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Global endpoint Strichartz estimates for Schrödinger equations on the cylinder $\mathbb{R}\times\mathbb{T}$[☆]

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ABSTRACT

We prove global in time Strichartz estimates on the semi-periodic cylinder with initial data in L^2. This extends the local results of Takaoka-Tzvetkov [13] and the global estimates with loss of derivatives of Hani-Pausader [9] and Barron [1].

1. Long-time, scaling-critical Strichartz estimates on $\mathbb{R}\times\mathbb{T}$

Define the norm on $\mathbb{R}\times\mathbb{R}\times\mathbb{T}=(\mathbb{Z}+[0,1))\times\mathbb{R}\times\mathbb{T}$:

$$\|u\|^a_{\ell^a L^b(\mathbb{R},L^c(\mathbb{R}\times\mathbb{T}))}:=\sum_{\gamma\in\mathbb{Z}}\left(\int_{s\in[0,1)}\left(\int_{x,y\in\mathbb{R}\times\mathbb{T}}|u(\gamma+s,x,y)|^c\,dxdy\right)^{\frac{b}{c}}ds\right)^{\frac{a}{b}}. \tag{1.1}$$

In this paper, we prove the following global in time Strichartz-type estimate:

Theorem 1.1. *There exists $C<\infty$ such that for all $f\in L^2(\mathbb{R}\times\mathbb{T})$,*

$$\|e^{it\Delta_{\mathbb{R}\times\mathbb{T}}}f\|_{\ell^8L^4(\mathbb{R},L^4(\mathbb{R}\times\mathbb{T}))}\le C\|f\|_{L^2(\mathbb{R}\times\mathbb{T})}. \tag{1.2}$$

This inequality is saturated[1] by two different families of functions of $(x,y)\in\mathbb{R}\times\mathbb{T}$:

$$F_n(x,y)=nG(n\sqrt{x^2+y^2})\mathbb{1}_{\{n(x^2+y^2)\le 1\}},\qquad f_n(x,y)=n^{-\frac{1}{2}}G(n^{-1}x), \tag{1.3}$$

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[1] In the sense that the quotient of both sides converges to a nonzero constant as $n\to\infty$.

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where $G(s) = e^{-s^2}$ is a Gaussian. These correspond respectively to saturators for Strichartz estimates in $2d$ and in $1d$ [12]. The exponents in (1.2) are optimal in the following sense: (i) on the one hand, since $e^{it\Delta_{\mathbb{R}\times\mathbb{T}}}f_n(x,y) = n^{-\frac{1}{2}}(e^{in^{-2}t\partial_{xx}}G)(n^{-1}x)$ behaves as a (low-frequency) solution of the Schrödinger equation on $\mathbb{R}$, the exponent 8 in (1.2) cannot be lowered; (ii) on the other hand, since $e^{it\Delta_{\mathbb{R}\times\mathbb{T}}}F_n$ behaves as a (high-frequency) solution of the Schrödinger equation in $\mathbb{R}^2$ (see e.g. [11, Lemma 4.2] for similar computations), the exponent 4 cannot be changed if the righthand side is measured in L^2.

Interpolating with the estimate[2] when $q = 4$ and $p = \infty$,

$$\|e^{it\Delta_{\mathbb{R}\times\mathbb{T}}}P_{\leq N}f\|_{\ell^4L^\infty(\mathbb{R},L^\infty(\mathbb{R}\times\mathbb{T}))} \lesssim N\|f\|_{L^2(\mathbb{R}\times\mathbb{T})}, \qquad \mathcal{F}\left\{P_{\leq N}f\right\}(\xi,k) = \varphi(\xi/N)\varphi(k/N)\widehat{f}(\xi,k),$$

where $\varphi \in C_c^\infty(\mathbb{R})$ is a smooth bump function, and using boundedness of the square function, we obtain the family of scaling invariant Strichartz estimates on $\mathbb{R}\times\mathbb{T}$:

$$\|e^{it\Delta_{\mathbb{R}\times\mathbb{T}}}f\|_{\ell^qL^p(\mathbb{R},L^p(\mathbb{R}\times\mathbb{T}))} \lesssim \|f\|_{H^s(\mathbb{R}\times\mathbb{T})}, \quad \frac{2}{q}+\frac{1}{p}=\frac{1}{2}, \quad 4<q\leq 8, \quad s = 1-\frac{4}{p}. \tag{1.4}$$

Strichartz-type inequalities with mixed norms in the time variable of the form (1.1) were introduced in [9] to study the asymptotic behavior of solutions to critical NLS on product spaces $\mathbb{R}^n\times\mathbb{T}^d$ which are examples of manifolds where the global dimension is smaller than the local dimension. Similar cases were later explored in [6,14,15] and the sharp results when $s > 0$ was obtained in [1] using results from ℓ^2-decoupling [4].

However, to study NLS with data in L^2, estimates with loss of derivatives are useless. This raised the question of whether a Strichartz-type inequality with no loss of derivatives could hold for Schrödinger equations on d-dimensional manifolds smaller at infinity than $\mathbb{R}^d$. For the torus $\mathbb{T}^d$, for instance, a lossless inequality like (1.2) does not hold, not even locally in time (that is, with $a = \infty$) as observed in [3]. In fact, for manifolds “smaller” than $\mathbb{R}^2$, the only estimate known to the authors is the result from [13] which obtains local version of (1.2) (with $a = \infty$ instead of $a = 8$). We refer e.g. to [2,5,7] for the study of Strichartz estimates without losses in the presence of trapped geodesic.

As for nonlinear applications of (1.2), one can easily show local well-posedness of the cubic NLS in $L^2(\mathbb{R}\times\mathbb{T})$, recovering the result in [13]. However, the long-time behavior is modified scattering as shown in [10], which requires more information (and stronger control on initial data) than L^2-Strichartz estimates and it remains a challenging open question as to whether nonlinear solutions satisfy global bounds of the type (1.2).

This leaves open some interesting questions:

(1) Can one extend this result to other semi-periodic settings, i.e., does an estimate like

$$\|e^{it\Delta_{\mathbb{R}^d\times\mathbb{T}^n}}f\|_{\ell^qL^p(\mathbb{R},L^p(\mathbb{R}^d\times\mathbb{T}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^d\times\mathbb{T}^n)}, \quad p = \frac{2(n+d+2)}{n+d}, \; q = \frac{2(n+d+2)}{d}.$$

hold? This is settled for $n+d \leq 2$, but for higher values, $p < 4$ and the problem is much more challenging.

(2) Can one understand and characterize optimizers of (1.2)? In principle, introducing a parameter for the length of the torus (or the local time interval), one may expect that optimizers should vary smoothly between the two families in (1.3).

(3) Can one obtain a good profile decomposition, i.e., study the defect of compactness of bounded sequences in $L^2(\mathbb{R}\times\mathbb{T})$?

[2] This follows from variants of classical TT^* estimates as in Ginibre-Velo [8], see [9, Section 3].

2. Proof of Theorem 1.1

Since the analysis is done purely in the frequency space, we pass to the Fourier transform and consider $f \in L^2(\mathbb{R}\times\mathbb{Z})$, which corresponds to the Fourier transform of the function in (1.2). By homogeneity, we may choose f to be of unit L^2 norm and by density we may assume that f is compactly supported so that all integrals below converge absolutely. We let $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$ and we define the Fourier transform on $\mathbb{R}\times\mathbb{Z}$

$$\widehat{f}(x,y)=\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} f(\xi,k)e^{ix\xi}e^{iky}dx,\qquad \check{g}(\xi,k)=\frac{1}{(2\pi)^2}\int_{\mathbb{R}}\int_{y=0}^{2\pi} g(x,y)e^{-ix\xi}e^{-iky}dydx.$$

Since we will take Fourier transforms, it will be convenient to replace the integral over $[0,1)$ in (1.1) by an integral over $\mathbb{R}$. To do this, we introduce a Gaussian cutoff in time and let

$$J_\gamma := \|e^{-\frac{1}{4}(t-\gamma)^2}e^{it\Delta_{\mathbb{R}\times\mathbb{T}}}\widehat{f}\|_{L^4_{x,y,t}(\mathbb{R}\times\mathbb{T}\times\mathbb{R})}. \tag{2.1}$$

To prove (1.2), it will suffice to control the ℓ^8-norm of J_γ. For simplicity of presentation, we let

$$\begin{aligned}
\vec{\xi}&=(\xi_1,\xi_2,\xi_3,\xi_4),\qquad \vec{k}=(k_1,k_2,k_3,k_4),\\
\langle\xi\rangle&=\xi_1-\xi_2+\xi_3-\xi_4=\langle\vec{\xi},(1,-1,1,-1)\rangle,\qquad \langle k\rangle=k_1-k_2+k_3-k_4,\\
f_j&=f(\xi_j,k_j),\ j\in\{1,3\},\qquad f_j=\overline{f}(\xi_j,k_j),\ j\in\{2,4\},\\
Q(\xi,k)&=|\xi_1|^2+|\xi_3|^2-|\xi_2|^2-|\xi_4|^2+|k_1|^2+|k_3|^2-|k_2|^2-|k_4|^2.
\end{aligned}$$

We substitute $t\to t+\gamma$ in (2.1) and expand J_γ^4 into

$$\begin{aligned}
J_\gamma^4&=\int_{x,y,t}\left[\sum_{k_1\dots k_4}\int_{\xi_1\dots\xi_4}\Pi_{j=1}^4 f_j\cdot e^{-t^2}e^{-i(t+\gamma)Q(\xi,k)}\cdot e^{ix\langle\xi\rangle}e^{iy\langle k\rangle}d\vec{\xi}\right]dxdydt\\
&=4\pi^{\frac{5}{2}}\sum_{k_1\dots k_4}\int_{\xi_1\dots\xi_4}\Pi_j f_j\cdot e^{-\frac{1}{4}(Q(\xi,k))^2}e^{-i\gamma Q(\xi,k)}\cdot\delta(\langle\xi\rangle)\delta(\langle k\rangle)d\vec{\xi}.
\end{aligned}$$

An argument of Takaoka-Tzevtkov [13] shows that each individual J_γ^4 is bounded, but we need to handle the sum in γ. We square J_γ^4 and sum over γ to get

$$\begin{aligned}
\mathcal{J}&:=\sum_{\gamma\in\mathbb{Z}}J_\gamma^8\\
&=16\pi^5\sum_{\substack{k_1\dots k_4\\ k_1'\dots k_4'}}\int_{\substack{\xi_1\dots\xi_4,\\ \xi_1'\dots\xi_4'}}\Pi_{j=1}^4 f_j\overline{\Pi_{l=1}^4 f_l'}\cdot e^{-\frac{1}{4}(Q(\xi,k))^2}e^{-\frac{1}{4}(Q(\xi',k'))^2}\cdot\sum_\gamma e^{-i\gamma\left[Q(\xi,k)-Q(\xi',k')\right]}\\
&\quad\cdot\delta(\langle\xi\rangle)\delta(\langle\xi'\rangle)\delta(\langle k\rangle)\delta(\langle k'\rangle)d\vec{\xi}d\vec{\xi}'.
\end{aligned}\tag{2.2}$$

Using Poisson summation in γ we observe that

$$\sum_{\gamma\in\mathbb{Z}}e^{-i\gamma\left[Q(\xi,k)-Q(\xi',k')\right]}=2\pi\sum_{\mu\in2\pi\mathbb{Z}}\delta(\mu-Q(\xi,k)+Q(\xi',k')).$$

Introducing the new notations

$$\begin{aligned}
\Xi&:=(\xi_1,\xi_3,\xi_2',\xi_4'),\qquad \Xi':=(\xi_2,\xi_4,\xi_1',\xi_3'),\\
K&:=(k_1,k_3,k_2',k_4'),\qquad K':=(k_2,k_4,k_1',k_3'),\\
F(\Xi,K)&:=f(\xi_1,k_1)f(\xi_3,k_3)f(\xi_2',k_2')f(\xi_4',k_4'),\quad F(\Xi',K'):=f(\xi_2,k_2)f(\xi_4,k_4)f(\xi_1',k_1')f(\xi_3',k_3'),\\
\phi_\mu&:=\mu-Q(\xi,k)+Q(\xi',k')=\mu-|\Xi|^2-|K|^2+|\Xi'|^2+|K'|^2,
\end{aligned}$$

we arrive at

$$\mathcal{J} = 32\pi^6 \sum_{K,K'\in\mathbb{Z}^4} \int_{\Xi,\Xi'} F(\Xi,K)\overline{F(\Xi',K')}\cdot\mathcal{K}(\Xi,K;\Xi',K')\cdot d\Xi d\Xi'$$
$$\mathcal{K}(\Xi,K;\Xi',K') := e^{-\frac{1}{4}\left[Q(\xi,k)^2+Q(\xi',k')^2\right]}\cdot\sum_{\mu\in 2\pi\mathbb{Z}}\delta(\phi_\mu)\delta(\langle\xi\rangle)\delta(\langle\xi'\rangle)\delta(\langle k\rangle)\delta(\langle k'\rangle).$$

Using the Schur test, the inequality (1.2) follows from the next lemma.

Lemma 2.1. *With the notations above,*

$$\sup_{(\Xi,K)\in\mathbb{R}^4\times\mathbb{Z}^4}\sum_{K'\in\mathbb{Z}^4}\int\mathcal{K}(\Xi,K;\Xi',K')d\Xi' < \infty.$$

Proof of Lemma 2.1. We need to bound

$$\sum_{\mu\in 2\pi\mathbb{Z}}\sum_{K'\in\mathbb{Z}^4}\int_{\Xi'\in\mathbb{R}^4} e^{-\frac{1}{4}\left[Q(\xi,k)^2+Q(\xi',k')^2\right]}\delta(\phi_\mu)\delta(\langle\xi\rangle)\delta(\langle\xi'\rangle)\delta(\langle k\rangle)\delta(\langle k'\rangle)d\Xi' \tag{2.3}$$

uniformly in $(\Xi,K)\in\mathbb{R}^4\times\mathbb{Z}^4$. Below we occasionally write $Q = Q(\xi,k)$ and $Q' = Q(\xi',k')$.

Using the polarization identity on the support of $\delta(\mu - Q + Q')$, we can bound

$$e^{-\frac{1}{4}\left[Q^2+(Q')^2\right]} = e^{-\frac{1}{8}\left[Q^2+(Q')^2\right]}e^{-\frac{1}{16}\left[(Q+Q')^2+(Q-Q')^2\right]} \le e^{-\frac{1}{16}\mu^2}e^{-\frac{1}{8}\left[Q^2+(Q')^2\right]}.$$

Moreover, when $\langle\xi\rangle = 0 = \langle k\rangle$ we can substitute

$$\xi_4 = \xi_1 - \xi_2 + \xi_3 \quad\text{and}\quad k_4 = k_1 - k_2 + k_3$$

into Q and then factor to obtain

$$Q(\xi,k) = -2\left[|(\xi_2 - c_x, k_2 - c_y)|^2 - R^2\right],$$
$$(c_x,c_y) = \left(\frac{\xi_1+\xi_3}{2},\frac{k_1+k_3}{2}\right),\qquad R^2 = \left(\frac{\xi_1-\xi_3}{2}\right)^2 + \left(\frac{k_1-k_3}{2}\right)^2.$$

A similar identity holds for Q' when $\langle\xi'\rangle = 0 = \langle k'\rangle$. Indeed, on the support of $\delta(\langle\xi'\rangle)\delta(\langle k'\rangle)$ we can substitute

$$\xi_3' = -\xi_1' + \xi_2' + \xi_4' \quad\text{and}\quad k_3' = -k_1' + k_2' + k_4'$$

into Q' and factor to obtain

$$Q(\xi',k') = 2\left[|(\xi_1' - c_x', k_1' - c_y')|^2 - (R')^2\right],$$
$$(c_x',c_y') = \left(\frac{\xi_2'+\xi_4'}{2},\frac{k_2'+k_4'}{2}\right),\qquad (R')^2 = \left(\frac{\xi_2'-\xi_4'}{2}\right)^2 + \left(\frac{k_2'-k_4'}{2}\right)^2.$$

With these substitutions made, notice that

$$\phi_\mu = \mu + 2[|(\xi_2 - c_x, k_2 - c_y)|^2 - R^2] + 2[|(\xi_1' - c_x', k_1' - c_y')|^2 - (R')^2]$$

and therefore

$$\delta(\phi_\mu) = \frac{1}{2}\delta(|(\xi_2 - c_x, k_2 - c_y)|^2 + |(\xi_1' - c_x', k_1' - c_y')|^2 - A_\mu),\qquad A_\mu = \frac{R^2+(R')^2-\mu}{2}.$$

Using these observations to estimate (2.3) we arrive at

$$(2.3) \leq \frac{1}{2}\sum_{\mu\in 2\pi\mathbb{Z}} e^{-\frac{1}{16}\mu^2}\sum_{k_2,k_1'}\int_{\mathbb{R}^2} e^{-\frac{1}{2}\left[\left[|(\xi_2-c_x,k_2-c_y)|^2-R^2\right]^2+\left[|(\xi_1'-c_x',k_1'-c_y')|^2-(R')^2\right]^2\right]}$$
$$\delta(|(\xi_2-c_x,k_2-c_y)|^2+|(\xi_1'-c_x',k_1'-c_y')|^2-A_\mu)d\xi_2 d\xi_1'$$

with c_x, c_y, c_x', c_y', and A_μ defined as above. Notice that R and R' only depend on $(\varXi, K)$, and these variables have been fixed. Since we also have exponential decay in μ it therefore suffices to bound the integral

$$\mathbf{I} := \sum_{\kappa,\kappa'}\int_{\zeta,\zeta'} e^{-\frac{1}{2}\left[||(\zeta,\kappa)-\vec{C}|^2-R^2|^2+||(\zeta',\kappa')-\vec{C}'|^2-(R')^2|^2\right]}\delta(|(\zeta,\kappa)-\vec{C}|^2+|(\zeta',\kappa')-\vec{C}'|^2-A)d\zeta d\zeta' \tag{2.4}$$

uniformly in $\vec{C},\vec{C}'\in\mathbb{R}^2$, $A,R,R'\in\mathbb{R}$. Moreover, since $2c_y$ and $2c_y'$ are both integers we can assume the second components of $\vec{C},\vec{C}'$ are in $\frac{1}{2}\mathbb{Z}$.

The integral in (2.4) is invariant with respect to translation on $(\mathbb{R}\times\mathbb{Z})\times(\mathbb{R}\times\mathbb{Z})$, and we may therefore assume that $\vec{C}=(0,c)$, $\vec{C}'=(0,c')$ for $c,c'\in\{0,\frac{1}{2}\}$. To control $\mathbf{I}$ we introduce sets where the exponential factors behave nicely. When $R\geq 50$, we let

$$\begin{aligned}\mathcal{S}_0 &:= \{|(\zeta,\kappa)-\vec{C}|-R|\leq R^{-1}\},\\ \mathcal{S}_j &:= \{|(\zeta,\kappa)-\vec{C}|-R|\in R^{-1}[j,j+1]\}, \qquad 1\leq j\leq R^{\frac{1}{2}}+1,\\ \mathcal{S}_\infty &:= \{|(\zeta,\kappa)-\vec{C}|-R|\geq R^{-\frac{1}{2}}\}\end{aligned} \tag{2.5}$$

and when $R\leq 50$, we let $S_j=\emptyset$ and $\mathcal{S}_\infty=\mathbb{R}\times\mathbb{Z}$. These satisfy

$$\begin{aligned}\mathbf{1}_{\mathcal{S}_j}(\zeta,\kappa)e^{-\frac{1}{2}\left[||(\zeta,\kappa)-\vec{C}|^2-R^2|^2\right]} &\lesssim e^{-\frac{1}{2}j^2}\mathbf{1}_{\mathcal{S}_j}(\zeta,\kappa), \quad 0\leq j\leq R^{\frac{1}{2}}+1,\\ \mathbf{1}_{\mathcal{S}_\infty}(\zeta,\kappa)e^{-\frac{1}{2}\left[||(\zeta,\kappa)-\vec{C}|^2-R^2|^2\right]} &\lesssim e^{-\frac{1}{2}|(\zeta,\kappa)-\vec{C}|}\mathbf{1}_{\mathcal{S}_\infty}(\zeta,\kappa).\end{aligned} \tag{2.6}$$

Indeed, the estimate on $\mathcal{S}_j$ in (2.6) follows by factoring the term in the exponential. To prove the estimate on $\mathcal{S}_\infty$ note that if $(\zeta,\kappa)\in\mathcal{S}_\infty$ and $R\geq 50$ then

$$||(\zeta,\kappa)-\vec{C}|^2-R^2|^2\geq\left[R^{-\frac{1}{2}}(|(\zeta,\kappa)-\vec{C}|+R)\right]^2\geq|(\zeta,\kappa)-\vec{C}|+R.$$

On the other hand

$$||(\zeta,\kappa)-\vec{C}|^2-R^2|^2\geq|(\zeta,\kappa)-\vec{C}|-R-2,$$

and the estimate in (2.6) in $\mathcal{S}_\infty$ follows if $R\leq 50$.

We first use (2.7) from Lemma 2.2 to control the contribution of $\mathcal{S}_\infty$ to (2.4). In particular

$$\begin{aligned}\mathbf{I}_{\infty\infty} &:= \sum_{\kappa,\kappa'}\iint \mathbf{1}_{\mathcal{S}_\infty}(\zeta,\kappa)\mathbf{1}_{\mathcal{S}_\infty}(\zeta',\kappa')e^{-\frac{1}{2}\left[||(\zeta,\kappa-c)|^2-R^2|^2+||(\zeta',\kappa'-c')|^2-(R')^2|^2\right]}\\ &\qquad\cdot\delta(|\zeta|^2+|\kappa-c|^2+|\zeta'|^2+|\kappa'-c'|^2-A)d\zeta d\zeta'\\ &\lesssim \sum_{\kappa,\kappa'}e^{-\frac{1}{2}(|\kappa-c|+|\kappa'-c'|)}\iint\delta(|\zeta|^2+|\kappa-c|^2+|\zeta'|^2+|\kappa'-c'|^2-A)d\zeta d\zeta'\\ &\lesssim \sup_{B\in\mathbb{R}}\iint\delta(|\zeta|^2+|\zeta'|^2-B)d\zeta d\zeta'\lesssim 1.\end{aligned}$$

Next, we consider

$$
\begin{aligned}
\mathbf{I}_{j\infty} &= \sum_{\kappa,\kappa'} \iint \mathbf{1}_{\mathcal{S}_j}(\zeta,\kappa)\mathbf{1}_{\mathcal{S}_\infty}(\zeta',\kappa')e^{-\frac{1}{2}\left[||(\zeta,\kappa)|^2-R^2|^2+||(\zeta',\kappa')|^2-(R')^2|^2\right]} \\
&\qquad \cdot \delta(|\zeta|^2+|\kappa-c|^2+|\zeta'|^2+|\kappa'-c'|^2-A)d\zeta d\zeta' \\
&\lesssim e^{-\frac{1}{2}j^2}\sum_{\kappa'} e^{-\frac{1}{2}|\kappa'-c'|}\sup_B \sum_\kappa \iint e^{-\frac{1}{2}|\zeta'|}\mathbf{1}_{\mathcal{S}_j}(\zeta,\kappa)\delta(|\zeta|^2+|\kappa-c|^2+|\zeta'|^2-B)d\zeta d\zeta'.
\end{aligned}
$$

We can split the integral above into two regions: (i) when $|\kappa|\in[R-10,R+2]$, the sum is only over a uniformly bounded number of κ and we can use (2.7); and (ii) when $|\kappa|\leq R-10$, in which case we use (2.8) and the rapid decay of $e^{-|\zeta'|}$. In both cases, we obtain a bounded contribution after summing over j.

Finally, by symmetry, it remains to consider:

$$
\begin{aligned}
\mathbf{I}_{jp} &= \sum_{\kappa,\kappa'} \iint \mathbf{1}_{\mathcal{S}_j}(\zeta,\kappa)\mathbf{1}_{\mathcal{S}_p}(\zeta',\kappa')\mathbf{1}_{\{|\zeta'|\leq|\zeta|\}}e^{-\frac{1}{2}\left[||(\zeta,\kappa-c)|^2-R^2|^2+||(\zeta',\kappa'-c')|^2-(R')^2|^2\right]} \\
&\qquad \cdot \delta(|(\zeta,\kappa-c,\zeta',\kappa'-c')|^2-A)d\zeta d\zeta'.
\end{aligned}
$$

Note that we may assume $R,R'\geq 50$ since otherwise $\mathcal{S}_j$ or $\mathcal{S}_p$ is empty. Using (2.6) we estimate

$$
\begin{aligned}
\mathbf{I}_{jp} &\leq 2e^{-\frac{1}{2}(j^2+p^2)}\left[\mathbf{J}^1_{jp}+\mathbf{J}^2_{jp}\right], \\
\mathbf{J}^1_{jp} &= \sum_{R-10\leq|\kappa|,|\kappa'|\leq R+10} \iint \mathbf{1}_{\mathcal{S}_j}\mathbf{1}_{\mathcal{S}_p}\delta(|\zeta|^2+|\kappa-c|^2+|\zeta'|^2+|\kappa'-c'|^2-A)d\zeta d\zeta', \\
\mathbf{J}^2_{jp} &= \sum_{\kappa'}\int_{\zeta'}\mathbf{1}_{\mathcal{S}_p}\left(\sum_{|\kappa|\leq R-10}\int_\zeta \mathbf{1}_{\mathcal{S}_j}\delta(|\zeta|^2+|\kappa-c|^2+|\zeta'|^2+|\kappa'-c'|^2-A)d\zeta\right)d\zeta'.
\end{aligned}
$$

For $\mathbf{J}^1_{jp}$, we observe that the sum is only over a uniformly bounded number of κ,κ' and we can use (2.7). For $\mathbf{J}^2_{jp}$, we can use (2.8) followed by Lemma 3.1. Summing over j,p, we obtain an acceptable contribution. □

In the proof above, we have use two simple bounds that allow us to cancel two integrals.

Lemma 2.2. *We have*

$$
\sup_{A\in\mathbb{R}}\iint_{\mathbb{R}^2}\delta(\zeta^2+\eta^2-A)d\zeta d\eta=\pi \tag{2.7}
$$

and, for $\mathcal{S}_j$ defined as in (2.5) and $R\geq 50$,

$$
\sup_{A\in\mathbb{R}}\sum_{|\kappa|\leq R-10}\int_{\mathbb{R}}\mathbf{1}_{\mathcal{S}_j}(\zeta,\kappa)\delta(\zeta^2-A)d\zeta\lesssim 1,\qquad 0\leq j\leq R^{\frac{1}{2}}+1. \tag{2.8}
$$

Proof of Lemma 2.2. The first bound is direct after passing to polar coordinates. To prove (2.8), we may assume $R\geq 50$. We first claim that

$$
(\zeta,\kappa)\in\mathcal{S}_j,\quad |\kappa|\leq R-10\quad\Rightarrow\quad |\zeta|\geq R^{\frac{1}{2}}(R-|\kappa|-1)^{\frac{1}{2}} \tag{2.9}
$$

Indeed, on $\mathcal{S}_j$, we see that $\zeta^2+(\kappa-c)^2\geq R^2-3\sqrt{R}$ for some $c\in\{0,\frac{1}{2}\}$ and

$$
\zeta^2\geq(R+|\kappa-c|)(R-|\kappa-c|)-3\sqrt{R}\geq R(R-|\kappa|-1)+R/2-3\sqrt{R}.
$$

Eliminating some terms and taking square roots give the result. To prove (2.8) we then apply a change of variables along with (2.9) to estimate

$$
\sum_{|\kappa|\leq R-10}\int\mathbf{1}_{\mathcal{S}_j}(\zeta,\kappa)\delta(\zeta^2-A)d\zeta\lesssim\sum_{|\kappa|\leq R-10}R^{-\frac{1}{2}}\left[R-|\kappa|-1\right]^{-\frac{1}{2}}\lesssim 1,
$$

which gives (2.8). □

3. On volumes of annuli in $\mathbb{R} \times \mathbb{Z}$

As we saw in the last section, the contribution of the integral $\mathbf{I}_{jp}$ is controlled by the following geometric lemma which says that the volume of a (large and thin) annulus in $\mathbb{R} \times \mathbb{Z}$ is proportional to its volume in $\mathbb{R}^2$. The result is essentially Lemma 2.1 from [13].

Lemma 3.1.

For $0 \leq w \leq 20 \leq R$ and $0 \leq |x| \leq 1/2$,

$$V(R, w) = |\mathbb{R}_\zeta \times \mathbb{Z}_\kappa \cap \{R^2 \leq \zeta^2 + (\kappa + x)^2 \leq (R + w)^2\}| \lesssim \sqrt{Rw} + Rw.$$

As a consequence, for the sets in (2.5) we have $|\mathcal{S}_j| \lesssim 1$ for $0 \leq j \leq R^{\frac{1}{2}} + 1$.

Proof of Lemma 3.1. Let

$$\ell(y) = \begin{cases} \sqrt{(R+w)^2 - y^2} & \text{if } R \leq |y| \leq R + w \\ \sqrt{(R+w)^2 - y^2} - \sqrt{R^2 - y^2} & \text{if } 0 \leq |y| \leq R \end{cases}$$

be the length of the horizontal segment in the annulus under consideration at ordinate y. This is maximized at $|y| = R$ when it is at most $\sqrt{3Rw}$. In addition, for $2^p \leq ||\kappa + x| - R| \leq 2^{p+1}$ and $32 \leq 2^p \leq R$, we can estimate

$$\ell(\kappa + x) \leq \frac{2Rw}{\sqrt{R}\sqrt{R - \kappa - 21}} \leq 4R^{\frac{1}{2}} 2^{-\frac{p}{2}} w.$$

Summing a bounded number of contributions when $\kappa + x \geq R - 50$ and the above bound otherwise, we conclude that the volume under consideration is at most

$$V \lesssim \sqrt{Rw} + R^{\frac{1}{2}} w \sum_p 2^{\frac{p}{2}} \lesssim \sqrt{Rw} + Rw. \quad \square$$

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