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GLOBAL REGULARITY OF SOLUTIONS OF THE EINSTEIN-KLEIN-GORDON SYSTEM: A REVIEW

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Abstract. In this article we consider the Einstein field equations of General Relativity for self-gravitating massive scalar fields (the Einstein-Klein-Gordon system). Our goal is to review the main results and ideas in our work [*The Einstein-Klein-Gordon coupled system: Global stability of the Minkowski solution*, preprint (2019)] on the global regularity, modified scattering, and asymptotic analysis of solutions of this system with initial data in a small neighborhood of the Minkowski space-time.

Contents

1. Introduction.

The Einstein field equations of General Relativity are a covariant geometric system that connect the Ricci tensor of a Lorentzian metric g on a manifold M to the energy-momentum tensor of the matter fields in the space-time, according to the equation

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}. \tag{1.1}$$

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Here $G_{\alpha\beta} = R_{\alpha\beta} - (1/2)Rg_{\alpha\beta}$ is the Einstein tensor, where $R_{\alpha\beta}$ is the Ricci tensor, R is the scalar curvature, and $T_{\alpha\beta}$ is the energy-momentum tensor of the matter in the space.

In this article we are concerned with the Einstein-Klein-Gordon coupled system, which describes the coupled evolution of an unknown Lorentzian metric g and a massive scalar field ψ. In this case the associated energy-momentum tensor $T_{\alpha\beta}$ is given by

$$T_{\alpha\beta} := \mathbf{D}_\alpha\psi\mathbf{D}_\beta\psi - (1/2)g_{\alpha\beta}\big(\mathbf{D}_\mu\psi\mathbf{D}^\mu\psi + \psi^2\big), \tag{1.2}$$

where $\mathbf{D}$ denotes covariant derivatives.

Our goal is to outline the main results of our work [31], which can be summarized as follows:

(1) a proof of global regularity (in wave coordinates) of solutions of the Einstein-Klein-Gordon coupled system, in the case of small, smooth, and localized perturbations of the stationary Minkowski solution $(g, \psi) = (m, 0)$;

(2) precise asymptotics of the metric components and the Klein-Gordon field as the time goes to infinity, including the construction of modified (nonlinear) scattering profiles and quantitative bounds for convergence;

(3) classical estimates on the solutions at null and timelike infinity, such as bounds on the metric components, peeling estimates of the Riemann curvature tensor, ADM and Bondi mass identities and estimates, and asymptotic description of null and timelike geodesics.

Our goal here is to present the main theorems in [31], together with some of the main ideas and ingredients in the proofs. We do not aim to present formal proofs, but we prefer instead to discuss and motivate the main concepts and constructions.

The general plan is to work in a standard gauge (in this case the classical wave coordinates) and transform the geometric Einstein-Klein-Gordon system (1.1)–(1.2) into a coupled system of quasilinear wave and Klein-Gordon equations. We then analyze this system in a framework inspired by the recent advances in the global existence theory for quasilinear dispersive models, such as plasma models and water waves.

More precisely, we rely on a combination of energy estimates and Fourier analysis. At a very general level one should think that energy estimates are used, in combination with vector-fields, to control high regularity norms of the solutions, while the Fourier analysis is used, mostly in connection with normal forms, analysis of resonant sets, and a special "designer" norm, to prove dispersion and decay in lower regularity norms.

A key advantage of our method over the classical physical space methods is the ability to easily identify and dispose of nonresonant nonlinear interactions, using integration by parts arguments in the Fourier space. This is very useful both for wave and Klein-Gordon evolutions, as well as for many other dispersive or hyperbolic evolutions.

1.1. *Elements of Lorentzian geometry.* We start by recalling some of the basic definitions and formulas of Lorentzian geometry.[1] Assume g is a sufficiently smooth Lorentzian metric in a 4 dimensional open set O. We assume that we are working in a system of coordinates x^0, x^1, x^2, x^3 in O. We define the connection coefficients $\mathbf{\Gamma}$ and the covariant

[1] At this stage, all the formulas are completely analogous to the Riemannian case, hold in any dimension, and the computations can be performed in local coordinates. A standard reference is the book of Wald [59].

derivative $\mathbf{D}$ by

$$\mathbf{\Gamma}_{\mu\alpha\beta} := g(\partial_\mu, \mathbf{D}_{\partial_\beta}\partial_\alpha) = \frac{1}{2}(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}), \tag{1.3}$$

where $\partial_\mu := \partial_{x^\mu}$, $\mu \in \{0,1,2,3\}$. Thus

$$\mathbf{D}_{\partial_\alpha}\partial_\beta = \mathbf{D}_{\partial_\beta}\partial_\alpha = \mathbf{\Gamma}^\nu{}_{\alpha\beta}\partial_\nu, \qquad \mathbf{\Gamma}^\nu{}_{\alpha\beta} := g^{\mu\nu}\mathbf{\Gamma}_{\mu\alpha\beta}, \tag{1.4}$$

where $g^{\alpha\beta}$ is the inverse of the matrix $g_{\alpha\beta}$, i.e., $g^{\alpha\beta}g_{\mu\beta} = \delta^\alpha_\mu$. For $\mu, \nu \in \{0,1,2,3\}$ let

$$\begin{aligned} \mathbf{\Gamma}^\nu &:= g^{\alpha\beta}\mathbf{\Gamma}^\nu{}_{\alpha\beta} = g^{\alpha\beta}g^{\mu\nu}(\partial_\alpha g_{\beta\mu} - \frac{1}{2}\partial_\mu g_{\alpha\beta}) = -\partial_\alpha g^{\alpha\nu} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\partial_\mu g_{\alpha\beta}, \\ \mathbf{\Gamma}_\mu &:= g_{\mu\nu}\mathbf{\Gamma}^\nu = g^{\alpha\beta}\partial_\alpha g_{\beta\mu} - \frac{1}{2}g^{\alpha\beta}\partial_\mu g_{\alpha\beta}. \end{aligned} \tag{1.5}$$

We record also the useful general identity

$$\partial_\alpha g^{\mu\nu} = -g^{\mu\rho}g^{\nu\lambda}\partial_\alpha g_{\rho\lambda}, \tag{1.6}$$

and the Jacobi formula

$$\partial_\alpha(\log|g|) = g^{\mu\nu}\partial_\alpha g_{\mu\nu}, \qquad \alpha \in \{0,1,2,3\}, \tag{1.7}$$

where $|g|$ denotes the determinant of the matrix $g_{\alpha\beta}$ in local coordinates.

Covariant derivatives can be calculated in local coordinates according to the general formula

$$\mathbf{D}_\alpha T_{\beta_1\ldots\beta_n} = \partial_\alpha T_{\beta_1\ldots\beta_n} - \sum_{j=1}^n \mathbf{\Gamma}^\nu{}_{\alpha\beta_j}T_{\beta_1\ldots\nu\ldots\beta_n} \tag{1.8}$$

for any covariant tensor T. In particular, for any scalar function f

$$\square_g f = g^{\alpha\beta}\mathbf{D}_\alpha\mathbf{D}_\beta f = \widetilde{\square}_g f - \mathbf{\Gamma}^\nu\partial_\nu f, \tag{1.9}$$

where $\widetilde{\square}_g := g^{\alpha\beta}\partial_\alpha\partial_\beta$ denotes the reduced wave operator.

The Riemann curvature tensor measures commutation of covariant derivatives according to the covariant formula

$$\mathbf{D}_\alpha\mathbf{D}_\beta\omega_\mu - \mathbf{D}_\beta\mathbf{D}_\alpha\omega_\mu = \mathbf{R}_{\alpha\beta\mu}{}^\nu\omega_\nu \tag{1.10}$$

for any form ω. The Riemann tensor R satisfies the symmetry properties

$$\begin{aligned} &\mathbf{R}_{\alpha\beta\mu\nu} = -\mathbf{R}_{\beta\alpha\mu\nu} = -\mathbf{R}_{\alpha\beta\nu\mu} = \mathbf{R}_{\mu\nu\alpha\beta}, \\ &\mathbf{R}_{\alpha\beta\mu\nu} + \mathbf{R}_{\beta\mu\alpha\nu} + \mathbf{R}_{\mu\alpha\beta\nu} = 0, \end{aligned} \tag{1.11}$$

and the Bianchi identities

$$\mathbf{D}_\rho\mathbf{R}_{\alpha\beta\mu\nu} + \mathbf{D}_\alpha\mathbf{R}_{\beta\rho\mu\nu} + \mathbf{D}_\beta\mathbf{R}_{\rho\alpha\mu\nu} = 0. \tag{1.12}$$

Its components can be calculated in local coordinates in terms of the connection coefficients according to the formula

$$\mathbf{R}_{\alpha\beta\mu}{}^\rho = -\partial_\alpha\mathbf{\Gamma}^\rho{}_{\beta\mu} + \partial_\beta\mathbf{\Gamma}^\rho{}_{\alpha\mu} - \mathbf{\Gamma}^\rho{}_{\alpha\nu}\mathbf{\Gamma}^\nu{}_{\beta\mu} + \mathbf{\Gamma}^\rho{}_{\beta\nu}\mathbf{\Gamma}^\nu{}_{\alpha\mu}. \tag{1.13}$$

Therefore, the Ricci tensor $R_{\alpha\mu} = g^{\beta\rho}\mathbf{R}_{\alpha\beta\mu\rho}$ is given by the formula

$$R_{\alpha\mu} = -\partial_\alpha\mathbf{\Gamma}^\rho{}_{\rho\mu} + \partial_\rho\mathbf{\Gamma}^\rho{}_{\alpha\mu} - \mathbf{\Gamma}^\rho{}_{\nu\alpha}\mathbf{\Gamma}^\nu{}_{\rho\mu} + \mathbf{\Gamma}^\rho{}_{\rho\nu}\mathbf{\Gamma}^\nu{}_{\alpha\mu}.$$

Simple calculations using the formulas (1.3) and (1.5) show that the Ricci tensor is given by

$$2R_{\alpha\mu} = -\widetilde{\Box}_g g_{\alpha\mu} + \partial_\alpha \mathbf{\Gamma}_\mu + \partial_\mu \mathbf{\Gamma}_\alpha + F^{\geq 2}_{\alpha\mu}(g, \partial g), \tag{1.14}$$

where $F^{\geq 2}_{\alpha\beta}(g, \partial g)$ is a quadratic semilinear expression,

$$\begin{aligned} F^{\geq 2}_{\alpha\beta}(g, \partial g) &= \frac{1}{2} g^{\rho\mu} g^{\nu\lambda} \{\partial_\nu g_{\rho\mu} \partial_\beta g_{\alpha\lambda} + \partial_\nu g_{\rho\mu} \partial_\alpha g_{\beta\lambda} - \partial_\nu g_{\rho\mu} \partial_\lambda g_{\alpha\beta}\} \\ &+ g^{\rho\mu} g^{\nu\lambda}\{ - \partial_\rho g_{\mu\lambda} \partial_\alpha g_{\beta\nu} - \partial_\rho g_{\mu\lambda} \partial_\beta g_{\alpha\nu} + \partial_\rho g_{\mu\lambda} \partial_\nu g_{\alpha\beta} + \partial_\alpha g_{\rho\lambda} \partial_\mu g_{\beta\nu} + \partial_\beta g_{\rho\lambda} \partial_\mu g_{\alpha\nu}\} \\ &- \frac{1}{2} g^{\rho\mu} g^{\nu\lambda} (\partial_\alpha g_{\nu\mu} + \partial_\nu g_{\alpha\mu} - \partial_\mu g_{\alpha\nu})(\partial_\beta g_{\rho\lambda} + \partial_\rho g_{\beta\lambda} - \partial_\lambda g_{\beta\rho}). \end{aligned} \tag{1.15}$$

1.2. *The Einstein-Klein-Gordon system.* We consider the Einstein field equations for an unknown space-time(M, g),[2]

$$G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = T_{\alpha\beta}, \tag{1.16}$$

where $T_{\alpha\beta}$ is the energy-momentum of a massive scalar field $\psi : M \to \mathbb{R}$,

$$T_{\alpha\beta} := \mathbf{D}_\alpha \psi \mathbf{D}_\beta \psi - \frac{1}{2} g_{\alpha\beta} \big(\mathbf{D}_\mu \psi \mathbf{D}^\mu \psi + \psi^2\big). \tag{1.17}$$

The covariant Bianchi identities $\mathbf{D}^\alpha G_{\alpha\beta} = 0$ can be used to derive an evolution equation for the massive scalar field ψ. The equation is

$$\Box_g \psi - \psi = 0. \tag{1.18}$$

Therefore the main unknowns in the problem are the metric tensor g and the scalar field ψ, which satisfy the covariant coupled system (1.16)–(1.18).

To construct solutions we need to fix a system of coordinates and transform the problem into a PDE problem. We work in *wave coordinates*, which is the condition

$$\mathbf{\Gamma}^\alpha = -\Box_g x^\alpha \equiv 0 \quad \text{for} \quad \alpha \in \{0, 1, 2, 3\}. \tag{1.19}$$

Our construction of global solutions of the Einstein-Klein-Gordon system is based on the following reduction.

PROPOSITION 1.1. Assume g is a Lorentzian metric in a 4 dimensional open set O, with induced covariant derivative $\mathbf{D}$ and Ricci curvature R, and $\psi : O \to \mathbb{R}$ is a scalar. Let x^0, x^1, x^2, x^3 denote a system of coordinates in O and let

$$\mathbf{\Gamma}^\nu := g^{\alpha\beta} \mathbf{\Gamma}^\nu{}_{\alpha\beta} = -\Box_g x^\nu = -\partial_\alpha g^{\alpha\nu} - \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} \partial_\mu g_{\alpha\beta}, \qquad \nu \in \{0, 1, 2, 3\}. \tag{1.20}$$

Let $\widetilde{\Box}_g$ denote the reduced wave operator

$$\widetilde{\Box}_g := g^{\alpha\beta} \partial_\alpha \partial_\beta. \tag{1.21}$$

(i) Assume that (g, ψ) satisfy the Einstein-Klein-Gordon system

$$\begin{aligned} &R_{\alpha\beta} - \mathbf{D}_\alpha \psi \mathbf{D}_\beta \psi - \frac{\psi^2}{2} g_{\alpha\beta} = 0, \\ &\Box_g \psi - \psi = 0, \end{aligned} \tag{1.22}$$

[2] For simplicity, we drop the factor of 8π from the energy-momentum tensor; compare with (1.1).

in O. Assume also that $\mathbf{\Gamma}^\mu \equiv 0$ in O, $\mu \in \{0,1,2,3\}$ (the harmonic gauge condition). Then

$$\begin{aligned}&\widetilde{\Box}_g g_{\alpha\beta} + 2\partial_\alpha\psi\partial_\beta\psi + \psi^2 g_{\alpha\beta} - F^{\geq 2}_{\alpha\beta}(g,\partial g) = 0,\\ &\widetilde{\Box}_g \psi - \psi = 0,\end{aligned} \tag{1.23}$$

where the quadratic semilinear terms $F^{\geq 2}_{\alpha\beta}(g,\partial g)$ are defined in (1.15).

(ii) Conversely, assume that the equations (1.23) (the reduced Einstein-Klein-Gordon system) hold in O. Then

$$\begin{aligned}&R_{\alpha\beta} - \partial_\alpha\psi\partial_\beta\psi - \frac{\psi^2}{2}g_{\alpha\beta} - \frac{1}{2}(\partial_\alpha\mathbf{\Gamma}_\beta + \partial_\beta\mathbf{\Gamma}_\alpha) = 0,\\ &\Box_g\psi - \psi + \mathbf{\Gamma}^\mu\partial_\mu\psi = 0,\end{aligned} \tag{1.24}$$

and the functions $\mathbf{\Gamma}_\beta = g_{\beta\nu}\mathbf{\Gamma}^\nu$ satisfy the reduced wave equations

$$\widetilde{\Box}_g \mathbf{\Gamma}_\beta = 2\mathbf{\Gamma}^\nu\partial_\nu\psi\partial_\beta\psi + g^{\rho\alpha}[\mathbf{\Gamma}^\nu{}_{\rho\alpha}(\partial_\nu\mathbf{\Gamma}_\beta + \partial_\beta\mathbf{\Gamma}_\nu) + \mathbf{\Gamma}^\nu{}_{\rho\beta}(\partial_\alpha\mathbf{\Gamma}_\nu + \partial_\nu\mathbf{\Gamma}_\alpha)] + \partial_\mu\mathbf{\Gamma}_\nu\partial_\beta g^{\mu\nu}. \tag{1.25}$$

In particular, the pair (g,ψ) solves the Einstein-Klein-Gordon system (1.22) if $\mathbf{\Gamma}_\mu \equiv 0$ in O.

The proposition can be proved by straightforward calculations: the identities (1.22) are easily seen to be equivalent to the system (1.16)–(1.17), the identities (1.23) follow from (1.22) and (1.14), while the identities (1.24) follow from (1.23) and (1.14).

The identities (1.25) are needed to prove the consistency of the wave coordinates condition (1.19), namely that it is propagated by the Einstein-Klein-Gordon flow provided that it is verified at the initial time. To prove them we start from the (covariant) Bianchi identity $\mathbf{D}^\alpha G_{\alpha\beta} = 0$. In coordinates, using also the formula (1.8), this gives

$$0 = g^{\rho\alpha}\mathbf{D}_\rho R_{\alpha\beta} - \frac{1}{2}\mathbf{D}_\beta R = g^{\rho\alpha}[\partial_\rho R_{\alpha\beta} - \mathbf{\Gamma}^\nu{}_{\rho\alpha}R_{\nu\beta} - \mathbf{\Gamma}^\nu{}_{\rho\beta}R_{\alpha\nu}] - \frac{1}{2}\partial_\beta(g^{\mu\nu}R_{\mu\nu}). \tag{1.26}$$

The desired identities (1.25) follow using also the identities (1.24) and the definitions.

Our basic strategy to construct global solutions of the Einstein-Klein-Gordon system is to use Proposition 1.1(ii). We construct first the pair (g,ψ) by solving the reduced Einstein-Klein-Gordon system (1.23) (regarded as a quasilinear wave-Klein-Gordon system) in the domain $\mathbb{R}^3 \times [0,\infty)$. In addition, we arrange that $\Gamma_\mu, \partial_t\Gamma_\mu$ vanish on the initial hypersurface, so they vanish in the entire open domain, as a consequence of the wave equations (1.25). Therefore the pair (g,ψ) solves the Einstein-Klein-Gordon system as desired.

In other words, the problem is reduced to constructing global solutions of the quasilinear system (1.23) for initial data compatible with the wave coordinates condition.

1.2.1. *Initial data sets.* To implement the strategy described above and use Proposition 1.1(ii) we need to prescribe suitable initial data. Let $\Sigma_0 = \{x \in O : t = x^0 = 0\}$. We assume that $\overline{g}, k$ are given symmetric tensors on Σ_0, such that $\overline{g}$ is a Riemannian metric on Σ_0. We assume also that $\psi_0, \psi_1 : \Sigma_0 \to \mathbb{R}$ are given initial data for the scalar field ψ.

We start by prescribing the metric components[3] on Σ_0

$$g_{ij} = \overline{g}_{ij}, \qquad g_{0i} = g_{i0} = 0, \qquad g_{00} = -1.$$

We also prescribe the time derivative of the metric tensor

$$\partial_t g_{ij} = -2k_{ij},$$

in such a way that k is the second fundamental form of the surface Σ_0, $k(X,Y) = -g(\mathbf{D}_X n, Y)$, where $n = \partial_0$ is the future-oriented unit normal vector-field on Σ_0. The condition $\mathbf{\Gamma}_0 = 0$ gives

$$0 = g^{\alpha\beta}\partial_\alpha g_{0\beta} - \frac{1}{2}g^{\alpha\beta}\partial_0 g_{\alpha\beta} = g^{00}\partial_0 g_{00} - \frac{1}{2}g^{ij}\partial_0 g_{ij} - \frac{1}{2}g^{00}\partial_0 g_{00} = -\frac{1}{2}\partial_0 g_{00} + \overline{g}^{ij}k_{ij},$$

where $\overline{g}^{ij}$ is the inverse of the matrix $\overline{g}_{ij}$, while the conditions $\mathbf{\Gamma}_n = 0$, $n \in \{1,2,3\}$, give

$$0 = g^{\alpha\beta}\partial_\alpha g_{n\beta} - \frac{1}{2}g^{\alpha\beta}\partial_n g_{\alpha\beta} = g^{00}\partial_0 g_{n0} + g^{ij}\partial_i g_{nj} - \frac{1}{2}g^{ij}\partial_n g_{ij}.$$

Therefore the full initial data for the pair (g,ψ) on the hypersurface Σ_0 is given by

$$\begin{aligned} &g_{ij} = \overline{g}_{ij}, \qquad g_{0i} = g_{i0} = 0, \qquad g_{00} = -1,\\ &\partial_t g_{ij} = -2k_{ij}, \qquad \partial_t g_{00} = 2\overline{g}^{ij}k_{ij}, \qquad \partial_t g_{n0} = \overline{g}^{ij}\partial_i\overline{g}_{jn} - \frac{1}{2}\overline{g}^{ij}\partial_n\overline{g}_{ij},\\ &\psi = \psi_0, \qquad \partial_t\psi = \psi_1. \end{aligned} \tag{1.27}$$

The remaining restrictions $\partial_0\mathbf{\Gamma}_\alpha = 0$ lead to the constraint equations. In view of (1.24) the constraint equations are equivalent to the conditions

$$R_{\alpha 0} - (1/2)Rg_{\alpha 0} = T_{\alpha 0}, \qquad \alpha \in \{0,1,2,3\}, \tag{1.28}$$

where $T_{\alpha\beta}$ is as in (1.17). These identities can be analyzed by considering the cases $\alpha = n \in \{1,2,3\}$ and $\alpha = 0$ and using the definitions. This leads to four constraint equations

$$\begin{aligned} \overline{\nabla}_n(\overline{g}^{ij}k_{ij}) - \overline{g}^{ij}\overline{\nabla}_j k_{in} &= \psi_1\overline{\nabla}_n\psi_0, \qquad n \in \{1,2,3\},\\ \overline{R} + \overline{g}^{ij}\overline{g}^{mn}(k_{ij}k_{mn} - k_{im}k_{jn}) &= \psi_1^2 + \overline{g}^{ij}\overline{\nabla}_i\psi_0\overline{\nabla}_j\psi_0 + \psi_0^2, \end{aligned} \tag{1.29}$$

where $\overline{\nabla}$ denotes the covariant derivative induced by the metric $\overline{g}$ on Σ_0, and $\overline{R}$ is the scalar curvature of the metric $\overline{g}$ on Σ_0.

1.3. *The main global regularity theorem.* Our first main theorem concerns the global regularity of the system (1.23) for small initial data $(\overline{g}_{ij}, k_{ij}, \psi_0, \psi_1)$. To state it precisely we need to introduce several Banach spaces of functions on $\mathbb{R}^3$.

Definition 1.2. For $a \geq 0$ let H^a denote the usual Sobolev spaces of index a on $\mathbb{R}^3$. We define also the Banach spaces $H^{a,b}_\Omega$, $a, b \in \mathbb{Z}_+$, by the norms

$$\|f\|_{H^{a,b}_\Omega} := \sum_{|\alpha|\leq b}\|\Omega^\alpha f\|_{H^a}. \tag{1.30}$$

[3]The conditions $g_{00} = -1$ and $g_{0i} = 0$ hold only on the initial hypersurface and are not propagated by the flow. They are imposed mostly for convenience and do not play a significant role in the analysis.

We also define the weighted Sobolev spaces $H^{a,b}_{S,wa}$ and $H^{a,b}_{S,kg}$ by the norms

$$\|f\|_{H^{a,b}_{S,wa}} := \sum_{|\beta'|\leq|\beta|\leq b} \|x^{\beta'}\partial^\beta f\|_{H^a}, \qquad \|f\|_{H^{a,b}_{S,kg}} := \sum_{|\beta|,|\beta'|\leq b} \|x^{\beta'}\partial^\beta f\|_{H^a}, \tag{1.31}$$

where $x^{\beta'} = x_1^{\beta'_1}x_2^{\beta'_2}x_3^{\beta'_3}$ and $\partial^\beta := \partial_1^{\beta_1}\partial_2^{\beta_2}\partial_3^{\beta_3}$. Notice that $H^{a,b}_{S,kg} \hookrightarrow H^{a,b}_{S,wa} \hookrightarrow H^{a,b}_{\Omega} \hookrightarrow H^a$.

We are now ready to state our first main theorem.

THEOREM 1.3. Let $\Sigma_0 := \{(x,t) \in \mathbb{R}^4 : t = 0\}$ and assume that we are given an initial data set $(\overline{g}_{ij}, k_{ij}, \psi_0, \psi_1)$ on Σ_0 satisfying the constraint equations (1.29) and the smallness conditions

$$\begin{aligned}\sum_{n=0}^{3}\sum_{i,j=1}^{3} \{\| |\nabla|^{1/2+\delta/4}(\overline{g}_{ij}-\delta_{ij})\|_{H^{N(n),n}_{S,wa}} + \| |\nabla|^{-1/2+\delta/4}k_{ij}\|_{H^{N(n),n}_{S,wa}}\} \\ + \sum_{n=0}^{3}\{\|\langle\nabla\rangle\psi_0\|_{H^{N(n),n}_{S,kg}} + \|\psi_1\|_{H^{N(n),n}_{S,kg}}\} \leq \varepsilon_0 \leq \overline{\varepsilon}.\end{aligned} \tag{1.32}$$

Here $N_0 := 40$, $d := 10$, $\delta := 10^{-10}$, $N(0) := N_0 + 16d$, $N(n) = N_0 - nd$ for $n \geq 1$, and $\overline{\varepsilon}$ is a sufficiently small constant.

(i) Then the reduced Einstein-Klein-Gordon system

$$\begin{aligned}\widetilde{\Box}_g g_{\alpha\beta} + 2\partial_\alpha\psi\partial_\beta\psi + \psi^2 g_{\alpha\beta} - F^{\geq 2}_{\alpha\beta}(g,\partial g) = 0, \\ \widetilde{\Box}_g\psi - \psi = 0,\end{aligned} \tag{1.33}$$

admits a unique global solution (g,ψ) in $M := \{(x,t) \in \mathbb{R}^4 : t \geq 0\}$, with initial data $(\overline{g}_{ij}, k_{ij}, \psi_0, \psi_1)$ on Σ_0 (as described in (1.27)). Here $F^{\geq 2}_{\alpha\beta}(g,\partial g)$ are as in (2.5) and $\widetilde{\Box}_g = g^{\mu\nu}\partial_\mu\partial_\nu$. The solution satisfies the harmonic gauge conditions

$$0 = \mathbf{\Gamma}_\mu = g^{\alpha\beta}\partial_\alpha g_{\beta\mu} - \frac{1}{2}g^{\alpha\beta}\partial_\mu g_{\alpha\beta}, \qquad \mu \in \{0,1,2,3\} \tag{1.34}$$

in $\mathbb{R}^3 \times [0,\infty)$. Moreover, the metric g stays close and converges as $t \to \infty$ to the Minkowski metric and ψ stays small and converges to 0 as $t \to \infty$ (in suitable norms).

(ii) In view of Proposition 1.1, the pair (g,ψ) is a global[4] solution in M of the Einstein-Klein-Gordon coupled system

$$R_{\alpha\beta} - \mathbf{D}_\alpha\psi\mathbf{D}_\beta\psi - \frac{\psi^2}{2}g_{\alpha\beta} = 0, \qquad \Box_g\psi - \psi = 0. \tag{1.35}$$

The system (1.33) is a quasilinear system of hyperbolic and dispersive equations. One of the key difficulties in the analysis comes from the fact that we have a genuine system in the sense that the linear evolution admits different speeds of propagation, corresponding to wave and Klein-Gordon propagation. As a result the set of "characteristics" is more involved and one has a more limited set of geometric symmetries (vector-fields).

The proof of Theorem 1.3 is based on a complex bootstrap argument, involving energy estimates, vector-fields, Fourier analysis, and nonlinear scattering. We outline some of the main elements of this argument in section 2 below.

[4]In our geometric context, globality means that all future directed timelike and null geodesics starting from points in M extend forever with respect to their affine parametrization.

1.4. *Remarks.* We conclude this section with some additional comments and references.

(1). *Small data global regularity theorems.* A classical question in evolution PDEs is the question of global stability of physical solutions of hyperbolic and dispersive systems. Several important techniques have been developed over the years in the study of such problems, starting with seminal contributions of John, Klainerman, Shatah, Simon, Christodoulou, Alinhac, and Delort [1,2,10,11,13,14,34,35,39–42,56,57]. These include the vector-field method, the normal form method, and the isolation of null structures.

In the last few years new methods have emerged in the study of global solutions of quasilinear evolutions, inspired mainly by the advances in semilinear theory. The basic idea is to combine the classical energy and vector-fields methods with refined analysis of the Duhamel formula, using the Fourier transform. This is the essence of the "method of space-time resonances" of Germain-Masmoudi-Shatah [22,23] and Gustafson-Nakanishi-Tsai [26], and the refinements by the authors and their collaborators in [15,16,24,25,28, 29,32,33,38], using atomic decompositions and more sophisticated norms.

According to this general philosophy, to prove Theorem 1.3 we work both in the physical space and the Fourier space. Our goal is to prove simultaneously high order energy estimates (including vector-fields), modified scattering, and decay of the solutions over time.

(2). *Vector-fields.* In the proof of Theorem 1.3 we use the Lorentz vector-fields Γ_a and the rotation vector-fields Ω_{ab}

$$\Gamma_a := x_a\partial_t + t\partial_a, \qquad \Omega_{ab} := x_a\partial_b - x_b\partial_a, \tag{1.36}$$

for $a, b \in \{1, 2, 3\}$. These vector-fields commute with both the wave operator and the Klein-Gordon operator in the flat Minkowski space (thus with the linear part of the system (1.33)). For any $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in (\mathbb{Z}_+)^3$ we define

$$\partial^\alpha := \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}, \qquad \Omega^\alpha := \Omega_{23}^{\alpha_1}\Omega_{31}^{\alpha_2}\Omega_{12}^{\alpha_3}, \qquad \Gamma^\alpha := \Gamma_1^{\alpha_1}\Gamma_2^{\alpha_2}\Gamma_3^{\alpha_3}. \tag{1.37}$$

For any $n, q \in \mathbb{Z}_+$ we define $\mathcal{V}_n^q$ as the set of differential operators of the form

$$\mathcal{V}_n^q := \big\{\mathcal{L} = \Gamma^a\Omega^b : |a| + |b| \le n,\, q(\mathcal{L}) := |a| \le q\big\}. \tag{1.38}$$

Here $q(\mathcal{L})$ denotes the number of vector-fields transversal to the surfaces $\Sigma_a := \{(x,t) \in \mathbb{R}^3 \times \mathbb{R} : t = a\}$. We remark that in our proof we distinguish between the Lorentz vector-fields Γ (which are transversal to the surfaces Σ_a and lead to slightly faster growth rates; see the definition (2.37)) and the rotational vector-fields Ω.

We also point out an important difference that appears when one considers a massive scalar field in that the *scaling vector-field*, $S = t\partial_t + x\cdot\nabla_x$ no longer satisfies nice commutation properties with the linearized system. Thus in this case, one has fewer vector-fields, which leads to weaker estimates and makes a method purely based on an energy estimate for vector-fields very challenging.

(3). *Stability results in General Relativity.* Global stability of physical solutions is an important topic in General Relativity. For example, the global nonlinear stability of the Minkowski space-time among solutions of the Einstein-vacuum equation is a central theorem in the field, due to Christodoulou-Klainerman [11]. See also the more recent proofs and extensions of Klainerman-Nicolò [43], Lindblad-Rodnianski [53], Bieri-Zipser [5], and Speck [58].

More recently, small data global regularity theorems have also been proved for other coupled Einstein field equations. For example, for the Einstein-Vlasov system this was done in recent work by Fajman-Joudioux-Smulevici [17] and Lindblad-Taylor [55], at least for certain classes of "restricted data" (see the remark below for a longer discussion).

The Einstein-Klein-Gordon system was also considered recently by LeFloch-Ma [49], who proved small data global regularity for restricted data, which agrees with a Schwarzschild solution with small mass outside a compact set. A similar result was announced by Wang [61].

Our main goals in this paper are (1) to work with general unrestricted small initial data, and (2) develop the full asymptotic analysis of the space-time. A similar global regularity result for general small data was announced recently by LeFloch-Ma [50].

In a different direction, one can also raise the question of linear and nonlinear stability of other physical solutions of the Einstein equations. Stability of the Kerr family of solutions has been under intense study in recent years, first at the linearized level (see for example [12] and the references therein) and more recently at the full nonlinear level (see [27] and [45]). In the case of the Einstein-Klein-Gordon system, we refer to [8,9,60] for recent results on stability of other space-times.

(4). *Restricted initial data.* In some cases one can simplify considerably the global analysis of wave and Klein-Gordon equations by considering initial data of compact support. The point is that the solutions have the finite speed of propagation, therefore they remain supported inside a light cone, and one can use the hyperbolic foliation method to analyze the evolution. See [47] for a recent account of this method and its refinements.

To implement this method one needs to first have control of the solution on an initial hyperboloid, and then propagate this control to the interior region. As a result, this approach is restricted to the case when one can establish such good control on an initial hyperboloid. Due to the finite speed of propagation, this is possible in the case of compactly supported data (in the case of systems of wave or Klein-Gordon equations), or data that agrees with the Schwarzschild solution S_m outside a compact set (in the case of the Einstein equations).

The use of "restricted initial data", sometimes coupled with the hyperbolic foliation method, leads to significant simplifications of the global analysis, particularly at the level of proving decay. In the context of the Einstein equations these ideas have been used by many authors, such as Friedrich [19], Lindblad-Rodnianski [53], Fajman-Joudioux-Smulevici [17], Lindblad-Taylor [55], LeFloch-Ma [49], Wang [61], and Klainerman-Szeftel [45].

(5). *Simplified wave-Klein-Gordon models.* The system (1.33) is complicated, but one can gain intuition by looking at simpler models. For example, to understand the nonlinear coupling of wave and Klein-Gordon fields, one can consider the simplified system

$$\begin{aligned} -\Box u &= A^{\alpha\beta}\partial_\alpha v\partial_\beta v + Dv^2, \\ (-\Box + 1)v &= uB^{\alpha\beta}\partial_\alpha\partial_\beta v + Euv, \end{aligned} \tag{1.39}$$

where u, v are real-valued functions, and $A^{\alpha\beta}$, $B^{\alpha\beta}$, D, and E are real constants. Without loss of generality one may assume that $A^{\alpha\beta} = A^{\beta\alpha}$ and $B^{\alpha\beta} = B^{\beta\alpha}$, $\alpha, \beta \in \{0, 1, 2, 3\}$.

The system (1.39) was derived by LeFloch-Ma [48] as a model for the full Einstein-Klein-Gordon system (1.33). Intuitively, the deviation of the Lorentzian metric g from the Minkowski metric is replaced by a scalar function u, and the massive scalar field ψ is replaced by v. The system (1.39) keeps the same linear structure as the reduced Einstein-Klein-Gordon system (1.33), but only keeps, schematically, quadratic interactions that involve the massive scalar field (the semilinear terms in the first equation and the quasilinear terms in the second equation coming from the reduced wave operator).

Small data global regularity for the system (1.39) was proved by LeFloch-Ma [48] in the case of compactly supported initial data (the restricted data case), using the hyperbolic foliation method. For general small initial data, global regularity was proved by the authors [30].

A similar system, the massive Maxwell-Klein-Gordon system, was analyzed recently by Klainerman–Wang–Yang [46], who also proved global regularity for general small initial data, using a different method.

(6). *Initial data assumptions.* The precise form of the smallness assumptions (1.32) on the metric initial data $\overline{g}_{ij}$ and k_{ij} is important. Indeed, in view of the positive mass theorem, one expects the metric components $\overline{g}_{ij} - \delta_{ij}$ to decay like $M/\langle x\rangle$ and the second fundamental form k to decay like $M/\langle x\rangle^2$, where $M \ll 1$ is the ADM mass. Capturing this type of decay, using L^2 based norms, is precisely the role of the homogeneous multipliers $|\nabla|^{1/2+\delta/4}$ and $|\nabla|^{-1/2+\delta/4}$ in (1.32). Notice that these multipliers are essentially sharp, up to the $\delta/4$ power.

Our assumptions on the metric are essentially of the type

$$g_{ij} = \delta_{ij} + \varepsilon_0 O(\langle x\rangle^{-1+\delta/4}), \qquad k_{ij} = \varepsilon_0 O(\langle x\rangle^{-2+\delta/4}), \tag{1.40}$$

at time $t = 0$. These are less restrictive than the assumptions used sometimes even in the vacuum case $\psi \equiv 0$, see for example [11], [43], or [54], in the sense that the initial data is not assumed to agree with the Schwarzschild initial data up to lower order terms. They are more restrictive, however, than the assumptions of Bieri [5] in the case the initial time slice is maximal, but we are able to prove more precise asymptotic bounds on the metric and the Riemann curvature tensor (see section 3), and make no additional assumption on the immersion $\Sigma_0 \hookrightarrow M$.

We remark also that our assumptions (1.32) allow for anisotropic initial data, possibly with different "masses" in different directions. For the vacuum case, initial data of this type, satisfying the constraint equations, have been constructed recently by Carlotto-Schoen [7].

(7). *The mini-bosons.* A general obstruction to small data global stability theorems is the presence of nondecaying "small" solutions, such as small solitons. A remarkable fact is that there are such small nondecaying solutions for the Einstein-Klein-Gordon system, namely the so-called mini-boson stars. These are time-periodic (therefore nondecaying) and spherically symmetric exact solutions of the Einstein-Klein-Gordon system. They were discovered numerically by physicists, such as Friedberg–Lee–Pang [18] (see also [51]), and then constructed rigorously by Bizoń–Wasserman [6].

These mini-bosons can be thought of as arbitrarily small (hence the name) in certain topologies, as explained in [6]. However, these solutions (in particular the Klein-Gordon component) are not small in the stronger topology we use here, as described by (1.32), so we can avoid them in our analysis.

2. Global dynamics and modified scattering. The global regularity conclusion of Theorem 1.3 is essentially a qualitative statement. To prove it we need to make it precise and quantitative. For this we need several ingredients: a Hodge decomposition of the metric tensor, the definition of linear profiles, suitable weighted norms and a special Z-norm, nonlinear phase corrections, and nonlinear profiles. We summarize these constructions in this section, and provide a more precise version of the main theorem.

2.1. *Decomposition of the metric tensor.* Let m denote the Minkowski metric and write

$$g_{\alpha\beta} = m_{\alpha\beta} + h_{\alpha\beta}, \qquad g^{\alpha\beta} = m^{\alpha\beta} + g^{\alpha\beta}_{\geq 1}. \tag{2.1}$$

It follows from (1.33) that for $\alpha, \beta \in \{0, 1, 2, 3\}$ we have

$$(\partial_0^2 - \Delta) h_{\alpha\beta} = \mathcal{N}^h_{\alpha\beta} := \mathcal{KG}_{\alpha\beta} + g^{\mu\nu}_{\geq 1}\partial_\mu\partial_\nu h_{\alpha\beta} - F^{\geq 2}_{\alpha\beta}(g, \partial g), \tag{2.2}$$

where $F^{\geq 2}_{\alpha\beta}(g, \partial g)$ are as in (2.5) and

$$\mathcal{KG}_{\alpha\beta} := 2\partial_\alpha\psi\partial_\beta\psi + \psi^2(m_{\alpha\beta} + h_{\alpha\beta}). \tag{2.3}$$

Moreover

$$(\partial_0^2 - \Delta + 1)\psi = \mathcal{N}^\psi := g^{\mu\nu}_{\geq 1}\partial_\mu\partial_\nu\psi. \tag{2.4}$$

For global analysis we need to understand well the terms $F^{\geq 2}_{\alpha\beta}(g, \partial g)$, which contain the semilinear wave interactions. These terms can be simplified and decomposed into classical null forms (most terms) and a small number of terms that only obey a weaker form of null structure. This structure is important in the global analysis of the wave equations for the metric components, as it is well known that semilinear quadratic terms that have no structure could lead to finite time blowup of solutions in 3D. The precise statement is the following.

LEMMA 2.1. In wave coordinates $\mathbf{\Gamma}_\mu = 0$ we have

$$F^{\geq 2}_{\alpha\beta}(g, \partial g) = Q_{\alpha\beta} + P_{\alpha\beta}, \tag{2.5}$$

where

$$\begin{aligned} Q_{\alpha\beta} = {} & g^{\rho\rho'}g^{\lambda\lambda'}(\partial_\alpha h_{\rho'\lambda'}\partial_\rho h_{\beta\lambda} - \partial_\rho h_{\rho'\lambda'}\partial_\alpha h_{\beta\lambda}) + g^{\rho\rho'}g^{\lambda\lambda'}(\partial_\beta h_{\rho'\lambda'}\partial_\rho h_{\alpha\lambda} - \partial_\rho h_{\rho'\lambda'}\partial_\beta h_{\alpha\lambda}) \\ & + \frac{1}{2}g^{\rho\rho'}g^{\lambda\lambda'}(\partial_{\lambda'} h_{\rho\rho'}\partial_\beta h_{\alpha\lambda} - \partial_\beta h_{\rho\rho'}\partial_{\lambda'} h_{\alpha\lambda}) + \frac{1}{2}g^{\rho\rho'}g^{\lambda\lambda'}(\partial_{\lambda'} h_{\rho\rho'}\partial_\alpha h_{\beta\lambda} - \partial_\alpha h_{\rho\rho'}\partial_{\lambda'} h_{\beta\lambda}) \\ & - g^{\rho\rho'}g^{\lambda\lambda'}(\partial_\lambda h_{\alpha\rho'}\partial_\rho h_{\beta\lambda'} - \partial_\rho h_{\alpha\rho'}\partial_\lambda h_{\beta\lambda'}) + g^{\rho\rho'}g^{\lambda\lambda'}\partial_{\rho'} h_{\alpha\lambda'}\partial_\rho h_{\beta\lambda} \end{aligned} \tag{2.6}$$

and

$$P_{\alpha\beta} = -\frac{1}{2}g^{\rho\rho'}g^{\lambda\lambda'}\partial_\alpha h_{\rho'\lambda'}\partial_\beta h_{\rho\lambda} + \frac{1}{4}g^{\rho\rho'}g^{\lambda\lambda'}\partial_\alpha h_{\rho\rho'}\partial_\beta h_{\lambda\lambda'}. \tag{2.7}$$

Notice that the quadratic part of $Q_{\alpha\beta}$, obtained by replacing g^{ab} with m^{ab} everywhere, is a sum of classical null forms in the variables $h_{\mu\nu}$.

The different components of the tensor $h_{\alpha\beta}$ evolve differently as $t \to \infty$, due to the weak null structure of the system (2.2). To identify and take advantage of this weak null structure we need to decompose the tensor $h_{\alpha\beta}$.

A standard way to decompose the metric tensor in General Relativity is based on the use of null frames (see for instance [11] or [53]). Here we use a different decomposition of the metric tensor, reminiscent of the div − curl decomposition of vector-fields in fluid models, which has the advantage of being more compatible with commutation with the vector-fields ∂_j, Γ_j, Ω_j. This decomposition is connected to the classical work of Arnowitt–Deser–Misner [3] on the Hamiltonian formulation of General Relativity.

More precisely, let $R_j = |\nabla|^{-1}\partial_j$, $j \in \{1,2,3\}$ denote the Riesz transforms on $\mathbb{R}^3$, and notice that $\delta_{jk}R_jR_k = -I$. We use a double Hodge decomposition for the metric tensor. Let

$$\begin{aligned} F &:= (1/2)[h_{00} + R_jR_kh_{jk}],\\ \underline{F} &:= (1/2)[h_{00} - R_jR_kh_{jk}],\\ \rho &:= R_jh_{0j},\\ \omega_j &:= \in_{jkl} R_kh_{0l},\\ \Omega_j &:= \in_{jkl} R_kR_mh_{lm},\\ \vartheta_{jk} &:= \in_{jmp}\in_{knq} R_mR_nh_{pq}. \end{aligned} \tag{2.8}$$

Notice that ω and Ω are divergence-free vector-fields,

$$R_j\omega_j = 0, \qquad R_j\Omega_j = 0, \tag{2.9}$$

and ϑ is a symmetric and divergence-free tensor-field,

$$\vartheta_{jk} = \vartheta_{kj}, \qquad R_j\vartheta_{jk} = 0, \qquad R_k\vartheta_{jk} = 0. \tag{2.10}$$

Moreover, using the general formula $\in_{mnk}\in_{pqk}= \delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np}$ one can recover the tensor h according to the formulas

$$\begin{aligned} &h_{00} = F + \underline{F},\\ &h_{0j} = -R_j\rho + \in_{jkl} R_k\omega_l,\\ &h_{jk} = R_jR_k(F - \underline{F}) - (\in_{klm} R_j + \in_{jlm} R_k)R_l\Omega_m + \in_{jpm}\in_{kqn} R_pR_q\vartheta_{mn}. \end{aligned} \tag{2.11}$$

We define also the associated nonlinearities

$$\begin{aligned} \mathcal{N}^F &:= (1/2)[\mathcal{N}^h_{00} + R_jR_k\mathcal{N}^h_{jk}],\\ \mathcal{N}^{\underline{F}} &:= (1/2)[\mathcal{N}^h_{00} - R_jR_k\mathcal{N}^h_{jk}],\\ \mathcal{N}^\rho &:= R_j\mathcal{N}^h_{0j},\\ \mathcal{N}^\omega_j &:= \in_{jkl} R_k\mathcal{N}^h_{0l},\\ \mathcal{N}^\Omega_j &:= \in_{jkl} R_kR_m\mathcal{N}^h_{lm},\\ \mathcal{N}^\vartheta_{jk} &:= \in_{jmp}\in_{knq} R_mR_n\mathcal{N}^h_{pq}, \end{aligned} \tag{2.12}$$

compare with the definitions (2.8), and notice that

$$(\partial_0^2 - \Delta)G = \mathcal{N}^G \qquad \text{for any } G \in \{F, \underline{F}, \rho, \omega_j, \Omega_j, \vartheta_{jk}\}. \tag{2.13}$$

As a consequence of the harmonic gauge condition, the main dynamical variables are $F, \underline{F}, \omega_j$ and the traceless part of ϑ_{jk}, while the variables ρ and Ω_j can be expressed elliptically in terms of these main variables, up to quadratic remainders (see Lemma 2.2 below).

More precisely, the harmonic gauge condition (1.34) gives

$$m^{\alpha\beta}\partial_\alpha h_{\beta\mu} - \frac{1}{2}m^{\alpha\beta}\partial_\mu h_{\alpha\beta} = E_\mu^{\geq 2} := -g_{\geq 1}^{\alpha\beta}\partial_\alpha h_{\beta\mu} + \frac{1}{2}g_{\geq 1}^{\alpha\beta}\partial_\mu h_{\alpha\beta}. \tag{2.14}$$

Let $R_0 := |\nabla|^{-1}\partial_t$ and

$$\tau := (1/2)[\delta_{jk}h_{jk} + R_j R_k h_{jk}] = -(1/2)\delta_{jk}\vartheta_{jk}, \qquad \mathcal{N}^\tau := -(1/2)\delta_{jk}\mathcal{N}_{jk}^\vartheta. \tag{2.15}$$

LEMMA 2.2. With the definitions (2.8), the variables ρ, Ω_j satisfy the elliptic-type identities

$$\begin{aligned} \rho &= R_0\underline{F} + R_0\tau + |\nabla|^{-1}E_0^{\geq 2}, \\ \Omega_j &= R_0\omega_j + |\nabla|^{-1} \in_{jlk} R_l E_k^{\geq 2}. \end{aligned} \tag{2.16}$$

Moreover, the variable τ is quadratic, i.e.,

$$\begin{aligned} 2|\nabla|^2\tau &= \partial_0 E_0^{\geq 2} + \partial_k E_k^{\geq 2} + \mathcal{N}^{\underline{F}} + \mathcal{N}^\tau, \\ 2|\nabla|\partial_0\tau &= -|\nabla|E_0^{\geq 2} + R_k\partial_0 E_k^{\geq 2} + \mathcal{N}^\rho. \end{aligned} \tag{2.17}$$

The identities in the lemma follow easily from the definitions (2.8) and (2.14). These identities are important in identifying the weak null structures of the metric nonlinearities $\mathcal{N}^G$ defined in (2.12), arising from the terms $P_{\alpha\beta}$ in (2.7).

2.2. *Linear and nonlinear profiles.* Much of our analysis is based on proving estimates on the *linear profiles* of the solutions. Profiles at time t are constructed by going forward in time up to time t according to the nonlinear evolution equations (2.2) and (2.4), and then going back in time using the linear flow. Therefore the linear profiles at time t measure the cumulative effect of the nonlinearities over the interval $[0, t]$.

More precisely, we define the *normalized solutions* $U^{h_{\alpha\beta}}$, U^F, $U^{\underline{F}}$, U^ρ, U^{ω_a}, U^{Ω_a}, $U^{\vartheta_{ab}}$, U^ψ and their associated *linear profiles* $V^{h_{\alpha\beta}}$, V^F, $V^{\underline{F}}$, V^ρ, V^{ω_a}, V^{Ω_a}, $V^{\vartheta_{ab}}$, V^ψ, $\alpha, \beta \in \{0, 1, 2, 3\}$, $a, b \in \{1, 2, 3\}$, by

$$\begin{aligned} &U^G(t) := \partial_t G(t) - i\Lambda_{wa}G(t), \quad V^G(t) := e^{it\Lambda_{wa}}U^G(t), \quad G \in \{h_{\alpha\beta}, F, \underline{F}, \rho, \omega_a, \Omega_a, \vartheta_{ab}\}, \\ &U^\psi(t) := \partial_t\psi(t) - i\Lambda_{kg}\psi(t), \qquad V^\psi(t) := e^{it\Lambda_{kg}}U^\psi(t), \end{aligned} \tag{2.18}$$

where $\Lambda_{wa} = |\nabla|$ and $\Lambda_{kg} = \sqrt{1 + |\nabla|^2}$. More generally, for $\mathcal{L} \in \mathcal{V}_3^3$ (see definition (1.38)) we define the *weighted linear profiles*

$$\begin{aligned} U^{\mathcal{L}h_{\alpha\beta}}(t) &:= (\partial_t - i\Lambda_{wa})(\mathcal{L}h_{\alpha\beta})(t), \qquad V^{\mathcal{L}h_{\alpha\beta}}(t) := e^{it\Lambda_{wa}}U^{\mathcal{L}h_{\alpha\beta}}(t), \\ U^{\mathcal{L}\psi}(t) &:= (\partial_t - i\Lambda_{kg})(\mathcal{L}\psi)(t), \qquad V^{\mathcal{L}\psi}(t) := e^{it\Lambda_{kg}}U^{\mathcal{L}\psi}(t). \end{aligned} \tag{2.19}$$

Notice that we only apply the differential operators $\mathcal{L}$ to the metric components $h_{\alpha\beta}$, but not to the variables $F, \underline{F}, \rho, \omega_a, \Omega_a, \vartheta_{ab}$. Also, for $* \in \{F, \underline{F}, \rho, \omega_a, \Omega_a, \vartheta_{ab}, \mathcal{L}h_{\alpha\beta}, \mathcal{L}\psi\}$, $\mathcal{L} \in \mathcal{V}_3^3$, we define

$$U^{*,+} := U^*, \qquad U^{*,-} := \overline{U^*}, \qquad V^{*,+} := V^*, \qquad V^{*,-} := \overline{V^*}. \tag{2.20}$$

The functions $\mathcal{L}h_{\alpha\beta}, \mathcal{L}\psi, F, \underline{F}, \rho, \omega_a, \Omega_a, \vartheta_{ab}$, can be recovered linearly from the normalized variables U^F, $U^{\underline{F}}$, U^ρ, U^{ω_a}, U^{Ω_a}, $U^{\vartheta_{ab}}$, $U^{\mathcal{L}h_{\alpha\beta}}$, $U^{\mathcal{L}\psi}$ by the formulas

$$\begin{aligned}&\partial_0 G = (U^G + \overline{U^G})/2, \qquad \Lambda_{wa} G = i(U^G - \overline{U^G})/2, \qquad G \in \{F, \underline{F}, \rho, \omega_a, \Omega_a, \vartheta_{ab}, \mathcal{L}h_{\alpha\beta}\},\\ &\partial_0 \mathcal{L}\psi = (U^{\mathcal{L}\psi} + \overline{U^{\mathcal{L}\psi}})/2, \qquad \Lambda_{kg} \mathcal{L}\psi = i(U^{\mathcal{L}\psi} - \overline{U^{\mathcal{L}\psi}})/2.\end{aligned} \tag{2.21}$$

2.2.1. *Renormalization and nonlinear profiles.* The linear profiles defined above are accurate enough for most estimates, but fail to converge as $t \to \infty$. For our proof it is important to understand this issue. We need to introduce an additional nonlinear correction, define nonlinear profiles, and prove *modified scattering*.

We start from the equation $\partial_t V^{h_{\alpha\beta}} = e^{it\Lambda_{wa}} \mathcal{N}^h_{\alpha\beta}$ for the profile $V^{h_{\alpha\beta}}$, which follows from (2.2) and the definitions. To extract the nonlinear phase correction we need to examine only the quasilinear quadratic part of the nonlinearity, which is

$$\mathcal{Q}^2_{\alpha\beta} := \{-h_{00}\Delta + 2h_{0j}\partial_0\partial_j - h_{jk}\partial_j\partial_k\} h_{\alpha\beta}. \tag{2.22}$$

Using the definitions, in the Fourier space this becomes

$$\begin{aligned}&e^{it\Lambda_{wa}(\xi)} \widehat{\mathcal{Q}^2_{\alpha\beta}}(\xi, t)\\ &\quad = \frac{1}{(2\pi)^3} \sum_{\pm} \int_{\mathbb{R}^3} i e^{it\Lambda_{wa}(\xi)} e^{\mp it\Lambda_{wa}(\xi-\eta)} \widehat{V^{h_{\alpha\beta},\pm}}(\xi-\eta, t) \mathfrak{q}_{wa,\pm}(\xi-\eta, \eta, t)\, d\eta\end{aligned} \tag{2.23}$$

where

$$\mathfrak{q}_{wa,\pm}(\rho, \eta, t) := \pm \widehat{h_{00}}(\eta, t) \frac{\Lambda_{wa}(\rho)}{2} + \widehat{h_{0j}}(\eta, t)\rho_j \pm \widehat{h_{jk}}(\eta, t) \frac{\rho_j \rho_k}{2\Lambda_{wa}(\rho)}. \tag{2.24}$$

The main contribution comes from low frequencies η and be approximated, heuristically, by

$$\begin{aligned}&\frac{1}{(2\pi)^3} \int_{|\eta| \ll \langle t\rangle^{-1/2}} i e^{it\Lambda_{wa}(\xi)} e^{-it\Lambda_{wa}(\xi-\eta)} \widehat{V^{h_{\alpha\beta},+}}(\xi-\eta, t) \mathfrak{q}_{wa,+}(\xi-\eta, \eta, t)\, d\eta\\ &\approx i \frac{\widehat{V^{h_{\alpha\beta},+}}(\xi, t)}{(2\pi)^3} \int_{|\eta| \ll \langle t\rangle^{-1/2}} e^{it\eta\cdot\nabla\Lambda_{wa}(\xi)} \Big\{ \widehat{h_{00}}(\eta, t) \frac{\Lambda_{wa}(\xi)}{2}\\ &\qquad\qquad + \widehat{h_{0j}}(\eta, t)\xi_j + \widehat{h_{jk}}(\eta, t) \frac{\xi_j\xi_k}{2\Lambda_{wa}(\xi)} \Big\}\, d\eta\\ &\approx i \widehat{V^{h_{\alpha\beta}}}(\xi, t) \Big\{ h^{low}_{00}(t\xi/\Lambda_{wa}(\xi), t) \frac{\Lambda_{wa}(\xi)}{2}\\ &\qquad\qquad + h^{low}_{0j}(t\xi/\Lambda_{wa}(\xi), t)\xi_j + h^{low}_{jk}(t\xi/\Lambda_{wa}(\xi), t) \frac{\xi_j\xi_k}{2\Lambda_{wa}(\xi)} \Big\}\end{aligned}$$

where $h^{low}_{\alpha\beta}$ are suitable low frequency components of $h_{\alpha\beta}$.

We can now define precisely our nonlinear phase correction and nonlinear profiles. Let

$$\widehat{h^{low}_{\alpha\beta}}(\rho, s) := \varphi_{\leq 0}(\langle s\rangle^{p_0}\rho) \widehat{h_{\alpha\beta}}(\rho, s), \qquad p_0 := 0.68. \tag{2.25}$$

The choice of p_0, slightly bigger than $2/3$, is important in the proof to justify these approximations rigorously. Then we define the wave phase correction

$$\Theta_{wa}(\xi,t) := \int_0^t \Big\{ h^{low}_{00}(s\xi/\Lambda_{wa}(\xi),s)\frac{\Lambda_{wa}(\xi)}{2} + h^{low}_{0j}(s\xi/\Lambda_{wa}(\xi),s)\xi_j + h^{low}_{jk}(s\xi/\Lambda_{wa}(\xi),s)\frac{\xi_j\xi_k}{2\Lambda_{wa}(\xi)} \Big\}\, ds \tag{2.26}$$

and the nonlinear (modified) metric profiles V^G_*, $G \in \{h_{\alpha\beta}, F, \omega_a, \vartheta_{ab}\}$ by

$$\widehat{V^G_*}(\xi,t) := e^{-i\Theta_{wa}(\xi,t)}\widehat{V^G}(\xi,t). \tag{2.27}$$

The construction is similar in the case of the Klein-Gordon field, so we define

$$\Theta_{kg}(\xi,t) := \int_0^t \Big\{ h^{low}_{00}(s\xi/\Lambda_{kg}(\xi),s)\frac{\Lambda_{kg}(\xi)}{2} + h^{low}_{0j}(s\xi/\Lambda_{kg}(\xi),s)\xi_j + h^{low}_{jk}(s\xi/\Lambda_{kg}(\xi),s)\frac{\xi_j\xi_k}{2\Lambda_{kg}(\xi)} \Big\}\, ds \tag{2.28}$$

and the nonlinear (modified) Klein-Gordon profile V^ψ_*

$$\widehat{V^\psi_*}(\xi,t) := e^{-i\Theta_{kg}(\xi,t)}\widehat{V^\psi}(\xi,t). \tag{2.29}$$

Geometrically, the two phase corrections Θ_{wa} and Θ_{kg} are obtained by integrating suitable low frequency components of the metric tensor along the characteristics of the wave and the Klein-Gordon linear flows. The nonlinear profiles are obtained by multiplying, in the Fourier space, the linear profiles by the oscillatory factors $e^{-i\Theta_{wa}(\xi,t)}$ and $e^{-i\Theta_{kg}(\xi,t)}$ (which are bounded since the phases Θ_{wa} and Θ_{kg} are real-valued).

The point of this construction is that the new nonlinear profiles V^F_*, $V^{\omega_a}_*$, $V^{\vartheta_{ab}}_*$, and V^ψ_* converge as the time goes to infinity to the *nonlinear scattering data* (see Theorem 2.4(ii)).

2.3. *Quantitative version of the main theorem.* To state a precise version of our main theorem we need a few parameters

$$N_0 := 40, \quad d := 10, \quad \kappa := 10^{-3}, \quad \delta := 10^{-10}, \quad \delta' := 2000\delta, \quad \gamma := \delta/4. \tag{2.30}$$

We define also the numbers $N(n)$ (which measure the number of Sobolev derivatives under control at the level of n vector-fields),

$$N(0) := N_0 + 16d, \qquad N(n) := N_0 - dn \text{ for } n \in \{1,2,3\}. \tag{2.31}$$

Let $|\xi|_{\leq 1}$ denote a smooth increasing radial function on $\mathbb{R}^3$ equal to $|\xi|$ if $|\xi| \leq 1/2$ and equal to 1 if $|\xi| \geq 2$. Let $|\nabla|^\theta_{\leq 1}$ denote the associated operator defined by the multiplier $\xi \to |\xi|^\theta_{\leq 1}$.

We are now ready to define the main Z-norms.

DEFINITION 2.3. For any $x \in \mathbb{R}$ let $x^+ = \max(x,0)$ and $x^- = \min(x,0)$. We define the spaces Z_{wa} and Z_{kg} by the norms

$$\|f\|_{Z_{wa}} := \sup_{k\in\mathbb{Z}} 2^{N_0 k^+} 2^{k^-(1+\kappa)} \|\widehat{P_k f}\|_{L^\infty} \tag{2.32}$$

and

$$\|f\|_{Z_{kg}} := \sup_{k\in\mathbb{Z}} 2^{N_0k^+}2^{k^-(1/2-\kappa)}\|\widehat{P_kf}\|_{L^\infty}, \tag{2.33}$$

where P_k denote Littlewood-Paley projections to frequencies of order 2^k on $\mathbb{R}^3$.

Finally, we are ready to state our main quantitative result in [31].

THEOREM 2.4. Assume that (g,ψ) is a global solution of the Einstein-Klein-Gordon system (1.33)-(1.34) as in Theorem 1.3.

(i) Define $U^G, U^{\mathcal{L}h_{\alpha\beta}}, U^{\mathcal{L}\psi}$ as in (2.18)–(2.19) and recall the definitions (1.38). Then,

$$\sup_{n\le 3,\,\mathcal{L}\in\mathcal{V}_n^q} \langle t\rangle^{-H(q,n)\delta}\{\|(\langle t\rangle|\nabla|_{\le 1})^\gamma|\nabla|^{-1/2}U^{\mathcal{L}h_{\alpha\beta}}(t)\|_{H^{N(n)}} + \|U^{\mathcal{L}\psi}(t)\|_{H^{N(n)}}\} \lesssim \varepsilon_0, \tag{2.34}$$

$$\begin{aligned}\sup_{n\le 2,\,\mathcal{L}\in\mathcal{V}_n^q}\ \sup_{k\in\mathbb{Z},\,l\in\{1,2,3\}} & 2^{N(n+1)k^+}\langle t\rangle^{-H(q+1,n+1)\delta}\\ &\{2^{k/2}(2^{k^-}\langle t\rangle)^\gamma\|P_k(x_lV^{\mathcal{L}h_{\alpha\beta}})(t)\|_{L^2} + 2^{k^+}\|P_k(x_lV^{\mathcal{L}\psi})(t)\|_{L^2}\} \lesssim \varepsilon_0,\end{aligned} \tag{2.35}$$

and

$$\|V^F(t)\|_{Z_{wa}} + \|V^{\omega_a}(t)\|_{Z_{wa}} + \|V^{\vartheta_{ab}}(t)\|_{Z_{wa}} + \langle t\rangle^{-\delta}\|V^{h_{\alpha\beta}}(t)\|_{Z_{wa}} + \|V^{\psi}(t)\|_{Z_{kg}} \lesssim \varepsilon_0 \tag{2.36}$$

for any $t\in[0,\infty)$, $\alpha,\beta\in\{0,1,2,3\}$, and $a,b\in\{1,2,3\}$. Here $\langle t\rangle := \sqrt{1+t^2}$ and

$$H(q,n) := \begin{cases} 1 & \text{if } q=0 \text{ and } n=0,\\ 60(n-1)+20 & \text{if } q=0 \text{ and } n\ge 1,\\ 200(n-1)+30 & \text{if } q=1 \text{ and } n\ge 1,\\ 100(q+1)(n-1) & \text{if } q\ge 2.\end{cases} \tag{2.37}$$

(ii) There are functions $V_\infty^F, V_\infty^{\omega_a}, V_\infty^{\vartheta_{ab}} \in Z_{wa}$, and $V_*^\psi \in Z_{kg}$ such that, for any $t\ge 0$,

$$\begin{aligned}\|V_*^F(t) - V_\infty^F\|_{Z_{wa}} + \sum_{a=1}^3 \|V_*^{\omega_a}(t) - V_\infty^{\omega_a}\|_{Z_{wa}} + \sum_{a,b=1}^3 \|V_*^{\vartheta_{ab}}(t) - V_\infty^{\vartheta_{ab}}\|_{Z_{wa}} &\lesssim \varepsilon_0\langle t\rangle^{-\delta/2},\\ \|V_*^\psi(t) - V_\infty^\psi\|_{Z_{kg}} &\lesssim \varepsilon_0\langle t\rangle^{-\delta/2}.\end{aligned} \tag{2.38}$$

Notice that our main theorem provides information on the solution (h,ψ) mostly in the Fourier space. In the next section we will show how to use this information to extract precise bounds on the metric g and the field ψ in the physical space, and prove some of the classical geometric conclusions, such as peeling estimates and Bondi mass estimates.

2.4. *Remarks.* We conclude this section with some discussion of the main theorem.

(1). The proof of Theorem 2.4 in [31] is based on a complex bootstrap argument, in which we assume slightly weaker control of the quantities in (2.34)–(2.36), and improve the bounds using the nonlinear equations.

A closer examination shows that we aim to control, simultaneously, three types of norms: (i) energy norms involving up to 3 vector-fields Γ_a and Ω_{ab}, measured in Sobolev spaces, (ii) weighted norms on the linear profiles $V^{\mathcal{L}h_{\alpha\beta}}$ and $V^{\mathcal{L}\psi}$, and (iii) the Z-norms on the undifferentiated profiles. We discuss each one of these norms in more detail below.

(2). The norms described in (2.34) are our main high order energy norms, using up to 3 vector-fields Γ_a and Ω_{ab} and measured in suitable Sobolev spaces. We notice that all of these energy norms are allowed to grow slowly in time. The energy bounds also have an important low frequency component, involving the operator $|\nabla|^{-1/2+\gamma}$, which is related to the low frequency assumption on the initial data in (1.32) and the expected $M/\langle x\rangle$ decay of the metric tensor as explained in the subsection 1.4.

(3). The function H defined in (2.37) is important, as it establishes a hierarchy of growth of the various energy norms. At the conceptual level this is needed because we define the weighted vector-fields Γ_a, Ω_a in terms of the coordinate functions x_j and t, thus we expect (at least logarithmic) losses as we apply more of these vector-fields.

At the technical level, the growth function H satisfies superlinear inequalities of the form

$$H(q_1, n_1) + H(q_2, n_2) \leq H(q_1 + q_2, n_1 + n_2) - 40,$$

when $n_1, n_2 \geq 1$ and $n_1 + n_2 \leq 3$, and more refined versions. These inequalities are helpful when estimating nonlinear interactions when the vector-fields Γ_a and Ω_a split among the different components.

(4). We notice also that we treat the two types of weighted vector-fields Γ_a and Ω_{ab} differently, in the sense that the application of the nontangential vector-fields Γ_a leads to more loss in terms of time growth than the application of the tangential vector-fields Ω_{ab} (for example $H(0,1) = 20 < H(1,1) = 30$ and similar inequalities hold for higher number of vector-fields). This is a subtle technical point to keep in mind, connected to a more general difficulty of estimating the effect of nontangential vector-fields.

(5). The change over time of the linear profiles $V^{\mathcal{L}h_{\alpha\beta}}$ and $V^{\mathcal{L}\psi}$ measures the accumulated effect of the nonlinearities. The weighted profile norms in (2.35) are an important component of our bootstrap argument. They imply pointwise decay estimates on solutions, of the form

$$\|P_k U^{\mathcal{L}h}(t)\|_{L^\infty} \lesssim \varepsilon_1 \langle t\rangle^{-1+\delta'} 2^{k^-} 2^{-N(n+1)k^+ + 2k^+} \min\{1, \langle t\rangle 2^{k^-}\}^{1-\delta}, \tag{2.39}$$

and

$$\|P_k U^{\mathcal{L}\psi}(t)\|_{L^\infty} \lesssim \varepsilon_1 \langle t\rangle^{-1+\delta'} 2^{k^-/2} 2^{-N(n+1)k^+ + 2k^+} \min\{1, 2^{2k^-}\langle t\rangle\} \tag{2.40}$$

for any $k \in \mathbb{Z}$, $t \geq 0$, $h \in \big\{h_{\alpha\beta} : \alpha, \beta \in \{0,1,2,3\}\big\}$, and $\mathcal{L} \in \mathcal{V}_n^q$, $n \leq 2$. These pointwise bounds and refined versions are useful in nonlinear estimates.

We emphasize, however, that weighted estimates on linear profiles are a lot stronger than pointwise decay estimates on solutions, and serve many other purposes. For example, space localization of the linear profiles gives us the main ingredient we need to decompose the various nonlinear contributions both in frequency and space.

(6). The Z-norm bounds in (2.36) provide the last piece of information needed to close the bootstrap argument. The Z-norm estimates are weaker than both the energy estimates (2.34) and the weighted estimates (2.35) in almost every way, except for one: the Z-norm estimates of some of the components are uniform and do not grow slowly in time.

We notice that the Z-norms are applied only to the undifferentiated profiles, without any of the weighted vector-fields Γ_a and Ω_{ab}. This is consistent with our intuition that application of the weighted vector-fields has to lose small powers of $\langle t\rangle$ because these vector-fields are defined using the Minkowski coordinate functions x and t.

(7). The nonlinear profiles V_*^G are important to understand the global dynamics of the problem. Thankfully, in our case they are not too far from the linear profiles V^G, and can be obtained from these linear profiles simply by multiplication in the Fourier space (see (2.27) and (2.29)). The nonlinear phases Θ_{wa} and Θ_{kg} defined in (2.26) and (2.28) may grow slowly in time as integrals along the characteristic flow.

As a result, the nonlinear profiles V_*^G satisfy similar bounds to the bounds (2.34)–(2.36) satisfied by the linear profiles V^G, possibly with additional $\langle t\rangle^{C\delta}$-type losses.

3. Asymptotic geometry. The results presented in this section are consequences of the global control on the solutions in Theorem 2.4 and in particular are not used in the proof of the main bootstrap argument. As a result, we can afford to be less careful with regularity and decay.

For the sake of clarity, we will always work up to a loss of $\langle t\rangle^\kappa$, instead of the more precise hierarchy of losses in the previous section (see (2.37)). We also simplify the statements of some of the main theorems. See [31] for more precise results and a longer discussion.

We introduce some vector fields and projections

$$L=\partial_t+\partial_r,\qquad \underline{L}=\partial_t-\partial_r,\qquad \Pi^{0\alpha}=\Pi^{\alpha 0}=0,\, \Pi^{jk}=\delta^{jk}-\frac{x^jx^k}{|x|^2},$$

and we have a first completeness property

$$m^{\alpha\beta}=-\frac{1}{2}\left\{L^\alpha\underline{L}^\beta+\underline{L}^\alpha L^\beta\right\}+\Pi^{\alpha\beta}. \tag{3.1}$$

Given a vector-field we associate the (flat) derivative operator $\partial_V=V^\alpha\partial_\alpha$. A key role will be played by the set of vector-fields which are tangential to the (Minkowski) light cone and we set

$$\mathcal{T}=\{L,e_j=\in_{jk}\Pi^{k\alpha}\}.$$

3.1. *Improved pointwise decay of the metric.* We first observe that the metric components $h_{\alpha\beta}$ have almost $\langle t\rangle^{-1}$ decay, and the "good" derivatives of the metric components have almost $\langle t\rangle^{-2}$ decay. In addition, it was observed in [53] that the harmonic gauge condition could be used to replace "bad" derivatives of some components with "good" derivatives.

PROPOSITION 3.1. Assume that (g,ψ) is a global solution of the Einstein-Klein-Gordon system given by Theorem 2.4 and let $r:=|x|$.

(i) The metric components have almost integrable decay, that is, if $h\in\{\mathcal{V}_2^2h_{\alpha\beta}\}$, then

$$|h(x,t)|\lesssim\varepsilon_0\langle t+r\rangle^{\kappa-1}.$$

(ii) If $h\in\{h_{\alpha\beta}\}$, then

$$|x||\partial_L h|+|\Omega_j h|\lesssim\varepsilon_0|x|\langle t+r\rangle^{\kappa-2},\qquad |\nabla_{x,t}h|\lesssim\varepsilon_0\langle t+r\rangle^{\kappa-1}\langle t-r\rangle^{-1}, \tag{3.2}$$

and some components of the metric also have favorable decay

$$|(\nabla_{x,t}h)_{L,T}| \lesssim \varepsilon_0 \langle t+r\rangle^{\kappa-2}, \qquad T\in\mathcal{T}. \tag{3.3}$$

(iii) The scalar field decays slightly faster but with limited improvement: for $\phi \in \mathcal{V}_1^1\psi$ we have

$$|\phi(x,t)| + |\partial_t\phi(x,t)| \lesssim \varepsilon_0\langle t+r\rangle^{\kappa-1}\langle r\rangle^{-1/2},$$
$$|\nabla_x\phi(x,t)| \lesssim \varepsilon_0\langle t+r\rangle^{\kappa-3/2}.$$

The estimate of the scalar field and metric and their first derivative follows from pointwise bounds such as (2.39)-(2.40) applied to various combinations of the vector-fields, together with the observation that L has a favorable structure:

$$(\partial_t+\partial_r)\,h = \Re\left\{(1+i\frac{x^j}{|x|}R_j)U\right\}.$$

The control of derivatives of some components follows from the harmonic gauge condition (1.34), which, expressed in the basis (3.1) gives

$$\frac{1}{2}L^\beta\partial_{\underline{L}}h_{\beta\mu} - \frac{1}{2}\partial_\mu h_{\alpha\beta}L^\alpha\underline{L}^\beta + \frac{1}{2}\Pi^{\alpha\beta}\partial_\mu h_{\alpha\beta} = \varepsilon_0 O(\langle t+r\rangle^{\kappa-2}). \tag{3.4}$$

Contracting with the basis (3.1) quickly leads to (3.3).

3.2. *Geodesics.* We now introduce a natural decomposition of the space-time into interior, null, and exterior regions. Given $f(t) = t^{1/10} = o(t)$ a sublinear function, we define

$$\begin{aligned}
\mathcal{I} &:= \{(x,t)\in M:\ |x|\le t-f(t),\\
\mathcal{C} &:= \{(x,t)\in M:\ t-f(t)\le |x|\le t+f(t)\},\\
\mathcal{E} &:= \{(x,t)\in M:\ |x|\ge t+f(t)\}.
\end{aligned} \tag{3.5}$$

The main dynamical relevance of this decomposition is contained in the following result.

Proposition 3.2. Let $\gamma(s) = x^\alpha(s)$ be an affinely parameterized causal geodesic (i.e., timelike or null); then there exists $\upsilon\in[0,\infty)$ and $\theta\in\mathbb{S}^2$ such that

$$x^j(s) = \upsilon\theta^j\cdot x^0(s) + O(|x|^\kappa), \qquad |x^0(s)|\to\infty \ \text{ as } s\to\infty$$

where $0\le\upsilon<1$ for timelike geodesics and $\upsilon=1$ for null geodesics. In particular, any timelike geodesic γ_t eventually lies in $\mathcal{I}$ and every null geodesic γ eventually lies in $\mathcal{C}$. Finally, geodesics extend to infinite affine parameter.

We also refer to [55] for a detailed study of timelike geodesics in a slightly different setting and when the metric is a perturbation of Schwarzschild at spatial infinity (in our case when we do not have an explicit form of the nonintegrable component of the metric). The main ingredient of the proof hinges on the fact that if Z is a Killing field, we have that $\mathbf{g}(Z,\dot\gamma)$ remains constant along a geodesic γ. In our case, we have no exact Killing field, but we will use this with the Minkowski Killing fields given by the Lorentz boosts Γ_j. Assuming for simplicity that $|\gamma(0)|\lesssim 1$, we can rewrite the geodesic equation as

$$0 = \frac{d}{ds}\left\{\mathbf{g}_{\alpha\beta}\dot x^\beta\right\} - \frac{1}{2}\left\{\dot x^0\dot x^0\partial_\alpha h_{00} + 2\dot x^0\dot x^j\partial_\alpha h_{0j} + \dot x^j\dot x^k\partial_\alpha h_{jk}\right\}. \tag{3.6}$$

Using this, we see that, for any vector-field V^α, there holds that

$$\frac{d}{ds}\left\{V^\alpha \mathbf{g}_{\alpha\beta}\dot{x}^\beta\right\} = \dot{V}^\alpha \mathbf{g}_{\alpha\beta}\dot{x}^\beta + \frac{1}{2}\partial_V h_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta. \tag{3.7}$$

Since the right hand side is almost integrable in time $O(\langle t\rangle^{\kappa-1})$, while the coefficients of V increase linearly in time, this forces the coordinates of $\dot{x}$ in the "basis" $\{S,\Gamma_1,\Gamma_2,\Gamma_3\}$, $S^\alpha = x^\alpha$ to be well behaved. In order to pass from arclength parameterization to time parameterization, we estimate the acceleration of the curve using the following interesting bounds which follow as in the proof of Proposition 3.1:

$$|\mathbf{\Gamma}_{0\alpha\beta}x^\alpha x^\beta| \lesssim \varepsilon_0\langle t+r\rangle^\kappa, \qquad |x^j\mathbf{\Gamma}_{j\alpha\beta}x^\alpha x^\beta| \lesssim \varepsilon_0 r\langle t+r\rangle^\kappa, \tag{3.8}$$

and correspond to gains of $1/t$ over trivial estimates.

3.3. *Almost optical functions and improved vector-fields.* In order to get precise information on the asymptotic behavior of the metric in physical space, we need to understand the bending of the light cones caused by the long-range effect of the nonlinearity (i.e., the modified scattering). In Minkowski space, the outgoing light cones correspond to level sets of $u^0 = r-t$, thus, we look for an almost optical function u close to $r-t$. Once again, the situation is simplified in the case of "restricted-like" data which are perturbations of Schwarzschild (see [52]), where one can choose

$$u(x,t) = r^* - t = |x| - t + M\ln|x|.$$

However, the metrics we consider have slower decay at infinity, see (1.40), and we expect a deviation that is both larger and not radially isotropic. In our setting, the almost-optical function is related to the functions Θ_{wa} from (2.26).

Lemma 3.3. There exists an almost optical function

$$u(x,t) = |x| - t + u^{cor}(x,t), \qquad g^{\alpha\beta}\partial_\alpha u\partial_\beta u = \varepsilon_0 O(\langle r\rangle^{-2}\langle t+r\rangle^\kappa) \tag{3.9}$$

with

$$\begin{aligned} &u^{cor},\, \Omega u^{cor} = \varepsilon_0 O(\langle t+r\rangle^\kappa),\\ &\partial_t u^{cor},\, \partial_r u^{cor} = \varepsilon_0 O(\langle t+r\rangle^{\kappa-1}), \qquad \nabla_{x,t}L^\alpha\partial_\alpha u^{cor} = \varepsilon_0 O(\langle t+r\rangle^{\kappa-2}). \end{aligned} \tag{3.10}$$

In addition, u^{cor} is close to Θ_{wa} close to the light cone:

$$\left|u^{cor}(x,t) - \frac{\Theta_{wa}(x,t)}{|x|}\right|\chi_{\mathcal{C}}(x,t) = \varepsilon_0 O\left(\langle t\rangle^{-1/100}\right) \qquad \text{when } |t-|x|| \le \langle t\rangle^{\frac{1}{10}}. \tag{3.11}$$

Linearizing the Eikonal equation around $u^0(x,t) = |x| - t$, we get

$$g^{\alpha\beta}\partial_\alpha u\partial_\beta u = \left\{2(\partial_t+\partial_r)u^{cor} + g^{\alpha\beta}\partial_\alpha u^0\partial_\beta u^0\right\} + 2g^{\alpha\beta}_{\ge 1}\partial_\alpha u^0\partial_\beta u^{cor} + g^{\alpha\beta}\partial_\alpha u^{cor}\partial_\beta u^{cor}.$$

In order to integrate the term in the bracket, we let

$$\begin{aligned} H_L &:= -\frac{1}{2}g^{\alpha\beta}_{\ge 1}\partial_\alpha u^0\partial_\beta u^0 = \frac{1}{2}L^\alpha L^\beta h_{\alpha\beta},\\ \gamma_{x,t}(s) &:= x + (s-t)\frac{x}{|x|}, \qquad \mathcal{F}\left\{\Pi_s^- f\right\}(\xi) := \varphi_{\le 0}(\langle s\rangle^{p_0}\xi)\widehat{f}(\xi),\\ H_L^+(x,t;s) &:= (1-\Pi_s^-)H_L(\gamma_{x,t}(s),s) + \left\{\Pi_s^- H_L(\gamma_{x,t}(s),s) - \Pi_s^- H_L(sx/|x|,s)\right\}. \end{aligned}$$

Then we define, for $|x| \leq t$ (similar formulas apply to the case $|x| \geq t$)

$$u^{cor}(x,t) := \int_0^t \Pi_s^- H_L(sx/|x|, s)\, ds - \int_t^\infty H_L^+(x,t;s)\, ds.$$

This informally integrates the low frequencies from the initial time slice and the high frequencies from ∞. Note that, close to the light cone, the low frequency contribution is close to the formula in (2.26).

REMARK 3.4. The level sets of the almost optical function u define our proxy for the outgoing light cones and as such are important in order to properly define the Friedlander field and the Bondi mass.

We can now look to improve on (3.1) with a frame adapted to g. The crux is to find an improved version of the null outgoing vector, $L = \nabla(r-t)$. Thus it makes sense to introduce the vector-fields

$$\mathcal{L}^\alpha := g^{\alpha\beta}\partial_\beta u, \qquad \underline{\mathcal{L}}^\alpha := (1 + \frac{1}{2}\partial_{\underline{L}} u^{cor})\underline{L}^\alpha + \frac{1}{4} g_{\underline{L}\underline{L}} L^\alpha. \tag{3.12}$$

3.3.1. *Friedlander fields and description of the metric at null infinity.* The main aspect of the asymptotic behavior of h at null infinity is related to its H^1-scattering through the Friedlander field defined below.

We define the asymptotic field to be

$$\mathcal{A}_G(u,\omega) = \frac{-i}{4\pi^2}\int_{\rho=0}^\infty e^{i\rho u}\widehat{V_\infty^G}(\rho\omega)\rho d\rho, \qquad G \in \{F, \omega_j, \Omega_{jk}, \vartheta_{ab}\}, \tag{3.13}$$

and we observe that

$$\begin{aligned} \int_{u\in\mathbb{R}}\int_{\omega\in\mathbb{S}^2} |\mathcal{A}_G(u,\omega)|^2 du d\omega = C\int_{\mathbb{R}^3} |\widehat{V}_\infty^G(\xi)|^2 d\xi, \\ \int_{\{a\leq u(x,t)\leq b\}} |\mathcal{A}_G(u(x,t), \frac{x}{|x|})|^2 \frac{dx}{|x|^2} = \int_{a\leq u\leq b}\int_{\theta\in\mathbb{S}^2} |\mathcal{A}_G(u,\theta)|^2 du d\theta. \end{aligned} \tag{3.14}$$

We can then describe the main order term of the metric and of the scalar field.

LEMMA 3.5. Assume that $(\mathbf{g}, \psi)$ is the solution from the Einstein-Klein-Gordon as in Theorem 1.3. We have the asymptotic description:

$$\begin{aligned} U^G(x,t) &= \frac{1}{|x|}\mathcal{A}_G(u(x,t), \frac{x}{|x|})\chi_{\mathcal{C}}(x,t) + U^G_{rem}(x,t), \qquad G \in \{F, \omega_j, \Omega_{jk}, \vartheta_{ab}\}, \\ U^\psi(x,t) &= C\left|\frac{\partial\nu_{kg}}{\partial x}(x,t)\right|^{\frac{1}{2}} e^{i\Phi_{kg}(x,\nu_{kg},t)}\widehat{V}_\infty^\psi(\nu_{kg}(x,t))\chi_{\mathcal{I}}(x,t) + U^\psi_{rem}(x,t), \end{aligned} \tag{3.15}$$

where the remainders satisfy

$$\|U^G_{rem}(t)\|_{L^2_x} + \|U^\psi_{rem}(t)\|_{L^2_x} \lesssim \varepsilon_0\langle t\rangle^{-\delta}.$$

Here, $\chi_{\mathcal{C}}$ and $\chi_{\mathcal{I}}$ are cut-off functions to the regions $\mathcal{C}$ and $\mathcal{I}$ as in (3.5) and $\nu_{kg}(x,t)$ is the stationary point of $v \mapsto \Phi_{kg}(x,v,t)$ where:

$$\begin{aligned} \Phi_{kg}(x,v,t) &:= \sqrt{1+|v|^2}\left\{-t + \frac{1}{2}\int_{s=1}^t H_{kg}(s\nabla\Lambda_{kg}(v), s)ds\right\} + \langle x, v\rangle, \\ H_{kg}(x,t) &:= \frac{1}{2}h^{low}_{\alpha\beta}(x,t)\mathfrak{n}_v^\alpha\mathfrak{n}_v^\beta, \qquad \mathfrak{n}_v^0 = 1, \quad \mathfrak{n}_v^j = \nabla_j\Lambda_{kg}(v). \end{aligned}$$

Lemma 3.5 is essentially an application of the stationary phase analysis to extract the main contribution of each term.

3.4. *Scattering of the mass.* We define the local density of mass to be

$$\mathbf{m} := \partial_k \{\partial_j g_{jk} - \partial_k g_{jj}\} = -2\Delta\tau,$$

where $\tau = -\vartheta_{jj}/2$ is defined in (2.15). The standard definition of the ADM mass of the time slice Σ_t is then (up to a multiplicative constant):

$$M_{ADM}(t) = \lim_{R\to\infty} \int_{\{|x|=R\}} (\partial_j h_{ij}(t) - \partial_i h_{jj}(t)) \cdot \frac{x^i}{|x|} dS;$$

see, e.g., [3, 4]. Using the Stokes theorem and the definition (2.15), one can therefore recast the ADM mass as an integral of $\Delta\tau$. We will see that in fact various integrals of $\Delta\tau$ on appropriate regions are nonnegative and obey suitable conditions. The key observation is the following which refines one of the estimates in Lemma 2.2 from the harmonic gauge condition (1.34).

Lemma 3.6. Assume that $(\mathbf{g}, \psi)$ satisfies the conclusion of Theorem 2.4, then, with the definitions in (2.15) and (2.18), there holds that

$$-2\Delta\tau = \frac{1}{4}\sum_{m,n} |U^{\vartheta_{mn}}|^2 + |U^\psi|^2 + \partial_j \mathcal{F}_j + \mathcal{E},$$
$$\|\mathcal{E}(t)\|_{L^1_x} = O(\langle t\rangle^{-\kappa}), \qquad \|\mathcal{F}(t)\|_{L^{\frac{3}{2}}_x} = O(\langle t\rangle^{-\kappa}).$$

In addition, there holds that

$$\Delta\partial_t\tau = \Delta\mathcal{F}', \qquad \|\mathcal{F}'(t)\|_{L^{\frac{3}{2}}} = O(\langle t\rangle^{-\kappa}). \tag{3.16}$$

The last assertion which implies that the ADM mass is constant in time is in fact a direct consequence of the equality (compare with (2.16))

$$0 = \Delta\big\{2\partial_t\tau + g^{\alpha\beta}_{\geq 1}(\partial_t h_{\alpha\beta} - 2\partial_\alpha h_{\beta 0}) + 2(\partial_t \underline{F} - |\nabla|\rho)\big\}.$$

The various mass loss properties are then easy consequences of Lemmas 3.5 and 3.6.

Theorem 3.7. (i) The ADM mass of a time-slice can be defined in terms of the Hodge decomposition:

$$M_{ADM}(t) = -\int_{\mathbb{R}^3} 2\Delta\tau dx = -2\lim_{R\to\infty}\int_{\mathbb{R}^3} (\chi_R\Delta\tau) dx, \tag{3.17}$$

where $\chi_R(x) = \chi(R^{-1}x)$ and $\chi \in C^\infty_c$ is equal to 1 in a neighborhood of 0. There holds that $M_{ADM}(t)$ is finite and, in fact independent of time (and of the defining function χ). In addition, $M_{ADM}(t) \geq 0$ and it vanishes only in case the space-time is the Minkowski space and the scalar field is 0. In fact, the mass is a quadratic form on the scattering data:[5]

$$M_{ADM} = \int_{\mathbb{R}^3} \left\{\frac{1}{4}|\nabla_{x,t}\vartheta^\infty|^2 + |\nabla_{x,t}\psi^\infty|^2 + |\psi^\infty|^2\right\} dx. \tag{3.18}$$

[5]Note that one can replace ϑ by $\underline{\vartheta}$, its traceless part.

(ii) Given a number $\bar{u} \in \mathbb{R}$, we define the null Bondi mass $M_B^{null}(\bar{u})$ to be

$$M_B^{null}(\overline{u}) = \lim_{t\to\infty} \int_{\{u(x,t)\le \overline{u}\}} -2\Delta\tau + \partial_j Y_j \, dx,$$

where $\partial_j Y_j$ is a suitable quadratic divergence (which can be defined covariantly; see Section 6.2 in [31] for details). The function M_B^{null} is well defined, continuous and nondecreasing for $-\infty < \bar{u} < \infty$. In fact, for $\bar{u}_1 < \bar{u}_2$, we have that

$$M_B^{null}(\bar{u}_1) - M_B^{null}(\bar{u}_2) = \frac{1}{4} \int_{u=\bar{u}_1}^{\bar{u}_2} \int_{\omega\in\mathbb{S}^2} \sum_{a,b} |\mathcal{A}_{\vartheta_{a,b}}(u,\omega)|^2 du d\omega,$$

$$\lim_{\bar{u}\to\infty} M_B^{null}(\bar{u}) = M_{ADM}, \qquad \lim_{\bar{u}\to-\infty} M_B^{null}(\bar{u}) = \|\psi^\infty\|_{L^2}^2 + \|\nabla_{x,t}\psi^\infty\|_{L^2}^2.$$

(iii) For any $R > 0$, define the timelike Bondi mass to be

$$M_B^{\mathcal{I}}(R) = \lim_{t\to\infty} \int_{\{|\nu_{kg}(x,t)|\le R\}} -2\Delta\tau + \partial_j Y_j \, dx.$$

Then, the limit exists, $M(R)$ is increasing in R and besides, we have an explicit formula through the scattering map:

$$M_B^{\mathcal{I}}(R) := \int_{\{|\xi|\le R\}} |\widehat{V}_\infty^\psi(\xi)|^2 d\xi.$$

In particular, we see that

$$\lim_{R\to\infty} M_B^{\mathcal{I}}(R) = \|\nabla_{x,t}\psi_\infty\|_{L^2}^2 + \|\psi_\infty\|_{L^2}^2, \qquad \lim_{R\to 0} M_B^{\mathcal{I}}(R) = 0,$$

and all the ADM-mass is accounted for:

$$M_{ADM} := \lim_{\overline{u}\to\infty} M_B^{null}(\overline{u}) - \lim_{\underline{u}\to\infty} M_B^{null}(\underline{u}) + \lim_{R\to\infty} M_B^{\mathcal{I}}(R) - \lim_{r\to 0} M_B^{\mathcal{I}}(r).$$

3.5. *Peeling estimates for the curvature.* The (weak) peeling estimates we prove here assert that certain components of the Riemann curvature tensor have improved decay property compared to the trivial estimate $\mathbf{R} = \varepsilon_0 O(\langle t+r\rangle^{\kappa-1}\langle t-r\rangle^{-2})$. We will decompose $\mathbf{R}$ using the basis (3.1); more precisely, we define

$$\begin{aligned}
\alpha^{pq} &:= \mathbf{R}_{a\beta b\rho}\Pi^{ap}\Pi^{bq}L^\beta L^\rho, & \underline{\alpha}^{pq} &:= \mathbf{R}_{a\beta b\rho}\Pi^{ap}\Pi^{bq}\underline{L}^\beta \underline{L}^\rho,\\
\beta^{p} &:= \mathbf{R}_{a\beta\mu\rho}\Pi^{ap}L^\beta \underline{L}^\mu L^\rho, & \underline{\beta}^{p} &:= \mathbf{R}_{a\beta\mu\rho}\Pi^{ap}\underline{L}^\beta \underline{L}^\mu L^\rho,\\
\varrho &:= \mathbf{R}_{\alpha\beta\mu\rho}\underline{L}^\alpha L^\beta \underline{L}^\mu L^\rho, & \sigma &:= \mathbf{R}_{\alpha\beta\mu\rho}M^{\alpha\beta}\underline{L}^\mu L^\rho, \quad M^{\alpha\beta} = \Pi^{\alpha\theta}\Pi^{\beta\gamma}\in_{\theta\gamma}.
\end{aligned} \tag{3.19}$$

This decomposition is slightly different from other decompositions such as the one in [43], most notably because of the lack of normalization and because it is done with respect to a frame which is only adapted to the Minkowski metric. However, the normalization factors would be bounded above and below in our case, and switching to adapted frame would not improve the decay in our case (with the exception of one component of signature 0 which is not listed in (3.19); see Section 6.2 in [31] for details).

THEOREM 3.8. For $r \geq t/10$ and $t \geq 1$ we have

$$\underline{\alpha}^{jk} = O(r^{\kappa-1}\langle t-r\rangle^{-2}),$$
$$\underline{\beta}^{j} = O(r^{\kappa-2}\langle t-r\rangle^{-1}),$$
$$|\sigma| + |\alpha^{jk}| + |\beta^j| + |\varrho| = O(r^{\kappa-3}).$$

REMARK 3.9. In view of the Einstein equations (1.22), we notice that the Ricci components

$$R_{\alpha\beta} \approx O(\psi, \nabla_{x,t}\psi)^2 \gtrsim O(\langle t\rangle^{-3}).$$

As a result we do not expect uniform estimates of order better than cubic for any components of the Riemann curvature tensor, so the weak peeling estimates in Theorem 3.8 seem to be optimal, at least up to r^κ losses.

The almost cubic decay for the null components $\alpha, \beta, \sigma, \varrho$ is also formally consistent with the weak peeling estimates of Klainerman–Nicolò [44, Theorem 1.2 (b)] in the setting of our more general metrics (one would need to formally take $\gamma = -1/2-$ and $\delta = 2+$ with the notation in [44], to match our decay assumptions (1.40), even though this range of parameters is not allowed in [44]).

The estimates in Theorem 3.8 follow from the formulas (3.19), precise bounds on first and second order derivatives of $h_{\alpha\beta}$, and the general identity

$$\mathbf{R}_{\alpha\beta\mu\rho} = -\partial_\alpha \mathbf{\Gamma}_{\rho\beta\mu} + \partial_\beta \mathbf{\Gamma}_{\rho\alpha\mu} + g^{\theta\lambda}\mathbf{\Gamma}_{\theta\beta\mu}\mathbf{\Gamma}_{\lambda\rho\alpha} - g^{\theta\lambda}\mathbf{\Gamma}_{\theta\alpha\mu}\mathbf{\Gamma}_{\lambda\rho\beta}$$

after carefully estimating each term.

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