Frame spectral pairs and exponential bases *

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Abstract

Given a domain $\Omega \subset \mathbb{R}^d$ with positive and finite Lebesgue measure and a discrete set $\Lambda \subset \mathbb{R}^d$, we say that (Ω, Λ) is a *frame spectral pair* if the set of exponential functions $\mathcal{E}(\Lambda) := \{e^{2\pi i\lambda\cdot x} : \lambda \in \Lambda\}$ is a frame for $L^2(\Omega)$. Special cases of frames include Riesz bases and orthogonal bases. In the finite setting $\mathbb{Z}_N^d$, $d, N \geq 1$, a frame spectral pair can be similarly defined. We show how to construct and obtain new classes of frame spectral pairs in $\mathbb{R}^d$ by "adding" frame spectral pairs in $\mathbb{R}^d$ and $\mathbb{Z}_N^d$. Our construction unifies the well-known examples of exponential frames for the union of cubes with equal volumes. We also remark on the link between the spectral property of a domain and sampling theory.

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ be a set with positive and finite Lebesgue measure, $0 < |\Omega| < \infty$, and let $\Lambda \subset \mathbb{R}^d$ be a discrete and countable set. Let $\mathcal{E}_\Omega(\Lambda)$ denote the set of exponentials, $\mathcal{E}_\Omega(\Lambda) = \{e_\lambda : \Omega \to S^1 \mid e_\lambda(x) := e^{2\pi i x\cdot\lambda}\}$. For simplicity, we will drop the subscript Ω in the sequel and write $\mathcal{E}(\Lambda)$. The set of exponentials $\mathcal{E}(\Lambda)$ is a frame for $L^2(\Omega)$ when there exist finite constants $c > 0$ and $C > 0$ such that for any $u \in L^2(\Omega)$

$$c\|u\|^2_{L^2(\Omega)} \leq \sum_{\lambda\in\Lambda} |\langle u, e_\lambda\rangle|^2 \leq C\|u\|^2_{L^2(\Omega)}. \tag{1}$$

The constants c and C are called *lower* and *upper frame constants* (see the classical references [4, 3] for the basic properties of frames). The set of exponentials $\mathcal{E}(\Lambda)$ is a *Riesz sequence* in $L^2(\Omega)$, if there are positive constants $c > 0$ and $C > 0$ such that for any finite set of scalars $\{c_\lambda\}_{\lambda\in F}$ ($F \subset \Lambda$ finite), the following inequalities hold:

$$c\sum_{\lambda\in F} |c_\lambda|^2 \leq \left\|\sum_{\lambda\in F} c_\lambda e_\lambda\right\|^2 \leq C\sum_{\lambda\in F} |c_\lambda|^2. \tag{2}$$

When only the right inequality in (2) holds, then $\mathcal{E}(\Lambda)$ is called a *Bessel sequence*. A Riesz sequence is called a *Riesz basis* if it is complete. Any Riesz basis is a frame, while the converse is not always true; see e.g. [4].

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Definition 1. *Given a set $\Omega \subset \mathbb{R}^d$ and discrete and countable set $\Lambda \subset \mathbb{R}^d$, we term (Ω, Λ) a* **frame spectral pair** *when the system of exponentials $\mathcal{E}(\Lambda)$ is a frame for $L^2(\Omega)$.*

For a given frame pair, when the frame is a Riesz basis, we shall call the pair a *Riesz spectral pair.* When it is an orthogonal basis, we shall simply call the pair a *spectral pair.* Here, we aim to study the following question:

Question A. *Given a frame spectral pair in $\mathbb{R}^d$, how can one construct a new frame spectral pair in $\mathbb{R}^d$, such that the frame is a Riesz basis or an orthogonal basis?*

The study of frame spectral pairs in this paper is motivated by the interpolation formula for bandlimited signals with structured spectra and the link between spectral properties of a domain to its sampling properties.

1.1 Main contributions

To address Question A., the simplest and perhaps the most natural way of constructing a new frame spectral pair in $\mathbb{R}^d$ is by adding two frame spectral pairs in both finite and continuous settings in 'an appropriate sense'. We shall answer the question in special cases.

Our main results are following:

Theorem 1. *Suppose that $\Omega_1 \subset \mathbb{R}^d$ is a set with positive and finite Lebesgue measure, $\Lambda_1 \subset \mathbb{R}^d$ is a discrete and countable set, and $N \geq 1$ is an integer. Let A be any subset of $\mathbb{Z}_N^d$ for which the translates of Ω_1 by A are disjoint, that is,*

$$|\Omega_1 + a \cap \Omega_1 + a'| = 0, \forall a, a' \in A, a \neq a'. \tag{3}$$

For a subset $J \subset \mathbb{Z}_N^d$ with $\sharp J \geq \sharp A$, define

$$\Omega := \Omega_1 + A, \quad \Lambda := \Lambda_1 + J/N, \tag{4}$$

where the sum is taken to be Minkowski addition. If (Ω_1, Λ_1) is frame spectral pair in $\mathbb{R}^d$ with frame constants α, β and (A, J) is a frame spectral pair in $\mathbb{Z}_N^d$ with frame constants c, C, then (Ω, Λ) is also a frame spectral pair in $\mathbb{R}^d$ with frame constants $\alpha c, \beta C$ if the following condition is satisfied:

$$e^{2\pi i \lambda_1 \cdot a} = 1, \qquad \forall a \in A, \forall \lambda_1 \in \Lambda_1. \tag{5}$$

Theorem 2. *With the assumptions of Theorem 1, $\mathcal{E}(\Lambda)$ is an Riesz basis for $L^2(\Omega)$ if $\mathcal{E}(\Lambda_1)$ is a Riesz basis for $L^2(\Omega_1)$ and $\mathcal{E}(J)$ is a basis for $\ell^2(A)$ and (5) holds. In this case, the frame constants are given as in Theorem 1.*

Theorem 3. *With the assumptions of Theorem 1, $\mathcal{E}(\Lambda)$ is an orthogonal basis for $L^2(\Omega)$ if $\mathcal{E}(\Lambda_1)$ is an orthogonal basis for $L^2(\Omega_1)$ and $\mathcal{E}(J)$ is an orthogonal basis for $\ell^2(A)$ and (5) holds.*

1.2 Comparison with existing work

Regarding Theorem 1, any bounded domain Ω with Lebesgue measure $0 < |\Omega| < \infty$ admits an exponential frame $\mathcal{E}(\Lambda)$ by projecting an exponential orthonormal basis of a cube (containing Ω)

onto Ω. In this method, the frame spectrum Λ is necessary a lattice. In Theorem 1, we provide an alternative method for the construction of exponential frames for bounded domains where the frame spectrum is not necessarily a lattice.

Notice that Theorem 2 illustrates how to construct a Riesz spectrum for the union of domains with equal measure. There are many cases where it is known that a set Ω admits a Riesz basis of exponential functions, such as multi-tiling (bounded and unbounded) domains in $\mathbb{R}^d$ [13, 22, 1, 12]. Recently, it was established in [32] that any convex polytope which is centrally symmetric and whose faces of all dimensions are also centrally symmetric, admits a Riesz basis of exponentials. For the existence of exponential Riesz bases in other special cases see, e.g., [26, 5] and the references therein. While there are known cases where Ω does not admit an orthogonal basis of exponentials, less is known about Riesz bases of exponentials. Finding a domain Ω that does not admit any Riesz basis of exponentials is still an unsolved problem.

Below, we compare our results with some well-known results for exponential Riesz bases. Exponential Riesz bases were constructed in [13], and later in [22] with fewer assumptions and a simpler proof. In [22], the author considers a multi-tiling domain in $\mathbb{R}^d$ which tiles the space by a lattice and proves that the domain admits an exponential Riesz basis. The authors obtain a Riesz spectrum using a finite union of translates of the dual lattice. The translation vectors e.g, in [22] depend on the lattice points. In our result, we pick a set Ω_1 with a (not necessarily a lattice) spectrum Λ_1 and consider a union of disjoint multi-integer translations of the set. The result is not necessarily a multi-tiling set. Then we prove the existence of a Riesz spectrum for this set by taking the union of translations of the spectrum Λ_1 by finitely many rational vectors in $[0, 1)^d$. Different than [22], in our proof these vectors do not depend on the spectrum Λ_1.

Another well-known example of exponential Riesz bases for union of co-mensurable cubes has been constructed by DeCarli [5] for the union of unit cubes. In her paper, the author takes a finite set of vectors in $\mathbb{R}^d$ with an arithmetic progression, and she proves that the union of shifts of $\mathbb{Z}^d$ by these vectors is a Riesz spectrum for the union of the cubes if and only if the evaluation matrix is invertible. Equivalently, for a given finite number of vectors in $\mathbb{R}^d$, the union of translations of $\mathbb{Z}^d$ by the vectors is a Riesz spectrum if the matrix is an invertible Vandermonde matrix. Theorem 2 requires fewer assumptions to establish the existence of Riesz bases for a finite union of unit cubes. More precisely, when Ω_1 is a d-dimensional cube, we construct a Riesz spectrum by taking the disjoint union of multi-rational shifts of $\mathbb{Z}^d$ with the sufficient condition that the matrix is invertible.

In Theorem 2, when Ω_1 is a cube in $\mathbb{R}^d$, the structure of the Riesz spectrum for the union of cubes is similar to the well-known example of sampling and interpolation sequence constructed by Lyubarskii and Seip [33] for the union of intervals of equal length, and by Manzo [28] in higher dimensions. In [33], the authors consider a union of p disjoint intervals with equal size and construct a sampling and interpolation sequence for the set using the p shifts of the spectrum of a single interval. Like in the current paper, the sufficient condition is the invertibility of an associated matrix (6). Our result in Theorem 2 provides a machinery for such constructions with more general domains beyond intervals (or cubes).

In recent years, research on orthogonal bases of exponentials has flourished in response to the development of exponential bases in Banach spaces, and in particular, to the growing interest in the Fuglede Conjecture [11]. The Fuglede Conjecture asserts that every domain of $\mathbb{R}^d$ with positive finite Lebesgue measure admits an orthogonal basis of exponentials if and only if it tiles $\mathbb{R}^d$ by translation. Although the Fuglede Conjecture is, in general, false, as shown by Tao in one direction [31] and by Kolountzakis and Matolcsi in the other [23, 24, 29], it has given rise to active investigations of the connections between orthogonal bases of exponentials and tilings in Euclidean space. The conjecture has been proved affirmative in special cases in various settings (continuous and discrete). See, for example, [14, 16, 18, 2, 7], and the references contained therein. There are many cases where it is known that a domain admits no orthogonal exponential bases. See, for example, [25, 17, 10, 15] and the references contained therein. The conjecture is still open in dimensions $d = 1, 2$. However, there are special cases in these dimensions where the conjecture has been proved affirmative (see e.g. [17, 27]).

Regarding Theorem 3, we shall point it out that when Ω_1 is a d-dimensional cube, the new spectrum set that we construct in Theorem 3 is different from the well-known spectrums in the literature, namely, the one for the union of cubes in $\mathbb{R}^d (d \geq 5)$ that was presented by Tao in [31]. The example is used to disprove the direction "spectral $\not\to$ tiling" of the Fuglede Conjecture. In the construction, Tao considers a union of translations of a spectral set in the finite domain $\mathbb{Z}_p^d$ and lifts it to a higher dimension. Our construction of a new spectral set is the result of adding a spectral set in $\mathbb{Z}_p^d$ to a spectral set in the continuous domain $\mathbb{R}^d$.

Another well-known example of a spectral set was constructed by Laba in dimension $d = 1$ [27]. In her paper, Laba characterizes the spectrum of the union of two co-measurable intervals. When the left endpoints of the intervals are integers, the spectrum of the union coincides with the construction of the spectrum we present in Theorem 3. In our result, we construct a spectral set for any (≥ 2) union of co-measurable intervals.

Outline. This paper is organized as follows. After introducing the notations and preliminaries in Section 2, we prove Theorem 1 in Section 3. In Section 4 we prove Theorem 2 and illustrate the explicit structure of biorthogonal dual Riesz bases along with some examples for a special subclass of Riesz spectral pairs and Riesz bases, using the techniques that were developed earlier by the first listed author and Okoudjou in [9]. Results and examples of dual bases in both the continuous and finite setting appear in Sections 4.1 and 4.2. In Section 5, we prove Theorem 3 followed by examples of spectral pairs. In Section 5.2 we remark on the link between the spectral property of a domain and sampling theory.

2 Notations and preliminaries

Throughout this paper, $\Omega \subset \mathbb{R}^d$ is a Lebesgue measurable set with measure $0 < |\Omega| < \infty$, and $\Lambda \subset \mathbb{R}^d$ is countable and discrete. The inner product of $u, v \in L^2(\Omega)$ is defined by $\langle u, v\rangle_{L^2(\Omega)} = \int_\Omega u(x)\overline{v(x)}dx$. In the sequel, we shall drop the subscripts and simply write $\langle f, g\rangle$ when the underlying Hilbert space is clear from the context.

For $f \in L^2(\mathbb{R}^d)$, we denote by $\hat{f}$ or $\mathcal{F}(f)$ the Fourier transform of f, and it is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}dx,\ \xi \in \mathbb{R}^d$, where, $x \cdot \xi = \sum_{i=1}^d x_i\xi_i$ is the scalar products of two vectors. The inverse Fourier transform, denoted by $\mathcal{F}^{-1}(\hat{f}) = f$, is then given by $f(x) = \int_{\mathbb{R}^d} \hat{f}(x)e^{2\pi i x\cdot\xi}dx,\ x \in \mathbb{R}^d$.

Here, and in the sequel, we denote the cardinality of a finite set F by $\sharp F$. For $d, N \geq 1$, $\mathbb{Z}_N^d$ denotes the d-dimensional vector space over the cyclic group $\mathbb{Z}_N$. For any functions $f, g \in \ell^2(\mathbb{Z}_N^d)$, the inner product is defined by $\langle f, g\rangle_{\ell^2(\mathbb{Z}_N^d)} = N^{-d}\sum_{x\in\mathbb{Z}_N^d} f(x)\overline{g(x)}$. In general, for a function $f : \mathbb{Z}_N^d \to \mathbb{C}$, $d \geq 1$, the cyclic Fourier transform is given by $\hat{f}(x) = N^{-d}\sum_{y\in\mathbb{Z}_N^d} e^{2\pi i y\cdot x/N}$.

A frame spectral pair in a finite setting can be defined in a fashion that is similar to the continuous setting. Definition 1 can also be expressed in terms of matrices, as follows: Let $d, N \geq 1$, and assume that $A, J \subseteq \mathbb{Z}_N^d$, with $\sharp J \geq \sharp A$. Let $\mathcal{F}$ denote the $N^d \times N^d$ discrete Fourier transform matrix over $\mathbb{Z}_N$. For a pair (A, J), let $\mathcal{F}_{A,J}$ denote the submatrix of $\mathcal{F}$ (or *evaluation matrix*) with columns indexed by J and rows indexed by A, i.e.

$$\mathcal{F}_{A,J} = \left[\omega^{j \cdot a}\right]_{j \in J, a \in A}, \tag{6}$$

where $\omega = e^{-\frac{2\pi i}{N}}$. We say that (A, J) is a *frame spectral pair in* $\mathbb{Z}_N^d$ if

$$c\|x\|_{\ell^2(A)}^2 \leq \|\mathcal{F}_{A,J}(x)\|_{\ell^2(J)}^2 \leq C\|x\|_{\ell^2(A)}^2, \quad \forall x \in \mathbb{C}^{\sharp A},$$

where $0 < c < C < \infty$. The submatrix $\mathcal{F}_{A,J}$ induces a Riesz basis (an orthogonal basis) for $\ell^2(A)$ if $\sharp A = \sharp J$ and it is invertible (unitary).

Notice that the frame spectral property is a symmetry property when the frame is a Riesz basis or an orthogonal basis. This can be easily verified using the associated submatrices of each system. More precisely, if $\mathcal{E}(J)$ is a Riesz (or orthogonal) basis for $\ell^2(A)$, then the submatrix $\mathcal{F}_{A,J}$ is an invertible (or unitary) matrix. Since the transpose matrix preserves the invertibility and unitary property, thus $\mathcal{E}(A)$ is a Riesz (or orthogonal) basis for $\ell^2(J)$, as claimed. The notion of symmetry discussed above loses its meaning in the infinite or continuous setting, e.g., $\mathbb{Z}_N^\infty$ or $\mathbb{R}^d$. These settings require a well-defined notion of symmetry, which is a challenging problem.

Notation: In the sequel, we use $e_\lambda(x) = e^{2\pi i x \cdot \lambda}$ for exponential functions in $\mathbb{R}^d$, and $E_j(z) = e^{2\pi i \frac{z \cdot j}{N}}$ in the finite setting $\mathbb{Z}_N^d$.

3 Proof of Theorem 1

Proof. Assume that Ω and Λ are given as in (4)$-$(3), and also assume that (5) holds. Let $u \in L^2(\Omega)$. Then

$$\begin{aligned}
\sum_{(\lambda,j)\in\Lambda_1\times J} \left|\langle u, e_{\lambda+j/N}\rangle_{L^2(\Omega_1+A)}\right|^2 &= \sum_{(\lambda,j)\in\Lambda_1\times J} \left|\sum_{a\in A}\langle u, e_{\lambda+j/N}\rangle_{L^2(\Omega_1+a)}\right|^2 \\
&= \sum_{(\lambda,j)\in\Lambda_1\times J} \left|\sum_{a\in A}\int_{\Omega_1} u(x+a)\overline{e_{\lambda+j/N}(x+a)}dx\right|^2 \\
&= \sum_{(\lambda,j)\in\Lambda_1\times J} \left|\int_{\Omega_1}\left(\sum_{a\in A} u(x+a)\overline{e_{j/N}(x+a)}\right)\overline{e_\lambda(x)}dx\right|^2.
\end{aligned} \tag{7}$$

The last line follows from (5). For any fixed $j \in J$, since (Ω_1, Λ_1) is a frame spectral pair and $u_j(x) := \sum_{a\in A} u(x+a)\overline{e_{j/N}(x+a)}$ is in $L^2(\Omega_1)$, there exists an upper frame constant $C > 0$ for which

$$\begin{aligned}
(7) \leq C\sum_{j\in J}\int_{\Omega_1}\left|\sum_{a\in A} u(x+a)\overline{e_{j/N}(x+a)}\right|^2 dx &= C\int_{\Omega_1}\sum_{j\in J}\left|\langle u(x+\cdot), e_{j/N}\rangle_{\ell^2(A)}\right|^2 dx. \\
&\leq C\beta\int_{\Omega_1}\|u(x+\cdot)\|_{\ell^2(A)}^2 dx = C\beta\|u\|_{L^2(\Omega)}^2,
\end{aligned}$$

where β is the upper frame constant for the frame spectral pair (A, J) in $\mathbb{Z}_N^d$. This completes the proof of the upper frame estimate for the pair (Ω, Λ) with the upper bound constant $C\beta$. The lower bound estimate is obtained similarly. The completeness of the frame sequence is obtained by the frame lower bound property. $\square$

We conclude this section with some examples.

Examples:

1. For any given $A, J \subseteq \mathbb{Z}_N^d$ and any frame pair (Ω_1, Λ_1) in $\mathbb{R}^d$, the set of exponentials $\mathcal{E}(\Lambda)$ is a Bessel sequence in $L^2(\Omega)$ with the Bessel constant $\sharp A \sharp J$. The Bessel sequence is a tight frame when $\sharp A = 1$.
2. Let $A \subseteq \mathbb{Z}_N^d$ such that the frame lower bound holds for the exponentials $\mathcal{E}(\Lambda_1)$ in $L^2(\Omega_1)$. Then the frame lower bound holds for $\mathcal{E}(\Lambda_1 + J)$ in $L^2(\Omega)$, for any $J \subset \mathbb{Z}_N^d$, $\Omega = \cup_{a\in A}\Omega_1 + a$.

The following result is of independent interest.

Proposition 1. *Suppose that $\Omega_1 \subset \mathbb{R}^d$ is a set with positive and finite Lebesgue measure, $\Lambda_1 \subset \mathbb{R}^d$ is a discrete and countable set, and $N \geq 1$ is an integer. Let A and J be any two subsets of $\mathbb{Z}_N^d$ with $|\Omega_1 + a \cap \Omega_1 + a'| = 0, \forall a' \in A, a' \neq a$, and define*

$$\Omega := \Omega_1 + A, \quad \Lambda := \Lambda_1 + J/N. \tag{8}$$

If $\mathcal{E}(\Lambda_1)$ is a complete set in $L^2(\Omega_1)$, then the family $\mathcal{E}(\Lambda)$ is also a complete set in $L^2(\Omega)$ if the following condition is satisfied for all $a \in A$: $e^{2\pi i \lambda_1 \cdot a} = 1$.

Proof. Let $u \in L^2(\Omega)$ and $\lambda = \lambda_1 + j/N \in \Lambda$. Then

$$\begin{aligned} \langle u, e_\lambda\rangle_{L^2(\Omega)} &= \sum_{a\in A} \langle u, e_{\lambda_1 + j/N}\rangle_{L^2(\Omega_1 + a)} \\ &= \sum_{a\in A} \langle u(\cdot + a), e_{\lambda_1+j/N}(\cdot + a)\rangle_{L^2(\Omega_1)} \\ &= \langle \sum_{a\in A} u(\cdot + a) e_{j/N}(\cdot + a), e_{\lambda_1}\rangle_{L^2(\Omega_1)}. \end{aligned} \tag{9}$$

Assume that $\langle u, e_\lambda\rangle_{L^2(\Omega)} = 0$ for all $\lambda = \lambda_1 + j \in \Lambda$, (i.e. for all $j \in J$ and $\lambda_1 \in \Lambda_1$). By the completeness of the pair (Ω_1, Λ_1), by (9) for all $j \in J$ we obtain

$$\sum_{a\in A} u(x + a) e_{j/N}(x + a) = 0 \quad a.e.\ x \in \Omega_1.$$

Or,

$$\sum_{a\in A} u(x + a) e_{j/N}(a) = 0 \quad a.e.\ x \in \Omega_1.$$

By the completeness of the pair (A, J) in the setting of $\mathbb{Z}_N^d$, the preceding equality implies that for all $a \in A$,

$$u(x + a) = 0 \quad a.e.\ x \in \Omega_1,$$

or equivalently,

$$u(x) = 0 \quad a.e.\ x \in \Omega_1 + a.$$

This proves that $u = 0$ in $L^2(\Omega_1 + a)$ for all $a \in A$ and therefore $u = 0$ in $L^2(\Omega)$. $\square$

4 Proof of Theorem 2

First we have a motivational result.

Proposition 2. *Assume that $\mathcal{E}(\Lambda_1)$ is a Bessel sequence in $L^2(\Omega_1)$ with a Bessel constant $C > 0$, and let $A, J \subset \mathbb{R}^d$ be any finite sets. Define*

$$\Omega := \Omega_1 + A \ , \ \ \Lambda := \Lambda_1 + J.$$

Then for the family of exponentials $\mathcal{E}(\Lambda)$ the Bessel inequality holds: For any finite set of scalars $\{c_{(\lambda,j)}\}_{(\lambda,j)\in F}$, $F = \Gamma \times I \subset \Lambda_1 \times J$, we have

$$\left\| \sum_{(\lambda,j)\in F} c_{(\lambda,j)} e_{\lambda+j} \right\|^2_{L^2(\Omega)} \leq B \sum_{(\lambda,j)\in F} |c_{(\lambda,j)}|^2, \tag{10}$$

where $B = \sharp A \sharp J |\Omega_1|$.

Proof. The proof is obtained using the Cauchy-Bunyakovsky-Schwartz inequality:

$$\begin{aligned} \left\| \sum_{(\lambda,j)\in F} c_{(\lambda,j)} e_{\lambda+j} \right\|^2_{L^2(\Omega)} &= \left\| \sum_{j\in I} \left(\sum_{\lambda\in\Gamma} c_{(\lambda,j)} e_\lambda \right) e_j \right\|^2_{L^2(\Omega)} \\ &\leq \sum_{a\in A} \left\| \sum_{j\in I} \left(\sum_{\lambda\in\Gamma} c_{(\lambda,j)} e_\lambda \right) e_j \right\|^2_{L^2(\Omega_1+a)} . \end{aligned} \tag{11}$$

For $a \in A$,

$$\begin{aligned} \int_{\Omega_1+a} \left| \sum_{j\in I} \left(\sum_{\lambda\in\Gamma} c_{(\lambda,j)} e_\lambda(x) \right) e_j(x) \right|^2 dx &\leq (\sharp I) \int_{\Omega_1+a} \sum_{j\in I} \left| \sum_{\lambda\in\Gamma} c_{(\lambda,j)} e_\lambda(x) \right|^2 dx \\ &\leq (\sharp J) \sum_{j\in I} \int_{\Omega_1+a} \left| \sum_{\lambda\in\Gamma} c_{(\lambda,j)} e_\lambda(x) \right|^2 dx \\ &= (\sharp J) \sum_{j\in I} \left\| \sum_{\lambda\in\Gamma} c_{(\lambda,j)} e_\lambda(x) \right\|^2_{L^2(\Omega_1+a)} \\ &\leq C(\sharp J) \sum_{(\lambda,j)\in F} |c_{(\lambda,j)}|^2. \end{aligned}$$

The last inequality is obtained using the Bessel property of $\mathcal{E}(\Lambda_1)$. Now, using the inequality in (11), the Bessel inequality (10) holds for $\mathcal{E}(\Lambda)$ in $L^2(\Omega)$ with a Bessel constant $C_1 = C(\sharp J)$. $\square$

Theorem 2 improves the result of Proposition 2 in the sense that a Bessel sequence is a Riesz sequence or a Riesz basis when additional assumptions on A, J and $\mathcal{E}(\Lambda_1)$ are satisfied.

Proof of Theorem 2. The completeness of the exponentials $\mathcal{E}(\Lambda)$ in $L^2(\Omega)$ is due to Theorem 1. Indeed, $\mathcal{E}(\Lambda)$ is a frame for $L^2(\Omega)$ and is therefore complete. It is then sufficient to prove the Riesz inequalities (2) for $\mathcal{E}(\Lambda)$. For this, let F be a finite index set as in Proposition 2 and $\{c_{(\lambda,j)}\}_{(\lambda,j)\in F}$ be any finite set of scalars. Using the equality in (11), we have

$$\begin{aligned}
\left\|\sum_{(\lambda,j)\in F} c_{(\lambda,j)} e_{\lambda+j/N}\right\|^2_{L^2(\Omega)} &= \sum_{a\in A}\left\|\sum_{j\in I}\left(\sum_{\lambda\in\Gamma} c_{(\lambda,j)} e_\lambda\right) e_{j/N}\right\|^2_{L^2(\Omega_1+a)} \\
&= \sum_{a\in A}\int_{\Omega_1+a}\left|\sum_{j\in I}\left(\sum_{\lambda\in\Gamma} c_{(\lambda,j)} e_\lambda(x)\right) e_{j/N}(x)\right|^2 dx \\
&= \int_{\Omega_1}\sum_{a\in A}\left|\sum_{j\in I}\left(\sum_{\lambda\in\Gamma} c_{(\lambda,j)} e_{\lambda+j/N}(x)\right) e_{j/N}(a)\right|^2 dx. \qquad (12)
\end{aligned}$$

The last equality is obtained after applying the assumption that $e_\lambda(a) = 1$, for all $\lambda \in \Lambda_1$ and $a \in A$.

For $j \in I$, define $d_j(x) := e_{j/N}(x)\sum_{\lambda\in\Gamma} c_{(\lambda,j)} e_\lambda(x)$ a.e. $x \in \Omega_1$. Then,

$$(12) = \int_{\Omega_1}\sum_{a\in A}\left|\sum_{j\in I} d_j(x) e_{j/N}(a)\right|^2 dx. \qquad (13)$$

Since $\mathcal{E}(J)$ is a basis for $\ell^2(A)$, it is a Riesz basis by a result co-authored by the second author in [8], and we have

$$\begin{aligned}
(13) &\leq \beta\int_{\Omega_1}\sum_{j\in I}|d_j(x)|^2 dx \\
&= \beta\sum_{j\in I}\int_{\Omega_1}\left|\sum_{\lambda\in\Gamma} c_{(\lambda,j)} e_\lambda(x)\right|^2 dx \\
&= \beta\sum_{j\in I}\left\|\sum_{\lambda\in\Gamma} c_{(\lambda,j)} e_\lambda\right\|^2_{L^2(\Omega_1)} \\
&= \beta C\sum_{j\in I}\sum_{\lambda\in\Gamma}|c_{(\lambda,j)}|^2.
\end{aligned}$$

Notice that we obtained the last inequality by the Riesz basis property of $\mathcal{E}(\Lambda_1)$ in $L^2(\Omega_1)$. A lower Riesz bound can be obtained with a similar calculation. This completes the proof. □

4.1 Explicit form of biorthogonal dual Riesz bases

It is known that any Riesz basis in a Hilbert space has a biorthogonal dual Riesz basis [4]. In the setting of exponential Riesz bases, this statement reads as follows.

Proposition 3. *Given any Riesz basis $\mathcal{E}(\Lambda)$ for $L^2(\Omega)$, there is a unique collection of functions $\{h_\lambda\}_{\lambda\in\Lambda}$ in $L^2(\Omega)$ such that the biorthogonality condition holds:*

$$\langle h_\lambda, e_{\lambda'}\rangle = |\Omega|\ \delta_\lambda(\lambda') \quad \forall\ \lambda, \lambda' \in \Lambda.$$

Moreover $\{h_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis for $L^2(\Omega)$ and any $u \in L^2(\Omega)$ can be represented uniquely as

$$h = |\Omega|^{-1} \sum_{\lambda\in\Lambda} \langle u, h_\lambda\rangle e_\lambda = |\Omega|^{-1} \sum_{\lambda\in\Lambda} \langle u, e_\lambda\rangle h_\lambda.$$

Here, the Dirac δ_t is given by

$$\delta_t(x) = \begin{cases} 1 & \text{if } x = t \\ 0 & \text{otherwise.} \end{cases}$$

The basis $\{h_\lambda\}$ is called *biorthogonal dual* Riesz basis for $\mathcal{E}(\Lambda)$. When $\mathcal{E}(\Lambda)$ is an orthogonal basis for $L^2(\Omega)$, the system of exponentials is self-dual, i.e. $e_\lambda = h_\lambda$, and in (2) we have $c = C = |\Omega|^{-1}$.

4.1.1 Finite case: $\mathbb{Z}_N^d$

Let (A, J) be a Riesz spectral pair in $\mathbb{Z}_N^d$, and suppose that $k = \#A$, that is, $A = \{a_r\}_{r=1}^k$, and $J = \{j_s\}_{s=1}^k$. This means that the $k \times k$ matrix $\mathcal{F}_{A,J}$ in (6) is invertible.

For $F \in \ell^2(A)$,

$$\langle F, E_{j_s}\rangle = \sum_{r=1}^k F(a_r)\omega^{a_r\cdot j_s},$$

where $\omega = e^{-2\pi i/N}$. The reconstruction of F using the Riesz basis $\mathcal{E}(J)$ is accomplished using the linear system

$$\begin{pmatrix} \langle F, E_{j_1}\rangle \\ \langle F, E_{j_2}\rangle \\ \vdots \\ \langle F, E_{j_k}\rangle \end{pmatrix} = \mathcal{F}_{A,J} \begin{pmatrix} F(a_1) \\ F(a_2) \\ \vdots \\ F(a_k) \end{pmatrix} \tag{14}$$

Assume that $\{G_{j_s}\}_{1\le s\le k}$ is the dual in $\ell^2(A)$. Then, we see that by biorthogonality $\langle G_{j_s}, E_{j_{s'}}\rangle = \#A\delta_s(s')$, we can recover the dual functions G_{j_s}. More precisley, for $F = G_{j_s}$ in (14) we get

$$\mathcal{F}_{A,J}^{-1}(k\mathbf{e}_s) = \begin{pmatrix} G_{j_s}(a_1) \\ G_{j_s}(a_2) \\ \vdots \\ G_{j_s}(a_k) \end{pmatrix},$$

where $\mathbf{e}_s$ is a k-dimensional vector with $(\mathbf{e}_s)_t = 1$ if $t = s$ and 0 elsewhere. In summary, the biorthogonal dual Riesz basis can be obtained using the sth column of $\mathcal{F}_{A,J}^{-1}$:

$$G_{j_s}(a_r) = k(\mathcal{F}_{A,J}^{-1})_{r,s} \qquad r = 1, \ldots, k. \tag{15}$$

Notice that by (15), the dual basis is a also set of exponentials only when $\mathcal{F}_{A,J}$ is a unitary matrix.

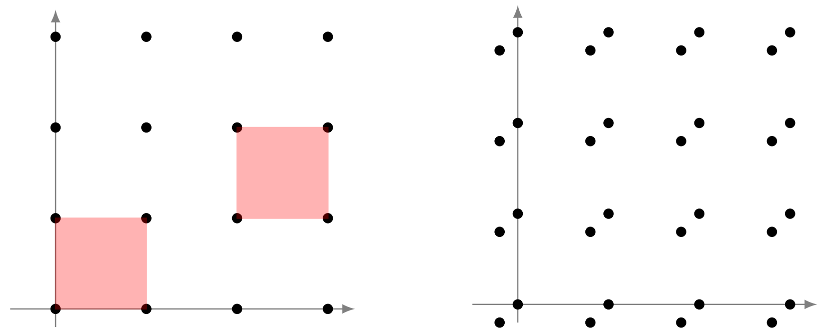

Figure 1: An example of a multi-tiling set of level $k = 2$ in dimention $d = 2$ and its Riesz spectrum.

4.1.2 Continuous case: multi-tiling domains

In this section, we shall illustrate an explicit form of the dual Riesz basis for a class of exponential Riesz bases for a bounded domain Ω given as in Theorem 2 (although the result extends to a class of unbounded domains).

Let Λ_1^* again be a full lattice with a fundamental domain $\Pi_{\Lambda_1^*}$ and dual lattice Λ_1. Then, let $A = \{a_r\}_{r=1}^k \subset \mathbb{Z}_N^d$, and let $J = \{j_s\}_{s=1}^k \subset \mathbb{Z}_N^d$ be a finite set of vectors such that the $k \times k$ matrix $\mathcal{F}_{A,J}$ in (6) is invertible. Then, (A, J) is a Riesz spectral pair. Assume that (Ω_1, Λ_1) is also a Riesz spectral pair in $\mathbb{R}^d$, such that the assumptions (5) and (3) hold. Then $\mathcal{E}(\Lambda)$ is a Riesz basis for $L^2(\Omega)$ as given in Theorem 2.

In the given situation, the associated biorthogonal Riesz basis to $\mathcal{E}(\Lambda)$ with $\Lambda = \Lambda_1 + \frac{1}{N}J$ has been recently illustrated explicitly and constructively by the first author and Okoudjou in [9]. The result is presented in [9] for a more general class of multi-tiling sets, but here we present the special case where Ω is a multi-rectangle in $\mathbb{R}^d$.

Then for $\lambda \in \Lambda_1$ and $1 \leq s \leq k$, the dual basis functions in $L^2(\Omega)$ are given by

$$g_{\lambda + j_s/N}(x) = e_{\lambda + j_s/N}(x) \langle G_{j_s}, E_{j_s} \chi_{\Pi_{\Lambda_1}}(x - \cdot) \rangle_{L^2(A)} \qquad \text{a.e. } x \in \Omega. \tag{16}$$

Here, $\chi_{\Pi_{\Lambda_1}}$ denotes the indicator function of the domain Π_{Λ_1}. For an illustration of a multi-tiling set in dimension $d = 2$ see Figure 3.

Remark 1. *If the system is self-dual, then this formula implies that* $(\mathcal{F}_{A,J})^{-1}_{rs} = \frac{1}{k} e^{2\pi i a_r \cdot j_s / N} = \frac{1}{k} \omega^{a_r \cdot j_s}$. *In this case,* (Ω, Λ) *is a spectral pair if and only if* $(\mathcal{F}_{A,J})^* (\mathcal{F}_{A,J}) = kI$ *meaning that* $\mathcal{F}_{A,J}$ *is a (log) Hadamard matrix [24].*

Remark 2. *A special case is a 1-tiling with repect to the full lattice* Λ_1^* *with a fundamental domain* $\Omega_1 = \Pi_{\Lambda_1}$ *and dual lattice* Λ_1*:*

$$\sum_{\lambda \in \Lambda_1} \langle f, e_\lambda \rangle e_\lambda(x) = vol(\Lambda_1) \sum_{\lambda^* \in \Lambda_1^*} f(x - \lambda^*) = vol(\Lambda_1) f(x).$$

Here we used the Poisson summation formula and the fact that Ω_1 *is a fundamental domain of* Λ_1^* *(so in the last line, the only value of* Λ^* *that gives a nonzero summand is* $\lambda^* = 0$*). This means that for* $f \in L^2(\Omega_1)$*,*

$$f(x) = \frac{1}{vol(\Lambda_1)} \sum_{\lambda \in \Lambda_1} \langle f, e_\lambda \rangle e_\lambda(x) = \sum_{\lambda \in \Lambda_1} \langle f, e_\lambda \rangle g_\lambda(x).$$

The biorthogonal dual Riesz basis is then given by

$$g_\lambda(x) = \frac{1}{vol(\Lambda_1)} e_\lambda(x).$$

4.2 Examples

In this section, we provide few examples of a Riesz spectral pair as well as the biorthogonal dual basis (16) in dimension $d = 2$. Let $\Lambda_1 = \mathbb{Z}^2$, and let $\Pi_{\Lambda_1} = \mathcal{Q}_2$ be the unit square. For a pair of distinct multi-integers $a_1, a_2 \in \Lambda_1$, let

$$\Omega := (\mathcal{Q}_2 + a_1) \cup (\mathcal{Q}_2 + a_2).$$

The set Ω multi-tiles $\mathbb{R}^2$ with the lattice $\mathbb{Z}^2$ at level $k = 2$. Let $N > 2$ be an integer and j_1, j_2 be any two vectors in $\mathbb{Z}^2$ such that the following matrix is invertible:

$$V = \mathcal{F}_{A,J} := \begin{pmatrix} \omega^{a_1 \cdot j_1} & \omega^{a_2 \cdot j_1} \\ \omega^{a_1 \cdot j_2} & \omega^{a_2 \cdot j_2} \end{pmatrix}. \tag{17}$$

Take

$$W = (\mathcal{F}_{A,J})^{-1} = \frac{1}{\det(V)} \begin{pmatrix} \omega^{a_2 \cdot j_2} & -\omega^{a_2 \cdot j_1} \\ -\omega^{a_1 \cdot j_2} & \omega^{a_1 \cdot j_1} \end{pmatrix}.$$

Since $\det(\mathcal{F}_{A,J}) \neq 0$, then the following system of exponential functions is a Riesz basis for $L^2(\Omega)$:

$$\{e_{n+j_1/N}(x)\}_{n \in \mathbb{Z}^d} \cup \{e_{n+j_2/N}(x)\}_{n \in \mathbb{Z}^d},$$

and by (16) the dual basis functions are given by

$$\{g_{n+j_1/N}(x)\}_{n \in \mathbb{Z}^d} \cup \{g_{n+j_2/N}(x)\}_{n \in \mathbb{Z}^d},$$

where

$$g_{n+j_1/N}(x) = 2e_{n+j_1/N}(x) \left(v_{11} w_{11} \chi_{\mathcal{Q}_2}(x - a_1) + v_{12} w_{21} \chi_{\mathcal{Q}_2}(x - a_2) \right) \quad n \in \mathbb{Z}^2,$$

and

$$g_{n+j_2/N}(x) = 2e_{n+j_2/N}(x) \left(v_{21} w_{12} \chi_{\mathcal{Q}_2}(x - a_1) + v_{22} w_{22} \chi_{\mathcal{Q}_2}(x - a_2) \right) \quad n \in \mathbb{Z}^2.$$

Example 1. *(case $d = 1$) Following the notation in Section 4.2, let $\{a_1 = 0, a_2 = 2\}$, and let $\{j_1 = 0, j_2 = 1\}$, for any integer number $N > 2$. Then the matrix V in (17) is invertible, so (Ω, Λ) is a Riesz spectral pair, where*

$$\Omega = [0, 1] \cup [2, 3]\ ,\ \Lambda = \mathbb{Z} \cup \mathbb{Z} + 1/N.$$

Then the exponential Riesz basis constructed above coincides with the exponential Riesz basis given as in Theorem 2, when $\Omega_1 = [0, 1], \Lambda_1 = \mathbb{Z}$, $A = \{0, 2\}$, $J = \{0, 1\}$, and $N > 2$. By the symmetry property, we obtain that $\Omega = [0, 2]$ and $\Lambda = \mathbb{Z} \cup \mathbb{Z} + 2/N$ is also a Riesz spectral pair. It can be readily verified that the system is an orthogonal basis only for $N = 4$; see Theorem 3. In this case, V is a unitary matrix with $V^*V = 2I$ and the basis is self-dual.

Example 2. *(case $d = 2$) As a concrete example in $d = 2$, let $N = 4$, $a_1 = (0,0)$, $a_2 = (2,0)$, $j_1 = (0,0)$, and $j_2 = (1,0)$. For this choice, the matrix V in (17) is invertible, so the following pair (Ω, Γ) builds a Riesz spectral pair:*

$$\Omega = \mathcal{Q}_2 \cup \mathcal{Q}_2 + (2,0)\ ,\ \ \Lambda = \mathbb{Z}^2 \cup \mathbb{Z}^2 + (1/4, 0).$$

By the symmetry property, for $\Omega = [0,2] \times [0,1]$ and $\Lambda = \mathbb{Z}^2 \cup \mathbb{Z}^2 + (1/2,0) = 2^{-1}\mathbb{Z} \times \mathbb{Z}$, the pair (Ω, Λ) is also a Riesz spectral pair, as we expected.

5 Proof of Theorem 3

Given positive integers $N, d \geq 1$, and sets $A, J \subseteq \mathbb{Z}_N^d$ of the same cardinality, we say that the family of exponentials $\{E_j(z) = e^{2\pi i j \cdot z/N}\}_{j \in J}$ is an orthogonal basis for $\ell^2(A)$ if the elements of $\mathcal{E}(J)$ are mutually orthogonal on A; that is

$$\langle E_j, E_{j'} \rangle_{\ell^2(A)} = \sum_{a \in A} e^{2\pi i (j-j')\cdot a/N} = 0 \quad \forall j, j' \in J,\ j \neq j', \quad \text{and,} \tag{18}$$

Notice that the orthogonality relation (18) can also be expressed as

$$\hat{\chi}_A(j - j') \equiv N^{-d} \sum_{a \in A} e^{2\pi i (j-j')\cdot a/N} = 0, \quad \forall j \neq j'$$

where $\hat{\chi}_A$ is the Fourier transform of the indicator function χ_A.

Proof of Theorem 3. To prove mutual orthogonality of the exponentials $\mathcal{E}(\Lambda)$ in $L^2(\Omega)$, let $\lambda_1 + j_1/N$ and $\lambda_2 + j_2/N$ be two distinct vectors in Λ. We have the following:

$$\begin{aligned}
\langle e_{\lambda_1+j_1/N}, e_{\lambda_2+j_2/N} \rangle_{L^2(\Omega)} &= \sum_{a \in A} \langle e_{\lambda_1+j_1/N}, e_{\lambda_2+j_2/N} \rangle_{L^2(\Omega_1+a)} \\
&= \left(\sum_{a \in A} e_{\lambda_1+j_1/N}(a) e_{\lambda_2+j_2/N}(-a) \right) \langle e_{\lambda_1+j_1/N}, e_{\lambda_2+j_2/N} \rangle_{L^2(\Omega_1)} \qquad (19) \\
&= \left(\sum_{a \in A} e_{j_1/N}(a) e_{j_2/N}(-a) \right) \langle e_{\lambda_1+j_1/N}, e_{\lambda_2+j_2/N} \rangle_{L^2(\Omega_1)} \qquad (20) \\
&= \langle e_{j_1}, e_{j_2} \rangle_{l^2(A)} \langle e_{\lambda_1+j_1/N}, e_{\lambda_2+j_2/N} \rangle_{L^2(\Omega_1)}. \qquad (21)
\end{aligned}$$

To pass from (19) to (20) we used the assumption that $e_\lambda(a) = 1$ for all $a \in A$ and $\lambda \in \Lambda_1$. If $j_1 \neq j_2$, by the orthogonality of $\mathcal{E}(J)$, the first inner product in (21) is zero. If $j_1 = j_2$, then by assumption $\lambda_1 \neq \lambda_2$ and by the orthogonality of $\mathcal{E}(\Lambda_1)$, $\langle e_{\lambda_1+j_1/N}, e_{\lambda_2+j_2/N} \rangle_{L^2(\Omega_1)} = \langle e_{\lambda_1}, e_{\lambda_2} \rangle_{L^2(\Omega_1)} = 0$. The completeness of $\mathcal{E}(\Lambda)$ in $L^2(\Omega)$ is obtained by a density condition (see [6]), or directly by Theorem 1 since $\mathcal{E}(\Lambda)$ is a frame for $L^2(\Omega)$.

□

Remark 3. *Note due to the symmetry property of spectral pairs in finite settings, by the assumption of Theorem 3, the domain $\Omega = \Omega_1 + J$ admits an orthogonal basis of exponentials $\mathcal{E}(\Lambda)$ with $\Lambda = \Lambda_1 + A/N$.*

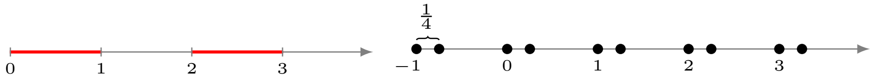

Figure 2: The figure illustrates $\Omega = [0,1] \cup [2,3]$ and its spectral set $\Lambda = \mathbb{Z} \cup \mathbb{Z} + 1/4$ in $d = 1$.

Remark 4. *Let (Ω_1, Λ_1) be a spectral pair. Assume that $A \subset \mathbb{Z}^d$ is finite and $B \subset \mathbb{R}^d$ is a finite set of rational numbers. Assume that $\sharp A = \sharp B = d$ and there exists an integer number $N \geq 1$ such that $NB \subset \mathbb{Z}^d$. Consider the $d \times d$ matrix $V = (v_{a,b})_{a\in A, b\in B}$ with entries given by*

$$v_{a,b} = e^{-2\pi i a\cdot b}.$$

If V is a unitary (or invertible) matrix, then $\mathcal{E}(\Lambda)$ is an orthogonal basis (or a Riesz basis) for $L^2(\Omega)$, where

$$\Omega = \Omega_1 + A, \ \ \Lambda = \Lambda_1 + B.$$

This can be readily obtained by Theorem 3 since (A, NB) is a spectral pair in $\mathbb{Z}_N^d$. The proof of the Riesz basis is due to Theorem 2 when V is an invertible matrix.

5.1 Examples of spectral pairs

The first example shows that the sufficient condition (5) is also a *necessary condition* in Theorem 3.

Example 3. *Let $\Omega_1 = [0,2]$. Take $N \in 6\mathbb{Z}$, and positive, $A = \{0,3\}$, $J = \{0, \alpha = N/6\}$. Then (A,J) is a spectral pair in $\mathbb{Z}_N^d$, while (Ω, Λ) fails to be a spectral pair in $\mathbb{R}^d$. Indeed, $\Lambda = 2^{-1}\mathbb{Z} \cup 2^{-1}\mathbb{Z} + 6^{-1}$, the exponentials $\mathcal{E}(\Lambda)$ are not mutually orthogonal on $\Omega = [0,2] \cup [3,5]$. Indeed, let*

$$Z_\Omega = \{0\} \cup \{\xi \in \mathbb{R} : \ \hat{\chi}_\Omega(\xi) = 0\}.$$

If $\mathcal{E}(\Lambda)$ is an orthogonal set in $L^2(\Omega)$, then $\Lambda \subseteq \Lambda - \Lambda \subseteq Z_\Omega$. On the other hand, we have $Z_\Omega = 2^{-1}\mathbb{Z} \cup 3^{-1}\mathbb{Z}$. This is a contradiction, thus the functions in $\mathcal{E}(\Lambda)$ are not orthogonal on Ω.

Example 4. *Let $N \geq 1$, and assume that (A,J) is a spectral pair in $\mathbb{Z}_N$. Take $\Omega := \cup_{a\in A}[a, a+1)$. Then, by Theorem 3, the pair (Ω, Λ) is spectral in $\mathbb{R}$ for $\Lambda = \mathbb{Z} + J/N$, if $|\Omega + j_1 \cap \Omega_1 + j_2| = 0$ for all $j \neq j'$.*

Example 5 ($d = 1$)**.** *Let $N = 4$, $A = \{0,2\}$ and $J = \{0,1\}$. Define the 2×2 evaluation matrix*

$$H = \frac{1}{\sqrt{2}}(\omega^{a\cdot j})_{a\in A, j\in J}.$$

Then, H is a unitary matrix, that is, $H^*H = I$. This proves that (A,J) is a spectral pair in $\mathbb{Z}_4$. By Theorem 3, for $\Omega = [0,1] \cup [2,3]$ and $\Lambda = \mathbb{Z} \cup \mathbb{Z} + 1/4$, the pair (Ω, Λ) is a spectral pair $\mathbb{R}$. This example is illustrated in Figure 2.

By the symmetry property, the pair (J, A) is also a spectral pair. Therefore, using the result of Theorem 3 we obtain the well-known spectral pair $(\Omega = [0,2], \Lambda = 2^{-1}\mathbb{Z})$.

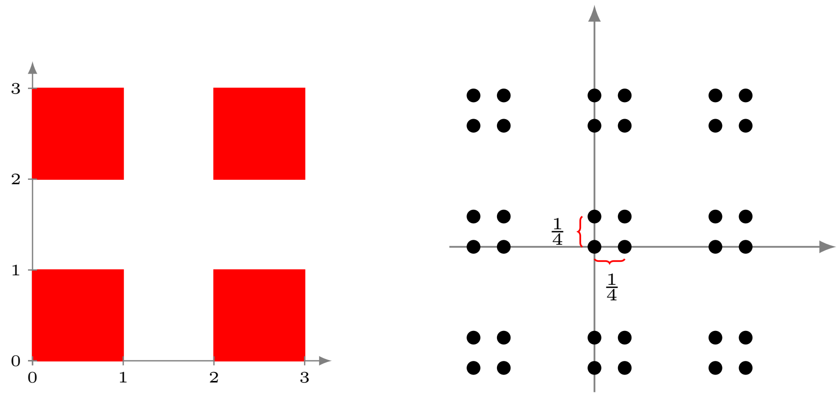

Figure 3: An example of a spectral set with its spectrum in $d = 2$

Example 6 ($d = 2$)**.** *To construct a spectral pair in higher dimensions, one possible technique is using the Cartesian product as in the results of Jorgensen and Pedersen in [21], Section 2. See also [19, 20] for more on the construction and 'closeness' of spectral pairs in Cartesian settings. An example of a spectral pair in dimension $d = 2$ is depicted in Figure 3. This example is constructed from the self Cartesian product of the spectral pair given in Example 5.*

5.2 Spectral set and function recovery

In this section, we remark on the link between spectral pairs and recovery problem, known as Shannon Sampling theorem. First, we need the following general result for any pair in $\mathbb{Z}_N$.

Lemma 1. *Define $\Omega = [0,1] + A$ and $\Lambda = \mathbb{Z} + J/N$ for any sets $A, J \subset \mathbb{Z}_N$, and suppose that f is a function in the Paley-Wiener space $PW_\Omega = \{f \in L^2(\mathbb{R}) \mid \hat{f}(\xi) = 0 \text{ a.e. } \xi \notin \Omega\}$. Then, the distribution*

$$F_s(t) = \sum_{\lambda \in \Lambda} f(\lambda)\delta_\lambda(t), \; t \in \mathbb{R} \tag{22}$$

has a Fourier transform given by

$$\hat{F}_s(\xi) = \sum_{k \in \mathbb{Z}} \hat{\chi}_J(k)\hat{f}(\xi - k) \quad \text{a.e. } \xi \in \Omega. \tag{23}$$

Proof. We express the distribution (general function) F_s as

$$F_s(t) = f(t)P(t) \tag{24}$$

where $P \in \mathscr{S}'(\mathbb{R})$ is a distribution, known as a *(Λ-sampling) pattern function*, given by

$$P(t) := \sum_{\lambda \in \Lambda} \delta_\lambda(t) = \sum_{n \in \mathbb{Z}} \delta_n(t) * \sum_{j \in J} \delta_{j/N}(t), \; t \in \mathbb{R}.$$

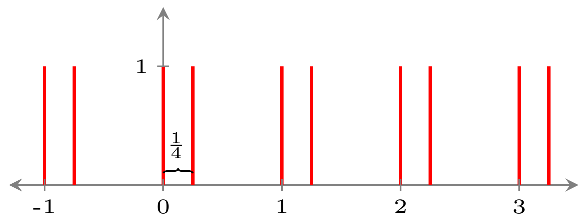

Figure 4: The figure illustrates the sampling pattern $\Lambda = \mathbb{Z} \cup \mathbb{Z} + 1/4$ in $d = 1$.

An example of such pattern is illustrated in Figure 4. By applying the Fourier transform to (24) we obtain

$$\hat{F}_s = \hat{f} * \hat{P}, \tag{25}$$

where

$$\hat{P}(\xi) = \mathcal{F}(\sum_{n\in\mathbb{Z}} \delta_n)(\xi)\mathcal{F}(\sum_{j\in J} \delta_{j/N})(\xi),\ \xi \in \mathbb{R}. \tag{26}$$

By applying the Poisson summation formula to the first term of (26), we get

$$\mathcal{F}(\sum_{n\in\mathbb{Z}} \delta_n)(\xi) = \sum_{n\in\mathbb{Z}} e^{-2\pi in\xi} = \sum_{k\in\mathbb{Z}} \delta_k(\xi),\ \xi \in \mathbb{R} \tag{27}$$

For the second term in (26), in terms of distributions we have

$$\mathcal{F}(\sum_{j\in J} \delta_{j/N})(\xi) = \sum_{j\in J} e^{-2\pi ij\xi/N},\ \xi \in \mathbb{R}. \tag{28}$$

Define the function[1] $\hat{\chi}_J(\xi) := \sum_{j\in J} e^{-2\pi ij\xi/N}$. By (27) and (28) in (26), we obtain

$$\hat{P}(\xi) = \hat{\chi}_J(\xi) \sum_{k\in\mathbb{Z}} \delta_k(\xi) = \sum_{k\in\mathbb{Z}} \hat{\chi}_J(k)\delta_k(\xi).$$

By substituting this expression in (25) we obtain

$$\hat{F}_s(\xi) = \hat{f} * \hat{P}(\xi) \tag{29a}$$

$$= \left(\hat{f} * \sum_{k\in\mathbb{Z}} \hat{\chi}_J(k)\delta_k\right)(\xi) \tag{29b}$$

$$= \sum_{k\in\mathbb{Z}} \hat{\chi}_J(k)\left(\hat{f} * \delta_k\right)(\xi) \tag{29c}$$

$$= \sum_{k\in\mathbb{Z}} \hat{\chi}_J(k)\hat{f}(\xi - k) \quad \forall \xi \in \Omega. \tag{29d}$$

This completes the proof. $\square$

[1] Note that the definition of this general function over $\mathbb{R}$, also called the *symbol* of J, coincides with the Fourier transform of the characteristic function χ_J over the cyclic group when the domain is restricted to $\mathbb{Z}_N$ up to a constant.

Note that the result of the lemma holds for any domain $\Omega \subset \mathbb{R}^d$ with finite measure and any $\Lambda = \Lambda_1 + J/N$, $J \subset \mathbb{Z}_N^d$, such that the Poisson summation holds for Λ_1.

As an application of the previous lemma, we have the following result.

Theorem 4. *Given a spectral pair (A, J) in $\mathbb{Z}_N$, let Ω and Λ be $\Omega := [0,1] + A$ and $\Lambda = \mathbb{Z} + J/N$. Let $f \in L^2(\mathbb{R})$ with $\hat{f}$ supported in Ω. Let F_s be as in (23). Then*

$$\hat{F}_s(\xi) = (\sharp J)\hat{f}(\xi), \ \ a.e. \xi \in \Omega.$$

Proof. For this, we need to show that all the terms in (29d) are zero except for $k = 0$. We do this by considering different cases for the values of k.

I: If $k = 0$, we have $\hat{\chi}_J(k)\hat{f}(\xi - k) = \sharp J \hat{f}(\xi)$.

II: If $k \neq 0$, the summation is known as *aliasing term* and we consider two cases as well:

(a) $k \in A - A$, $k \neq 0$. Then, $k = a - a'$ for some distinct $a, a' \in A$. By the symmetry property of the spectral pairs in $\mathbb{Z}_N$ (and more generally in $\mathbb{Z}_N^d$), A is a spectral set for the set J and we obtain $\hat{\chi}_J(k) = \hat{\chi}_J(a - a') = 0$.

(b) $k \in \mathbb{Z} \backslash \{A - A\}$, $k \neq 0$. In this case, we claim that $|\Omega \cap \Omega + k| = 0$. If not, then there must be $a, a' \in A$ such that $|[a + k, a + k + 1) \cap [a', a' + 1)| \neq 0$. Since $a \in \mathbb{Z}$ and each interval has length unit 1, then we must have $k = a' - a$, which is a contradiction. Thus the sum over all $k \notin A - A$ is equal to zero.

In summary, we obtain $\hat{F}_s(\xi) = \sharp J \hat{f}(\xi)$ for $\xi \in \Omega$, or,

$$\hat{f}(\xi) = (\sharp J)^{-1}\hat{F}_s(\xi). \tag{30}$$

□

Notice that, by (29d), the Fourier transform of F_s is equal to a sum of translated copies of the Fourier transform of f on k-shifts of Ω multiplied with coefficients $\hat{\chi}_J(k)$. The theorem proves that the Fourier transform of F_s over Ω is the exact Fourier transform of f up to some constant, and the translations do not overlap. In the language of signal processing, this means that the *aliasing term* is zero.

Remark 5. *Given a spectral pair (A, J) in $\mathbb{Z}_N$, let Ω and Λ be $\Omega := [0,1] + A$ and $\Lambda = \mathbb{Z} + J/N$. Let $f \in L^2(\mathbb{R})$ with $\hat{f}$ supported in Ω. Then for any $f \in PW_\Omega$, the reconstruction formula holds:*

$$\hat{f}(\xi) = (\sharp J)^{-1} \sum_{\lambda \in \Lambda} f(\lambda) e^{2\pi i \lambda \xi} \quad a.e. \ \xi \in \Omega$$

.

Proof. By the previous theorem, we have

$$\hat{f}(\xi) = (\sharp J)^{-1}\hat{F}_s(\xi). \tag{31}$$

By the definition of F_s in (23), we obtain

$$\hat{F}_s(\xi) = \int_{\mathbb{R}} \left(\sum_{\lambda\in\Lambda} f(\lambda)\delta_\lambda(t) \right) e^{-2\pi i t\xi} dt = \sum_{\lambda\in\Lambda} f(\lambda) e^{-2\pi i\lambda\xi} \quad a.e.\ \xi \in \Omega.$$

Substituting this in (30) we obtain

$$\hat{f}(\xi) = (\sharp J)^{-1} \sum_{\lambda\in\Lambda} f(\lambda) e^{-2\pi i\lambda\xi} \quad a.e.\ \xi \in \Omega.$$

This completes the proof. □

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