i that is pulled out of each sum, the complex conjugation and presence of the Levi Civita tensor ensure that all of these terms are real.

C. Relative flux contributions from the mass quadrupole at next order in ν

We next need to consider the linear-order-in- ν contributions to (2.47), i.e., the $\mathcal{R}_i^{\text{MQ1}}$ and $\mathcal{Z}_i^{\text{MQ1}}$ terms. Fortunately, much of the procedure is identical to that in the previous subsection, with only minor modifications to generate the corresponding reductions. One difference lies in the fact that there can be no appearance of ν at 0PN order. Thus, contributions from the radial frequency and magnetic factor p (that is, of type $\mathcal{R}_1^{\text{MQ02}}$ and $\mathcal{R}_1^{\text{MQ03}}$, respectively) will not recur here. This eliminates two potential types of terms that involve the Newtonian portion of the mass quadrupole. However, a sum involving the Newtonian mass quadrupole (termed $\mathcal{R}_i^{\text{MQ12}}$) will still manifest at $\mathcal{O}(\nu)$ in all 1PN logarithms except $\mathcal{R}_1$, first appearing in the 2.5PN tail through a factor of the ADM mass $\mathcal{M} = M(1 - \nu x/2 + \mathcal{O}(x^2))$.

As a result, the 1PN order ν flux terms are straightforward, containing only the $\mathcal{O}(\nu)$ correction induced by the corresponding portion of the quadrupole moment. All aspects of their derivations are functionally identical to those of $\mathcal{R}_1^{\text{MQ01}}$ and $\mathcal{Z}_1^{\text{MQ01}}$ at ν^0 , with the simple substitution $I^{10} \rightarrow I^{11}$. We find

$$\begin{aligned}\mathcal{R}_1^{\text{MQ11}} &= \frac{1}{16} \sum_{n=1}^{\infty} n^6 [\hat{I}_{ij}^{00} \hat{I}_{ij}^{11*} + \hat{I}_{ij}^{11} \hat{I}_{ij}^{00*}], \\ \mathcal{Z}_1^{\text{MQ11}} &= -\frac{i}{8} \epsilon_{3jl} \sum_{n=-\infty}^{\infty} n^5 [\hat{I}_{ja}^{00} \hat{I}_{la}^{11*} + \hat{I}_{ja}^{11} \hat{I}_{la}^{00*}] \hat{z},\end{aligned}\quad (3.30)$$

with $\mathcal{R}_1^{\text{MQ12}} = \mathcal{Z}_1^{\text{MQ12}} = 0$. The square bracket in the second sum has the same anti-Hermitian behavior as before.

The 2.5PN terms are only slightly more involved. The first contribution emerges from the same substitution in $\mathcal{R}_{5/2}^{\text{MQ01}}$, with $I^{10} \rightarrow I^{11}$. However, now there is a second term that comes from the 1PN correction to the ADM mass. We find

$$\begin{aligned}\mathcal{R}_{5/2}^{\text{MQ11}} &= \frac{\pi}{8} \sum_{n=1}^{\infty} n^7 [\hat{I}_{ij}^{00} \hat{I}_{ij}^{11*} + \hat{I}_{ij}^{11} \hat{I}_{ij}^{00*}], \\ \mathcal{R}_{5/2}^{\text{MQ12}} &= -\frac{\pi}{16} \sum_{n=1}^{\infty} n^7 |\hat{I}_{ij}^{00}|^2, \\ \mathcal{Z}_{5/2}^{\text{MQ11}} &= -\frac{\pi i}{4} \epsilon_{3jl} \sum_{n=1}^{\infty} n^6 [\hat{I}_{ja}^{00} \hat{I}_{la}^{11*} + \hat{I}_{ja}^{11} \hat{I}_{la}^{00*}] \hat{z}, \\ \mathcal{Z}_{5/2}^{\text{MQ12}} &= \frac{\pi i}{8} \epsilon_{3jl} \sum_{n=1}^{\infty} n^6 \hat{I}_{ja}^{00} \hat{I}_{la}^{00*} \hat{z}.\end{aligned}\quad (3.31)$$

These 2.5PN contributions to the flux can be summed together and normalized to generate the enhancement functions $\theta(e_t)$ and $\tilde{\theta}(e_t)$ defined in [14,25,32].

D. Eccentricity singular factors and full flux functions

The various sums over Fourier amplitude products derived above will produce, when added together, the power series in eccentricity for the full flux contributions at 1PN and 2.5PN. As with the leading logarithms [8], each such sum will have an associated eccentricity singular factor governing its divergent behavior as $e_t \rightarrow 1$. For each of the separated parts (except one) its own singular behavior is easily determined using the asymptotic analysis developed in [7] (specific examples are given in Appendix A). The exception is the term labeled MQ01, which involves the quadrupole components with unevaluated integrals (2.46). Because this part does not have a clean representation in terms of Bessel functions, it is not amenable to the exact same asymptotic analysis technique. Nevertheless, its divergent behavior appears to adhere to the same patterns, and we have demonstrated apparent convergence through 22PN (see Appendix A for more details) and verified the behavior with a new all-analytic perturbation code [18]. The conclusion is that the terms in the various 1PN log sequences have the following singular behavior:

$$\begin{aligned}\mathcal{R}_{(3k+1)L(k)} &\sim \frac{1}{(1-e_t^2)^{k+9/2}}, & \mathcal{R}_{(3k+5/2)L(k)} &\sim \frac{1}{(1-e_t^2)^{k+6}}, \\ \mathcal{Z}_{(3k+1)L(k)} &\sim \frac{1}{(1-e_t^2)^{k+3}}, & \mathcal{Z}_{(3k+1)L(k)} &\sim \frac{1}{(1-e_t^2)^{k+9/2}}.\end{aligned}\quad (3.32)$$

With the divergent behavior understood, the remaining eccentricity dependence is found to be closed-form (polynomial) expressions for the integer-order 1PN logarithms and convergent power series at the half-integer orders.

Putting all of these elements together involves summing the results of the previous sections and extracting the appropriate overall eccentricity singular factor. Focusing on low PN order, we can rederive the known energy and angular momentum flux functions. This frequency domain approach leads to the well-known closed-form expressions at 1PN

$$\begin{aligned}\mathcal{R}_1(e_t, \nu) &= \frac{1}{(1-e_t^2)^{9/2}} \left(-\frac{1247}{336} + \frac{10475e_t^2}{672} \right. \\ &\quad \left. + \frac{10043e_t^4}{384} + \frac{2179e_t^6}{1792} \right) \\ &\quad - \frac{\nu}{(1-e_t^2)^{9/2}} \left(\frac{35}{12} + \frac{1081e_t^2}{36} + \frac{311e_t^4}{12} + \frac{851e_t^6}{576} \right), \\ \mathcal{Z}_1(e_t, \nu) &= \frac{1}{(1-e_t^2)^3} \left(-\frac{1247}{336} + \frac{3019e_t^2}{336} + \frac{8399e_t^4}{2688} \right) \\ &\quad - \frac{\nu}{(1-e_t^2)^3} \left(\frac{35}{12} + \frac{335e_t^2}{24} + \frac{275e_t^4}{96} \right),\end{aligned}\quad (3.33)$$

which (being purely instantaneous) were previously derived through time domain analysis [20,36].

The 2.5PN flux functions on the other hand do not have closed-form representations. The original work in [14,25] showed numerical results and presented expansions in eccentricity only through e_t^4 . Forseth *et al.* [7] used a frequency-domain procedure similar to the present one to generate $\mathcal{R}_{5/2}$ to e_t^{70} and developed the asymptotic analysis

$$\begin{aligned}\mathcal{R}_{5/2}(e_t, \nu) &= \frac{1}{(1 - e_t^2)^6} \left(-\frac{8191}{672} + \frac{36067e_t^2}{336} + \frac{19817891e_t^4}{43008} + \frac{62900483e_t^6}{387072} + \frac{26368199e_t^8}{7077888} - \frac{1052581e_t^{10}}{34406400} + \dots \right) \\ &\quad + \frac{\nu}{(1 - e_t^2)^6} \left(-\frac{583}{24} - \frac{717733e_t^2}{2016} - \frac{21216061e_t^4}{32256} - \frac{78753305e_t^6}{387072} - \frac{208563695e_t^8}{37158912} + \frac{46886227e_t^{10}}{3715891200} + \dots \right), \\ \mathcal{Z}_{5/2}(e_t, \nu) &= \frac{1}{(1 - e_t^2)^{9/2}} \left(-\frac{8191}{672} + \frac{108551e_t^2}{1344} + \frac{5055125e_t^4}{43008} + \frac{4125385e_t^6}{774144} - \frac{11065099e_t^8}{49545216} + \frac{68397463e_t^{10}}{2477260800} + \dots \right) \\ &\quad + \frac{\nu}{(1 - e_t^2)^{9/2}} \left(-\frac{583}{24} - \frac{32821e_t^2}{168} - \frac{1566125e_t^4}{10752} - \frac{712219e_t^6}{96768} + \frac{457507e_t^8}{12386304} - \frac{792569e_t^{10}}{309657600} + \dots \right).\end{aligned}\quad (3.34)$$

As $e_t \rightarrow 1$, these series approach approximately $(722.1524014 - 1247.1117956\nu)/(1 - e_t^2)^6$ and $(191.2520614 - 372.6399916\nu)/(1 - e_t^2)^{9/2}$, respectively (see discussion in Sec. III C of [9] regarding prior tabulated numerical values [25] of these series in the vicinity of $e_t = 1$).

IV. HIGHER-ORDER ELEMENTS OF THE 1PN LOG SEQUENCES

With the derivations in the previous sections, plus the leading logarithm series [8] and numerical input from BHPT, we now have enough information to generalize to the form of the 1PN logarithm series for all PN orders. As in Paper I, this process will involve incrementing powers of n within sums over products of the Fourier amplitudes and determining the correct rational-number prefactor at each order.

A. Mass octupole and current quadrupole contributions to higher-order 1PN log terms

We begin with the two 1PN source multipole moments (mass octupole and current quadrupole) that can be calculated (for present purposes) using Newtonian dynamics. These moments give rise to the spectra $h(n, e_t)$ and $k(n, e_t)$. Sums over these multipole spectra with higher powers of n lead to their contributions to the higher-order 1PN log fluxes, much as sums over the Newtonian mass quadrupole spectra did in contributing to the higher-order leading logs as shown in Paper I. For integers $k \geq 0$, the mass octupole contributions to the 1PN log (energy) fluxes are given by

to investigate the behavior as $e_t \rightarrow 1$ at lowest order in ν . Later, Loutrel and Yunes [32] also derived asymptotics of these functions as $e_t \rightarrow 1$ and for both orders in ν . We have now calculated the terms in the power series to e_t^{120} using the methods described above, with the ability to push to much higher order should it prove necessary. These two series have leading behavior

$$\mathcal{R}_{(3k+1)L(k)}^{\text{MO}} = (1 - 4\nu) \left(-\frac{26}{21} \right)^k \left(\frac{1}{k!} \right) \sum_{n=1}^{\infty} n^{2k} h(n, e_t), \quad (4.1)$$

$$\mathcal{R}_{(3k+5/2)L(k)}^{\text{MO}} = (1 - 4\nu) \left(-\frac{26}{21} \right)^k \left(\frac{2\pi}{k!} \right) \sum_{n=1}^{\infty} n^{2k+1} h(n, e_t). \quad (4.2)$$

The current quadrupole series are even closer in appearance to the leading logarithms of Paper I, taking the following forms:

$$\mathcal{R}_{(3k+1)L(k)}^{\text{CQ}} = (1 - 4\nu) \left(-\frac{214}{105} \right)^k \left(\frac{1}{k!} \right) \sum_{n=1}^{\infty} n^{2k} k(n, e_t), \quad (4.3)$$

$$\mathcal{R}_{(3k+5/2)L(k)}^{\text{CQ}} = (1 - 4\nu) \left(-\frac{214}{105} \right)^k \left(\frac{2\pi}{k!} \right) \sum_{n=1}^{\infty} n^{2k+1} k(n, e_t). \quad (4.4)$$

In each case, the angular momentum analog $\mathcal{Z}_i$ is obtained by simply substituting $h \rightarrow \tilde{h}$ or $k \rightarrow \tilde{k}$, as appropriate.

B. Mass quadrupole (at lowest order in ν) contributions to higher-order 1PN log terms

1. The energy flux

At lowest order in the mass ratio, three separate sums over Fourier amplitudes must be handled. The simplest of

the three to derive (though the hardest to compute explicitly), $\mathcal{R}_{(3k+1)L(k)}^{\text{MQ01}}$, comes from the correction to the mass quadrupole itself. Careful inspection reveals that this term must be identical in form to the leading logarithm series, except with the Newtonian part of the mass quadrupole supplanted by its 1PN counterpart. Thus, the prefactor must be the same, and we can simply adjust the result of Paper I to get the following energy flux contributions:

$$\mathcal{R}_{(3k+1)L(k)}^{\text{MQ01}} = \frac{1}{16(k!)} \left(-\frac{214}{105} \right)^k \sum_{n=1}^{\infty} n^{2k+6} [\hat{I}_{ij}^{00} \hat{I}_{ij}^{10*} + \hat{I}_{ij}^{10} \hat{I}_{ij}^{00*}], \quad (4.5)$$

$$\mathcal{R}_{(3k+5/2)L(k)}^{\text{MQ01}} = \frac{\pi}{8(k!)} \left(-\frac{214}{105} \right)^k \sum_{n=1}^{\infty} n^{2k+7} [\hat{I}_{ij}^{00} \hat{I}_{ij}^{10*} + \hat{I}_{ij}^{10} \hat{I}_{ij}^{00*}]. \quad (4.6)$$

(We note again that in these and all sums in this section, k refers to any non-negative integer, rather than the ratio of frequencies $k = \Omega_\phi/\Omega_r - 1$.)

The next sum type, $\mathcal{R}_i^{\text{MQ02}}$, which in our scheme involves the 1PN correction to Ω_r , can be found in a similar manner. The portion of the quadrupole moment involved is just the Newtonian part and the k -dependent coefficient follows from a binomial expansion of powers of $\Omega_r = \Omega_\phi(1 - 3x/(1 - e_t^2))$ to 1PN order. We find

$$\mathcal{R}_{(3k+1)L(k)}^{\text{MQ02}} = -\frac{3k+9}{8(k!)(1 - e_t^2)} \left(-\frac{214}{105} \right)^k \sum_{n=1}^{\infty} n^{2k+6} |\hat{I}_{ij}^{00}|^2, \quad (4.7)$$

$$\mathcal{R}_{(3k+5/2)L(k)}^{\text{MQ02}} = -\frac{3\pi(2k+7)}{8(k!)(1 - e_t^2)} \left(-\frac{214}{105} \right)^k \sum_{n=1}^{\infty} n^{2k+7} |\hat{I}_{ij}^{00}|^2. \quad (4.8)$$

Finally, the third sum type is $\mathcal{R}_k^{\text{MQ03}}$, whose definition involves the magnetic factor p with $\hat{I}_{ij}^{00}$. We find (and illustrate in the discussion below) that the k -dependent coefficient prefacing this summation is equal and opposite to that of $\mathcal{R}^{\text{MQ02}}$, or

$$\mathcal{R}_{(3k+1)L(k)}^{\text{MQ03}} = \frac{3k+9}{8(k!)(1 - e_t^2)} \left(-\frac{214}{105} \right)^k \times \sum_{n=1}^{\infty} \sum_{p=-2,2} n^{2k+5} p |\hat{I}_{ij}^{00}|^2, \quad (4.9)$$

$$\mathcal{R}_{(3k+5/2)L(k)}^{\text{MQ03}} = \frac{3\pi(2k+7)}{8(k!)(1 - e_t^2)} \left(-\frac{214}{105} \right)^k \times \sum_{n=1}^{\infty} \sum_{p=-2,2} n^{2k+6} p |\hat{I}_{ij}^{00}|^2. \quad (4.10)$$

2. The angular momentum flux

As seen throughout Sec. III, the contributions to the angular momentum flux are nearly identical in form, only requiring minor adjustments in the moments and prefactors. The first sum mirrors that of the leading logarithm series, giving

$$\begin{aligned} \mathcal{Z}_{(3k+1)L(k)}^{\text{MQ01}} &= -\frac{i}{8(k!)} \left(-\frac{214}{105} \right)^k \epsilon_{3jl} \\ &\times \sum_{n=1}^{\infty} n^{2k+5} [\hat{I}_{ja}^{00} \hat{I}_{la}^{10*} + \hat{I}_{ja}^{10} \hat{I}_{la}^{00*}] \hat{z}, \\ \mathcal{Z}_{(3k+5/2)L(k)}^{\text{MQ01}} &= -\frac{\pi i}{4(k!)} \left(-\frac{214}{105} \right)^k \epsilon_{3jl} \\ &\times \sum_{n=1}^{\infty} n^{2k+6} [\hat{I}_{ja}^{00} \hat{I}_{la}^{10*} + \hat{I}_{ja}^{10} \hat{I}_{la}^{00*}] \hat{z}. \end{aligned} \quad (4.11)$$

The second sum type, $\mathcal{Z}_i^{\text{MQ02}}$, has one lower power of Ω_r than the corresponding energy flux term, $\mathcal{R}_i^{\text{MQ02}}$, and is found to be

$$\begin{aligned} \mathcal{Z}_{(3k+1)L(k)}^{\text{MQ02}} &= \frac{3(2k+5)i}{8(k!)(1 - e_t^2)} \left(-\frac{214}{105} \right)^k \epsilon_{3jl} \\ &\times \sum_{n=1}^{\infty} n^{2k+5} \hat{I}_{ja}^{00} \hat{I}_{la}^{00*} \hat{z}, \\ \mathcal{Z}_{(3k+5/2)L(k)}^{\text{MQ02}} &= \frac{3\pi(k+3)i}{2(k!)(1 - e_t^2)} \left(-\frac{214}{105} \right)^k \epsilon_{3jl} \\ &\times \sum_{n=1}^{\infty} n^{2k+6} \hat{I}_{ja}^{00} \hat{I}_{la}^{00*} \hat{z}, \end{aligned} \quad (4.12)$$

with the antisymmetry and factor of i guaranteeing the flux is real. Finally, terms of the third sum type emerge with identical k -dependent factors (up to sign), and are found to be

$$\begin{aligned} \mathcal{Z}_{(3k+1)L(k)}^{\text{MQ03}} &= -\frac{3(2k+5)i}{8(k!)(1 - e_t^2)} \left(-\frac{214}{105} \right)^k \epsilon_{3jl} \\ &\times \sum_{n=1}^{\infty} \sum_{p=-2,2} n^{2k+4} p \hat{I}_{ja}^{00} \hat{I}_{la}^{00*} \hat{z}, \\ \mathcal{Z}_{(3k+5/2)L(k)}^{\text{MQ03}} &= -\frac{3\pi(k+3)i}{2(k!)(1 - e_t^2)} \left(-\frac{214}{105} \right)^k \epsilon_{3jl} \\ &\times \sum_{n=1}^{\infty} \sum_{p=-2,2} n^{2k+5} p \hat{I}_{ja}^{00} \hat{I}_{la}^{00*} \hat{z}. \end{aligned} \quad (4.13)$$

C. Mass quadrupole (next order in ν) contributions to higher-order 1PN log terms

There is an expected contribution at next order in ν to the flux in each higher-order 1PN log term, just as there was with the base terms of these sequences: $\mathcal{R}_1$, $\mathcal{Z}_1$, $\mathcal{R}_{5/2}$, and $\mathcal{Z}_{5/2}$. These contributions emerge from two summations—one involving the 1PN part of the quadrupole moment, $\hat{I}^{11}$, and one containing its Newtonian counterpart, $\hat{I}^{00}$. From the earlier discussion of the 1PN and 2.5PN relative order fluxes, we can see that the coefficients for $\mathcal{R}^{\text{MQ11}}$ in the 1PN log sequence must exactly match those of their $\mathcal{R}^{\text{MQ01}}$ counterparts in the previous subsection.

The k -dependent factor preceding the sum for $\mathcal{R}^{\text{MQ12}}$ is less straightforward. This sum involves the Newtonian-order mass quadrupole and is of a form that did not make an appearance in $\mathcal{R}_1$. Instead, it first shows up with the ADM mass in the 2.5PN tail. The appearance of the ADM mass in the known hereditary flux terms is fairly regular: Each higher-order tail merely sees an increment in the power of $\mathcal{M}$ [see, for example, Eq. (4.8) of [37]], making the tail portion of $\mathcal{R}^{\text{MQ12}}$ calculable to high PN order. Moreover, in Paper I we used a combination of BHPT and PN results to show that for leading logarithms (starting with $\mathcal{R}_{3L}$), all instantaneous contributions uniformly equal a factor of $-2/3$ of their hereditary counterparts. A similar line of reasoning might be applied to 1PN log terms at $\mathcal{O}(\nu^0)$. However, because that argument relied upon information from BHPT, which is presently limited to first order in the mass ratio, it cannot be extended as written for next order in ν [i.e., $\mathcal{O}(\nu^1)$] results.

Nevertheless, the PN regularization parameter r_0 [11], which exists in all hereditary integrals but which must cancel in the overall flux and thus implies corresponding factors in the instantaneous flux, lends strong credence to the notion that the simple relationship also exists at $\mathcal{O}(\nu^1)$. For the time being we conjecture that this is the case and present the results that follow from this assumption. If the conjecture is correct, then the coefficients on the $\mathcal{R}_i^{\text{MQ12}}$ terms become nearly identical to those of $\mathcal{R}_i^{\text{MQ11}}$, except the binomial expansion of $\mathcal{M}^q = M^q(1 - \nu x/2)^q$ introduces a factor of $-q/2$ for the $(q+1)$ th element of the 1PN log series. We are led to the following expected forms of the next order in ν eccentricity-dependent flux functions:

$$\begin{aligned}\mathcal{R}_{(3k+1)L(k)}^{\text{MQ11}} &= \frac{1}{16(k!)} \left(-\frac{214}{105}\right)^k \sum_{n=1}^{\infty} n^{2k+6} [\hat{I}_{ij}^{00} \hat{I}_{ij}^{11*} + \hat{I}_{ij}^{11} \hat{I}_{ij}^{00*}], \\ \mathcal{R}_{(3k+5/2)L(k)}^{\text{MQ11}} &= \frac{\pi}{8(k!)} \left(-\frac{214}{105}\right)^k \sum_{n=1}^{\infty} n^{2k+7} [\hat{I}_{ij}^{00} \hat{I}_{ij}^{11*} + \hat{I}_{ij}^{11} \hat{I}_{ij}^{00*}], \\ \mathcal{R}_{(3k+1)L(k)}^{\text{MQ12}} &= -\frac{1}{16(k-1)!} \left(-\frac{214}{105}\right)^k \sum_{n=1}^{\infty} n^{2k+6} |\hat{I}_{ij}^{00}|^2, \\ \mathcal{R}_{(3k+5/2)L(k)}^{\text{MQ12}} &= -\frac{\pi(2k+1)}{16(k!)} \left(-\frac{214}{105}\right)^k \sum_{n=1}^{\infty} n^{2k+7} |\hat{I}_{ij}^{00}|^2,\end{aligned}\tag{4.14}$$

and

$$\begin{aligned}\mathcal{Z}_{(3k+1)L(k)}^{\text{MQ11}} &= -\frac{i}{8(k!)} \left(-\frac{214}{105}\right)^k \epsilon_{3jl} \\ &\quad \times \sum_{n=1}^{\infty} n^{2k+5} [\hat{I}_{ja}^{00} \hat{I}_{la}^{11*} + \hat{I}_{ja}^{11} \hat{I}_{la}^{00*}], \\ \mathcal{Z}_{(3k+5/2)L(k)}^{\text{MQ11}} &= -\frac{\pi i}{4(k!)} \left(-\frac{214}{105}\right)^k \epsilon_{3jl} \\ &\quad \times \sum_{n=1}^{\infty} n^{2k+6} [\hat{I}_{ja}^{00} \hat{I}_{la}^{11*} + \hat{I}_{ja}^{11} \hat{I}_{la}^{00*}], \\ \mathcal{Z}_{(3k+1)L(k)}^{\text{MQ12}} &= \frac{i}{8(k-1)!} \left(-\frac{214}{105}\right)^k \epsilon_{3jl} \sum_{n=1}^{\infty} n^{2k+5} \hat{I}_{ja}^{00} \hat{I}_{la}^{00*}, \\ \mathcal{Z}_{(3k+5/2)L(k)}^{\text{MQ12}} &= \frac{(2k+1)\pi i}{8(k!)} \left(-\frac{214}{105}\right)^k \epsilon_{3jl} \sum_{n=1}^{\infty} n^{2k+6} \hat{I}_{ja}^{00} \hat{I}_{la}^{00*}.\end{aligned}\tag{4.15}$$

Unfortunately, if the above conjecture were to break down for some k , the representations for $\mathcal{R}^{\text{MQ12}}$ and $\mathcal{Z}^{\text{MQ12}}$ would cease to hold. However, we would expect that the MQ11 summations, as well as all components of $\mathcal{R}^{\text{MQ0}}$ and $\mathcal{Z}^{\text{MQ0}}$, would continue to remain valid.

D. Assembling the complete 1PN log sequences

We now draw together all of the preceding computations into compact expressions for the terms in each 1PN logarithm sequence. To make this assembly for, say, the integer-order energy flux terms involve the following sum of terms:

$$\begin{aligned}\mathcal{R}_{(3k+1)L(k)} &= \mathcal{R}_{(3k+1)L(k)}^{\text{MQ01}} + \mathcal{R}_{(3k+1)L(k)}^{\text{MQ02}} + \mathcal{R}_{(3k+1)L(k)}^{\text{MQ03}} + \nu(\mathcal{R}_{(3k+1)L(k)}^{\text{MQ11}} + \mathcal{R}_{(3k+1)L(k)}^{\text{MQ12}}) \\ &\quad + \mathcal{R}_{(3k+1)L(k)}^{\text{MO}} + \mathcal{R}_{(3k+1)L(k)}^{\text{CQ}}.\end{aligned}\tag{4.16}$$

The full expressions for the integer-order and half-integer-order energy fluxes are given by

$$\begin{aligned}
\mathcal{R}_{(3k+1)L(k)} = & \frac{1}{16(k!)} \left(-\frac{214}{105} \right)^k \sum_{n=1}^{\infty} \left[\left(\frac{6k+18}{1-e_t^2} \right) \left(\sum_{p=-2,2} n^{2k+5} p |\hat{I}_{ij}^{00}|^2 \right) + n^{2k+6} (\hat{I}_{ij}^{00} \hat{I}_{ij}^{10*} + \hat{I}_{ij}^{10} \hat{I}_{ij}^{00*}) \right. \\
& - \left(\frac{6k+18}{1-e_t^2} \right) (n^{2k+6} |\hat{I}_{ij}^{00}|^2) - \nu (n^{2k+6} (k) |\hat{I}_{ij}^{00}|^2) + \nu n^{2k+6} (\hat{I}_{ij}^{00} \hat{I}_{ij}^{11*} + \hat{I}_{ij}^{11} \hat{I}_{ij}^{00*}) \left. \right] \\
& + (1-4\nu) \left(-\frac{26}{21} \right)^k \left(\frac{1}{k!} \right) \sum_{n=1}^{\infty} n^{2k} h(n, e_t) + (1-4\nu) \left(-\frac{214}{105} \right)^k \left(\frac{1}{k!} \right) \sum_{n=1}^{\infty} n^{2k} k(n, e_t), \quad (4.17)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{R}_{(3k+5/2)L(k)} = & \frac{\pi}{8(k!)} \left(-\frac{214}{105} \right)^k \sum_{n=1}^{\infty} \left[\left(\frac{6k+21}{1-e_t^2} \right) \left(\sum_{p=-2,2} n^{2k+6} p |\hat{I}_{ij}^{00}|^2 \right) + n^{2k+7} (\hat{I}_{ij}^{00} \hat{I}_{ij}^{10*} + \hat{I}_{ij}^{10} \hat{I}_{ij}^{00*}) \right. \\
& - \left(\frac{6k+21}{1-e_t^2} \right) (n^{2k+7} |\hat{I}_{ij}^{00}|^2) - \frac{\nu}{2} (n^{2k+7} (2k+1) |\hat{I}_{ij}^{00}|^2) + \nu n^{2k+7} (\hat{I}_{ij}^{00} \hat{I}_{ij}^{11*} + \hat{I}_{ij}^{11} \hat{I}_{ij}^{00*}) \left. \right] \\
& + (1-4\nu) \left(-\frac{26}{21} \right)^k \left(\frac{2\pi}{k!} \right) \sum_{n=1}^{\infty} n^{2k+1} h(n, e_t) + (1-4\nu) \left(-\frac{214}{105} \right)^k \left(\frac{2\pi}{k!} \right) \sum_{n=1}^{\infty} n^{2k+1} k(n, e_t). \quad (4.18)
\end{aligned}$$

In these expressions (and in the angular momentum analogs that will follow), we emphasize once again that the validity of the portion from MQ12, which determines in part the linear-in- ν piece of the flux, depends on the conjecture made in the previous subsection. If that supposition were to fail at some PN order, these expressions would not be accurate at first order in ν but would, of course, continue to be valid for the $\mathcal{O}(\nu^0)$ portion.

The last essential consideration when using these expressions to generate high-order eccentricity functions or power series is that of their eccentricity singular behavior. As mentioned in Sec. III D, past work [7–9,18,32] shows that each 1PN logarithm will be characterized by a divergence as $e_t \rightarrow 1$ in the form of an eccentricity singular factor. For

PN order r , that singular factor will have the form $(1 - e_t^2)^{-(r+7/2)}$. In fact, once we account for the presence of a singular factor $(1 - e_t^2)^{-(3k+9/2)}$, we find closed-form expressions for the integer-order terms $\mathcal{R}_{(3k+1)L(k)}$. The half-integer sequence $\mathcal{R}_{(3k+5/2)L(k)}$ almost surely admits no closed representations. However, here too the removal of the singular factor $(1 - e_t^2)^{-(3k+6)}$ is beneficial, and leads to a remaining power series that is convergent as $e_t \rightarrow 1$. We have demonstrated convergence in these terms to 22PN through direct eccentricity expansion to high order.

Returning to the assembly of the entire flux terms, the terms in the angular momentum 1PN log sequences are given by

$$\mathcal{Z}_i = \mathcal{Z}_i^{\text{MQ01}} + \mathcal{Z}_i^{\text{MQ02}} + \mathcal{Z}_i^{\text{MQ03}} + \nu (\mathcal{Z}_i^{\text{MQ11}} + \mathcal{Z}_i^{\text{MQ12}}) + \mathcal{Z}_i^{\text{MO}} + \mathcal{Z}_i^{\text{CQ}}, \quad (4.19)$$

which for the integer-order sequence can be shown to be

$$\begin{aligned}
\mathcal{Z}_{(3k+1)L(k)} = & \frac{-i}{8(k!)} \left(-\frac{214}{105} \right)^k \epsilon_{3jl} \sum_{n=1}^{\infty} \left[\left(\frac{6k+15}{1-e_t^2} \right) \left(\sum_{p=-2,2} n^{2k+4} p \hat{I}_{ja}^{00} \hat{I}_{la}^{00*} \right) + n^{2k+5} (\hat{I}_{ja}^{00} \hat{I}_{la}^{10*} + \hat{I}_{ja}^{10} \hat{I}_{la}^{00*}) \right. \\
& - \left(\frac{6k+15}{1-e_t^2} \right) (n^{2k+5} \hat{I}_{ja}^{00} \hat{I}_{la}^{00*}) - \nu (n^{2k+5} (k) \hat{I}_{ja}^{00} \hat{I}_{la}^{00*}) + \nu n^{2k+5} (\hat{I}_{ja}^{00} \hat{I}_{la}^{11*} + \hat{I}_{ja}^{11} \hat{I}_{la}^{00*}) \left. \right] \\
& + (1-4\nu) \left(-\frac{26}{21} \right)^k \left(\frac{1}{k!} \right) \sum_{n=1}^{\infty} n^{2k} \tilde{h}(n, e_t) + (1-4\nu) \left(-\frac{214}{105} \right)^k \left(\frac{1}{k!} \right) \sum_{n=1}^{\infty} n^{2k} \tilde{k}(n, e_t), \quad (4.20)
\end{aligned}$$

and for the half-integer-order sequence becomes

$$\begin{aligned}
\mathcal{Z}_{(3k+5/2)L(k)} = & \frac{-\pi i}{4(k!)} \left(-\frac{214}{105}\right)^k \epsilon_{3jl} \sum_{n=1}^{\infty} \left[\left(\frac{6k+18}{1-e_t^2}\right) \left(\sum_{p=-2,2} n^{2k+5} p \hat{I}_{ja}^{00} \hat{I}_{la}^{00*} \right) + n^{2k+6} (\hat{I}_{ja}^{00} \hat{I}_{la}^{10*} + \hat{I}_{ja}^{10} \hat{I}_{la}^{00*}) \right. \\
& - \left. \left(\frac{6k+18}{1-e_t^2}\right) (n^{2k+6} \hat{I}_{ja}^{00} \hat{I}_{la}^{00*}) - \frac{\nu}{2} (n^{2k+6} (2k+1) \hat{I}_{ja}^{00} \hat{I}_{la}^{00*}) + \nu n^{2k+6} (\hat{I}_{ja}^{00} \hat{I}_{la}^{11*} + \hat{I}_{ja}^{11} \hat{I}_{la}^{00*}) \right] \\
& + (1-4\nu) \left(-\frac{26}{21}\right)^k \left(\frac{2\pi}{k!}\right) \sum_{n=1}^{\infty} n^{2k+1} \tilde{h}(n, e_t) + (1-4\nu) \left(-\frac{214}{105}\right)^k \left(\frac{2\pi}{k!}\right) \sum_{n=1}^{\infty} n^{2k+1} \tilde{k}(n, e_t). \quad (4.21)
\end{aligned}$$

To reduce further, the relevant singular factors, which are respectively $(1-e_t^2)^{-(3k+3)}$ and $(1-e_t^2)^{-(3k+9/2)}$, would be pulled out. While it is difficult to see until after that step and after the source multipoles are inserted and expanded, the integer-order flux terms all produce residual polynomials in e_t^2 while the half-integer-order terms have residual convergent power series.

E. Some explicit results from the 1PN log sequences

These formulas can now be utilized to generate explicit eccentricity functions or power series for higher-order members of the 1PN log sequences. In fact, each term from 4PN to 8.5PN at lowest order in ν has already been calculated to high order in Darwin e in a companion paper

[9] to this one and Paper I. Those results were obtained by combining BHPT numerical calculations with the partial sum of least squares integer-relation algorithm on an lmn mode basis to extract the coefficients in analytic form. The eccentricity functions in that paper (upon conversion from e to e_t) provide a valuable check on our results. Unfortunately, the portions at next order in ν cannot be similarly validated by BHPT yet and thus remain a conjecture as discussed in the previous two subsections.

We consider first the pair of fluxes at 4PN log order, $\mathcal{R}_{4L}$ and $\mathcal{Z}_{4L}$, which are the second elements in the integer-order 1PN log sequences. With the appropriate eccentricity singular function removed, we find that each provides a closed-form expression

$$\begin{aligned}
\mathcal{R}_{4L}(e_t, \nu) = & \frac{1}{(1-e_t^2)^{15/2}} \left(\frac{232597}{8820} - \frac{1020553e_t^2}{5880} - \frac{85136197e_t^4}{35280} - \frac{194295169e_t^6}{70560} - \frac{570319469e_t^8}{1128960} - \frac{1677429e_t^{10}}{250880} \right) \\
& + \frac{\nu}{(1-e_t^2)^{15/2}} \left(\frac{34889}{735} + \frac{7327462e_t^2}{6615} + \frac{1218464e_t^4}{315} + \frac{10895825e_t^6}{3528} + \frac{89571401e_t^8}{169344} + \frac{284187e_t^{10}}{31360} \right), \quad (4.22)
\end{aligned}$$

$$\begin{aligned}
\mathcal{Z}_{4L}(e_t, \nu) = & \frac{1}{(1-e_t^2)^6} \left(\frac{232597}{8820} - \frac{1761619e_t^2}{8820} - \frac{6412241e_t^4}{7056} - \frac{22800487e_t^6}{70560} - \frac{803961e_t^8}{125440} \right) \\
& + \frac{\nu}{(1-e_t^2)^6} \left(\frac{34889}{735} + \frac{2961167e_t^2}{4410} + \frac{4210447e_t^4}{3528} + \frac{12389779e_t^6}{35280} + \frac{413339e_t^8}{47040} \right). \quad (4.23)
\end{aligned}$$

The order ν^0 part of $\mathcal{R}_{4L}(e_t)$ was previously discovered and described in [7] [actually as a closed-form function $\mathcal{L}_{4L}(e)$ in e which is easily converted from e to e_t to compare to $\mathcal{R}_{4L}(e_t)$]. The order ν^0 part of $\mathcal{Z}_{4L}(e_t)$ was also effectively previously found [38] [again as a closed-form function $\mathcal{J}_{4L}(e)$ in e , convertible to $\mathcal{Z}_{4L}(e_t)$].

Turning next to the 5.5PN log fluxes, which are the second elements in the half-integer-order 1PN log sequences, we find a pair of convergent infinite series that begin with

$$\begin{aligned}
\mathcal{R}_{11/2L}(e_t, \nu) = & \frac{1}{(1-e_t^2)^9} \left(\frac{177293}{2352} - \frac{33467083e_t^2}{35280} - \frac{55188741467e_t^4}{2257920} - \frac{41837812001e_t^6}{677376} - \frac{90806230499893e_t^8}{2601123840} \right. \\
& - \frac{52599122781397e_t^{10}}{13005619200} - \frac{743899262983e_t^{12}}{18874368000} - \frac{1035559450883e_t^{14}}{32774160384000} + \dots \Big) \\
& + \frac{\nu}{(1-e_t^2)^9} \left(\frac{2880637}{8820} + \frac{168033239e_t^2}{15120} + \frac{107321219093e_t^4}{1693440} + \frac{132390912895e_t^6}{1354752} + \frac{86038451049547e_t^8}{1950842880} \right. \\
& + \frac{956641092501793e_t^{10}}{195084288000} + \frac{223041145823171e_t^{12}}{3745618329600} + \frac{493441625881693e_t^{14}}{3670705963008000} + \dots \Big),
\end{aligned}$$

$$\begin{aligned}
\mathcal{Z}_{11/2L}(e_t, \nu) = & \frac{1}{(1 - e_t^2)^{15/2}} \left(\frac{177293}{2352} - \frac{31011679e_t^2}{23520} - \frac{8782000063e_t^4}{752640} - \frac{11351372383e_t^6}{903168} - \frac{661373524417e_t^8}{289013760} \right. \\
& - \frac{102517313999e_t^{10}}{3715891200} - \frac{8739859767749e_t^{12}}{8323596288000} + \frac{70533378963781e_t^{14}}{349591044096000} + \dots \Big) \\
& + \frac{\nu}{(1 - e_t^2)^{15/2}} \left(\frac{2880637}{8820} + \frac{349633e_t^2}{49} + \frac{127083053e_t^4}{5376} + \frac{9184388503e_t^6}{508032} + \frac{647130113243e_t^8}{216760320} \right. \\
& + \frac{199406895011e_t^{10}}{4064256000} + \frac{214645510549e_t^{12}}{2080899072000} - \frac{3051343498747e_t^{14}}{152946081792000} + \dots \Big). \quad (4.24)
\end{aligned}$$

The third elements in the integer-order 1PN log sequences are the 7PN $\log^2(x)$ fluxes. These flux contributions also have closed-form expressions, as anticipated

$$\begin{aligned}
\mathcal{R}_{7L2}(e_t, \nu) = & \frac{1}{(1 - e_t^2)^{21/2}} \left(-\frac{52525903}{617400} - \frac{3278576291e_t^2}{3704400} + \frac{74771270471e_t^4}{2116800} + \frac{6205145514833e_t^6}{29635200} + \frac{1870078245281e_t^8}{6773760} \right. \\
& + \frac{6721297402199e_t^{10}}{67737600} + \frac{728355846523e_t^{12}}{90316800} + \frac{976909819e_t^{14}}{16859136} \Big) \\
& + \frac{\nu}{(1 - e_t^2)^{21/2}} \left(\frac{146507731}{463050} + \frac{11066346794e_t^2}{694575} + \frac{32900565808e_t^4}{231525} + \frac{8296487145703e_t^6}{22226400} + \frac{401207614757e_t^8}{1209600} \right. \\
& + \frac{164075647087e_t^{10}}{1693440} + \frac{30213176177e_t^{12}}{4064256} + \frac{812183599e_t^{14}}{12644352} \Big), \\
\mathcal{Z}_{7L2}(e_t, \nu) = & \frac{1}{(1 - e_t^2)^9} \left(-\frac{52525903}{617400} + \frac{1285805299e_t^2}{1852200} + \frac{346615894427e_t^4}{14817600} + \frac{449325541709e_t^6}{7408800} + \frac{327681660679e_t^8}{9483264} \right. \\
& + \frac{189470991041e_t^{10}}{47416320} + \frac{1602170813e_t^{12}}{42147840} \Big) + \frac{\nu}{(1 - e_t^2)^9} \left(-\frac{146507731}{463050} - \frac{9851216819e_t^2}{926100} \right. \\
& - \frac{22077622729e_t^4}{3704400} - \frac{332158696507e_t^6}{3704400} - \frac{463335196631e_t^8}{11854080} - \frac{97794785317e_t^{10}}{23708160} - \frac{161911009e_t^{12}}{3512320} \Big). \quad (4.25)
\end{aligned}$$

At order ν^0 these functions and power series show complete agreement with those found using BHPT fitting. The convergent power series for $\mathcal{R}_{11/2L}$ and $\mathcal{Z}_{11/2L}$ were verified to e_t^{30} in the power series expansion and those for $\mathcal{R}_{17/2L2}$ and $\mathcal{Z}_{17/2L2}$ were checked and verified to order e_t^{20} . Additionally, we extended the validation to 22PN at the level e_t^{10} by combining BHPT results with Johnson-McDaniel's S_{lmn} factorization [39] (see Sec. IVD of Paper I), again at $\mathcal{O}(\nu^0)$. As we will explain in the next subsection, we now have the means to compute these and all other members of the 1PN log series to at least e_t^{120} with manageable computational cost.

F. Discussion

To summarize, despite an increase in calculational complexity, the pair of 1PN log sequences (shown in blue in Fig. 1) are determined in their entirety by a few low-order source multipoles—namely, the Newtonian mass octupole and current quadrupole moments and the 1PN-order mass

quadrupole moment. This behavior is exactly analogous to, if more complicated than, the way the Newtonian quadrupole moment provided all the information necessary to derive all elements of the leading-log sequences (as shown in Paper I). The Fourier amplitudes of these moments appear in sums as complex products weighted by successively higher powers of n , the harmonics of orbital frequency that are present in eccentric motion. As such, these terms represent in the time domain higher and higher order time derivatives of the low-order source multipole moments.

The greater complexity is due in part to the fact that the 1PN quadrupole moment gives rise to five different sums over squares of Fourier amplitudes. In compensation, however, simplifying patterns emerge amongst these sums. For example, we found an exact correspondence between the higher-order quadrupole sums MQ01 and MQ11 and the sums over the Newtonian-order quadrupole moment in the leading-logarithm sequence. Specifically, the substitution $I^{00} \rightarrow I^{10}$ or $I^{00} \rightarrow I^{11}$ in terms where the former

appears, along with changes in the normalization, leads to parts of the flux at 1PN order higher. Secondly, a relationship exists between the sums we denoted by MQ02 and MQ03, which are related to the 1PN correction in the frequency Ω_r and the “magnetic” harmonics, p , respectively. The k -dependent prefactors on these sums turn out to be the additive inverse of each other. The reason for this symmetry is that the harmonics (as defined and manipulated in Sec. III B) ultimately satisfy $m = -n$ and $s = -p$, given orthogonality, and so Ω_r and p only appear in the combination $\pm\Omega_r(n + pk)$. Through 1PN order this can be rewritten as $\pm\Omega_\phi[n + (p - n)k]$, which means that a 1PN contribution will emerge with $p - n$ times the rest of the quadrupole factors. We had simply split this into two separate sums originally, with otherwise identical forms.

The open question concerns the sum that we labeled MQ12, which involves the appearance of the I^{00} (Newtonian quadrupole) at next order in the mass ratio and which first arises with the ADM mass at 2.5PN order. As we mentioned in Sec. IV C, in PN theory it is expected that progressively higher powers of the ADM mass will appear in progressively higher corrections to the tail. Thus, we expect that this will lead to a simple factor from the relevant binomial expansion of $(1 - \nu x/2)^q$. However, it is not clear how else the Newtonian quadrupole might manifest at this order in ν . If, for instance, the ADM mass in the tail were the sole appearance of this type of sum, then the partial cancellation between instantaneous and hereditary contributions discussed in [8] would not occur, enhancing any orders with both types of flux by a factor of 3. According to [21], this would include all orders 3PN and above. However, this would leave an unphysical normalization constant r_0 in the full flux (see, for example, Sec. V E), which cannot exist. Therefore, the likeliest possibility is that a corresponding summation exists on the instantaneous side and the cancellation seen at $\mathcal{O}(\nu^0)$ does continue here, leading to the result above. Regardless, further developments in full PN theory or second-order BHPT should soon be able to resolve this question definitively. At that point, even if our conjecture of Sec. IV C fails to hold, the Fourier infrastructure presented here should be able to provide accurate $\mathcal{O}(\nu)$ expansions in eccentricity for all elements of the 1PN log sequences once the correct prefactor is supplied by other means.

Equally important to the generation of high-order expansions is the question of computational implementation and cost. The procedures we describe in this paper turn out to be quite manageable computationally, though the calculation of complete flux terms tends to be more than an order of magnitude more time consuming than the leading logarithm calculations of Paper I. Of the seven required sums, three (MQ02, MQ12, CQ) are roughly equal in expense to the corresponding leading logarithms. Three (MQ03, MQ11, MO) are 1.5–4 times more expensive to compute, owing to their lengthier Bessel function

representations. In any event, calculation of all of these terms only amounts to a matter of at most minutes for computation to hundreds of orders in e_t on an average laptop in *Mathematica*.

However, the remaining summation MQ01, with the 1PN amplitudes ${}_{(n)}I_{ij}^{10}$, is the ultimate bottleneck. As noted in Sec. II D, these Fourier coefficients cannot be expressed cleanly in terms of Bessel functions, and the unevaluated integral in (2.46) is cumbersome to handle. We had partial success in handling it by expanding the integrand in e_t directly before integrating. However, the arctangent function with its complicated argument remained a prime source of difficulty, leading to a series of integrals that can require hours to expand, as well as require large quantities of memory, on cluster computers that support *Mathematica*. We found that a convenient way to proceed was to precompute the expansion of this arctangent function on the University of North Carolina cluster KillDevil to e_t^{120} , a task which required about 1.5 hours and 20 GB of RAM. Once this expansion was calculated, the rest of the process became much more manageable. Indeed, with the arctangent series in hand, we are now able to expand any element in the 1PN log sequences to e_t^{120} via laptop in only a few minutes. This process was used in particular to expand $\psi(e_t)$ and $\tilde{\psi}(e_t)$ to e_t^{120} , enhancement functions which are discussed in [9]. Another difficult function, $\mathcal{R}_4^\chi$ (described below), can also be obtained to e_t^{120} in this manner.

V. DERIVING AN ESSENTIAL PART OF THE 4PN TAIL

Up to now we have focused on the 1PN log sequences of gravitational wave fluxes (depicted by the blue lines in Fig. 1). Drawing upon the frequency domain multipole analysis in Sec. II, we rederived the known 1PN and 2.5PN relative fluxes in Sec. III. We then used that frequency domain approach in Sec. IV to detail the analytic dependence of elements in those sequences to all higher PN orders. What remains, for this section and Sec. VI, is to apply a similar approach to the 4PN log sequences (i.e., the orange lines in Fig. 1).

Like the subleading log sequences of Paper I (what we call here the 3PN logs), the derivation of the form of the 4PN logs requires an assist from BHPT. As Paper I showed, it is possible to find a theoretical explanation for part of each subleading log term (even absent a full PN calculation) that is based merely on knowledge of the Newtonian quadrupole moment. The remaining part of each subleading log term can then in principle be determined, at lowest order in ν , by BHPT. A similar useful split carries over to the elements in the 4PN log sequences, though it requires the 1PN source multipoles.

Because the process is somewhat involved, we focus primarily on illustrating how it is applied to the 4PN nonlog fluxes, $\mathcal{R}_4(e_t)$ and $\mathcal{Z}_4(e_t)$, the first elements in the

integer-order 4PN log sequences. (Section VI also briefly touches on the 5.5PN nonlog term, which is the first element in the half-integer-order 4PN log sequence.) We find that an essential tail portion of these 4PN terms is theoretically determined by the same 1PN source multipoles that were discussed in Sec. II. Deriving that tail portion is the subject of this section. Once this essential 4PN tail portion is known, we combine it with knowledge of the 4PN log flux from Sec. IV and results [9] from BHPT to determine the entire analytic form of the 4PN nonlog fluxes $\mathcal{R}_4(e_t)$ and $\mathcal{Z}_4(e_t)$ to high order in an expansion in eccentricity. This result is timely, as it will provide a

valuable check for those working to extend PN theory to a full description of the orbital mechanics and radiative losses at 4PN.

The portion of the 4PN tail to be addressed provides the 1PN correction to the 3PN enhancement function $\chi(e_t)$ [14]. This portion of the full tail is provided by the sum of the tail² and tail-of-tails corrections to the flux, and is determined by the 1PN source multipoles. The mass octupole and current quadrupole orbital computations will remain at Newtonian order, mirroring the derivation of $\chi(e_t)$ itself in [14]. However, as usual, the mass quadrupole part requires extension to 1PN, as discussed in Sec. II.

A. Mass octupole

For the mass octupole the quadratic in $\mathcal{M}$ portions of the energy flux tail have the following time domain expressions:

$$\mathcal{P}_\infty^{\text{MO(tail)}^2} = \frac{4\mathcal{M}^2}{189} \left\{ \int_0^\infty I_{ijk}^{(6)}(t-\tau) \left[\log\left(\frac{\tau}{2r_0}\right) + \frac{97}{60} \right] d\tau \right\}^2, \quad (5.1)$$

$$\mathcal{P}_\infty^{\text{MO(tail-of-tails)}} = \frac{4\mathcal{M}^2}{189} I_{ijk}^{(4)}(t) \int_0^\infty I_{ijk}^{(7)}(t-\tau) \left[\log\left(\frac{\tau}{2r_0}\right)^2 + \frac{183}{70} \log\left(\frac{\tau}{2r_0}\right) + \frac{13283}{8820} \right] d\tau, \quad (5.2)$$

where the tail-of-tail coefficients were taken from Eq. (4.9a) in [37] with constant b set to r_0 . Note that a factor of 2 is pulled from their equation, with another factor of 2 coming from the polynomial product $U_L U_L$.

The MO part of the tail² term can be evaluated using the integral identity (3.24). Then, because $k = \Omega_\varphi/\Omega_r - 1 = 0$ and $\mathcal{M} = M$ for a Newtonian orbit, the time average of the MO tail² term can be simplified to

$$\langle \mathcal{P}_\infty^{\text{MO(tail)}^2} \rangle = \frac{8M^2}{189} \sum_{n=1}^\infty (\Omega_\varphi)^{10} n^{10} |I_{ijk}^{00}|_{(n)}^2 \left[\frac{\pi^2}{4} + \left(\log(2\Omega_r|n|r_0) + \gamma_E - \frac{97}{60} \right)^2 \right]. \quad (5.3)$$

The tail-of-tails term requires a bit more work. First, the $\log^2$ piece must be handled using the following integral identity [8,14,32]:

$$\int_0^\infty e^{i(n-sk)\Omega_r\tau} \log\left(\frac{\tau}{2r_0}\right)^2 d\tau = \frac{i}{(n-sk)\Omega_r} \left[\frac{\pi^2}{6} + \left(\frac{\pi i}{2} \text{sign}(-n) + \log(2\Omega_r|n-sk|r_0) + \gamma_E \right)^2 \right] \quad (5.4)$$

$$\approx \frac{i}{n\Omega_r} \left[-\frac{\pi^2}{12} + (\log(2\Omega_\varphi|n|r_0) + \gamma_E)^2 + \pi i \text{sign}(-n)(\log(2\Omega_\varphi|n|r_0) + \gamma_E) \right], \quad (5.5)$$

where in the second line we set $k = 0$ and made a lowest-order PN expansion. When the various factors of i and n are considered, it becomes clear that the last term in (5.5) cancels in a sum over positive and negative n . Once combined with the rest of the integral, the total tail-of-tails contribution has the following time average:

$$\langle \mathcal{P}_\infty^{\text{MO(tail-of-tails)}} \rangle = \frac{8M^2}{189} \sum_{n=1}^\infty (\Omega_\varphi)^{10} n^{10} |I_{ijk}^{00}|_{(n)}^2 \left[\frac{\pi^2}{12} - (\log(2\Omega_\varphi|n|r_0) + \gamma_E)^2 + \frac{183}{70} (\log(2\Omega_\varphi|n|r_0) + \gamma_E) - \frac{13283}{8820} \right]. \quad (5.6)$$

Then, (5.3) and (5.6) are summed to yield the complete mass octupole flux contribution

$$\langle \mathcal{P}_\infty^{\text{MO(tail)}^2 + \text{(tail-of-tails)}} \rangle = \frac{8M^2}{189} (\Omega_\varphi)^{10} \sum_{n=1}^\infty n^{10} |I_{ijk}^{00}|_{(n)}^2 \left[\frac{\pi^2}{3} - \frac{13}{21} (\log(2\Omega_\varphi|n|r_0) + \gamma_E) + \frac{21709}{19600} \right]. \quad (5.7)$$

Likewise, the angular momentum expressions have the following time-dependent forms:

$$\mathcal{G}_\infty^{\text{MO}(\text{tail})^2} = \frac{4\mathcal{M}^2}{63} \epsilon_{3jl} \left\{ \int_0^\infty I_{jab}^{(5)}(t-\tau) \left[\log\left(\frac{\tau}{2r_0}\right) + \frac{97}{60} \right] d\tau \right\} \times \left\{ \int_0^\infty I_{lab}^{(6)}(t-\tau) \left[\log\left(\frac{\tau}{2r_0}\right) + \frac{97}{60} \right] d\tau \right\} \hat{z}, \quad (5.8)$$

$$\begin{aligned} \mathcal{G}_\infty^{\text{MO}(\text{tail-of-tails})} &= \frac{2\mathcal{M}^2}{63} \epsilon_{3jl} \left\{ I_{jab}^{(3)} \int_0^\infty I_{lab}^{(7)}(t-\tau) \left[\log\left(\frac{\tau}{2r_0}\right)^2 + \frac{183}{70} \log\left(\frac{\tau}{2r_0}\right) + \frac{13283}{8820} \right] d\tau \right. \\ &\quad \left. + I_{lab}^{(4)} \int_0^\infty I_{jab}^{(6)}(t-\tau) \left[\log\left(\frac{\tau}{2r_0}\right)^2 + \frac{183}{70} \log\left(\frac{\tau}{2r_0}\right) + \frac{13283}{8820} \right] d\tau \right\} \hat{z}. \end{aligned} \quad (5.9)$$

These merge together in the same way to generate the complete mass octupole flux contribution

$$\langle \mathcal{G}_\infty^{\text{MO}(\text{tail})^2 + (\text{tail-of-tails})} \rangle = -\frac{8M^2 i}{63} (\Omega_\varphi)^9 \epsilon_{3jl} \sum_{n=1}^\infty n^9 I_{jab}^{00} I_{lab}^{00*} \left[\frac{\pi^2}{3} - \frac{13}{21} (\log(2\Omega_\varphi |n| r_0) + \gamma_E) + \frac{21709}{19600} \right] \hat{z}. \quad (5.10)$$

B. Current quadrupole

The next component of the quadratic-in- $\mathcal{M}$ 4PN tail stems from the Newtonian current quadrupole. The energy and angular momentum time domain representations are

$$\begin{aligned} \mathcal{P}_\infty^{\text{CQ}(\text{tail})^2} &= \frac{64\mathcal{M}^2}{45} \left\{ \int_0^\infty J_{ij}^{(5)}(t-\tau) \left[\log\left(\frac{\tau}{2r_0}\right) + \frac{7}{6} \right] d\tau \right\}^2, \\ \mathcal{P}_\infty^{\text{CQ}(\text{tail-of-tails})} &= \frac{64\mathcal{M}^2}{45} J_{ij}^{(3)}(t) \int_0^\infty J_{ij}^{(6)}(t-\tau) \left[\log\left(\frac{\tau}{2r_0}\right)^2 + \frac{46}{35} \log\left(\frac{\tau}{2r_0}\right) - \frac{26254}{22050} \right] d\tau, \end{aligned} \quad (5.11)$$

and

$$\mathcal{G}_\infty^{\text{CQ}(\text{tail})^2} = \frac{128\mathcal{M}^2}{45} \epsilon_{3jl} \left\{ \int_0^\infty J_{ja}^{(4)}(t-\tau) \left[\log\left(\frac{\tau}{2r_0}\right) + \frac{7}{6} \right] d\tau \right\} \times \left\{ \int_0^\infty J_{la}^{(5)}(t-\tau) \left[\log\left(\frac{\tau}{2r_0}\right) + \frac{7}{6} \right] d\tau \right\} \hat{z}, \quad (5.12)$$

$$\begin{aligned} \mathcal{G}_\infty^{\text{CQ}(\text{tail-of-tails})} &= \frac{64\mathcal{M}^2}{45} \epsilon_{3jl} \left\{ J_{ja}^{(2)} \int_0^\infty J_{la}^{(6)}(t-\tau) \left[\log\left(\frac{\tau}{2r_0}\right)^2 + \frac{46}{35} \log\left(\frac{\tau}{2r_0}\right) - \frac{26254}{22050} \right] d\tau \right. \\ &\quad \left. + J_{la}^{(3)} \int_0^\infty J_{ja}^{(5)}(t-\tau) \left[\log\left(\frac{\tau}{2r_0}\right)^2 + \frac{46}{35} \log\left(\frac{\tau}{2r_0}\right) - \frac{26254}{22050} \right] d\tau \right\} \hat{z}, \end{aligned} \quad (5.13)$$

respectively. Again, the particular forms the two tails-of-tails were adapted from [37]. The time averaged fluxes are then found to be

$$\langle \mathcal{P}_\infty^{\text{CQ}(\text{tail})^2 + (\text{tail-of-tails})} \rangle = \frac{128M^2}{45} \sum_{n=1}^\infty (\Omega_\varphi)^8 n^8 |I_{ij}^{00}|^2 \left[\frac{\pi^2}{3} - \frac{107}{105} (\log(2\Omega_\varphi |n| r_0) + \gamma_E) + \frac{7517}{44100} \right], \quad (5.14)$$

$$\langle \mathcal{G}_\infty^{\text{CQ}(\text{tail})^2 + (\text{tail-of-tails})} \rangle = -\frac{256M^2}{45} i \epsilon_{3jl} \sum_{n=1}^\infty (\Omega_\varphi)^7 n^7 I_{ja}^{00} I_{la}^{00*} \left[\frac{\pi^2}{3} - \frac{107}{105} (\log(2\Omega_\varphi |n| r_0) + \gamma_E) + \frac{7517}{44100} \right]. \quad (5.15)$$

C. Mass quadrupole, lowest order in ν

The remaining order $\mathcal{M}^2$ part of the 1PN correction to the tail² and the tail-of-tails terms comes from the 1PN correction to the mass quadrupole moment.

1. The energy flux tail²

In the time domain, the mass quadrupole part of the tail² is given by [14]

$$\mathcal{P}_{\infty}^{\text{MQ(tail)}^2} = \frac{4\mathcal{M}^2}{5} \left\{ \int_0^{\infty} I_{ij}^{(5)}(t-\tau) \left[\log\left(\frac{\tau}{2r_0}\right) + \frac{11}{12} \right] d\tau \right\}^2. \quad (5.16)$$

When the quadrupole moment is taken to leading (Newtonian) order, this term contributes to the 3PN hereditary flux. By taking the calculation to one PN order higher approximation, we can obtain its contribution to the 4PN flux. To do so, we plug in the biperiodic Fourier expansion for the quadrupole moment along with the expansion for the ADM mass, replace the time derivatives with powers of the frequency, and take the time average. An intermediate step in the calculation is

$$\begin{aligned} \mathcal{P}_{\infty}^{\text{MQ(tail)}^2} &= \frac{4M^2}{5} \sum_{n=-\infty}^{\infty} \sum_{p=-2}^2 (\Omega_r)^{10} (n^{10} + 10n^9 pk) I_{ij}^{(n,p)} I_{ij}^{(-n,-p)} \left\{ \int_0^{\infty} e^{i(n+pk)\Omega_r\tau} \left[\log\left(\frac{\tau}{2r_0}\right) + \frac{11}{12} \right] d\tau \right\} \\ &\times \left\{ \int_0^{\infty} e^{-i(n+pk)\Omega_r\tau} \left[\log\left(\frac{\tau}{2r_0}\right) + \frac{11}{12} \right] d\tau \right\}. \end{aligned} \quad (5.17)$$

The product of integrals can be simplified through a double application of (3.24). Collecting the results of an expansion through first order reduces the product of integrals to

$$\frac{1}{n^2\Omega_r^2} \left(\frac{\pi^2}{4} + \beta_0^2 \right) - \frac{1}{2n^2\Omega_r^2} \left[4\beta_0 + \frac{p}{n} (\pi^2 + 4\beta_0^2 - 4\beta_0) \right] k, \quad (5.18)$$

where we define $\beta_0 \equiv \log(2\Omega_{\varphi}|n|r_0) + \gamma_E - 11/12$. The final result reduces to three compact sums:

$$\langle \mathcal{P}_{\infty}^{\text{MQ01(tail)}^2} \rangle = \frac{2M^2(\Omega_{\varphi})^8}{5} \sum_{n=1}^{\infty} (n^8) (I_{ij}^{10} I_{ij}^{00*} + I_{ij}^{00} I_{ij}^{10*}) (\pi^2 + 4\beta_0^2), \quad (5.19)$$

$$\langle \mathcal{P}_{\infty}^{\text{MQ02(tail)}^2} \rangle = -\frac{48M^2x(\Omega_{\varphi})^8}{5(1-e_t^2)} \sum_{n=1}^{\infty} n^8 |I_{ij}^{00}|^2 (\pi^2 + 4\beta_0^2 + \beta_0), \quad (5.20)$$

$$\langle \mathcal{P}_{\infty}^{\text{MQ03(tail)}^2} \rangle = \frac{48M^2x(\Omega_{\varphi})^8}{5(1-e_t^2)} \sum_{n=1}^{\infty} \sum_p n^7 p |I_{ij}^{00}|^2 (\pi^2 + 4\beta_0^2 + \beta_0). \quad (5.21)$$

2. The energy flux tail-of-tails

The mass quadrupole part of the tail-of-tails time-dependent flux is given by [14]

$$\mathcal{P}_{\infty}^{\text{MQ(tail-of-tails)}} = \frac{4\mathcal{M}^2}{5} I_{ij}^{(3)} \int_0^{\infty} I_{ij}^{(6)}(t-\tau) \left[\log\left(\frac{\tau}{2r_0}\right)^2 + \frac{57}{70} \log\left(\frac{\tau}{2r_0}\right) + \frac{124627}{44100} \right] d\tau. \quad (5.22)$$

When the quadrupole moment is calculated to Newtonian order, this gives a hereditary contribution to the 3PN flux. The 4PN contribution we seek comes from considering 1PN orbital dynamics and the mass quadrupole through 1PN order. The usual Fourier simplifications lead to

$$\frac{4M^2}{5} \sum_{n,p} (\Omega_r)^9 i(n^9 + 9n^8 p) I_{ij}^{(n,p)} I_{ij}^{(-n,-p)} \int_0^{\infty} e^{i(n+pk)\Omega_r\tau} \left[\log\left(\frac{\tau}{2r_0}\right)^2 + \frac{57}{70} \log\left(\frac{\tau}{2r_0}\right) + \frac{124627}{44100} \right] d\tau. \quad (5.23)$$

To handle the $\log^2$ term, we expand the integral identity (5.4) to first order in k , giving

$$\int_0^\infty e^{i(n+pk)\Omega_r\tau} \log\left(\frac{\tau}{2r_0}\right)^2 d\tau = \frac{i}{(n+pk)\Omega_r} \left[\frac{\pi^2}{6} + \left(\frac{\pi i}{2} \text{sign}(-n) + \log(2\Omega_r|n+pk|r_0) + \gamma_E \right)^2 \right] \quad (5.24)$$

$$\approx \frac{i}{n\Omega_r} \left[\alpha_0^2 - \frac{\pi^2}{12} - \pi i \text{sign}(n) \alpha_0 \right] + \frac{i}{n^2\Omega_r} \left[\left(\frac{\pi^2}{12} - \alpha_0^2 + 2\alpha_0 - (1 - \alpha_0)\pi i \text{sign}(n) \right) p - 2n\alpha_0 + n\pi i \text{sign}(n) \right] k, \quad (5.25)$$

where $\alpha_0 \equiv \log(2\Omega_\varphi|n|r_0) + \gamma_E$. The rest of the integral can be found using (3.24). In all cases the terms with $\text{sign}(\pm n)$ will vanish in sums over positive and negative n , so those are dropped in what follows. We combine what is left with the other terms in the integrand of (5.23) to get the total contribution from that integral

$$\frac{i}{n\Omega_r} \left(\alpha_0^2 - \frac{57}{70} \alpha_0 - \frac{\pi^2}{12} + \frac{124627}{44100} \right) + \frac{i}{n\Omega_r} \left[-2\alpha_0 + \frac{57}{70} - \frac{p}{n} \left(\alpha_0^2 - \frac{197}{70} \alpha_0 - \frac{\pi^2}{12} + \frac{160537}{44100} \right) \right] k. \quad (5.26)$$

With this factor reinserted in the expression for the flux, the tail-of-tails can be separated at 1PN order into the now-familiar three sums

$$\langle \mathcal{P}_\infty^{\text{MQ01(tail-of-tails)}} \rangle = \frac{8M^2(\Omega_\varphi)^8}{5} \sum_{n=1}^\infty n^8 \begin{pmatrix} I_{ij}^{10} & I_{ij}^{00*} \\ (n) & (n) \end{pmatrix} \begin{pmatrix} I_{ij}^{00} & I_{ij}^{10*} \\ (n) & (n) \end{pmatrix} \left[\frac{\pi^2}{12} - \alpha_0^2 + \frac{57}{70} \alpha_0 - \frac{124627}{44100} \right], \quad (5.27)$$

$$\langle \mathcal{P}_\infty^{\text{MQ02(tail-of-tails)}} \rangle = -\frac{24M^2x}{5(1-e_t^2)} \sum_{n=1}^\infty (\Omega_\varphi)^8 n^8 |I_{ij}^{00}|^2 \left[\frac{2\pi^2}{3} - 8\alpha_0^2 + \frac{158}{35} \alpha_0 - \frac{480553}{22050} \right], \quad (5.28)$$

$$\langle \mathcal{P}_\infty^{\text{MQ03(tail-of-tails)}} \rangle = \frac{24M^2x(\Omega_\varphi)^8}{5(1-e_t^2)} \sum_{n=1}^\infty \sum_p n^7 p |I_{ij}^{00}|^2 \left[\frac{2\pi^2}{3} - 8\alpha_0^2 + \frac{158}{35} \alpha_0 - \frac{480553}{22050} \right]. \quad (5.29)$$

3. Summing the tail² and tail-of-tails

We can now combine the sums of corresponding type from the tail-of-tail and tail² parts into one set of 1PN mass quadrupole contributions. We find that upon fusing the tail pieces all of the $\log^2$ terms (i.e., α_0^2 terms) vanish. The result is

$$\langle \mathcal{P}_\infty^{\text{MQ01(tail)}^2 + \text{(tail-of-tails)}} \rangle = M^2(\Omega_\varphi)^8 \sum_{n=1}^\infty n^8 \begin{pmatrix} I_{ij}^{10} & I_{ij}^{00*} \\ (n) & (n) \end{pmatrix} \begin{pmatrix} I_{ij}^{00} & I_{ij}^{10*} \\ (n) & (n) \end{pmatrix} \left[\frac{8\pi^2}{15} - \frac{856}{525} (\log(2\Omega_\varphi|n|r_0) + \gamma_E) - \frac{116761}{36750} \right], \quad (5.30)$$

$$\langle \mathcal{P}_\infty^{\text{MQ02(tail)}^2 + \text{(tail-of-tails)}} \rangle = -\frac{M^2x(\Omega_\varphi)^8}{(1-e_t^2)} \sum_{n=1}^\infty n^8 |I_{ij}^{00}|^2 \left[\frac{64\pi^2}{5} - \frac{6848}{175} (\log(2\Omega_\varphi|n|r_0) + \gamma_E) - \frac{497004}{6125} \right], \quad (5.31)$$

$$\langle \mathcal{P}_\infty^{\text{MQ03(tail)}^2 + \text{(tail-of-tails)}} \rangle = \frac{M^2x(\Omega_\varphi)^8}{(1-e_t^2)} \sum_{n=1}^\infty \sum_p n^7 p |I_{ij}^{00}|^2 \left[\frac{64\pi^2}{5} - \frac{6848}{175} (\log(2\Omega_\varphi|n|r_0) + \gamma_E) - \frac{497004}{6125} \right]. \quad (5.32)$$

4. The angular momentum tail flux

On the angular momentum side, the time-dependent tail² and tail-of-tails fluxes take the following forms:

$$\mathcal{G}_\infty^{\text{MQ(tail)}^2} = \frac{8\mathcal{M}}{5} \epsilon_{3jl} \left\{ \int_0^\infty I_{ja}^{(4)}(t-\tau) \left[\log\left(\frac{\tau}{2r_0}\right) + \frac{11}{12} \right] d\tau \right\} \times \left\{ \int_0^\infty I_{la}^{(5)}(t-\tau) \left[\log\left(\frac{\tau}{2r_0}\right) + \frac{11}{12} \right] d\tau \right\} \hat{z}, \quad (5.33)$$

$$\mathcal{G}_\infty^{\text{MQ}(\text{tail-of-tails})} = \frac{4\mathcal{M}^2}{5} \epsilon_{3jl} \left\{ I_{ja}^{(2)} \int_0^\infty I_{la}^{(6)}(t-\tau) \left[\log\left(\frac{\tau}{2r_0}\right)^2 + \frac{57}{70} \log\left(\frac{\tau}{2r_0}\right) + \frac{124627}{44100} \right] d\tau \right. \\ \left. + I_{la}^{(3)} \int_0^\infty I_{ja}^{(5)}(t-\tau) \left[\log\left(\frac{\tau}{2r_0}\right)^2 + \frac{57}{70} \log\left(\frac{\tau}{2r_0}\right) + \frac{124627}{44100} \right] d\tau \right\} \dot{z}. \quad (5.34)$$

The simplification procedure is nearly identical to that in the energy case, so we jump straight to the three sums that give this essential part of the tail flux

$$\langle \mathcal{G}_\infty^{\text{MQ01}(\text{tail})^2+(\text{tail-of-tails})} \rangle = -M^2(\Omega_\varphi)^7 i \epsilon_{3jl} \sum_{n=1}^\infty n^7 \left(I_{ja}^{10} I_{la}^{00*} + I_{ja}^{00} I_{la}^{10*} \right) \left[\frac{16\pi^2}{15} - \frac{1712}{525} (\log(2\Omega_\varphi |n| r_0) + \gamma_E) - \frac{116761}{18375} \right], \quad (5.35)$$

$$\langle \mathcal{G}_\infty^{\text{MQ02}(\text{tail})^2+(\text{tail-of-tails})} \rangle = \frac{M^2 x(\Omega_\varphi)^7 i}{(1-e_t^2)} \epsilon_{3jl} \sum_{n=1}^\infty n^7 I_{ja}^{00} I_{la}^{00*} \left[\frac{112\pi^2}{5} - \frac{1712}{25} (\log(2\Omega_\varphi |n| r_0) + \gamma_E) - \frac{17903}{125} \right], \quad (5.36)$$

$$\langle \mathcal{G}_\infty^{\text{MQ03}(\text{tail})^2+(\text{tail-of-tails})} \rangle = -\frac{M^2 x(\Omega_\varphi)^7 i}{(1-e_t^2)} \epsilon_{3jl} \sum_{n=1}^\infty \sum_p n^6 p I_{ja}^{00} I_{la}^{00*} \left[\frac{112\pi^2}{5} - \frac{1712}{25} (\log(2\Omega_\varphi |n| r_0) + \gamma_E) - \frac{17903}{125} \right]. \quad (5.37)$$

D. Mass quadrupole, next order in ν

As noted in an earlier section of the paper, only minor adjustments to the above results are required to obtain these parts of the 4PN tail at $\mathcal{O}(\nu)$. The ν correction to the quadrupole moment itself can again be found by simple substitution, and the ν correction to the ADM mass simply provides a factor of (-1) . Thus, these essential parts of the 4PN tail at order ν become

$$\langle \mathcal{P}_\infty^{\text{MQ11}(\text{tail})^2+(\text{tail-of-tails})} \rangle = \nu x M^2(\Omega_\varphi)^8 \sum_{n=1}^\infty (n^8) \left(I_{ij}^{11} I_{ij}^{00*} + I_{ij}^{00} I_{ij}^{11*} \right) \left[\frac{8\pi^2}{15} - \frac{856}{525} (\log(2\Omega_\varphi |n| r_0) + \gamma_E) - \frac{116761}{36750} \right], \quad (5.38)$$

$$\langle \mathcal{P}_\infty^{\text{MQ12}(\text{tail})^2+(\text{tail-of-tails})} \rangle = -\nu x M^2(\Omega_\varphi)^8 \sum_{n=1}^\infty (n^8) |I_{ij}^{00}|^2 \left[\frac{8\pi^2}{15} - \frac{856}{525} (\log(2\Omega_\varphi |n| r_0) + \gamma_E) - \frac{116761}{36750} \right] \quad (5.39)$$

for the energy flux and

$$\langle \mathcal{G}_\infty^{\text{MQ11}(\text{tail})^2+(\text{tail-of-tails})} \rangle = -\nu x M^2(\Omega_\varphi)^7 i \epsilon_{3jl} \sum_{n=1}^\infty (n^7) \left(I_{ja}^{11} I_{la}^{00*} + I_{ja}^{00} I_{la}^{11*} \right) \left[\frac{16\pi^2}{15} - \frac{1712}{525} (\log(2\Omega_\varphi |n| r_0) + \gamma_E) - \frac{116761}{18375} \right], \quad (5.40)$$

$$\langle \mathcal{G}_\infty^{\text{MQ12}(\text{tail})^2+(\text{tail-of-tails})} \rangle = \nu x M^2(\Omega_\varphi)^7 i \epsilon_{3jl} \sum_{n=1}^\infty (n^7) I_{ja}^{00} I_{la}^{00*} \left[\frac{16\pi^2}{15} - \frac{1712}{525} (\log(2\Omega_\varphi |n| r_0) + \gamma_E) - \frac{116761}{18375} \right] \quad (5.41)$$

for the angular momentum flux.

E. Putting the essential part of the 4PN tail together

We are now in a position to assemble the entire order- $\mathcal{M}^2$ part of the 4PN tail. We will focus on the energy flux case first. This net tail flux comes from summing together (5.30), (5.31), (5.32), (5.38), and (5.39). Since this is a 4PN energy flux, we pull out the circular-orbit limit and an extra factor of x^4 to define a tail enhancement function $\mathcal{R}_4^{\text{tail}}(e_t, \nu)$:

$$\left\langle \frac{dE}{dt} \right\rangle_{4L}^{\text{tail}} = \frac{32}{5} \nu^2 x^9 \mathcal{R}_4^{\text{tail}}(e_t, \nu). \quad (5.42)$$

With $\mathcal{R}_4^{\text{tail}}(e_t, \nu)$ defined, we then make a new separation of this function by grouping on common factors like π^2 , rational numbers, a variant of the eulerlog function [8,9], and $\log(\frac{n}{2})$, all of which appear in (5.30), (5.31), (5.32), (5.38), and (5.39). Then these separate groupings are each expanded in power series in e^2 .

We draw attention first to the grouping on the $\log(\frac{n}{2})$ term within the sums, which defines a new function that we call $\mathcal{R}_4^x(e_t, \nu)$. This function is reminiscent of the 3PN function

$\chi(e_t)$ [7,8,14] that leads to the related relative flux function $\mathcal{R}_3^x(e_t)$,

$$\mathcal{R}_3^x(e_t) = -\frac{1712}{105}\chi(e_t) = -\frac{1712}{105}\sum_{n=1}^{\infty}\frac{n^2}{4}\log\left(\frac{n}{2}\right)g(n, e_t). \quad (5.43)$$

In turn, $\chi(e_t)$ is related to an infinite sequence of functions $\Lambda_k(e_t)$ that we defined in Sec. IV of Paper I. With these connections in mind, the definition for $\mathcal{R}_4^x(e_t, \nu)$, along with its power series expansion, is found to be

$$\begin{aligned} \mathcal{R}_4^x(e_t, \nu) &= \left(\frac{107}{420}\right)\sum_{n=1}^{\infty}\log\left(\frac{n}{2}\right)\left[\left(\frac{24}{1-e_t^2}\right)\left(n^8|\hat{I}_{ij}^{00}|^2 - \sum_{p=-2,2}n^7p|\hat{I}_{ij}^{00}|^2\right) - n^8(\hat{I}_{ij}^{00}\hat{I}_{ij}^{10*} + \hat{I}_{ij}^{10}\hat{I}_{ij}^{00*})\right. \\ &\quad \left.+ \nu n^8|\hat{I}_{ij}^{00}|^2 - \nu n^8(\hat{I}_{ij}^{00}\hat{I}_{ij}^{11*} + \hat{I}_{ij}^{11}\hat{I}_{ij}^{00*})\right] - (1-4\nu)\sum_{n=1}^{\infty}n^2\log\left(\frac{n}{2}\right)\left[\left(\frac{52}{21}\right)h(n, e_t) + \left(\frac{428}{105}\right)k(n, e_t)\right] \\ &= -3\mathcal{R}_{4L}\log(1-e_t^2) + \frac{1}{(1-e_t^2)^{15/2}}\left[\frac{133771\log(2)}{4410} - \frac{47385\log(3)}{1568} + \left(-\frac{232597}{2940} - \frac{19405829\log(2)}{8820}\right.\right. \\ &\quad \left.+\frac{15792327\log(3)}{15680}\right)e_t^2 + \left(\frac{1414531}{2940} + \frac{365627093\log(2)}{17640} + \frac{459923913\log(3)}{71680} - \frac{15869140625\log(5)}{903168}\right)e_t^4 \\ &\quad \left.+\left(\frac{13183159}{1764} + \frac{30959613721\log(2)}{317520} - \frac{276844091571\log(3)}{1003520} + \frac{2189439453125\log(5)}{16257024}\right)e_t^6 + \dots\right] \\ &\quad + \frac{\nu}{(1-e_t^2)^{15/2}}\left[\left(-\frac{267542\log(2)}{2205} + \frac{47385\log(3)}{392}\right) + \left(-\frac{34889}{245} + \frac{7321852\log(2)}{945} - \frac{11424159\log(3)}{3920}\right)e_t^2\right. \\ &\quad \left.+\left(-\frac{2993785}{882} - \frac{241129100\log(2)}{1323} + \frac{3196747377\log(3)}{125440} + \frac{15869140625\log(5)}{225792}\right)e_t^4\right. \\ &\quad \left.+\left(-\frac{29356142}{2205} + \frac{52459170329\log(2)}{26460} + \frac{3605227461\log(3)}{7168} - \frac{4796728515625\log(5)}{4064256}\right)e_t^6 + \dots\right]. \quad (5.44) \end{aligned}$$

While this function has no overall closed form, it *does* have an isolated closed-form part that involves the 1PN-log-sequence function $\mathcal{R}_{4L}(e_t, \nu)$ (4.22). The reappearance of this 1PN log function within a 4PN log function is exactly analogous to the way a leading log function, $F(e_t)$, reappears in the 3PN function $\chi(e_t)$ [7] (see also Paper I, Sec. IV A). Its appearance aids in isolating the singular behavior (as $e_t \rightarrow 1$)

of $\mathcal{R}_4^x$ into two parts—one with algebraic divergence and one with a dual logarithmic/algebraic divergence.

The remaining groupings on the other factors (π^2 , rational numbers, and a variant of the eulerlog function) lead to the remarkable behavior that all of the rest of $\mathcal{R}_4^{\text{tail}}$ has a closed-form appearance. We find

$$\begin{aligned} \mathcal{R}_4^{\text{tail}} &= \frac{1}{(1-e_t^2)^{15/2}}\left[\left(\frac{5887504939}{22226400} + \frac{105800809423e_t^2}{44452800} - \frac{12538208629e_t^4}{29635200} - \frac{778013619983e_t^6}{177811200} - \frac{2645724108523e_t^8}{2844979200}\right.\right. \\ &\quad \left.-\frac{1498169789e_t^{10}}{210739200}\right) + \nu\left(-\frac{1488040411}{5556600} - \frac{2854515929e_t^2}{555660} - \frac{1030726283e_t^4}{66150} - \frac{103160580401e_t^6}{8890560} - \frac{289778969059e_t^8}{142248960}\right. \\ &\quad \left.-\frac{1066256753e_t^{10}}{26342400}\right)\left] - \frac{1}{(1-e_t^2)^6}\left(\frac{1712}{35} + \frac{24503e_t^2}{70} + \frac{34989e_t^4}{140} + \frac{7383e_t^6}{560}\right) + \frac{\pi^2}{(1-e_t^2)^{15/2}}\left[\left(-\frac{1369}{126} + \frac{62107e_t^2}{252}\right.\right. \\ &\quad \left.+\frac{1011881e_t^4}{504} + \frac{715759e_t^6}{336} + \frac{693593e_t^8}{1792} + \frac{19389e_t^{10}}{3584}\right) + \nu\left(-\frac{3566}{63} - \frac{237442e_t^2}{189} - \frac{38219e_t^4}{9} - \frac{280195e_t^6}{84}\right. \\ &\quad \left.-\frac{6922325e_t^8}{12096} - \frac{141e_t^{10}}{14}\right)\left] + 2\left(\gamma_E + 2\log 2 + \frac{3}{2}\log x\right)\mathcal{R}_{4L}(e_t, \nu) + \mathcal{R}_4^x(e_t, \nu). \quad (5.45) \end{aligned}$$

Note that the eulerlog function becomes a coefficient on another appearance of $\mathcal{R}_{4L}(e_t, \nu)$, with a form that exactly matches the predictions laid out in Sec. IV E of Paper I. (This is only part of the appearance of $\log x$ at 4PN order; the remainder arises in the instantaneous 4PN term, which is not calculated here.) All the other terms involve polynomials once the relevant eccentricity singular factors are removed.

In turning to the case of the angular momentum flux, all of the steps made for energy flux carry over almost identically. At the end of the process we find that the order- $\mathcal{M}^2$ part of the 4PN tail in angular momentum flux is

$$\begin{aligned} \mathcal{Z}_4^{\text{tail}} = & \frac{1}{(1-e_t^2)^6} \left[\left(\frac{5887504939}{22226400} + \frac{7325116643e_t^2}{7408800} - \frac{27567910067e_t^4}{29635200} - \frac{98001030431e_t^6}{177811200} - \frac{125213141e_t^8}{21073920} \right) \right. \\ & + \left(-\frac{1488040411}{5556600} - \frac{2260694969e_t^2}{740880} - \frac{71048901401e_t^4}{14817600} - \frac{125477205683e_t^6}{88905600} - \frac{36791617e_t^8}{878080} \right) \Big] \\ & - \frac{1}{(1-e_t^2)^{9/2}} \left(\frac{1712}{35} + \frac{10379e_t^2}{70} + \frac{749e_t^4}{40} \right) + \frac{\pi^2}{(1-e_t^2)^6} \left[\left(-\frac{1369}{126} + \frac{26519e_t^2}{126} + \frac{366503e_t^4}{504} + \frac{84109e_t^6}{336} + \frac{9417e_t^8}{1792} \right) \right. \\ & + \left. \nu \left(-\frac{3566}{63} - \frac{47785e_t^2}{63} - \frac{330481e_t^4}{252} - \frac{194269e_t^6}{504} - \frac{3359e_t^8}{336} \right) \right] + 2 \left(\gamma_E + 2 \log 2 + \frac{3}{2} \log x \right) \mathcal{Z}_{4L}(e_t, \nu) + \mathcal{Z}_4^{\chi}(e_t, \nu). \end{aligned} \quad (5.46)$$

Here, $\mathcal{Z}_{4L}(e_t, \nu)$ is the second element in the integer-order 1PN log sequence defined in (4.23). The remaining part of the above expression is a new function defined by

$$\begin{aligned} \mathcal{Z}_4^{\chi}(e_t, \nu) = & \left(\frac{107i}{210} \right) \epsilon_{3jl} \sum_{n=1}^{\infty} \log \left(\frac{n}{2} \right) \left[\left(\frac{-21}{1-e_t^2} \right) \left(n^7 \hat{I}_{ja}^{00} \hat{I}_{la}^{00*} - \sum_{p=-2,2} n^6 p \hat{I}_{ja}^{00} \hat{I}_{la}^{00*} \right) + n^7 (\hat{I}_{ja}^{00} \hat{I}_{la}^{10*} + \hat{I}_{ja}^{10} \hat{I}_{la}^{00*}) \right. \\ & - \nu (n^7 \hat{I}_{ja}^{00} \hat{I}_{la}^{00*}) + \nu n^7 (\hat{I}_{ja}^{00} \hat{I}_{la}^{11*} + \hat{I}_{ja}^{11} \hat{I}_{la}^{00*}) \Big] - (1-4\nu) \sum_{n=1}^{\infty} n^2 \log \left(\frac{n}{2} \right) \left[\left(\frac{52}{21} \right) \tilde{h}(n, e_t) + \left(\frac{428}{105} \right) \tilde{k}(n, e_t) \right] \\ = & -3\mathcal{Z}_{4L} \log(1-e_t^2) + \frac{1}{(1-e_t^2)^6} \left[\frac{133771 \log(2)}{4410} - \frac{47385 \log(3)}{1568} + e_t^2 \left(-\frac{232597}{2940} - \frac{6267421 \log(2)}{4410} \right. \right. \\ & + \left. \frac{426465 \log(3)}{784} \right) + \left(\frac{3290641}{5880} + \frac{54495991 \log(2)}{5880} + \frac{3298439961 \log(3)}{501760} - \frac{3173828125 \log(5)}{301056} \right) e_t^4 \\ & + \left(\frac{15117563}{5040} + \frac{19854591727 \log(2)}{317520} - \frac{78260015511 \log(3)}{501760} + \frac{88952734375 \log(5)}{1161216} \right) e_t^6 + \dots \Big] \\ & + \frac{\nu}{(1-e_t^2)^6} \left[\left(-\frac{267542 \log(2)}{2205} + \frac{47385 \log(3)}{392} \right) + \left(-\frac{34889}{245} + \frac{1944359 \log(2)}{315} - \frac{2496339 \log(3)}{980} \right) e_t^2 \right. \\ & + \left(-\frac{1532917}{735} - \frac{10104547 \log(2)}{84} + \frac{2481686343 \log(3)}{125440} + \frac{3173828125 \log(5)}{75264} \right) e_t^4 \\ & + \left. \left(-\frac{3893383}{840} + \frac{180364779509 \log(2)}{158760} + \frac{5720993955 \log(3)}{25088} - \frac{1302119140625 \log(5)}{2032128} \right) e_t^6 + \dots \right], \end{aligned} \quad (5.47)$$

where the second equality provides its power series expansion. The full order- $\mathcal{M}^2$ tail functions, $\mathcal{R}_4^{\text{tail}}(e_t, \nu)$ and $\mathcal{Z}_4^{\text{tail}}(e_t, \nu)$, can now be used with an assist from BHPT to determine the flux terms $\mathcal{R}_4$ and $\mathcal{Z}_4$, at lowest order in ν .

VI. THE COMPLETE 4PN FLUXES AT LOWEST ORDER IN ν

A. 1PN correction to $\chi(e_t)$ and compact expressions for $\mathcal{L}_4(e)$ and $\mathcal{J}_4(e)$

We demonstrated in Paper I how the threefold combination of (i) knowledge of the leading logarithm sequence,

(ii) theoretical understanding of the role of the $\Lambda_k(e_t)$ sequence of functions [analogs of the function $\chi(e_t)$], and (iii) use of BHPT and fitting to finite-order expansions in eccentricity was sufficient to determine completely the integer-order 3PN log sequence at lowest order in the mass ratio. This procedure involved, first, converting a given leading-log term and its associated $\Lambda_k(e_t)$ function from expressions and expansions in e_t into expansions in Darwin e , the natural eccentricity for BHPT calculations. Then, these known functions were incorporated into a model for the eccentricity power series dependence at the given PN order. Thirdly, high accuracy BHPT numerical results, or a

fully analytic BHPT calculation, were used to determine the remaining, most-often rational, coefficients in the model. Finally, the result was then transformed back from e to e_t . In this way, the leading-log (0PN log) sequence was used to assist in finding terms in the 3PN log sequence at corresponding PN order. This is a connection between the red and the green lines in Fig. 1. We used this process to determine the $\mathcal{R}_{6L}(e_t)$ and $\mathcal{Z}_{6L}(e_t)$ terms in their entirety, aided by knowledge of the leading logs $\mathcal{R}_{6L2}(e_t)$ and $\mathcal{Z}_{6L2}(e_t)$.

A similar process appears to hold in being able to use terms in the 1PN log sequence to aid in determining the form of the corresponding term in the 4PN log sequence (i.e., a connection between the blue and orange lines in the figure), which we demonstrate with the first element in the 4PN log sequence—the 4PN nonlog flux itself. The derivations in Sec. IV provided one key component of this process—the closed-form expressions for the second elements in the integer-order 1PN log sequences, $\mathcal{R}_{4L}(e_t)$ and $\mathcal{Z}_{4L}(e_t)$. Then the analysis in Sec. V provided a second key component—the analytic form (including one infinite series) for the energy and angular momentum χ -like tail fluxes, $\mathcal{R}_4^\chi(e_t)$ and $\mathcal{Z}_4^\chi(e_t)$, the analogs at 4PN of the 3PN tail functions, $\chi(e_t)$ and $\tilde{\chi}(e_t)$. Knowing how these functions make an appearance in the full 4PN nonlog fluxes was sufficient to allow BHPT fitting to determine closed-form dependence for the rest of the 4PN nonlog fluxes at lowest order in ν .

Beginning with the energy flux, we require first a high-order eccentricity expansion for $\mathcal{L}_4^\chi(e)$, which like $\mathcal{R}_4^\chi(e_t)$ will be an infinite series. The process to obtain $\mathcal{L}_4^\chi(e)$ is straightforward. We start with $\mathcal{R}_4^\chi(e_t, \nu = 0)$, which can be isolated from (5.44). This function is expanded in e_t to e_t^{30} . Then, $\mathcal{R}_4^\chi(e_t)$ must be converted to $\mathcal{L}_4^\chi(e)$, that is, from a function of time eccentricity e_t to one of Darwin eccentricity e . This is achieved by expressing e_t in terms of e , to sufficient approximation, as $e_t = e(1 - 3x + \mathcal{O}(x^2))$, substituting into the full energy flux expansion, and letting the post-Newtonian difference between e and e_t ripple through the flux expressions. Then we collect all relevant results at 4PN order. The post-Newtonian corrections not only come from switching from e_t to e in $\mathcal{R}_4^\chi$ but also from

a correction to $\mathcal{R}_3^\chi$ (5.43). The result is that $\mathcal{L}_4^\chi(e)$ is calculated by taking

$$x^4 \mathcal{L}_4^\chi(e) = \left(-\frac{1712}{105} x^3 \chi(e - 3xe) + x^4 \mathcal{R}_4^\chi(e) \right)^{4\text{PN}}, \quad (6.1)$$

where the superscript “4PN” on the right side means expand out and then collect and retain the $\mathcal{O}(x^4)$ terms.

We could perform a similar procedure to generate $\mathcal{L}_{4L}(e)$ from $\mathcal{R}_{3L}(e_t)$ and $\mathcal{R}_{4L}(e_t)$, but there is no need since we can simply use the expression already found in [7,9,38] via fitting. The closed-form expression is

$$\begin{aligned} \mathcal{L}_{4L}(e) = & \frac{1}{(1-e^2)^{15/2}} \left(\frac{232597}{8820} + \frac{4923511e^2}{5880} \right. \\ & + \frac{142278179e^4}{35280} + \frac{318425291e^6}{70560} \\ & \left. + \frac{1256401651e^8}{1128960} + \frac{7220691e^{10}}{250880} \right). \end{aligned} \quad (6.2)$$

With those two functions, $\mathcal{L}_{4L}(e)$ and $\mathcal{L}_4^\chi(e)$, determined, the procedure now closely follows that of Paper I. The tail part $\mathcal{L}_4^\chi(e)$ is expected to appear directly as a term in $\mathcal{L}_4(e)$, while the function $\mathcal{L}_{4L}(e)$ appears also but only after having been multiplied by a particular function containing γ_E and a log term. The sum of these two terms is expanded in a power series to e^{30} . The model for the entire behavior of $\mathcal{L}_4(e)$, similar to one assumed in Paper I for $\mathcal{L}(e)$, includes these two parts as well as a power series in e^2 with rational coefficients and a second power series in e^2 with rational coefficients that is multiplied by π^2 . The starting point for these two power series is actually three closed-form expressions with relevant eccentricity singular factors. We subtract the known part in this model due to $\mathcal{L}_{4L}(e)$ and $\mathcal{L}_4^\chi(e)$ from the numerical 4PN nonlog flux data provided by BHPT. The modified numerical data should be represented by the remaining two rational-coefficient power series in this model. We then progressively solve for the remaining unknown (rational) coefficients. This process is successful, meaning the model was a correct ansatz, and yields

$$\begin{aligned} \mathcal{L}_4(e) = & \frac{1}{(1-e^2)^{15/2}} \left[\frac{18510752431}{44144100} - \frac{40934075709731e^2}{6356750400} - \frac{131458534402891e^4}{2542700160} - \frac{3215698875850801e^6}{50854003200} \right. \\ & - \frac{586522182193681e^8}{31294771200} - \frac{3028139270269e^{10}}{45203558400} + \frac{670101511e^{12}}{14057472} + \sqrt{1-e^2} \left(-\frac{1654225499}{3175200} + \frac{2426725501e^2}{3175200} \right. \\ & + \frac{186636561079e^4}{12700800} + \frac{72788261801e^6}{8467200} - \frac{16274063783e^8}{7526400} - \frac{4982855e^{10}}{18432} \left. \right) - \frac{1369\pi^2}{126(1-e^2)^{15/2}} \left(1 + \frac{104549e^2}{2738} \right. \\ & \left. + \frac{1113487e^4}{5476} + \frac{2644503e^6}{10952} + \frac{10829823e^8}{175232} + \frac{573939e^{10}}{350464} \right) + 2 \left[\gamma_E + \log \left(\frac{8(1-e^2)}{1+\sqrt{1-e^2}} \right) \right] \mathcal{L}_{4L}(e) + \mathcal{L}_4^\chi(e). \end{aligned} \quad (6.3)$$

The result matches the expansion for $\mathcal{L}_4(e)$ found to e^{30} in [9,40].

The 4PN nonlog angular momentum flux follows precisely the same procedure, and yields

$$\begin{aligned} \mathcal{J}_4(e) = & \frac{1}{(1-e^2)^6} \left[\frac{139774944409}{1059458400} - \frac{35619868663789e^2}{12713500800} - \frac{284430057678037e^4}{25427001600} - \frac{353931345220951e^6}{50854003200} \right. \\ & - \frac{66321815297809e^8}{67805337600} - \frac{3159752887e^{10}}{147603456} + \sqrt{1-e^2} \left(-\frac{370844347}{1587600} + \frac{2259332951e^2}{3175200} + \frac{2594164919e^4}{846720} \right. \\ & \left. \left. + \frac{921635651e^6}{2822400} - \frac{572575e^8}{8064} \right) \right] - \frac{1369\pi^2}{126(1-e^2)^6} \left(1 + \frac{22495e^2}{1369} + \frac{259969e^4}{5476} + \frac{268179e^6}{10952} + \frac{193455e^8}{175232} \right) \\ & + 2 \left[\gamma_E + 3 \log(2) + \log \left(\frac{1-e^2}{1+\sqrt{1-e^2}} \right) \right] \mathcal{J}_{4L}(e) + \mathcal{J}_4^\chi(e), \end{aligned} \quad (6.4)$$

where the second element in the angular momentum (integer-order) 1PN log sequence is

$$\mathcal{J}_{4L}(e) = \frac{1}{(1-e^2)^6} \left(\frac{232597}{8820} + \frac{3482879e^2}{8820} + \frac{34971299e^4}{35280} + \frac{6578731e^6}{14112} + \frac{2503623e^8}{125440} \right). \quad (6.5)$$

The result in (6.4) also matches the expansion found by fitting given in [9,40] but provides a deeper, though partial, theoretical explanation.

B. Transforming from $\mathcal{L}_4(e)$ and $\mathcal{J}_4(e)$ to $\mathcal{R}_4(e_t)$ and $\mathcal{Z}_4(e_t)$

In order to convert these flux terms to functions in terms of e_t [i.e., $\mathcal{R}_4(e_t)$ and $\mathcal{Z}_4(e_t)$] in the modified harmonic gauge, we require the relationship between e and e_t to 4PN order at lowest order in the mass ratio. With that restriction, the expansions relating e and e_t can be calculated to *any* PN order by analyzing geodesic motion on a Schwarzschild background [18]. We quote the result through the necessary order

$$\begin{aligned} \frac{e^2}{e_t^2} = & 1 + 6x + \frac{(17 - 21e_t^2 + 15\sqrt{1-e_t^2})x^2}{1-e_t^2} + \frac{(26 + 54e_t^4 + 150\sqrt{1-e_t^2} - e_t^2(107 + 90\sqrt{1-e_t^2}))x^3}{(1-e_t^2)^2} \\ & - \frac{(880e_t^6 - 10e_t^4(367 + 240\sqrt{1-e_t^2}) - 2(865 + 3167\sqrt{1-e_t^2}) + e_t^2(6120 + 6265\sqrt{1-e_t^2}))x^4}{8(1-e_t^2)^3} + \mathcal{O}(\nu, x^5). \end{aligned} \quad (6.6)$$

We then construct the net flux by combining $\mathcal{L}_0(e)$, $\mathcal{L}_1(e)$, $\mathcal{L}_2(e)$, $\mathcal{L}_3(e)$, and $\mathcal{L}_4(e)$ and replacing e with its relationship to e_t given in (6.6) along the lines done in (6.1). The result is expanded, allowing the PN corrections to ripple through to the 4PN term, giving at last

$$\begin{aligned} \mathcal{R}_4(e_t) = & \frac{1}{(1-e_t^2)^{15/2}} \left[\frac{20670029551}{44144100} + \frac{90592819680523e_t^2}{6356750400} + \frac{45374652958109e_t^4}{1589187600} + \frac{215773793118089e_t^6}{50854003200} \right. \\ & - \frac{139754682191e_t^8}{2844979200} + \frac{4853373238601e_t^{10}}{45203558400} - \frac{12776867e_t^{12}}{14057472} + \sqrt{1-e_t^2} \left(-\frac{1809538139}{3175200} \right. \\ & - \frac{30429943463e_t^2}{3175200} - \frac{103455982193e_t^4}{12700800} + \frac{67397848199e_t^6}{8467200} + \frac{863341943e_t^8}{501760} - \frac{308515e_t^{10}}{64512} \left. \right) \\ & - \frac{1369\pi^2}{126(1-e_t^2)^{15/2}} \left(1 - \frac{62107e_t^2}{2738} - \frac{1011881e_t^4}{5476} - \frac{2147277e_t^6}{10952} - \frac{6242337e_t^8}{175232} - \frac{174501e_t^{10}}{350464} \right) \\ & \left. + 2 \left[\gamma_E + 3 \log(2) + \log \left(\frac{(1-e_t^2)}{1+\sqrt{1-e_t^2}} \right) \right] \mathcal{R}_{4L}(e_t) + \mathcal{R}_4^\chi(e_t). \right] \end{aligned} \quad (6.7)$$

We follow precisely the same procedure in combining $\mathcal{J}_0(e)$, $\mathcal{J}_1(e)$, $\mathcal{J}_2(e)$, $\mathcal{J}_3(e)$, and $\mathcal{J}_4(e)$ to obtain

$$\begin{aligned}
\mathcal{Z}_4(e_t) = & \frac{1}{(1-e_t^2)^6} \left[\frac{191597595289}{1059458400} + \frac{99527954953927e_t^2}{12713500800} + \frac{191377070535107e_t^4}{25427001600} - \frac{101432063662609e_t^6}{50854003200} \right. \\
& - \frac{13890223720171e_t^8}{67805337600} + \frac{9732011e_t^{10}}{21086208} + \sqrt{1-e_t^2} \left(-\frac{448500667}{1587600} - \frac{14027009779e_t^2}{3175200} - \frac{1764770893e_t^4}{846720} \right. \\
& \left. \left. + \frac{235209407e_t^6}{352800} - \frac{81965e_t^8}{16128} \right) \right] - \frac{1369\pi^2}{126(1-e_t^2)^6} \left(1 - \frac{26519e_t^2}{1369} - \frac{366503e_t^4}{5476} - \frac{252327e_t^6}{10952} - \frac{84753e_t^8}{175232} \right) \\
& + 2 \left[\gamma_E + 3 \log(2) + \log \left(\frac{1-e_t^2}{1+\sqrt{1-e_t^2}} \right) \right] \mathcal{Z}_{4L}(e_t) + \mathcal{Z}_4^\chi(e_t). \tag{6.8}
\end{aligned}$$

Note that the polynomial attached to π^2 in each of these expressions now perfectly matches the corresponding result obtained through analysis of the 4PN tail in Sec. V E.

C. General structure in the 4PN log sequences and a simplified form for $\mathcal{L}_{11/2}(e)$

The first part of this section has developed compact expressions for the first elements in the two integer-order 4PN log sequences, namely the terms $\mathcal{L}_4(e)$ and $\mathcal{J}_4(e)$, by combining 1PN source multipoles in formulas for tail fluxes and using perturbation theory to find rational number coefficients in the remaining (closed-form) functions of e . This is simply an extrapolation to the 4PN log sequences (solid orange line in Fig. 1) of the procedure used in Paper I (Sec. IV) to find comparable expressions for the 3PN log sequences (solid green line).

We went about this by deriving the form of $\mathcal{R}_4^\chi(e_t)$ and $\mathcal{Z}_4^\chi(e_t)$ directly using the 4PN hereditary contributions. However, strictly speaking this approach was not necessary. At lowest order in the mass ratio, the analysis of Sec. IV E in Paper I still holds, meaning that the form of the χ -like contribution to *any* 4PN logarithm can be ascertained *a priori*. Using the results of that section in Paper I, we see that the expressions for $\mathcal{R}_{(3k+1)L(k)}^\chi(e_t)$ and $\mathcal{Z}_{(3k+1)L(k)}^\chi(e_t)$ can be derived from the summations for $\mathcal{R}_{(3k+1)L(k)}(e_t)$ and $\mathcal{Z}_{(3k+1)L(k)}(e_t)$, which appear in (4.17)

and (4.20), respectively, by including in each a factor of $(2k) \log(n/2)$. In this way we can not only reproduce the results for the 4PN nonlog flux but also generalize to arbitrarily higher order terms in the integer-order 4PN log sequence. The more general χ -like functions will have dual logarithmic and algebraic divergent parts, with the former attached to the corresponding 1PN log term. With these higher PN order χ -like functions determined, we might then rely upon BHPT to determine the remaining functional dependence in these higher (integer) order 4PN log terms. The next of these, $\mathcal{L}_{7L}(e)$, would have a model with a set of closed-form functions with unknown rational coefficients. Those functions would need to be expanded in a power series to e^{34} and then BHPT would be used to fit for the rational coefficients to that order.

Unfortunately, the terms in the half-integer 4PN log sequences cannot be manipulated into expressions that are quite as compact, with closed-form parts. However, each term in these sequences can still be reduced to a remaining infinite series with rational coefficients, once the roles of the mass quadrupole, mass octupole, and current quadrupole 1PN moments are understood. As an example, take the first half-integer (energy) 4PN logarithm, $\mathcal{R}_{11/2}(e_t)$. By applying the procedure above, the counterpart function in e , $\mathcal{L}_{11/2}(e)$, can be given the following form:

$$\begin{aligned}
\mathcal{L}_{11/2}(e) = & \frac{\pi}{(1-e^2)^9} \left(\frac{8399309750401}{101708006400} - \frac{6431125434321667e^2}{203416012800} - \frac{347369943176265227e^4}{813664051200} - \frac{15186120717515117243e^6}{11716762337280} \right. \\
& - \frac{18230625005177349698411e^8}{14997455791718400} - \frac{66989953560049801996499e^{10}}{249957596528640000} - \frac{997758112480120369559e^{12}}{3856488632156160000} \\
& - \frac{18489702206162114169107e^{14}}{1476312054497280000} - \frac{58073989289629682554885336291e^{16}}{5418088860997889556480000} \\
& \left. - \frac{7460925685163379777975573791651e^{18}}{877730395481658108149760000} - \frac{46468844430394780206366046707113e^{20}}{6751772272935831601152000000} + \dots \right) \\
& + 2 \left[\gamma_E + 3 \log 2 + \log \left(\frac{1-e^2}{1+\sqrt{1-e^2}} \right) \right] \mathcal{L}_{11/2L}(e) + \mathcal{L}_{11/2}^\chi(e). \tag{6.9}
\end{aligned}$$

As mentioned, the initial part of this expression is an infinite series with rational coefficients. We have calculated the series to e^{30} [9,40] but omitted here the last few coefficients for brevity. The remainder of the expression involves two functions that together capture all of the transcendental and logarithmic constants. The first of these is the 1PN log function itself at 5.5PN order, $\mathcal{L}_{11/2L}(e)$, which can be derived from the expression for $\mathcal{R}_{11/2L}(e)$ given earlier in this paper or found by consulting [9]. The second is the 5.5PN χ -like function, $\mathcal{L}_{11/2}^\chi(e)$, which can be found by the process described above.

VII. CONCLUSIONS AND OUTLOOK

This paper extended an approach found in Paper I for determining the eccentricity dependence of the leading log sequences of flux terms (depicted as solid and dashed red lines in Fig. 1) and the 3PN log sequences (green lines) to additional strips in the higher-order PN structure. The new

strips considered here are the 1PN log sequences (blue lines) and 4PN log sequences (orange lines). In the earlier paper we developed a complete understanding of the terms in the leading log sequences in terms of the Newtonian quadrupole moment Fourier spectrum $g(n, e_t)$. The integer-order leading logs were found to have closed-form expressions and the half-integer-order leading logs were shown to be infinite series in e^2 with calculable rational coefficients. The 3PN log terms, at lowest order in the mass ratio, were shown to have part of their functional dependence given by the quadrupole spectrum, with the rest involving series with rational coefficients that could be determined with the assistance of BHPT fitting.

In this paper we showed that a mirror image of those procedures could be found which would allow us to calculate the 1PN and 4PN log sequences, provided we use PN theory to calculate additional, 1PN multipole moment spectra (i.e., the mass octupole, current quadrupole, and 1PN mass quadrupole moments) along with

TABLE I. State of knowledge of eccentricity dependence of high PN order flux terms. The second column indicates whether a closed form exists or to what order in e the power series expansion is known. The closed-form result for $\mathcal{L}_{4L}$ was previously found in Forseth *et al.* [7]. All other results come from this paper and its companions, Paper I [8] and Munna *et al.* [9]. Flux terms labeled as “all orders” are infinite series in e^2 but with coefficients that can now be analytically calculated to arbitrary order. Other terms are only known in analytic form up to order e^{30} (or in a few cases less). The fourth column gives the number of PN corrections to the leading-logs which must be calculated to derive the term fully. The fifth column indicates the number of leading log [and $\Lambda(e_t)/\Xi(e_t)$] corrections which must be calculated to extract the term to all orders in e in the manner of Sec. VI. A superset of these terms allows for the separation of transcendental contributions in the same way, as shown in column six. Above 5PN it is more difficult to apply these methods (labeled by asterisk). The last two rows represent all further leading and 1PN logarithms.

Term	Known order in e	Original source	PN order beyond LL	Order for fitting extraction	Order to find transcendental part
$L_{7/2}$	Fitted to e^{30}	Munna <i>et al.</i>	2PN	...	...
$\mathcal{L}_4$	All orders	This paper	4PN	1PN	1PN
$\mathcal{L}_{4L}$	Closed form	Forseth <i>et al.</i>	1PN	...	...
$\mathcal{L}_{9/2}$	Fitted to e^{30}	Munna <i>et al.</i>	3PN	...	0PN
$\mathcal{L}_{9/2L}$	All orders	Paper I	...	...	...
$\mathcal{L}_5$	Fitted to e^{30}	Munna <i>et al.</i>	5PN	2PN	2PN
$\mathcal{L}_{5L}$	Closed form	Munna <i>et al.</i>	2PN	...	...
$\mathcal{L}_{11/2}$	Fitted to e^{30}	Munna <i>et al.</i>	4PN	...	1PN
$\mathcal{L}_{11/2L}$	All orders	this paper	1PN	...	...
$\mathcal{L}_6$	Fitted to e^{20}	Munna <i>et al.</i>	6PN	3PN*	3PN*
$\mathcal{L}_{6L}$	All orders	Paper I	3PN	0PN	0PN
$\mathcal{L}_{6L2}$	Closed form	Paper I	...	...	...
$\mathcal{L}_{13/2}$	Fitted to e^{30}	Munna <i>et al.</i>	5PN	...	2PN
$\mathcal{L}_{13/2L}$	Fitted to e^{30}	Munna <i>et al.</i>	2PN	...	...
$\mathcal{L}_7$	Fitted to e^{12}	Munna <i>et al.</i>	7PN	4PN*	4PN*
$\mathcal{L}_{7L}$	Fitted to e^{26}	Munna <i>et al.</i>	4PN	1PN	1PN
$\mathcal{L}_{7L2}$	Closed form	This paper	1PN	...	...
$\mathcal{L}_{15/2}$	Fitted to e^{12}	Munna <i>et al.</i>	6PN	...	3PN*
$\mathcal{L}_{15/2L}$	Fitted to e^{26}	Munna <i>et al.</i>	3PN	...	0PN
$\mathcal{L}_{15/2L2}$	All orders	Paper I	...	...	...
$\mathcal{L}_{(3k)L(k)}$	Closed form	Paper I	...	...	...
$\mathcal{L}_{(3k+3/2)L(k)}$	All orders	Paper I	...	...	...
$\mathcal{L}_{(3k+1)L(k)}$	Closed form	This paper	1PN	...	...
$\mathcal{L}_{(3k+5/2)L(k)}$	All orders	This paper	1PN	...	...

somewhat higher order in e^2 BHPT fitting. In the case of the 1PN log sequences, the PN calculation provides as a bonus next-order-in- ν parts of the fluxes, with only some remaining uncertainty whether the $\mathcal{O}(\nu)$ part of the MQ12 terms [see Eq. (4.14)] is complete. Without a full PN theory calculation, the conjecture that the order ν part is complete can only be verified by an (as yet unavailable) second-order BHPT comparison. We used the procedure to detail explicitly the 4PN log, 5.5PN log, and 7PN $\log^2$ terms. However, our computational infrastructure allows us to compute any integer 1PN logarithm as a closed-form expression and permits the rapid expansion of all half-integer (nonclosed) 1PN logs to at least e_t^{120} .

In addition to the 1PN logarithms, our approach allowed for the computation of the 1PN correction to the $\Lambda_k(e_t)$ and $\Xi_k(e_t)$ set of functions of Paper I. The specific 1PN correction to $\Lambda_1(e_t) = \chi(e_t)$ allowed for the extraction of the full 4PN nonlog fluxes at lowest order in ν , as well as the isolation of all transcendental contributions in the 5.5PN nonlog term, $\mathcal{R}_{11/2}(e_t)$.

To extend the procedures of Paper I and this paper further, we would need to calculate the Fourier spectra of the 2PN source multipoles and use even higher-order BHPT fitting. The algorithmic complexity and cost would increase, and there may be additional hereditary-term integrals that are more difficult to compute. This 2PN

extension, to the 2PN log sequences and the 5PN log sequences, may be the subject of future work.

We conclude by presenting an update of a table found in Paper I that summarizes the state of knowledge of the eccentricity dependence of high PN order (lowest order in ν) flux terms. This is given in Table I.

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APPENDIX A: COMPONENT SUMS FOR 1PN LOGARITHMS

We provide here some low-order examples of the component sums that, once added together, produce the corresponding 1PN log sequence term. These illustrate (i) some of the steps in the procedure, (ii) the particular eccentricity dependence of individual terms, (iii) how different source multipoles contribute, and (iv) the presence of next-order-in- ν dependence. At 1PN order itself, we find

$$\begin{aligned}\mathcal{R}_1^{\text{MQ01}} &= \frac{1}{(1-e_t^2)^{9/2}} \left(\frac{271}{21} + \frac{1705e_t^2}{28} + \frac{2555e_t^4}{96} + \frac{1189e_t^6}{1344} \right) - \frac{1}{(1-e_t^2)^3} \left(18 + \frac{63e_t^2}{4} \right), \\ \mathcal{R}_1^{\text{MQ02}} &= \frac{1}{(1-e_t^2)^{9/2}} \left(-18 - \frac{219e_t^2}{4} - \frac{111e_t^4}{16} \right), \quad \mathcal{R}_1^{\text{MQ03}} = \frac{1}{(1-e_t^2)^3} \left(18 + \frac{63e_t^2}{4} \right), \\ \mathcal{R}_1^{\text{MQ11}} &= \frac{1}{(1-e_t^2)^{9/2}} \left(\frac{55}{21} + \frac{3907e_t^2}{504} - \frac{e_t^4}{96} - \frac{307e_t^6}{2016} \right), \quad \mathcal{R}_1^{\text{CQ}} = \frac{1-4\nu}{(1-e_t^2)^{9/2}} \left(\frac{1}{36} + \frac{19e_t^2}{72} + \frac{23e_t^4}{96} + \frac{e_t^6}{64} \right).\end{aligned}\quad (\text{A1})$$

The mass octupole portion is given in (2.37). The Newtonian moments match their expected forms, but the mass quadrupole functions are more interesting, with the pieces displaying somewhat distinct singular behavior as $e_t \rightarrow 1$. We have confirmed that a similar pattern exists in all integer-order 1PN log terms through 22PN, with

$$\begin{aligned}\mathcal{R}_{(3k+1)L(k)}^{\text{MQ01}} &= \frac{1}{(1-e_t^2)^{3k+9/2}} f_k^{(1)}(e_t) - \frac{1}{(1-e_t^2)^{3k+3}} f_k^{(2)}(e_t), \quad \mathcal{R}_{(3k+1)L(k)}^{\text{MQ02}} = \frac{1}{(1-e_t^2)^{3k+9/2}} f_k^{(3)}(e_t), \\ \mathcal{R}_{(3k+1)L(k)}^{\text{MQ03}} &= \frac{1}{(1-e_t^2)^{3k+3}} f_k^{(2)}(e_t), \quad \mathcal{R}_{(3k+1)L(k)}^{\text{MQ11}} = \frac{1}{(1-e_t^2)^{3k+9/2}} f_k^{(4)}(e_t),\end{aligned}\quad (\text{A2})$$

where $f_k^{(1)}(e_t), f_k^{(2)}(e_t), f_k^{(3)}(e_t), f_k^{(4)}(e_t)$ are polynomials in e_t . It is not difficult to prove that the trends in singular behavior continue to all orders for MQ02, MQ03, MQ11, MO, and CQ using the methods of asymptotic analysis laid out in [7,9,32]. Unfortunately, a similar proof for MQ01 has remained elusive, though there are overlapping reasons to believe that the same behavior arises in this term as well,

including the fact that all divergences as $e_t \rightarrow 1$ must vanish in a PN expansion that uses $1/p$ (the semilatus rectum) as the compactness parameter instead of x (see [9]). Nearly identical trends exist in the integer-order 1PN angular momentum log sequence terms.

At half-integer orders, the component terms are not closed in form. For future reference, at 2.5PN we find

$$\begin{aligned}
\mathcal{R}_{5/2}^{\text{MQ01}} &= \frac{1}{(1-e_t^2)^6} \left(\frac{1336}{21} + \frac{29083e_t^2}{48} + \frac{137933e_t^4}{192} + \frac{3704005e_t^6}{27648} + \frac{3902585e_t^8}{1548288} - \frac{54803587e_t^{10}}{619315200} + \dots \right) - \mathcal{R}_{5/2}^{\text{MQ03}}, \\
\mathcal{R}_{5/2}^{\text{MQ02}} &= \frac{1}{(1-e_t^2)^6} \left(-84 - \frac{9625e_t^2}{16} - \frac{27545e_t^4}{64} - \frac{70049e_t^6}{3072} - \frac{16247e_t^8}{73728} + \frac{1664999e_t^{10}}{29491200} - \frac{1280041e_t^{12}}{353894400} + \dots \right), \\
\mathcal{R}_{5/2}^{\text{MQ03}} &= \frac{1}{(1-e_t^2)^{9/2}} \left(84 + \frac{2037e_t^2}{8} + \frac{1029e_t^4}{32} - \frac{343e_t^6}{1536} - \frac{763e_t^8}{12288} - \frac{17969e_t^{10}}{4915200} + \frac{32543e_t^{12}}{58982400} + \dots \right), \\
\mathcal{R}_{5/2}^{\text{MQ11}} &= \frac{1}{(1-e_t^2)^6} \left(\frac{220}{21} + \frac{9841e_t^2}{144} + \frac{16891e_t^4}{576} - \frac{216235e_t^6}{27648} - \frac{2088109e_t^8}{4644864} + \frac{4380643e_t^{10}}{371589120} - \frac{33875507e_t^{12}}{22295347200} + \dots \right), \\
\mathcal{R}_{5/2}^{\text{MQ12}} &= \frac{1}{(1-e_t^2)^6} \left(-2 - \frac{1183e_t^2}{96} + \frac{1565e_t^4}{384} + \frac{178873e_t^6}{18432} + \frac{237847e_t^8}{442368} + \frac{1166257e_t^{10}}{176947200} - \frac{3037147e_t^{12}}{2123366400} + \dots \right), \\
\mathcal{R}_{5/2}^{\text{MO}} &= \frac{1-4\nu}{(1-e_t^2)^6} \left(\frac{16403}{2016} + \frac{34163e_t^2}{336} + \frac{21836233e_t^4}{129024} + \frac{57821777e_t^6}{1161216} + \frac{67599745e_t^8}{49545216} + \frac{241631e_t^{10}}{132710400} + \dots \right), \\
\mathcal{R}_{5/2}^{\text{CO}} &= \frac{1-4\nu}{(1-e_t^2)^6} \left(\frac{1}{18} + \frac{4e_t^2}{3} + \frac{2041e_t^4}{576} + \frac{7991e_t^6}{5184} + \frac{2989e_t^8}{49152} - \frac{6307e_t^{10}}{16588800} + \frac{11669e_t^{12}}{212336640} + \dots \right). \tag{A3}
\end{aligned}$$

In each infinite series, the coefficients drop off rapidly in magnitude with power of e_t , indicating likely convergence as $e_t \rightarrow 1$. As with integer orders, similar singular behavior can be proven to hold to all orders in each type of sum except for that of MQ01. Nevertheless, we have used high-order expansions to demonstrate apparent convergence for the MQ01 sums (and the rest) through 20.5PN order. There is nearly identical structure observed again in the angular momentum flux case.

APPENDIX B: FOURIER SUM IDENTITIES

In this section, we briefly provide a couple of the Fourier series identities used in the various 1PN mass quadrupole derivations. We start with sums of the following form:

$$\sum_{n=-\infty}^{\infty} \sum_{p,s=-2}^2 n^{2r} p \begin{matrix} I_{ij} \\ (n,p) \end{matrix} \begin{matrix} I_{ij} \\ (-n,s) \end{matrix} = \sum_{n=-\infty}^{\infty} \sum_{p=-2}^2 n^{2r} p \begin{matrix} I_{ij} \\ (n,p) \end{matrix} \begin{matrix} I_{ij} \\ (-n,-p) \end{matrix}, \tag{B1}$$

where r is an integer, and where on the right-hand side we noted that only terms with $s = -p$ will survive. Then,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \sum_p n^{2r} p \begin{matrix} I_{ij} \\ (n,p) \end{matrix} \begin{matrix} I_{ij} \\ (-n,-p) \end{matrix} &= \sum_{n=-\infty}^{\infty} n^{2r} [2 \begin{matrix} I_{ij} \\ (n,2) \end{matrix} \begin{matrix} I_{ij} \\ (-n,-2) \end{matrix} - 2 \begin{matrix} I_{ij} \\ (n,-2) \end{matrix} \begin{matrix} I_{ij} \\ (-n,2) \end{matrix}] \\
&= \sum_{n=1}^{\infty} n^{2r} [2 \begin{matrix} I_{ij} \\ (n,2) \end{matrix} \begin{matrix} I_{ij} \\ (-n,-2) \end{matrix} - 2 \begin{matrix} I_{ij} \\ (n,-2) \end{matrix} \begin{matrix} I_{ij} \\ (-n,2) \end{matrix}] + \sum_{n=-1}^{-\infty} n^{2r} [2 \begin{matrix} I_{ij} \\ (n,2) \end{matrix} \begin{matrix} I_{ij} \\ (-n,-2) \end{matrix} - 2 \begin{matrix} I_{ij} \\ (n,-2) \end{matrix} \begin{matrix} I_{ij} \\ (-n,2) \end{matrix}] \\
&= \sum_{n=1}^{\infty} n^{2r} [2 \begin{matrix} I_{ij} \\ (n,2) \end{matrix} \begin{matrix} I_{ij} \\ (-n,-2) \end{matrix} - 2 \begin{matrix} I_{ij} \\ (n,-2) \end{matrix} \begin{matrix} I_{ij} \\ (-n,2) \end{matrix}] + \sum_{n=1}^{\infty} n^{2r} [2 \begin{matrix} I_{ij} \\ (-n,2) \end{matrix} \begin{matrix} I_{ij} \\ (n,-2) \end{matrix} - 2 \begin{matrix} I_{ij} \\ (-n,-2) \end{matrix} \begin{matrix} I_{ij} \\ (n,2) \end{matrix}] = 0. \tag{B2}
\end{aligned}$$

In the same way, we can prove that

$$\sum_{n=-\infty}^{\infty} \sum_{p,s=-2}^2 n^{2r+1} \text{sign}(n) p \begin{matrix} I_{ij} \\ (n,p) \end{matrix} \begin{matrix} I_{ij} \\ (-n,s) \end{matrix} = \epsilon_{3jl} \sum_{n=-\infty}^{\infty} \sum_{p,s=-2}^2 n^{2r+1} p \begin{matrix} I_{ja} \\ (n,p) \end{matrix} \begin{matrix} I_{la} \\ (-n,s) \end{matrix} = \epsilon_{3jl} \sum_{n=-\infty}^{\infty} \sum_{p,s=-2}^2 n^{2r} \text{sign}(n) p \begin{matrix} I_{ja} \\ (n,p) \end{matrix} \begin{matrix} I_{la} \\ (-n,s) \end{matrix} = 0,$$

all vanish and

$$\begin{aligned}
& \sum_{n=-\infty}^{\infty} \sum_{p,s=-2}^2 n^{2r+1} p I_{ij}^{(n,p)} I_{ij}^{(-n,s)}, \\
& \sum_{n=-\infty}^{\infty} \sum_{p,s=-2}^2 n^{2r} \text{sign}(n) p I_{ij}^{(n,p)} I_{ij}^{(-n,s)}, \\
& \epsilon_{3jl} \sum_{n=-\infty}^{\infty} \sum_{p,s=-2}^2 n^{2r} p I_{ja}^{(n,p)} I_{la}^{(-n,s)}, \\
& \epsilon_{3jl} \sum_{n=-\infty}^{\infty} \sum_{p,s=-2}^2 n^{2r+1} \text{sign}(n) p I_{ja}^{(n,p)} I_{la}^{(-n,s)},
\end{aligned}$$

all gain a factor of 2 when expressed in terms of positive- n sums.

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