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ORIGINAL ARTICLE

Scattering Map for the Vlasov–Poisson System

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Abstract
We construct (modified) scattering operators for the Vlasov–Poisson system in three dimensions, mapping small asymptotic dynamics as $t \to -\infty$ to asymptotic dynamics as $t \to +\infty$. The main novelty is the construction of modified wave operators, but we also obtain a new simple proof of modified scattering. Our analysis is guided by the Hamiltonian structure of the Vlasov–Poisson system. Via a pseudo-conformal inversion, we recast the question of asymptotic behavior in terms of local in time dynamics of a new equation with singular coefficients which is approximately integrated using a generating function.

Keywords Vlasov–Poisson · Modified scattering · Modified wave operators · Scattering map

Mathematics Subject Classification 35Q83 · 35B40 · 35Q70

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1 Introduction

The three-dimensional Vlasov–Poisson system describes the evolution of a particle distribution[1] $\mu(t,x,v) : \mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}$ satisfying

$$\big(\partial_t + v\cdot\nabla_x\big)\mu + \lambda\nabla_x\psi\cdot\nabla_v\mu = 0, \quad \Delta_x\psi(t,x) = \rho(t,x), \quad \rho(t,x) = \int_{\mathbb{R}^3}\mu^2(t,x,v)\mathrm{d}v. \tag{1.1}$$

This is a model for a continuum limit of a classical many-body problem with Newtonian self-interactions through a force field $\nabla_x\psi$ that can be attractive ($\lambda = -1$) as in a galactic setting, or repulsive ($\lambda = 1$) as in a plasma or ion gas, and which is generated by the spatial density $\rho(t,x)$ of the particle distribution.

The mathematical theory for the initial value problem associated with (1.1) is classical and guarantees the global existence of unique solutions under suitable assumptions on the initial data [1, 24, 32, 33]. In recent years, there has been progress in understanding the long time asymptotic behavior: sharp decay rates of the density and force field are known in some settings [17, 19, 29, 31, 36, 38], and it has been shown that for sufficiently small initial data μ_0 the problem (1.1) exhibits a modified scattering dynamic [6, 20] defined in terms of a limit distribution μ_∞ and an asymptotic force field $E_\infty[\mu_\infty]$, defined by inverting the roles of x and v:

$$E_\infty[\mu](v) := \frac{1}{4\pi}\iint\frac{v-w}{|v-w|^3}\cdot\mu^2(y,w)\mathrm{d}y\mathrm{d}w. \tag{1.2}$$

In this paper, using pseudo-conformal inversion, we prove the converse statement, namely that any solution of the asymptotic dynamic arises in a unique way as a limit of a solution to (1.1), i.e., we construct the wave operator $\mu_\infty \mapsto \mu_0$. Thus, we obtain the existence of a scattering operator linking the asymptotic behavior in the past to the asymptotic behavior in the future ($\mu_{-\infty} \mapsto \mu_0 \mapsto \mu_{+\infty}$).

Our main results can be summarized as follows:

Theorem 1.1 *There exists $\varepsilon > 0$ such that:*

(i) (*Global existence and modified scattering*) *Given $\mu_1(x,v)$ satisfying*

$$\|\mu_1\|_{L^2_{x,v}} + \big\|\langle x-v\rangle^2\mu_1\big\|_{L^\infty_{x,v}} + \|\nabla_{x,v}\mu_1\|_{L^\infty_{x,v}} \le \varepsilon, \tag{1.3}$$

there exists a unique global strong solution μ of the initial value problem for (1.1) with $\mu(1,x,v) = \mu_1(x,v)$. In addition, there exist $\mu_\infty(x,v)$ and $E_\infty = E_\infty[\mu_\infty]$ as in (1.2) such that, locally uniformly in (x,v),

[1] To be precise, the physically relevant quantity $f(t,x,v) = \mu^2(t,x,v)$ is the square of our unknown μ—see also [20].

$$\mu(t, x + tv - \lambda \ln(t) E_\infty(v), v) \to \mu_\infty(x, v), \quad t \to +\infty. \tag{1.4}$$

(ii) (*Existence of modified wave operators*) *Given* $\mu_\infty \in W^{2,\infty}(\mathbb{R}^3_x \times \mathbb{R}^3_v)$ *and* $E_\infty = E_\infty[\mu_\infty] \in W^{3,\infty}(\mathbb{R}^3)$ *as in* (1.2) *satisfying*

$$\begin{aligned} &\|\mu_\infty\|_{L^2_{x,v}} + \left\|\langle x\rangle^5 \mu_\infty\right\|_{L^\infty_{x,v}} + \|\langle x\rangle \nabla_{x,v} \mu_\infty\|_{L^\infty_{x,v}} + \left\|\langle x\rangle^2 \nabla^2_{x,v} \mu_\infty\right\|_{L^\infty_{x,v}} \\ &\quad + \|E_\infty\|_{W^{3,\infty}} < \infty, \end{aligned} \tag{1.5}$$

there exists a unique strong global solution μ *of* (1.1) *for which* (1.4) *holds.*

(iii) (*Scattering map*) *For any asymptotic state* $\mu_{-\infty}$ *with* $E_{-\infty} = E_\infty[\mu_{-\infty}] \in W^{3,\infty}(\mathbb{R}^3)$ *as in* (1.2),

$$\|\mu_{-\infty}\|_{L^2_{x,v}} + \left\|\langle x, v\rangle^5 \mu_{-\infty}\right\|_{L^\infty_{x,v}} + \|\langle x\rangle \nabla_{x,v} \mu_{-\infty}\|_{L^\infty_{x,v}} + \left\|\langle x\rangle^2 \nabla^2_{x,v} \mu_{-\infty}\right\|_{L^\infty_{x,v}} \le \varepsilon,$$

there exist a unique strong solution μ *of* (1.1), $\mu_{+\infty} \in L^2_{x,v} \cap L^\infty_{x,v}$ *and* $E_{+\infty} = E_\infty[\mu_{+\infty}]$ *such that*

$$\mu(t, x + tv \mp \lambda \ln(\langle t\rangle) E_{\pm\infty}(v), v) \to \mu_{\pm\infty}(x, v), \quad t \to \pm\infty. \tag{1.6}$$

We call the map defined in a neighborhood of the origin in the Schwartz space through (iii) above,

$$\mathcal{S} : \mu_{-\infty} \mapsto \mu_{+\infty}, \tag{1.7}$$

the *Scattering map*. We refer to Theorem 3.1 for a more precise statement of our results for (i) and to Theorem 4.2 for a more precise statement of (ii). In particular, we note that the force field has optimal decay $|\nabla \psi| \lesssim \langle t\rangle^{-2}$ in all cases.

Remark 1.2 We comment on some points:

(1) The main novelty of this work is the construction of the wave operator (ii). While the small data modified scattering dynamic (1.4) was already obtained in [20], the present result (i) is also of interest since it is stronger and the approach, while less generalizable, leads to a simple derivation of the asymptotic dynamic. We also refer to [30] for yet another point of view on the modified scattering as arising from mixing.
(2) Our topology for small data/modified scattering in (1.5) is weaker than in all other works on asymptotic behavior that we are aware of [1, 6, 17, 20, 29, 36, 38]. It is unclear what the optimal topology is, but to get almost Lipschitz bounds on the force field, by (1.8), one cannot work in a much weaker setting than ours.
(3) We also obtain propagation of regularity: assuming more regularity on the initial data we obtain higher regularity on the final (scattering) data and vice versa.
(4) Our initial data for scattering may have infinite energy and momentum; in addition, a simple modification also allows for initial data of infinite mass. It is unclear which role (if any) the physical conservation laws play for the asymptotic behavior.

(5) It is worth noting a curious fact: our proof can be adapted directly to the case of a plasma of two species (ions and electrons). In this case, using (ii), one can construct solutions for which the asymptotic electric field profile $E_\infty \equiv 0$ vanishes and the solutions *scatter linearly*. In this case, the same equation allows two different asymptotic behaviors. It remains to be understood to which extent the linear scattering is nongeneric (say in case the total charge vanishes).

1.1 About the Proof of Theorem 1.1

In the spirit of the prior work [20] (see also [9, 10, 22, 23]), we build on parallels between kinetic and dispersive equations. In particular, the Hamiltonian structure of (1.1) guides our analysis.

The simplest case for asymptotic behavior of a nonlinear equation is *linear scattering* when the nonlinearity can simply be neglected to model asymptotic dynamics. For the Vlasov–Poisson system, this happens in the setting of Landau damping [2, 11, 27], the ion/screened problem [3, 14], and in higher dimensions [36], where solutions asymptotically satisfy $\mathcal{T}(\mu) = 0$ with $\mathcal{T}$ defined in (1.9). The asymptotic behavior of *modified scattering* as in (1.4) and (1.6) can be viewed as a manifestation of the unrelenting relevance of nonlinear interactions in (1.1) throughout time. In (1.1) the nonlinear, long-range interactions are governed by a force field which does not decay fast enough to produce only a finite correction as time tends to infinity and produces the logarithmic corrections identified in the above theorem—see also [6, 20, 30] for the Vlasov–Poisson setting, and [15, 16, 18, 21, 28] for related results on other equations.

To understand the asymptotic behavior, we need to (i) identify a mechanism for decay (here dispersion), (ii) prove global existence, (iii) isolate an asymptotic dynamic and (iv) prove convergence to it. We offload the dispersion to the pseudo-conformal transform $\mathcal{I}$ which compactifies time and reduces global existence to local existence for a singular equation in the transformed unknown

$$\gamma(s,q,p) := \mu\Big(\frac{1}{s}, \frac{q}{s}, q - sp\Big), \quad (s,q,p) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3,$$

see also [4, 5, 7, 37] for similar ideas. At this point, the problem merely reduces to establishing convergence at the image of infinity, $s = 0$, where, however, the equation has a violent singularity. We extend the force field $E = -\nabla\psi$ via a variant of the continuity equation:

$$\partial_s E + \nabla\Delta^{-1}\mathrm{div}(\mathbf{j}) = 0, \quad \mathbf{j}(s,q) = \int p\gamma^2(s,q,p)\mathrm{d}p, \tag{1.8}$$

which does not involve the (singular) acceleration and provides good control of E so long as we control some moments of γ. Once we obtain convergence of E to a fixed asymptotic field E_0, the equation becomes a simple perturbation of transport by a shear term:

$$\big(\partial_s + \lambda s^{-1} E_0(q) \cdot \nabla_p\big)\gamma = O(1),$$

which is easily integrated to recover the dynamic originally isolated in [20]. To make this rigorous, we need to propagate mild control on appropriate norms. This is done through a bootstrap that allows some deterioration over time in different ways depending on the scenarios: growth of nonconvergent norms in the case of modified scattering and loss of moment in the case of wave operators (where we start from the singular time $s = 0$).

The proof of part (i) shows how natural the pseudo-conformal inversion $\mathcal{I}$ is to study asymptotics of (1.1): working with only moments that are conserved in the linear evolution of (1.1) one directly obtains global solutions in a bootstrap argument. Additional regularity as in (1.3) is easily propagated to yield unique strong solutions and to recover the asymptotic behavior (1.4)—see Sect. 3.

Part (ii) is proved using a canonical change of variables in (1.12) to mitigate the strong singularity at $s = 0$—see Sect. 4. The Cauchy problem for the resulting equations (4.5) can in fact be (locally) solved starting from $s = 0$ for a sufficiently large class of initial data as in (1.5). Again, moments are easily bootstrapped, while propagating derivatives requires us to identify a proper weighted norm which compensates for the *ill-conditioned* Hessian of the new Hamiltonian by allowing one loss of moment. Since via $\mathcal{I}$ this corresponds to a strong solution on $[T, \infty)$ for some $T > 0$, classical theory as in [24] then gives a global solution.

Finally (iii) follows simply by combining (ii) (backwards in time) to go from past-asymptotic data to initial data and (i) to go from initial data to future asymptotic data.

While it may be less intuitive, using the pseudo-conformal transformation simplifies the presentation over the physical space analysis as in [20], and quickly leads to the natural modified scattering behavior. It also sheds new light on some classical decay estimates like (1.13).

1.2 Open Questions

We list some open questions which remain outstanding:

- Is there a topology that makes the scattering operator in (1.7) an endomorphism?
- In the plasma case $\lambda = +1$, what is the asymptotic behavior for large data? Solutions are global, there are no nontrivial equilibriums and the wave operators are defined for large data, so it is tempting to believe that Theorem 1.1 may be extended to all solutions (see [19, 31, 34] and references therein for general results in this direction, and [29, 35] for the case of more symmetric data).
- In the gravitational case $\lambda = -1$, is there a "ground state", i.e., a smallest solution which does not scatter? Are there solutions which satisfy some form of modified scattering towards a nonzero stationary solution (of which there are many, see, e.g., [12, 22, 26])? This appears very challenging, but we note [30] for an example of a stability result around a nonzero equilibrium in a related setting and [8, 13] for related works.

1.3 Pseudo-conformal Inversion

We define the involution of $\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3$ given by the *pseudo-conformal inversion* (see also [22])

$$\mathcal{I}:(t,x,v)\mapsto\left(\frac{1}{t},\frac{x}{t},x-tv\right).$$

This transformation interacts favorably with free streaming,

$$\mathcal{T}:=\partial_t+v\cdot\nabla_x, \tag{1.9}$$

since heuristically it exchanges the role of v with that of $x-tv$, both of which are conserved along the evolution (i.e., commute with $\mathcal{T}$). Indeed, one can observe that if $(s,q,p)=\mathcal{I}(t,x,v)$,

$$\partial_s=-s^{-2}\big(\partial_t+q\cdot\nabla_x\big)-p\cdot\nabla_v,\quad \nabla_q=s^{-1}\nabla_x+\nabla_v,\quad \nabla_p=-s\nabla_v,$$

and

$$\mathcal{T}(f\circ\mathcal{I})=-s^{-2}\mathcal{T}(f)\circ\mathcal{I}.$$

so that composition with $\mathcal{I}$ preserves the class of solutions of free streaming $\mathcal{T}f=0$. The transformation $\mathcal{I}$ is almost symplectic in the sense that $\mathrm{d}q\wedge\mathrm{d}p=-\mathrm{d}x\wedge\mathrm{d}v$, and in particular the total charge is preserved:

$$\iint(f\circ\mathcal{I})^2\mathrm{d}q\mathrm{d}p=\iint f^2\mathrm{d}x\mathrm{d}v.$$

1.3.1 Recasting Vlasov–Poisson

Given a solution $\mu(t,x,v)$ of (1.1), we let $\gamma=\mu\circ\mathcal{I}$, so that

$$\gamma(s,q,p):=\mu\left(\frac{1}{s},\frac{q}{s},q-sp\right),\quad \mu(t,x,v)=\gamma\left(\frac{1}{t},\frac{x}{t},x-tv\right). \tag{1.10}$$

The Vlasov–Poisson system involves a perturbation of free streaming (1.9) by a force field (in this paper, we stick to the plasma terminology and refer to it as the "Electric field"):

$$E[\mu](t,x):=\nabla_x\Delta_x^{-1}\int\mu^2(t,x,v)\mathrm{d}v=\frac{1}{4\pi}\iint\frac{x-y}{|x-y|^3}\cdot\mu^2(t,y,v)\mathrm{d}v\mathrm{d}y, \tag{1.11}$$

which also transforms naturally:

$$E[\mu](t,tx)=\frac{1}{t^2}E[\gamma]\left(\frac{1}{t},x\right),$$

and we see that μ solves (1.1) on $0 \le T_* \le t \le T^*$ if and only if γ satisfies for $0 \le (T^*)^{-1} \le s \le (T_*)^{-1}$,

$$\big(\partial_s + p \cdot \nabla_q\big)\gamma + \lambda s^{-1} E[\gamma] \cdot \nabla_p \gamma = 0. \tag{1.12}$$

Remark 1.3 The natural energy estimate for (1.12) is

$$-s^2 \frac{\mathrm{d}}{\mathrm{d}s}\left(\iint |p|^2 \gamma^2(s,q,p)\mathrm{d}q\mathrm{d}p + \frac{\lambda}{s}\int |E[\gamma](q)|^2 \mathrm{d}q \right) = \lambda \int |E[\gamma](q)|^2 \mathrm{d}q, \tag{1.13}$$

which, after rescaling, recovers one of the main integral estimates in [19, 31] and leads, for $\lambda > 0$, to the optimal control of $E[\gamma] \in L^\infty_s L^2_q$.[2]

2 The Force Field and the Continuity Equation

To prove both the modified scattering and wave operator theorems, we require general estimates on the electric field E defined in (1.11). In Lemma 2.1, we prove fixtime bounds on the operator $\gamma \mapsto E$. In Lemma 2.3 we obtain dynamic bounds for an electric field $E = E[\gamma]$ provided γ satisfies (2.8), a slight strengthening of the continuity equation.

Lemma 2.1 *Let $\gamma = \gamma(q,p)$ be such that $\gamma \in L^2_{q,p}$, $\langle p\rangle^2\gamma \in L^\infty_{q,p}$ and $\nabla_q\gamma \in L^\infty_{q,p}$ and $E = E[\gamma]$ defined by (1.11). For all $A > 0$ and $\kappa \in (0, \frac{1}{3})$ we have*

$$\begin{aligned} \|E\|_{L^\infty_q} &\lesssim A\Big[\|\gamma\|^2_{L^2_{q,p}} + \|\gamma\|^2_{L^\infty_{q,p}}\Big] + A^{-1}\||p|^2\gamma\|^2_{L^\infty_{q,p}}, \\ \|\nabla_q E(s)\|_{L^\infty_q} &\lesssim A\|\gamma\|^2_{L^2_{q,p}} + A^{-\frac{\kappa}{3}}\||p|^2\gamma\|^2_{L^\infty_{q,p}} + A^{\kappa-\frac{1}{3}}\|\gamma\|_{L^\infty_{q,p}}\|\nabla_q\gamma\|_{L^\infty_{q,p}}. \end{aligned} \tag{2.1}$$

In fact, we will mostly make use of the second line of (2.1) corresponding to the choice $A = \langle \ln(s)\rangle^4$, $\kappa = \frac{1}{30}$, i.e., the bound

$$\|\nabla_q E(s)\|_{L^\infty_q} \lesssim \langle \ln(s)\rangle^4 \|\gamma\|^2_{L^2_{q,p}} + \||p|^2\gamma\|^2_{L^\infty_{q,p}} + \langle \ln(s)\rangle^{-\frac{6}{5}}\|\gamma\|_{L^\infty_{q,p}}\|\nabla_q\gamma\|_{L^\infty_{q,p}}. \tag{2.2}$$

Remark 2.2 In the estimates of this section, up to minor modifications, one may alternatively work with the $\langle p\rangle^{-1}L^4_{q,p}$ norm of γ, rather than its $L^2_{q,p}$ norm. This allows to consider initial data with infinite mass—see also Remark 1.2 (4).

Proof of Lemma 2.1 We decompose the electric field on different scales using a radially symmetric function $\chi \in C^\infty_c(\{\frac{1}{2} \le |y| \le 2\})$ with $\int_{\mathbb{R}^3}\chi(y)\mathrm{d}y = 1$, namely

[2] This in turn implies the optimal decay rate of $\|E[\mu](t)\|_{L^2_x} \lesssim \langle t\rangle^{-1/2}$ in the original variables.

$$E^j[\gamma](q) = c\int_{R=0}^{\infty} E^j_R(q)\frac{\mathrm{d}R}{R^2},$$
$$E^j_R[\gamma](q) := \iint R^{-1}\{\partial_{q^j}\chi\}(R^{-1}(q-r))\cdot\gamma^2(r,u)\mathrm{d}r\mathrm{d}u,$$

and we directly obtain the following elementary bounds

$$|E^j_R| \lesssim R^{-1}\|\gamma\|^2_{L^2_{q,p}}, \quad |\partial_q E^j_R| \lesssim R^{-2}\|\gamma\|^2_{L^2_{q,p}}, \tag{2.3}$$

which is enough for large R. To go further, we introduce

$$E^j_{R,V}[\gamma](q) := \iint R^{-1}\{\partial_{q^j}\chi\}(R^{-1}(q-r))\cdot\chi(V^{-1}u)\cdot\gamma^2(r,u)\mathrm{d}r\mathrm{d}u,$$

with $E^j[\gamma](q) = c\int_{R=0}^{\infty}\int_{V=0}^{\infty} E^j_{R,V}(q)\frac{\mathrm{d}V}{V}\frac{\mathrm{d}R}{R^2}$ and we estimate

$$\begin{aligned}
&|E^j_{R,V}| \lesssim R^2\min\{V^3\|\gamma\|^2_{L^\infty_{q,p}}, V^{-1}\||p|^2\gamma\|^2_{L^\infty_{q,p}}\},\\
&|\partial_q E^j_{R,V}| \lesssim R\min\{V^{-1}\||p|^2\gamma\|^2_{L^\infty_{q,p}}, RV^3\|\nabla_q\gamma\|_{L^\infty_{q,p}}\|\gamma\|_{L^\infty_{q,p}}\}.
\end{aligned} \tag{2.4}$$

From this, we deduce that

$$\begin{aligned}
|E^j[\gamma]| &\lesssim \int_{R=A}^{\infty}|E^j_R|\frac{\mathrm{d}R}{R^2} + \int_{R=0}^{A}\int_{V=0}^{B}|E^j_{R,V}|\frac{\mathrm{d}R}{R^2}\frac{\mathrm{d}V}{V} + \int_{R=0}^{A}\int_{V=B}^{\infty}|E^j_{R,V}|\frac{\mathrm{d}R}{R^2}\frac{\mathrm{d}V}{V}\\
&\lesssim A^{-2}\|\gamma\|^2_{L^2_{q,p}} + AB^3\|\gamma\|^2_{L^\infty_{q,p}} + AB^{-1}\||p|^2\gamma\|^2_{L^\infty_{q,p}}
\end{aligned}$$

and choosing $A = B^{-1}$, we obtain the first line of (2.1). Similarly, we see that for $\kappa \in (0, \frac{1}{3})$

$$\begin{aligned}
|\partial_q E^j| &\lesssim \int_{R=A}^{\infty}|\partial_q E^j_R|\frac{\mathrm{d}R}{R^2}\\
&\quad + \int_{R=0}^{A}\int_{V=0}^{R^{-\kappa}}|\partial_q E^j_{R,V}|\frac{\mathrm{d}V}{V}\frac{\mathrm{d}R}{R^2} + \int_{R=0}^{A}\int_{V=R^{-\kappa}}^{\infty}|\partial_q E^j_{R,V}|\frac{\mathrm{d}V}{V}\frac{\mathrm{d}R}{R^2}\\
&\lesssim A^{-3}\|\gamma\|^2_{L^2_{q,p}} + A^{\kappa}\||p|^2\gamma\|^2_{L^\infty_{q,p}} + A^{1-3\kappa}\|\gamma\|_{L^\infty_{q,p}}\|\nabla_q\gamma\|_{L^\infty_{q,p}}.
\end{aligned}$$

After substituting A with $A^{-1/3}$, this gives the second line of (2.1). □

Lemma 2.3 *Fix $0 < s_0 < s_1$ and let $\gamma \in L^\infty_s([s_0, s_1]; L^2_{q,p})$.*

(i) *Assuming that $E = E[\gamma]$ satisfies (1.8), we see that*

$$\begin{aligned}
\|E(s_1) - E(s_0)\|_{L^\infty_q} &\lesssim \langle\ln(s_1 - s_0)\rangle(s_1 - s_0)\|\mathbf{j}\|_{L^\infty_{s,q}}\\
&\quad + (s_1 - s_0)^2\Big[\|\langle p\rangle^2\gamma\|^2_{L^\infty_{s,q,p}} + \|\gamma\|^2_{L^\infty_s L^2_{q,p}}\Big].
\end{aligned} \tag{2.5}$$

We also have the corresponding estimate for $\nabla_q E = \nabla_q E[\gamma]$*:*

$$\begin{aligned}\|\nabla_q E(s_1) - \nabla_q E(s_0)\|_{L^\infty_q} \lesssim \langle \ln(s_1 - s_0)\rangle (s_1 - s_0)\|\nabla_q \mathbf{j}\|_{L^\infty_{s,q}} \\ + (s_1 - s_0)^2 \Big[\|\langle p\rangle^4 \gamma\|^2_{L^\infty_{s,q,p}} \\ + \|\nabla_q \gamma\|^2_{L^\infty_{s,q,p}} + \|\gamma\|^2_{L^\infty_s L^2_{q,p}} \Big],\end{aligned} \tag{2.6}$$

from which we deduce

$$\begin{aligned}\|\nabla_q E(s_1) - \nabla_q E(s_0)\|_{L^\infty_q} \lesssim \langle \ln(s_1 - s_0)\rangle (s_1 - s_0) \Big[\|\langle p\rangle^5 \gamma\|^2_{L^\infty_{s,q,p}} \\ + \|\nabla_q \gamma\|^2_{L^\infty_{s,q,p}} + \|\gamma\|^2_{L^\infty_s L^2_{q,p}} \Big].\end{aligned} \tag{2.7}$$

(ii) *If* γ *satisfies a slight strengthening of the continuity Eq.* (1.8)*, namely*

$$\partial_s \{\gamma^2\} + \mathrm{div}_q \{p\gamma^2\} + \mathrm{div}_p \{F\gamma^2\} = 0 \tag{2.8}$$

for some force field $F(s, q)$*, then for* $E = E[\gamma]$ *there holds that*

$$\begin{aligned}\|E(s_1) - E(s_0)\|_{L^\infty_q} \lesssim \langle \ln(s_1 - s_0)\rangle^2 (s_1 - s_0)\||p|^2 \gamma\|^2_{L^\infty_{s,q,p}} \\ + (s_1 - s_0)^2 \Big[\|\gamma\|^2_{L^\infty_{s,q,p}} + \|\gamma\|^2_{L^\infty_s L^2_{q,p}} \Big] \\ + (s_1 - s_0)^3 \left\langle \ln\left(\frac{s_1}{s_0}\right) \right\rangle \|\langle p\rangle^2 \gamma\|^2_{L^\infty_{s,q,p}} \|sF\|_{L^\infty_{s,q}}.\end{aligned}$$

Proof We start with (ii): using (2.3) and (2.4), we see that for $s \in \{s_0, s_1\}$,

$$\int_{R=A^{-1}}^{\infty} |E_R(s)| \frac{\mathrm{d}R}{R^2} \lesssim A^2 \|\gamma(s)\|^2_{L^2_{q,p}}, \quad \int_{R=0}^{A^2} |E_R(s)| \frac{\mathrm{d}R}{R^2} \lesssim A^2 \|\langle p\rangle^2 \gamma(s)\|^2_{L^\infty_{q,p}}, \tag{2.9}$$

and

$$\begin{aligned}\int_{R=0}^{A^{-1}} \int_{V=0}^{B} |E_{R,V}(s)| \frac{\mathrm{d}R}{R^2} \frac{\mathrm{d}V}{V} \lesssim A^{-1} B^3 \|\gamma(s)\|^2_{L^\infty_{q,p}}, \\ \int_{R=0}^{A^{-1}} \int_{V=B^{-3}}^{\infty} |E_{R,V}(s)| \frac{\mathrm{d}R}{R^2} \frac{\mathrm{d}V}{V} \lesssim A^{-1} B^3 \||p|^2 \gamma(s)\|^2_{L^\infty_{q,p}},\end{aligned}$$

and we conclude that

$$\begin{aligned}\left| E(s) - \int_{R=A^2}^{A^{-1}} E_{R,a}(s) \frac{\mathrm{d}R}{R^2} \right| \le A^{-1} B^3 \|\langle p\rangle^2 \gamma\|^2_{L^\infty_{q,p}} + A^2 \Big[\|\gamma\|^2_{L^2_{q,p}} + \|\langle p\rangle^2 \gamma\|^2_{L^\infty_{q,p}} \Big], \\ E_{R,a} := \iint R^{-1} \{\partial_{q^j} \chi\}(R^{-1}(q - r)) \cdot \chi_{\{B \le \cdot \le B^{-3}\}}(u) \cdot \gamma^2(r, u) \mathrm{d}r \mathrm{d}u,\end{aligned}$$

where

$$\chi_{\{B\le\cdot\le B^{-3}\}}(u) = \int_{\{B\le V\le B^{-3}\}} \chi(V^{-1}u)\frac{\mathrm{d}V}{V}.$$

On the other hand, using Eq. (2.8), we find that

$$\begin{aligned}
0 &= \int_{s=s_0}^{s_1} \iint R^{-1}\{\partial_{q^j}\chi\}(R^{-1}(q-r))\cdot\chi_{\{B\le\cdot\le B^{-3}\}}(u)\ \cdot\left\{\partial_s\gamma^2 + \mathrm{div}_r(\gamma^2 u) + \mathrm{div}_u(F\gamma^2)\right\}\mathrm{d}r\mathrm{d}u\mathrm{d}s,\\
&= E_{R,a}(s_1) - E_{R,a}(s_0)\\
&\quad + \int_{s=s_0}^{s_1} \iint R^{-2}u^k\{\partial_{q^j}\partial_{q^k}\chi\}(R^{-1}(q-r))\cdot\chi_{\{B\le\cdot\le B^{-3}\}}(u)\cdot\gamma^2(s,r,u)\mathrm{d}r\mathrm{d}u\mathrm{d}s,\\
&\quad - \int_{s=s_0}^{s_1} \iint R^{-1}\partial_{q^j}\chi(R^{-1}(q-r))\cdot\gamma^2(s,r,u)\cdot(F\cdot\nabla_u)\chi_{\{B\le\cdot\le B^{-3}\}}(u)\mathrm{d}r\mathrm{d}u\mathrm{d}s.
\end{aligned} \tag{2.10}$$

Since

$$\left|\nabla_u\chi_{\{B\le\cdot\le B^{-3}\}}(u)\right| \lesssim B^{-1}\mathbf{1}_{\{|u|\le 2B\}} + B^3\mathbf{1}_{\{|u|\ge B^{-3}/2\}},$$

we see that

$$\begin{aligned}
&\left|\iint R^{-1}\partial_{q^j}\chi(R^{-1}(q-r))\cdot\gamma^2(r,u)\cdot(F\cdot\nabla_u)\chi_{\{B\le\cdot\le B^{-3}\}}(u)\mathrm{d}r\mathrm{d}u\right|\\
&\qquad\lesssim \|sF\|_{L^\infty_q}\cdot s^{-1}R^2\cdot\left[B^2\|\gamma\|^2_{L^\infty_{q,p}} + B^6\||p|^2\gamma\|^2_{L^\infty_{q,p}}\right],
\end{aligned}$$

and using a crude bound for the second integral in (2.10), we find that

$$\begin{aligned}
\left|\int_{R=A^2}^{A^{-1}}\left\{E_{R,a}(s_1) - E_{R,a}(s_0)\right\}\frac{\mathrm{d}R}{R^2}\right| &\lesssim (s_1-s_0)\cdot\||u|^2\gamma\|^2_{L^\infty_{s,r,u}}\cdot\int_{R=A^2}^{A^{-1}}\frac{\mathrm{d}R}{R}\cdot\int_{V=B}^{B^{-3}}\frac{\mathrm{d}V}{V}\\
&\quad + \left\langle\ln\left(\frac{s_1}{s_0}\right)\right\rangle\cdot\|sF\|_{L^\infty_{s,q}}\cdot A^{-1}B^2\|\langle p\rangle^2\gamma\|^2_{L^\infty_{s,q,p}}.
\end{aligned}$$

Letting $B = A^2 = (s_1-s_0)^2$, we obtain the result. For the variant (2.5), we do not localize in u. In this case, we need only use (2.9) and the last term in (2.10) simplifies. We detail this in the similar analysis of $\nabla_q E$ in (i) below.

For (i) we use a similar analysis without localizing in u. Passing the derivative onto γ gives

$$\begin{aligned}
\int_{R=A^{-\frac{2}{3}}}^{\infty}|\nabla_q E_R(s)|\frac{\mathrm{d}R}{R^2} &\lesssim A^2\|\gamma(s)\|^2_{L^2_{q,p}},\\
\int_{R=0}^{A^2}|\nabla_q E_R(s)|\frac{\mathrm{d}R}{R^2} &\lesssim A^2\|\nabla_q\gamma(s)\|_{L^\infty_{q,p}}\cdot\|\langle p\rangle^4\gamma(s)\|_{L^\infty_{q,p}},
\end{aligned}$$

and the continuity Eq. (2.8) gives

$$
\begin{aligned}
0 &= \int_{s=s_0}^{s_1} \iint R^{-1}\{\partial_{q^j}\chi\}(R^{-1}(q-r))\cdot\partial_j\big\{\partial_s\gamma^2+\mathrm{div}_r(\gamma^2 u)\big\}\mathrm{d}r\mathrm{d}u\mathrm{d}s\\
&= \partial_j E_R(s_1)-\partial_j E_R(s_0)\\
&\quad + 2\int_{s=s_0}^{s_1}\iint R^{-2}u^k\{\partial_{q^j}\partial_{q^k}\chi\}(R^{-1}(q-r))\cdot\gamma\cdot\nabla_q\gamma(r,u)\mathrm{d}r\mathrm{d}u\mathrm{d}s,
\end{aligned}
$$

from which we deduce that

$$
\|\nabla_q E_R(s_1)-\nabla_q E_R(s_0)\|_{L^\infty_{q,p}} \lesssim (s_1-s_0)\cdot R\cdot\|\langle p\rangle^5\gamma\|_{L^\infty_{s,q,p}}\|\nabla_q\gamma\|_{L^\infty_{s,q,p}}
$$

and integrating in $A^2\le R\le A^{-1}$, we obtain (2.7). □

Finally, we collect the modifications of Lemmas 2.1 and 2.3 above needed to consider smoother solutions. The proofs are similar (passing the derivative through the density) and are omitted.

Lemma 2.4 *There holds that for all* $\kappa\in(0,\frac{1}{3})$,

$$
\|\nabla_q^2 E\|_{L^\infty_q} \lesssim A\|\gamma\|^2_{L^2_{q,p}} + A^{-\frac{\kappa}{4}}\||p|^4\gamma\|_{L^\infty_{q,p}}\|\nabla_q\gamma\|_{L^\infty_{q,p}} + A^{\frac{3\kappa-1}{4}}\|\gamma\|_{L^\infty_{q,p}}\|\nabla_q^2\gamma\|_{L^\infty_{q,p}},
$$

and

$$
\begin{aligned}
\|\nabla_q^2 E(s_1)-\nabla_q^2 E(s_0)\|_{L^\infty_q} \lesssim{}& \langle\ln(s_1-s_0)\rangle(s_1-s_0)\|\nabla_q^2\mathbf{j}\|_{L^\infty_{s,q}}\\
&+(s_1-s_0)^2\Big[\|\gamma\|^2_{L^\infty_s L^2_{q,p}}+\|\langle p\rangle^5\gamma\|_{L^\infty_{s,q,p}}\|\nabla_q^2\gamma\|_{L^\infty_{s,q,p}}\\
&+\|\langle p\rangle^{2.1}\nabla_q\gamma\|^2_{L^\infty_{s,q,p}}\Big].
\end{aligned}
$$

3 Modified Scattering

While we only need to study (1.12) on a compact time interval, this equation is now time dependent with a violent singularity at $s=0$. This can be mitigated since the singular terms

$$
\big(\partial_s+\lambda s^{-1}E(s,q)\cdot\nabla_p\big)\gamma=\text{l.o.t.}
$$

can be integrated to main order:

$$
\begin{aligned}
\Gamma(s,q,p) &= \gamma\bigg(s,q,p+\lambda\int_{s'=1}^{s}E(s',q)\frac{\mathrm{d}s'}{s'}\bigg),\\
\gamma(s,q,p) &= \Gamma\bigg(s,q,p-\lambda\int_{s'=1}^{s}E(s',q)\frac{\mathrm{d}s'}{s'}\bigg).
\end{aligned}
\tag{3.1}
$$

Since Γ satisfies an equivalent but more cumbersome equation, we prefer to work with (1.12) to bootstrap control of the norms, but a variant of (3.1) leads quickly to the modified dynamics (3.4) once E is shown to converge.

The main result of this section is the following statement about modified scattering:

Theorem 3.1 *There exists $\varepsilon > 0$ such that if $\gamma_1(q,p)$ satisfies*

$$\|\gamma_1\|_{L^2_{q,p}} + \left\|\langle p\rangle^2\gamma_1\right\|_{L^\infty_{q,p}} + \|\nabla_{p,q}\gamma_1\|_{L^\infty_{q,p}} \leq \varepsilon_0 \leq \varepsilon, \tag{3.2}$$

then there exists a unique solution γ of (1.12) with "initial" data $\gamma(s=1)=\gamma_1$ for all times $0 < s \leq 1$, and $\gamma \in L^\infty_s((0,1], L^\infty_{q,p} \cap L^2_{q,p})$ satisfies

$$\left\|\langle p\rangle^2\gamma(s)\right\|_{L^\infty_{q,p}} \lesssim \varepsilon_0\langle \ln(s)\rangle^2, \quad \|\nabla_{p,q}\gamma(s)\|_{L^\infty_{q,p}} \lesssim \varepsilon_0\langle \ln(s)\rangle^5.$$

If in addition

$$\|\langle p\rangle\nabla_{p,q}\gamma_1\|_{L^\infty_{q,p}} \leq \varepsilon_0, \tag{3.3}$$

then $\|\langle p\rangle\nabla_{p,q}\gamma(s)\|_{L^\infty_{q,p}} \lesssim \varepsilon_0\langle \ln(s)\rangle^6$ and there exist $E_0 = E[\gamma_0] \in L^\infty_q$ and $\gamma_0 \in L^\infty_{q,p}$ such that, uniformly in q, p,

$$\gamma(s, q + ps + \lambda s\ln(s)E_0(q), p + \lambda\ln(s)E_0(q)) \to \gamma_0(q,p), \quad s \to 0. \tag{3.4}$$

Remark 3.2 We comment on some points of interest:

(1) In fact, as we will show below one can obtain global solutions in a bootstrap argument involving only the moments $\langle p\rangle^2\gamma$. The higher regularity of (3.2) is only used to make sense of the equations in a stronger sense.

(2) The assumption (3.3) is used to guarantee the convergence (3.4). We note that this statement is slightly different from the one in Theorem 1.1, in that in (3.3) we start with *uniform* control of one additional moment in p on the gradients and obtain *uniform* (rather than local) convergence in (3.4). The proofs are easily adapted to establish the corresponding local statement under local assumptions as in Theorem 1.1.

(3) Our proof of Theorem 3.1 shows that control of higher moments (in both p and q) as well as higher regularity can be propagated. For higher moments in p, this is explicitly done in Proposition 3.4, and from this the propagation of moments in q follows by the commutation relations (3.6). For higher regularity, by (3.6) one needs control of derivatives of the electric field; these in turn can be directly bounded by derivatives of γ via an adaptation of Lemmas 2.1 and 2.3 (see, e.g., Lemma 2.4 for one additional derivative). As a consequence, given more regularity and/or moments on a solution, the convergence (3.4) can then be shown to hold in a correspondingly strengthened topology.

(4) The convergence (3.4) implies the asymptotic dynamic (1.4) of Theorem 1.1: Letting

$$\mathcal{A} : (s, q, p) \mapsto (s, q + ps + \lambda s \ln(s) E_0(q), p + \lambda \ln(s) E_0(q)),$$

by $\mathcal{I}^2 = \mathrm{Id}$ there holds that

$$\gamma \circ \mathcal{A}(s, q, p) = \mu \circ (\mathcal{I} \circ \mathcal{A})(s, q, p) = \mu\Big(\frac{1}{s}, \frac{q}{s} + p + \lambda \ln(s) E_0(q), q\Big),$$

which gives (1.4) with $\mu_\infty(x, v) = \gamma_0(v, x)$ by relabeling the arguments.

The proof of Theorem 3.1 makes frequent use of the fact that (1.12) is a transport equation and we can propagate uniform bounds using the maximum principle along the characteristics. In particular, writing

$$\mathcal{L} := \partial_s + p \cdot \nabla_q + \lambda s^{-1} E \cdot \nabla_p, \quad \mathcal{L}[f] = \partial_s f + \mathrm{div}_{q,p}\big\{(p, \lambda s^{-1} E(q)) \cdot f\big\}$$

we have that if h is a strong solution in a neighborhood of $s = 1$ to

$$\mathcal{L}[h] = F(s, q, p)$$

with $h(1) \in L^r_{q,p}$ for some $r \geq 1$, then since the transport field is divergence free, there holds that

$$\|h(s)\|_{L^r_{q,p}} \leq \|h(1)\|_{L^r_{q,p}} + \int_s^1 \|F(s')\|_{L^r_{q,p}} \mathrm{d}s' \tag{3.5}$$

for all $0 \leq s \leq 1$ in the interval of existence.

3.1 Commutation Relations

Now consider a solution γ to (1.12), i.e., $\mathcal{L}[\gamma] = 0$. To decide which equation we want to use, it will be convenient to compute some commutation relations: For any $m, n \in \{1, 2, 3\}$, we have

$$\begin{aligned} &\mathcal{L}[q^m \gamma] = \mathcal{L}[q^m]\gamma = p^m \gamma, \quad \mathcal{L}[p^m \gamma] = \lambda s^{-1} E^m \gamma, \\ &\mathcal{L}[\partial_{q^m} \gamma] = \partial_{q^m}(\mathcal{L}[\gamma]) - (\partial_{q^m} \mathcal{L})[\gamma] = -\lambda s^{-1} \partial_{q^m} E^j \partial_{p^j} \gamma, \quad \mathcal{L}[\partial_{p^m} \gamma] = -\partial_{q^m} \gamma, \end{aligned} \tag{3.6}$$

and we also remark that

$$\begin{aligned} \mathcal{L}[p^m \partial_{q^n} \gamma] &= -\lambda s^{-1} p^m \partial_{q^n} E^j \partial_{p^j} \gamma + \lambda s^{-1} E^m \partial_{q^n} \gamma, \\ \mathcal{L}[p^m \partial_{p^n} \gamma] &= -p^m \partial_{q^n} \gamma + \lambda s^{-1} E^m \partial_{p^n} \gamma. \end{aligned} \tag{3.7}$$

3.2 Bootstrap and Global Existence

As a first step, we see that as long as the electric field remains bounded, we can propagate all the moments we want.

Lemma 3.3 *Let γ be a strong solution of (1.12) on $T^* \le s \le 1$ with "initial" data $\gamma(s=1) = \gamma_1$. Assume that γ_1 satisfies for some $a \in \mathbb{N}$, $r \in [2, \infty]$ that*

$$\|\langle p\rangle^a \gamma_1\|_{L^r_{q,p}} \le \varepsilon_0,$$

and that

$$|E(s,q)| \le D, \quad T^* \le s \le 1.$$

Then there holds that

$$\begin{aligned} &\|\gamma(s)\|_{L^r_{q,p}} \le \varepsilon_0, \\ &\|\langle p\rangle^a \gamma(s)\|_{L^r_{q,p}} \le \varepsilon_0 + aD\varepsilon_0 \langle \ln(s)\rangle^a. \end{aligned}$$

Proof The proof follows by applying (3.6) and (3.5) inductively to $p^\beta \gamma$, $\beta \in \mathbb{N}_0^3$, with $|\beta| \le a$. □

Proposition 3.4 *Let $0 < \varepsilon_0 \le \varepsilon_1 \ll 1$, and let γ be a solution of (1.12) on $T^* \le s \le 1$ with "initial" data $\gamma(s=1) = \gamma_1$ satisfying*

$$\|\gamma_1\|_{L^2_{q,p}} + \|\gamma_1\|_{L^\infty_{q,p}} \le \varepsilon_0. \tag{3.8}$$

(1) (*Moments and the electric field*) *If there holds that*

$$\|\langle p\rangle \gamma_1\|_{L^2_{q,p}} + \left\|\langle p\rangle^m \gamma_1\right\|_{L^\infty_{q,p}} \le \varepsilon_1, \quad m \ge 2 \tag{3.9}$$

then the electric field $E(s)$ remains bounded and the solution satisfies the bounds

$$\begin{aligned} &\|\langle p\rangle \gamma(s)\|_{L^2_{q,p}} \lesssim \varepsilon_1 \langle \ln s\rangle, \\ &\left\|\langle p\rangle^a \gamma(s)\right\|_{L^\infty_{q,p}} \lesssim \varepsilon_1 \langle \ln s\rangle^a, \quad 0 \le a \le m. \end{aligned} \tag{3.10}$$

Moreover, there exists $C > 0$ (independent of T^*) such that for any $T^* \le s_1 \le s_2 \le 1$ there holds

$$|E(s_1,q) - E(s_2,q)| \le C\varepsilon_1^2 \langle \ln(s_1)\rangle^4 \langle \ln(s_2 - s_1)\rangle^2 (s_2 - s_1). \tag{3.11}$$

(2) (*Derivatives*) *Assume additionally that for some $b \in \{0,1\}$ there holds that*

$$\left\|\langle p\rangle^b \nabla_{p,q} \gamma_1\right\|_{L^\infty_{q,p}} \le \varepsilon_1. \tag{3.12}$$

Then, we have the bounds

$$\begin{aligned} &\left\|\langle p\rangle^a \nabla_p \gamma(s)\right\|_{L^\infty_{q,p}} \lesssim \varepsilon_1 \langle \ln s\rangle^a, \quad 0 \le a \le b, \\ &\left\|\langle p\rangle^a \nabla_q \gamma(s)\right\|_{L^\infty_{q,p}} \lesssim \varepsilon_1 \langle \ln s\rangle^{5+a}, \quad 0 \le a \le b. \end{aligned} \tag{3.13}$$

Proof We start by establishing claim (1). Let $C > 0$ be a constant larger than all the implied constants appearing in Sect. 2 and let ε_1 be small enough so that

$$4C^2\varepsilon_1^2 \leq 1. \tag{3.14}$$

We make the following bootstrap assumption: Let $I \subset [T^*, 1]$ be such that for $s \in I$, there holds

$$\|E(s)\|_{L^\infty_q} \leq 2C^2\varepsilon_1^2. \tag{3.15}$$

By the first line of (2.1) (with $A = 1$) and the assumptions (3.8), (3.9) we have that $1 \in I \neq \emptyset$, and by continuity I is closed in $[T^*, 1]$. To establish the claim it then suffices to prove that (3.15) holds with strictly smaller constants, implying that I is also open in $[T^*, 1]$.

To this end, note that by Lemma 3.3 we have that for $0 \leq a \leq m$,

$$\|\langle p\rangle^a \gamma(s)\|_{L^\infty_{q,p}} \leq \varepsilon_1(1 + aC^2\varepsilon_1^2)\langle \ln(s)\rangle^a, \quad s \in I. \tag{3.16}$$

By Lemma 2.3 and (3.14), it then follows for $T^* \leq s_1 \leq s_2 \leq 1$ that

$$\big\|E(s_1) - E(s_2)\big\|_{L^\infty_q} \leq 4C\varepsilon_1^2\left[\langle \ln(s_2 - s_1)\rangle^2 + \varepsilon_1^2\left\langle \ln\left(\frac{s_2}{s_1}\right)\right\rangle\right]\langle \ln(s_1)\rangle^4(s_2 - s_1)$$

and (3.11) is proved. In particular, when $2^{-k} \leq s_1 \leq s_2 \leq 2^{1-k}$, $k \geq 1$,

$$\big\|E(s_1) - E(s_2)\big\|_{L^\infty_q} \leq 10C\varepsilon_1^2 k^6 2^{-k} \tag{3.17}$$

and since by (2.1) we have $\|E(1)\|_{L^\infty_q} \leq 2C\varepsilon_1^2$, we see that for $s \in I$,

$$\|E(s)\|_{L^\infty_q} \leq \|E(1)\|_{L^\infty_q} + 10C\varepsilon_1^2 \sum_{k\geq 1} k^6 2^{-k} \ll C^2\varepsilon_1^2,$$

provided C is large enough.

To prove (2), we use a similar bootstrap argument based on the assumptions

$$\begin{aligned} \|\langle p\rangle^b \nabla_p \gamma(s)\|_{L^\infty_{q,p}} &\leq 2C^4\varepsilon_1\langle \ln(s)\rangle^b, \\ \|\langle p\rangle^b \nabla_q \gamma(s)\|_{L^\infty_{q,p}} &\leq 2C^2\varepsilon_1\langle \ln(s)\rangle^{5+b}. \end{aligned} \tag{3.18}$$

Using the commutation relations (3.6) and (3.5), we deduce from (3.12) and (3.18) that

$$\|\nabla_p \gamma(s)\|_{L^\infty_{q,p}} \leq \|\nabla_p \gamma(1)\|_{L^\infty_{q,p}} + \int_s^1 \|\nabla_q \gamma(s')\|_{L^\infty_{q,p}} \mathrm{d}s' \leq C^3 2\varepsilon_1, \tag{3.19}$$

provided $C > 0$ is large enough.

From the transport bounds and the commutation relations (3.6), we then deduce the estimate for $\nabla_q \gamma$: From (2.2) we have under our assumption (3.15) and with (3.16) and (3.19) that

$$\begin{aligned}\|\nabla_q\gamma(s)\|_{L^\infty_{q,p}} &\le \|\nabla_q\gamma(1)\|_{L^\infty_{q,p}} + \int_s^1 \|\nabla_q E\|_{L^\infty_q}\|\nabla_p\gamma\|_{L^\infty_{q,p}}\frac{\mathrm{d}s'}{s'}\\ &\le \varepsilon_1 + \int_s^1 \Big[\langle \ln s'\rangle^4\|\gamma\|^2_{L^2_{q,p}} + \||p|^2\gamma\|^2_{L^\infty_{q,p}}\\ &\quad + \langle \ln s'\rangle^{-\frac{6}{5}}\|\gamma\|_{L^\infty_{q,p}}\|\nabla_q\gamma\|_{L^\infty_{q,p}}\Big]\|\nabla_p\gamma\|_{L^\infty_{q,p}}\frac{\mathrm{d}s'}{s'}\\ &\le \varepsilon_1 + \langle \ln s\rangle^5\big(\varepsilon_0^2+4\varepsilon_1^2\big)\cdot 2C^4\varepsilon_1 + 2C^4\varepsilon_0\varepsilon_1\int_s^1\langle \ln s'\rangle^{-\frac{6}{5}}\|\nabla_q\gamma(s')\|_{L^\infty_{q,p}}\frac{\mathrm{d}s'}{s'},\end{aligned}$$

so that by Grönwall's lemma there holds that

$$\|\nabla_q\gamma(s)\|_{L^\infty_{q,p}} \le 10\varepsilon_1\langle \ln s\rangle^5, \tag{3.20}$$

provided ε_1 is small enough. A similar argument using (3.7), (3.15) and (3.18) shows that

$$\begin{aligned}\|p^m\partial_{p^n}\gamma(s)\|_{L^\infty_{q,p}} &\le \||p|\nabla_p\gamma(1)\|_{L^\infty_{q,p}}\\ &\quad + \int_s^1\||p|\nabla_q\gamma(s')\|_{L^\infty_{q,p}}\mathrm{d}s' + \int_s^1\|E\|_{L^\infty_q}\|\nabla_p\gamma(s')\|_{L^\infty_{q,p}}\frac{\mathrm{d}s'}{s'}\\ &\le C^3\varepsilon_1\langle \ln s\rangle.\end{aligned} \tag{3.21}$$

For the last bound, we see from (3.7) that we need a bound on the derivative of the electric field. Using (2.2), (3.16) and (3.20), we find that

$$\begin{aligned}\|\nabla_q E(s)\|_{L^\infty_q} &\le C\Big[\langle \ln s\rangle^4\|\gamma(s)\|^2_{L^2_{q,p}} + \big\|p^2\gamma(s)\big\|^2_{L^\infty_{q,p}}\\ &\quad + \langle \ln s\rangle^{-\frac{6}{5}}\|\gamma(s)\|_{L^\infty_{q,p}}\|\nabla_q\gamma(s)\|_{L^\infty_{q,p}}\Big]\\ &\le C\big(\varepsilon_0^2+4\varepsilon_1^2\big)\langle \ln(s)\rangle^4 + 10C\varepsilon_0\varepsilon_1\langle \ln s\rangle^{5-\frac{6}{5}}\\ &\le 10C\varepsilon_1^2\langle \ln s\rangle^4,\end{aligned} \tag{3.22}$$

so that (3.7) with (3.5), (3.21), (3.15), (3.20) and (3.22) gives

$$\begin{aligned}\|p^m\nabla_q\gamma(s)\|_{L^\infty_{q,p}} &\le \||p|\nabla_q\gamma(1)\|_{L^\infty_{q,p}}\\ &\quad + \int_s^1\Big(\|\nabla_q E\|_{L^\infty_q}\||p|\nabla_p\gamma\|_{L^\infty_{q,p}} + \|E\|_{L^\infty_q}\|\nabla_q\gamma\|_{L^\infty_{q,p}}\Big)\frac{\mathrm{d}s'}{s'}\\ &\le \varepsilon_1 + \varepsilon_1\int_s^1\Big(10C^4\varepsilon_1^3\langle \ln s'\rangle^5 + 20C^2\varepsilon_1^3\langle \ln s'\rangle^5\Big)\frac{\mathrm{d}s'}{s'}\\ &\le \varepsilon_1\langle \ln s\rangle^6.\end{aligned}$$

This closes the bootstrap (3.18). □

3.3 Asymptotic Behavior

From (3.11), we can deduce that the electric field has an asymptotic limit:

Corollary 3.5 *Let γ be a solution of (1.12) as in Proposition 3.4, which is moreover defined for $s \in (0, 1]$. Then, the limit*

$$E_0(q) := \lim_{s \to 0} E(s, q)$$

exists and is bounded

$$\|E_0\|_{L^\infty_q} \lesssim \varepsilon_1^2.$$

In addition, we have the following convergence rate: if $0 \le s_1 \le s_2 \le 1$, there holds

$$\|E(s_1) - E(s_2)\|_{L^\infty_q} \lesssim \varepsilon^2 \langle \ln(s_2) \rangle^6 s_2. \tag{3.23}$$

The rate of convergence (3.23) is linked to the topology we choose through the continuity Eq. (1.8). Our assumptions scale like $\mathbf{j} \in L^\infty_q$ and we obtain almost Lipschitz bounds in time.

Proof It follows from (3.17) in the proof above that $E(2^{-k})$ is Cauchy in L^∞_q. Summing again (3.17) gives (3.23). □

Now, we are in the position to give the proof of the modified scattering result:

Proof of Theorem 3.1 From Proposition 3.4 we obtain a global solution γ on $(0, 1]$, which satisfies (3.10), (3.13) and (3.23). Next, we define

$$\nu(s, q, p) := \gamma(s, q + ps + \lambda s \ln(s) E_0(q), p + \lambda \ln(s) E_0(q)),$$

which satisfies

$$\begin{aligned} \partial_s \nu &= \partial_s \gamma + p \cdot \nabla_q \gamma + \lambda(1 + \ln(s)) E_0(q) \cdot \nabla_q \gamma + \lambda s^{-1} E_0(q) \cdot \nabla_p \gamma \\ &= \lambda E_0(q) \cdot \nabla_q \gamma + \lambda s^{-1} [E_0(q) - E(s, q + ps + \lambda s \ln(s) E_0(q))] \cdot \nabla_p \gamma, \end{aligned}$$

where

$$\begin{aligned} &s^{-1} \big| E_0(q) - E(s, q + ps + \lambda s \ln(s) E_0(q)) \big| \\ &\quad \lesssim s^{-1} |E_0(q) - E(s, q)| + |p + \ln(s) E_0(q)| \|\nabla E(s)\|_{L^\infty_q}. \end{aligned}$$

Hence by (2.2), Corollary 3.5 and (3.22) we have that

$$\|\partial_s \nu\|_{L^\infty_{q,p}} \lesssim \varepsilon_1^2 \|\nabla_q \gamma\|_{L^\infty_{q,p}} + \varepsilon_1^2 \langle \ln(s) \rangle^4 \|p \nabla_p \gamma\|_{L^\infty_{q,p}} + \varepsilon_1^2 \langle \ln(s) \rangle^6 \|\nabla_p \gamma\|_{L^\infty_{q,p}},$$

which is integrable over $0 \le s \le 1$. □

4 Wave Operators and Cauchy Problem at Infinity

Using the symplectic structure, Eq. (1.12) can be written as

$$\gamma_s + \{\gamma, \mathcal{H}\} = 0, \quad \{f, g\} := \nabla_q f \cdot \nabla_p g - \nabla_p f \cdot \nabla_q g$$

with the Hamiltonian

$$\mathcal{H}(s, q, p) := \frac{|p|^2}{2} - \lambda s^{-1} \phi(s, q),$$

where $\Delta\phi(s, q) = \int \gamma^2(s, q, p)\mathrm{d}p$. We wish to find a new coordinate system (w, z) for which the Cauchy problem at $s = 0$ can be solved. For this, we introduce the type-3 generating function[3]

$$S(s, w, p) := w \cdot p + \frac{|p|^2}{2} s - \lambda \ln(s)\phi_0(w),$$

where $\phi_0(q) = \phi(0, q)$. This gives rise to the canonical change of coordinates

$$\begin{aligned} z &= \nabla_w S(w, p) = p - \lambda \ln(s)\nabla\phi_0(w), \\ q &= \nabla_p S(w, p) = w + ps = w + zs + \lambda s \ln(s)\nabla\phi_0(w), \end{aligned}$$

or

$$\begin{aligned} q &= w + sz + \lambda s \ln(s)\nabla\phi_0(w), \quad & w &= q - sp, \\ p &= z + \lambda \ln(s)\nabla\phi_0(w), & z &= p - \lambda \ln(s)\nabla\phi_0(q - sp), \end{aligned} \tag{4.1}$$

with Jacobian matrix

$$\frac{\partial(w, z)}{\partial(q, p)} = \begin{pmatrix} \mathrm{Id} & -s\mathrm{Id} \\ -\lambda \ln(s)\nabla E_0 & \mathrm{Id} + \lambda s \ln(s)\nabla E_0 \end{pmatrix}, \tag{4.2}$$

with the usual notation $E = \nabla\phi$, $E_0 = \nabla\phi_0$. This corresponds to the new Hamiltonian

$$\mathcal{K}(s, w, z) := \mathcal{H}(s, q, p) - \partial_s S(s, w, p) = \lambda s^{-1}\left[\phi_0(w) - \phi(s, q)\right]$$

and vector field

$$\nabla_w \mathcal{K} = -\lambda s^{-1}\left\{E(s, q) - E_0(w)\right\} - \lambda^2 \ln(s) E(s, q) \cdot \nabla E_0(w), \quad \nabla_z \mathcal{K} = -\lambda E(q). \tag{4.3}$$

It follows that

$$\sigma(s, w, z) := \gamma(s, q, p) \tag{4.4}$$

solves

[3] See, e.g., [25, Chapter 8].

$$0 = \partial_s \sigma + \{\sigma, \mathcal{K}\} = \partial_s \sigma + \nabla_w \sigma \cdot \nabla_z \mathcal{K} - \nabla_z \sigma \cdot \nabla_w \mathcal{K}. \tag{4.5}$$

Remark 4.1 We note that the new variables (w, z) have a simple interpretation in terms of the original variables in (1.1): $w = v$, $z = x - tv - \lambda \ln(t) E_0(v)$, which are the variables in which the modified scattering of Theorem 1.1 and [20] is expressed.

The main result of this section then is the following:

Theorem 4.2 *Assume that initial data σ_0 and E_0*[4] *satisfy*

$$\|E_0\|_{W^{3,\infty}} \leq c_0^2, \tag{4.6}$$

and

$$\|\sigma_0\|_{L^2_{w,z}} + \|\langle z\rangle^5 \sigma_0\|_{L^\infty_{w,z}} + \sum_{0\leq m+n\leq 2} \|\langle z\rangle^m \nabla_z^m \nabla_w^n \sigma_0\|_{L^\infty_{w,z}} \leq c_0. \tag{4.7}$$

Then there exists $T^* = T^*(c_0) > 0$ and a unique solution $\sigma \in C^0_s([0, T^*) : L^2_{w,z})$ of (4.5) with "initial" data $\sigma(s = 0) = \sigma_0$, and such that $s\partial_s \sigma$, $\nabla_{w,z}\sigma \in C^0_{s,w,z}$. Moreover, for $0 \leq s < T^*$ we have that for any $\ell \in \mathbb{N}$,

$$\begin{aligned} &\|\sigma(s)\|_{L^2_{w,z}} + \big\|\langle z\rangle^5 \sigma(s)\big\|_{L^\infty_{w,z}} + \|\nabla_{w,z}\sigma(s)\|_{L^\infty_{w,z}} \lesssim c_0, \\ &\|\langle w, z\rangle^\ell \sigma(s)\|_{L^\infty_{w,z}} \lesssim \|\langle w, z\rangle^\ell \sigma_0\|_{L^\infty_{w,z}}, \end{aligned} \tag{4.8}$$

and if c_0 is sufficiently small we may take $T^* = 1$.

The proof of Theorem 4.2 is given below in Sect. 4.3, after we have established some a priori estimates on the propagation of moments and derivatives for the system (4.5) in Sects. 4.1 and 4.2.

4.1 Commutation Relations

Writing $\mathfrak{L} = \partial_s + \{\cdot, \mathcal{K}\}$, for moments in w, z we have the commutation relations

$$\begin{aligned} &\mathfrak{L}[w_j \sigma] = -\lambda E_j(s, q)\sigma, \\ &\mathfrak{L}[z_j \sigma] = \big(\lambda s^{-1}[E_j(s, q) - E_{0,j}(w)] + \lambda^2 \ln(s) E(s, q) \cdot \nabla_w E_{0,j}(w)\big)\sigma. \end{aligned} \tag{4.9}$$

For the derivatives, we have

$$\mathfrak{L}(\partial_1 \sigma) = \{\partial_1 \mathcal{K}, \sigma\}, \quad \mathfrak{L}(\partial_2 \partial_1 \sigma) = \{\partial_1 \mathcal{K}, \partial_2 \sigma\} + \{\partial_2 \mathcal{K}, \partial_1 \sigma\} + \{\partial_2 \partial_1 \mathcal{K}, \sigma\} \tag{4.10}$$

and this gives in block diagonal form

[4] These are linked through (4.16).

$$\mathfrak{L}\begin{pmatrix}\nabla_w\sigma\\ \nabla_z\sigma\end{pmatrix}=\begin{pmatrix}-\nabla_w\nabla_z\mathcal{K} & \nabla_w^2\mathcal{K}\\ -\nabla_z^2\mathcal{K} & \nabla_w\nabla_z\mathcal{K}\end{pmatrix}\begin{pmatrix}\nabla_w\sigma\\ \nabla_z\sigma\end{pmatrix}, \tag{4.11}$$

with

$$\begin{aligned}\nabla^2_{w^jw^k}\mathcal{K} &= -\lambda s^{-1}\partial_j[E_k(s,q)-E_{0,k}(w)]\\ &\quad -\lambda^2\ln(s)\big\{\partial_jE(s,q)\cdot\partial_kE_0(w)\\ &\quad +\partial_kE(s,q)\cdot\partial_jE_0(w)+\partial_j\partial_kE_0(w)\cdot E(s,q)\big\}\\ &\quad -\lambda^3 s\ln(s)^2\partial_kE_{0,a}(w)\partial_jE_{0,b}(w)\cdot\partial_aE_b(s,q),\\ \nabla_w\nabla_z\mathcal{K} &= -\lambda\nabla E(s,q)-\lambda^2 s\ln(s)(\nabla E(s,q)\cdot\nabla)E_0(w),\\ \nabla_z^2\mathcal{K} &= -\lambda s\nabla E(s,q).\end{aligned} \tag{4.12}$$

We note that the matrix $\nabla^2_{w,z}\mathcal{K}$ is *ill-conditioned*, and to mitigate this effect, we introduce a weight on the gradient:

$$\theta(s,z):=\frac{\langle z\rangle}{1+s\langle z\rangle},\quad \frac{1}{2}\min\{\langle z\rangle,s^{-1}\}\le\theta(s,z)\le\min\{\langle z\rangle,s^{-1}\},$$

which is linked to the vector field through (4.15) and satisfies nice differential equalities

$$\partial_s\theta=-\theta^2,\quad \nabla_z\theta=\left(\frac{z}{\langle z\rangle^3}\right)\cdot\theta^2.$$

4.2 A Priori Estimates

The goal of this section is to bootstrap the following assumptions: given c_0 as in (4.6), we assume that for $0\le s\le T(c_0)$ there holds that

$$\begin{aligned}&\|\sigma(s)\|_{L^2_{w,z}}+\|\langle z\rangle^5\sigma(s)\|_{L^\infty_{w,z}}\le A\le 4c_0,\\ &\|\nabla_w\sigma(s)\|_{L^\infty_{w,z}}+\|\theta\nabla_z\sigma(s)\|_{L^\infty_{w,z}}\le B\le 4c_0,\\ &\|\nabla^2_{w,w}\sigma(s)\|_{L^\infty_{w,z}}+\|\theta\nabla^2_{w,z}\sigma(s)\|_{L^\infty_{w,z}}+\|\theta^2\nabla^2_{z,z}\sigma(s)\|_{L^\infty_{w,z}}\le C\le 4c_0.\end{aligned} \tag{4.13}$$

As we will show below in Sect. 4.2.1, this implies in particular that

$$\begin{aligned}&|\nabla_z\mathcal{K}(w,z)|\le 2c_0^2,\\ &|\nabla_w\mathcal{K}(w,z)|\le c_0^2\big\{\min\{s^{-1},|z|\}+\langle\ln(s)\rangle^3\big\},\end{aligned} \tag{4.14}$$

and that we have the derivative bounds

$$\|\nabla_w\nabla_z\mathcal{K}\|_{L^\infty_{w,z}}+\|\theta\nabla_z\nabla_z\mathcal{K}\|_{L^\infty_{w,z}}+\|\theta^{-1}\nabla_w\nabla_w\mathcal{K}\|_{L^\infty_{w,z}}\le c_0^2\langle\ln(s)\rangle^4. \tag{4.15}$$

These in turn can then be used to close the bootstrap for (4.13), as in Sect. 4.2.2.

4.2.1 A Priori Control on the Electric Field

Here we consider a particle density $\sigma \in C^0([0,s] : L^2_{w,z})$ such that $\sigma(0) = \sigma_0$ and which satisfies the bounds (4.13). This creates an electric field $E(s)$ through the formula

$$\begin{aligned} E[\sigma](s,Q) = E(s,Q) &= \iint \frac{Q - q(s,w,z)}{|Q - q(s,w,z)|^3} \sigma^2(s,w,z) \mathrm{d}w\mathrm{d}z \\ &= \iint \frac{Q-q}{|Q-q|^3} \gamma^2(s,q,p) \mathrm{d}q\mathrm{d}p, \end{aligned} \tag{4.16}$$

where σ and γ are related through (4.4). Simple bounds give uniformly in $R > 0$:

$$|E(s_2,Q) - E(s_1,Q)| \lesssim R \|\langle z\rangle^4 \sigma\|^2_{L^\infty_{s,w,z}} + R^{-2} \|\sigma(s)\|_{L^\infty_s L^2_{w,z}} \|\sigma(s_2) - \sigma(s_1)\|_{L^2_{w,z}},$$

which ensures that E is continuous in time. In the next lemma, we adapt the bounds from Sect. 2 to obtain stronger control as in (4.14) and (4.15).

Lemma 4.3 *Let $\sigma \in C^0([0,T], L^2_{w,z})$ with $\sigma(0) = \sigma_0$ such that $E[\sigma]$ satisfies the continuity equation (1.8) and E_0 satisfies (4.6). Then there exists $T^*(c_0) \in (0,T]$ such that*

(i) *Assuming the first line of (4.13) holds, we obtain (4.14) for $0 \le s \le T^*$.*
(ii) *Assuming the first two lines of (4.13) hold, we obtain (4.15) for $0 \le s \le T^*$.*

Proof (i) To use Lemma 2.3, we observe that

$$|p - z| \le c_0^2 \langle \ln(s)\rangle, \quad |q - w| \le s|z| + c_0^2 s \langle \ln(s)\rangle \tag{4.17}$$

and that the change of variable (4.1) preserves volume, so that

$$\begin{aligned} \|\gamma(s)\|_{L^r_{q,p}} &= \|\sigma(s)\|_{L^r_{w,z}}, \\ \||p|^\alpha \gamma(s)\|_{L^r_{q,p}} &\lesssim \||z|^\alpha \gamma(s)\|_{L^r_{q,p}} + c_0^{2\alpha} \langle \ln(s)\rangle^\alpha \|\gamma(s)\|_{L^r_{q,p}} \\ &\lesssim c_0^{2\alpha} \langle \ln(s)\rangle^\alpha \|\sigma(s)\|_{L^r_{w,z}} + \||z|^\alpha \sigma(s)\|_{L^r_{w,z}}. \end{aligned}$$

In addition, since (see (4.2)) $\frac{\partial z}{\partial p} = \mathrm{Id} + O(c_0 s \langle \ln(s)\rangle)$ has bounded Jacobian, we see that

$$\|\mathbf{j}(s)\|_{L^\infty_q} \le \left\| \int \left[|z| + c_0^2 \langle \ln(s)\rangle\right] \sigma^2 \mathrm{d}z \right\|_{L^\infty_w} \le c_0^2 \langle \ln(s)\rangle \|\langle z\rangle^3 \sigma\|^2_{L^\infty_{w,z}},$$

and using (2.5), we obtain that for $2^{-k-1} \le s_2 \le s_1 \le 2^{-k}$,

$$\|E(s_2,q) - E(s_1,q)\|_{L^\infty_q} \lesssim \langle c_0\rangle^2 k^2 2^{-k} \|\langle z\rangle^3 \sigma\|^2_{L^\infty_{s,w,z}} + 2^{-2k} \|\sigma\|^2_{L^\infty_s L^2_{w,z}}$$

and summing we see that $E(2^{-k})$ is Cauchy in L^∞_q and that

$$\|E(s,q)-E_0(q)\|_{L^\infty_q} \lesssim \langle c_0\rangle^2 s\langle \ln(s)\rangle^2 \|\langle z\rangle^3\sigma\|^2_{L^\infty_{s,w,z}} + \langle c_0\rangle^2 s^2 \|\sigma\|^2_{L^\infty_s L^2_{w,z}}. \quad (4.18)$$

Using the formulas in (4.3), the control on $\nabla_z \mathcal{K}$ follows directly, while we see that

$$\nabla_w \mathcal{K}(w,z) = -\lambda s^{-1}\big\{E(s,q)-E_0(q)\big\} - \lambda s^{-1}\big\{E_0(q)-E_0(w)\big\} + O(c_0^2\langle \ln(s)\rangle)$$

and with (4.18), (4.13), (4.6) and (4.17), we obtain (4.14).

(ii) We want to use (2.6), which requires some additional control on the derivatives. From (4.2), we see that

$$\nabla_q \gamma = \nabla_w \sigma - \lambda \ln(s)\nabla E_0 \cdot \nabla_z \sigma$$

so that

$$\|\nabla_q \gamma\|_{L^r_{q,p}} \lesssim c_0^2\langle \ln(s)\rangle \|\nabla_{w,z}\sigma\|_{L^r_{w,z}}$$

and

$$\begin{aligned}\|\nabla_q \mathbf{j}(s)\|_{L^\infty_q} &\lesssim c_0^2\langle \ln(s)\rangle \Big\| \int \big[|z| + c_0^2\langle \ln(s)\rangle\big]|\sigma|\cdot|\nabla_{w,z}\sigma|\mathrm{d}z\Big\|_{L^\infty_w}\\ &\lesssim c_0^4\langle \ln(s)\rangle^2\Big[\big\|\langle z\rangle^5\sigma\big\|^2_{L^\infty_{w,z}} + \|\nabla_{w,z}\sigma\|^2_{L^\infty_{w,z}}\Big].\end{aligned}$$

For $2^{-k-1}\le s_2\le s_1\le 2^{-k}$, this gives by (2.6) that

$$\begin{aligned}\|\nabla_q E(s_2,q)-\nabla_q E(s_1,q)\|_{L^\infty_q} &\lesssim \langle c_0\rangle^4 k^3 2^{-k}\cdot\Big(\big\|\langle z\rangle^5\sigma\big\|^2_{L^\infty_{s,w,z}} + \|\nabla_{w,z}\sigma\|^2_{L^\infty_{s,w,z}}\Big)\\ &\quad + c_0^{10}2^{-\frac{3k}{2}}\Big[\big\|\langle z\rangle^5\sigma\big\|^2_{L^\infty_{s,w,z}} + \|\sigma\|^2_{L^\infty_s L^2_{w,z}}\Big],\end{aligned}$$

and applying similar arguments as before we obtain

$$\begin{aligned}\|\nabla_q E(s,q)-\nabla_q E_0(q)\|_{L^\infty_q} &\lesssim \langle c_0\rangle^2 s\langle \ln(s)\rangle^3\cdot\Big(\big\|\langle z\rangle^5\sigma\big\|^2_{L^\infty_{s,w,z}} + \|\nabla_{w,z}\sigma(s,w,z)\|^2_{L^\infty_{s,w,z}}\Big)\\ &\quad + c_0^{10}s^{\frac{3}{2}}\Big[\big\|\langle z\rangle^5\sigma\big\|^2_{L^\infty_{s,w,z}} + \|\sigma\|^2_{L^\infty_s L^2_{w,z}}\Big]\\ &\le c_0^2 s\langle \ln(s)\rangle^4\end{aligned} \quad (4.19)$$

up to choosing $T(c_0)>0$ small enough. Using the formulas in (4.12), we directly see that

$$\begin{aligned}\theta|\nabla^2_{z,z}\mathcal{K}| &\le |\nabla E|\cdot s\min\{s^{-1},|z|\}\le c_0^2,\\ |\nabla^2_{w,z}\mathcal{K}| &\le |\nabla E|(1+s\langle \ln(s)\rangle|\nabla E_0|)\le 2c_0^2.\end{aligned}$$

Moreover, using (4.6) and (4.19), we find that, up to choosing $T(c_0)>0$ smaller,

$$
\begin{aligned}
|\nabla^2_{w,w}\mathcal{K}| &\le s^{-1}|\nabla E_0(q) - \nabla E_0(w)| + s^{-1}|\nabla E(s,q) - \nabla E_0(w)| \\
&\quad + \langle \ln(s)\rangle \big[2|\nabla E_0|^2|\nabla E| + |\nabla^2 E_0||E|\big] + s\langle \ln(s)\rangle^2 |\nabla E_0|^2|\nabla E| \\
&\le c_0^2 \min\{s^{-1}, |z|\} + c_0^2\langle \ln(s)\rangle^4,
\end{aligned}
$$

from which we deduce (4.15). □

4.2.2 A Priori Estimates on the Particle Density

Here, we close the bootstrap of (4.13):

Lemma 4.4 *Assume that $\sigma \in C^0([0,T], L^2_{w,z})$ satisfies (4.5), for some Hamiltonian $\mathcal{K}$ (not necessarily related to σ) satisfying (4.14) and (4.15). If $\sigma_0 = \sigma(0)$ satisfies (4.7), there exists $T(c_0) > 0$ such that (4.13) holds for $A = B = C = 2c_0$.*

Proof We first close the bootstrap for A, then for A, B. Finally, we adapt the argument for A, B, C. The control follows from the commutation relations (compare with (4.11)):

$$
\begin{aligned}
\mathfrak{L}(\langle z\rangle^m \sigma) &= \sigma\{\langle z\rangle^m, \mathcal{K}\} = -m\sigma\langle z\rangle^{m-2} z\cdot \nabla_w \mathcal{K}, \\
\mathfrak{L}(\theta \nabla_z \sigma) &= (\mathfrak{L}\ln\theta)\cdot \theta\nabla_z\sigma + \theta\{\nabla_z\mathcal{K}, \sigma\} \\
&= (\mathfrak{L}\ln\theta)\cdot\theta\nabla_z\sigma + \theta\nabla_z\sigma\cdot\nabla_w\nabla_z\mathcal{K} - \nabla_w\sigma\cdot\theta\nabla_z\nabla_z\mathcal{K}, \\
\mathfrak{L}(\nabla_w\sigma) &= \{\nabla_w\mathcal{K}, \sigma\} = \theta\nabla_z\sigma\cdot\theta^{-1}\nabla_w\nabla_w\mathcal{K} - \nabla_w\sigma\cdot\nabla_z\nabla_w\mathcal{K}.
\end{aligned}
\tag{4.20}
$$

As in (3.5), we find that

$$
\big\|\langle z\rangle^m\sigma(s)\big\|_{L^r_{w,z}} \le \big\|\langle z\rangle^m\sigma_0\big\|_{L^r_{w,z}} + m\int_0^s \big\|\langle z\rangle^{-1}\nabla_w\mathcal{K}(s')\big\|_{L^\infty_{w,z}} \big\|\langle z\rangle^m\sigma(s')\big\|_{L^r_{w,z}}\mathrm{d}s',
$$

and we can easily propagate the first line of (4.13).

For the derivatives, we also need to control θ. On the one hand, we can bound from above (note that $\mathfrak{L}(\ln\theta)$ can be very negative)

$$
\mathfrak{L}(\ln\theta) = -\left(1 + \frac{z}{\langle z\rangle^3}\nabla_w\mathcal{K}\right)\theta \lesssim c_0^2 + c_0^2\langle \ln(s)\rangle^3
$$

and we deduce from (4.20), (3.5) and (4.15) that

$$
\begin{aligned}
\|\theta\nabla_z\sigma(s)\|_{L^r_{w,z}} &\le \|\theta\nabla_z\sigma_0\|_{L^r_{w,z}} + c_0^2\int_0^s \langle \ln(s')\rangle^4 \Big\{\|\theta\nabla_z\sigma(s')\|_{L^r_{w,z}} + \|\nabla_w\sigma(s')\|_{L^r_{w,z}}\Big\}\mathrm{d}s', \\
\|\nabla_w\sigma(s)\|_{L^r_{w,z}} &\le \|\nabla_w\sigma_0\|_{L^r_{w,z}} + c_0^2\int_0^s \langle \ln(s')\rangle^4 \Big\{\|\theta\nabla_z\sigma(s')\|_{L^r_{w,z}} + \|\nabla_w\sigma(s')\|_{L^r_{w,z}}\Big\}\mathrm{d}s',
\end{aligned}
$$

and this allows us to propagate the second line of (4.13) for short time.

We now propagate higher-order derivatives to bound the bootstrap for C. First by interpolation in (4.13), we observe that

$$\|\langle z\rangle^{2.1}\nabla_{w,z}\sigma\|_{L^\infty_{w,z}} \leq A + C.$$

We will use the weight θ to control the ∂_z derivatives. Using (4.10), we find that

$$\begin{aligned}
\mathfrak{L}(\partial_{w^j}\partial_{w^k}\sigma) &= \theta^{-1}\nabla_w\partial_{w^j}\mathcal{K}\cdot(\theta\nabla_z\partial_{w^k}\sigma) + \theta^{-1}\nabla_w\partial_{w^k}\mathcal{K}\cdot(\theta\nabla_z\partial_{w^j}\sigma) - \nabla_z\partial_{w^j}\mathcal{K}\cdot\nabla_w\partial_{w^k}\sigma\\
&\quad - \nabla_z\partial_{w^k}\mathcal{K}\cdot\nabla_w\partial_{w^j}\sigma + \theta^{-1}\nabla_w\partial_{w^j}\partial_{w^k}\mathcal{K}\cdot(\theta\nabla_z\sigma) - \nabla_z\partial_{w^j}\partial_{w^k}\mathcal{K}\cdot\nabla_w\sigma,\\
\mathfrak{L}(\theta\partial_{z^j}\partial_{w^k}\sigma) &= \mathfrak{L}(\ln\theta)\cdot\theta\partial_{z^j}\partial_{w^k}\sigma + \theta^{-1}\nabla_w\partial_{w^k}\mathcal{K}\cdot(\theta^2\nabla_z\partial_{z^j}\sigma) - \nabla_z\partial_{w^k}\mathcal{K}\cdot(\theta\nabla_w\partial_{z^j}\sigma)\\
&\quad + \nabla_w\partial_{z^j}\mathcal{K}\cdot(\theta\nabla_z\partial_{w^k}\sigma) - (\theta\nabla_z\partial_{z^j}\mathcal{K})\cdot\nabla_w\partial_{w^k}\sigma\\
&\quad + \nabla_w\partial_{z^j}\partial_{w^k}\mathcal{K}\cdot(\theta\nabla_z\sigma) - \theta\nabla_z\partial_{z^j}\partial_{w^k}\mathcal{K}\cdot\nabla_w\sigma,\\
\mathfrak{L}(\theta^2\partial_{z^j}\partial_{z^k}\sigma) &= 2\mathfrak{L}(\ln\theta)\cdot\theta^2\partial_{z^j}\partial_{z^k}\sigma + \nabla_w\partial_{z^k}\mathcal{K}\cdot(\theta^2\nabla_z\partial_{z^j}\sigma) - \theta\nabla_z\partial_{z^k}\mathcal{K}\cdot(\theta\nabla_w\partial_{z^j}\sigma)\\
&\quad + \nabla_w\partial_{z^j}\mathcal{K}\cdot(\theta^2\nabla_z\partial_{z^k}\sigma) - (\theta\nabla_z\partial_{z^j}\mathcal{K})\cdot(\theta\nabla_w\partial_{z^k}\sigma)\\
&\quad + \theta\nabla_w\partial_{z^j}\partial_{z^k}\mathcal{K}\cdot(\theta\nabla_z\sigma) - \theta^2\nabla_z\partial_{z^j}\partial_{z^k}\mathcal{K}\cdot\nabla_w\sigma,
\end{aligned}$$

and we can proceed as for the case of one derivative once we control the new terms

$$\|\theta^{-1}\nabla^3_{w,w,w}\mathcal{K}\|_{L^\infty_{w,z}} + \|\nabla^3_{w,w,z}\mathcal{K}\|_{L^\infty_{w,z}} + \|\theta\nabla^3_{w,z,z}\mathcal{K}\|_{L^\infty_{w,z}} + \|\theta^2\nabla^3_{z,z,z}\mathcal{K}\|_{L^\infty_{w,z}} \leq c_0^2\langle\ln(s)\rangle^5. \tag{4.21}$$

It remains to prove (4.21). Starting from

$$\nabla_z\mathcal{K} = -\lambda E(q), \quad \frac{\partial q^k}{\partial z^j} = s\delta^k_j, \quad \frac{\partial q^k}{\partial w^j} = \delta^k_j + \lambda s\ln(s)\partial_j\partial_k\phi_0(w),$$

we deduce

$$\begin{aligned}
\theta^2|\nabla^3_{z,z,z}\mathcal{K}| &= (s\theta)^2|\nabla^2E(q)|,\\
\theta|\nabla^3_{w,z,z}\mathcal{K}| &\leq (s\theta)\cdot\big[1 + s\langle\ln(s)\rangle|\nabla E_0|\big]\cdot|\nabla^2E(q)|,\\
|\nabla^3_{w,w,z}\mathcal{K}| &\leq \big[1 + s\langle\ln(s)\rangle|\nabla E_0|\big]^2\cdot|\nabla^2E(q)|\\
&\quad + \big[1 + s\langle\ln(s)\rangle|\nabla E_0|\big]\cdot\big[1 + s\langle\ln(s)\rangle|\nabla^2E_0|\big]\cdot|\nabla E(q)|,
\end{aligned}$$

and finally, from (4.12), we obtain that

$$\begin{aligned}
\theta^{-1}|\nabla^3_{w,w,w}\mathcal{K}| &\leq \big[s^{-1} + \langle\ln(s)\rangle\cdot|\nabla E_0|\big]\cdot|\nabla^2E(s,q) - \nabla^2E_0(w)|\\
&\quad + \langle\ln(s)\rangle\cdot\big[|\nabla^2E|\cdot|\nabla E_0| + |\nabla E|\cdot|\nabla^2E_0| + |\nabla^3E_0|\cdot|E|\big]\\
&\quad + s\langle\ln(s)\rangle^2\cdot\big[|\nabla^2E|\cdot|\nabla E_0|^2 + |\nabla E|\cdot|\nabla E_0|\cdot|\nabla^2E_0|\big]\\
&\quad + s^2\langle\ln(s)\rangle^3\cdot\big[|\nabla^2E|\cdot|\nabla E_0|^3\big].
\end{aligned}$$

Independently, we find that

$$\|\nabla^2\mathbf{j}(s)\|_{L^\infty_q} \lesssim c_0^2\langle \ln(s)\rangle^2 \Big\| \int \big[|z| + c_0^2\langle \ln(s)\rangle\big] \cdot \Big[|\sigma| \cdot |\nabla^2_{w,z}\sigma| + |\nabla_{w,z}\sigma|^2\Big] \mathrm{d}z \Big\|_{L^\infty_w}$$
$$\lesssim c_0^4\langle \ln(s)\rangle^3 \Big[\|\langle z\rangle^5\sigma\|_{L^\infty_{w,z}} \|\nabla^2_{w,z}\sigma\|_{L^\infty_{w,z}} + \|\langle z\rangle^{2.1}\nabla_{w,z}\sigma\|^2_{L^\infty_{w,z}}\Big].$$

Now using Lemma 2.4 and the bootstrap assumptions, we obtain that

$$\|\nabla^2 E(s,q) - \nabla^2 E_0(w)\|_{L^\infty_{w,z}} \le c_0^2\langle \ln(s)\rangle^5 + c_0^2 \min\{s^{-1}, |z|\},$$

which easily leads to (4.21). □

4.3 Local Solutions

We construct local solutions for the singular Eq. (4.5) via Picard iteration.

Proof of Theorem 4.2 We proceed in two steps.

Step 1: A priori estimates. We construct a sequence of approximate solutions on a time interval $[0, T]$ (with $T > 0$ to be chosen later) via Picard iteration: We define $\sigma_{(0)}(s,w,z) := \sigma_0(w,z)$, and given $\sigma_{(n)} \in C^0_s([0,T], C^1_{w,z})$ satisfying (4.13) with $A = B = C = 4c_0$, we let $\sigma_{(n+1)} \in C^0_s([0,T], C^1_{w,z})$ be the solution of

$$\partial_s\sigma_{(n+1)} + \{\sigma_{(n+1)}, \mathcal{K}_n\} = 0, \quad \sigma_{(n+1)}(0) = \sigma_0,$$
$$\mathcal{K}_n := \lambda s^{-1}(\phi_0(w) - \phi_n(s,q)), \quad \Delta\phi_n = \int \gamma^2_{(n)}(s,q,p)\mathrm{d}p,$$

where $\gamma_{(n)}$ and $\sigma_{(n)}$ are related through (4.4). Using Lemma 4.3, we see that $\mathcal{K}_n$ satisfies (4.14) and (4.15). Using Lemma 4.4, we see that $\sigma_{(n+1)}$ satisfies (4.13) with $A = B = C = 2c_0$. We deduce that (4.13) holds uniformly in n with $A = B = C = 2c_0$ on a fixed time interval $0 \le s \le T(c_0)$.

In addition using the commutation relations (4.9), we easily propagate (4.8) uniformly in n.

Step 2: Contraction in $L^\infty_{s,w,z}$. Let

$$\delta_{(n)} := \sigma_{(n+1)} - \sigma_{(n)}, \quad \delta\mathcal{K}_{(n)} := \mathcal{K}_n - \mathcal{K}_{n-1}, \quad \mathfrak{L}_n := \partial_s + \{\cdot, \mathcal{K}_n\}, \quad \delta\mathfrak{L}_n = \{\cdot, \delta\mathcal{K}_{(n)}\},$$

so that

$$\mathfrak{L}_n\delta_{(n)} = \delta\mathfrak{L}_n\sigma_{(n)}, \tag{4.22}$$

and we can express

$$\nabla_z\delta\mathcal{K}_{(n)} = -\lambda(E_n(s,q) - E_{n-1}(s,q)),$$
$$\nabla_w\delta\mathcal{K}_{(n)} = -\lambda s^{-1}(E_n(s,q) - E_{n-1}(s,q)) - \lambda^2 \ln(s)(E_n(s,q) - E_{n-1}(s,q)) \cdot \nabla E_0(q). \tag{4.23}$$

Invoking the uniform bounds for $\sigma_{(n)}$, we will prove below that

$$\|\nabla_{w,z}\delta\mathcal{K}_{(n)}(s)\|_{L^\infty_{w,z}} \leq c_0\langle \ln(s)\rangle^6 \left\|\delta_{(n-1)}(s)\right\|_{L^\infty_{w,z}}. \tag{4.24}$$

In combination with (4.22), we find that

$$\begin{aligned}\|\delta_{(n)}(s)\|_{L^\infty_{w,z}} &\lesssim \int_0^s \|\nabla_{w,z}\delta\mathcal{K}_{(n)}(s')\|_{L^\infty_{w,z}}\|\nabla_{w,z}\sigma_{(n)}(s')\|_{L^\infty_{w,z}}\mathrm{d}s' \\ &\lesssim c_0^2\int_0^s \langle \ln(s')\rangle^6\|\delta_{(n-1)}(s')\|_{L^\infty_{w,z}}\mathrm{d}s',\end{aligned}$$

from which we deduce that, possibly taking $T(c_0) > 0$ smaller, $\sigma_{(n)}$ form a Cauchy sequence in $L^\infty_{s,w,z}$, and thus $\sigma_{(n)} \to \sigma \in L^\infty_{s,w,z}$ as $n \to \infty$. Interpolation gives convergence in the other topologies. In particular,

$$\|\nabla_{w,z}\delta_{(n)}\|_{L^\infty_{s,w,z}} \lesssim \|\delta_{(n)}\|^{\frac{1}{2}}_{L^\infty_{s,w,z}}\left[\left\|\nabla^2_{w,z}\sigma_{(n+1)}\right\|_{L^\infty_{s,w,z}} + \left\|\nabla^2_{w,z}\sigma_{(n)}\right\|_{L^\infty_{s,w,z}}\right]^{\frac{1}{2}},$$

so that $\sigma_{(n)}$ is Cauchy in $C^0_s C^1_{w,z}$ and the other bounds follow by Fatou's Lemma or by conservation. In particular, (4.8) follows by pointwise convergence. Finally we note that if c_0 is sufficiently small, the arguments give a contraction for any $T \leq 2$.

It remains to show (4.24). The main point is that E is quadratic in γ, so that in the estimates for $\delta\mathcal{K}_{(n)}$, we can always factor out the difference $\delta_{(n)}$ in $L^\infty_{w,z}$. The bound on $\nabla_z\delta\mathcal{K}_{(n)}$ follows from adaptation to Lemma 2.1 and this also allows to control all but the first term in $\nabla_w\delta\mathcal{K}_{(n)}$ as in (4.23). These then follow from (2.5) using the difference continuity equation

$$\partial_s(E_n - E_{n-1}) + \nabla\Delta^{-1}\mathrm{div}_q\{\delta\mathbf{j}_n\}, \quad \delta\mathbf{j}_n = \int_{\mathbb{R}^3_p}(\gamma_n + \gamma_{n-1})(\gamma_n - \gamma_{n-1})\cdot p\mathrm{d}p$$

with

$$\|\delta\mathbf{j}_n\|_{L^\infty_q} \lesssim \|\delta_{(n-1)}\|_{L^\infty_{w,z}}\cdot\left[\left\|\langle p\rangle^5\gamma_n\right\|_{L^\infty_{q,p}} + \left\|\langle p\rangle^5\gamma_{n-1}\right\|_{L^\infty_{q,p}}\right]$$

and simple adaptations of Lemma 2.3. □

Finally, we prove the main theorem.

Proof of Theorem 1.1 For (i), using (1.10), the assumption (1.3) leads to (3.2) in Theorem 3.1 and the local convergence is easily adapted (see Remark 3.2). For (ii), the assumption (1.5) leads to (4.6), (4.7) and the conclusion follows from that of Theorem 4.2. Finally, for (iii), we can apply Theorem 4.2 to $\mu_{-\infty}(x,-v)$ to get, using (4.4), (1.10) and (4.8) a solution for $-\infty < t \leq -1$ such that

$$\|\mu(-1)\|_{L^2_{x,v}} + \left\|\langle x,v\rangle^5\mu(-1)\right\|_{L^\infty_{x,v}} + \|\nabla_{x,v}\mu(-1)\|_{L^\infty_{x,v}} \lesssim \varepsilon.$$

By local existence, we can extend these bounds for $-1 \leq t \leq 1$, at which point we can simply apply (i). □

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