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Competition models for plant stems

Alberto Bressan [a,*], Sondre T. Galtung [b], Audun Reigstad [b], Johanna Ridder [a]

[a] *Department of Mathematics, Penn State University, University Park, PA 16802, USA*
[b] *Department of Mathematical Sciences, NTNU – Norwegian University of Science and Technology, NO-7491 Trondheim, Norway*

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Abstract

The models introduced in this paper describe a uniform distribution of plant stems competing for sunlight. The shape of each stem, and the density of leaves, are designed in order to maximize the captured sunlight, subject to a cost for transporting water and nutrients from the root to all the leaves. Given the intensity of light, depending on the height above ground, we first solve the optimization problem determining the best possible shape for a single stem. We then study a competitive equilibrium among a large number of similar plants, where the shape of each stem is optimal given the shade produced by all others. Uniqueness of equilibria is proved by analyzing the two-point boundary value problem for a system of ODEs derived from the necessary conditions for optimality.

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1. Introduction

Optimization problems for tree branches have recently been studied in [3,5]. In these models, optimal shapes maximize the total amount of sunlight gathered by the leaves, subject to a cost for

* Corresponding author.
E-mail addresses: axb62@psu.edu (A. Bressan), sondre.galtung@ntnu.no (S.T. Galtung), audun.reigstad@ntnu.no (A. Reigstad), johanna@jridder.de (J. Ridder).

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building a network of branches that will bring water and nutrients from the root to all the leaves. Following [2,8,11,13,14], this cost is defined in terms of a ramified transport.

In the present paper we consider a competition model, where a large number of similar plants compete for sunlight. To make the problem tractable, instead of a tree-like structure we assume that each plant consists of a single stem. As a first step, assuming that the intensity of light $I(\cdot)$ depends only on the height above ground, we determine the corresponding optimal shape of the stem. This will be a curve $\gamma(\cdot)$ which can be found by classical techniques of the Calculus of Variations or optimal control [4,6,7]. In turn, given the density of plants (i.e., the average number of plants growing per unit area), if all stems have the same shape $\gamma(\cdot)$ one can compute the intensity of light $I(h)$ that reaches a point at height h.

An equilibrium configuration is now defined as a fixed point of the composition of the two maps $I(\cdot) \mapsto \gamma(\cdot)$ and $\gamma(\cdot) \mapsto I(\cdot)$. A major goal of this paper is to study the existence and properties of these equilibria, where the shape of each stem is optimal subject to the presence of all other competing plants.

In Section 2 we introduce our two basic models. In the first model, the length ℓ of the stems and the thickness (i.e., the density of leaves along each stem) are assigned a priori. The only function to optimize is thus the curve $\gamma : [0, \ell] \mapsto \mathbb{R}^2$ describing the shape of the stems. In the second model, also the length and the thickness of the stems are allowed to vary, and optimal values for these variables need to be determined.

In Section 3, given a light intensity function $I(\cdot)$, we study the optimization problem for Model 1, proving the existence of an optimal solution and deriving necessary conditions for optimality. We also give a condition which guarantees the uniqueness of the optimal solution. A counterexample shows that, in general, if this condition is not satisfied multiple solutions can exist. In Section 4 we consider the competition of a large number of stems, and prove the existence of an equilibrium solution. In this case, the common shape of the plant stems can be explicitly determined by solving a particular ODE.

The subsequent sections extend the analysis to a more general setting (Model 2), where both the length and the thickness of the stems are to be optimized. In Section 5 we prove the existence of optimal stem configurations, and derive necessary conditions for optimality, while in Section 6 we establish the existence of a unique equilibrium solution for the competitive game, assuming that the density (i.e., the average number of stems growing per unit area) is sufficiently small. The key step in the proof is the analysis of a two-point boundary value problem, for a system of ODEs derived from the necessary conditions.

In the above models, the density of stems was assumed to be uniform on the whole space. As a consequence, the light intensity $I(h)$ depends only of the height h above ground. Section 7, on the other hand, is concerned with a family of stems growing only on the positive half line. In this case the light intensity $I = I(h, x)$ depends also on the spatial location x, and the analysis becomes considerably more difficult. Here we only derive a set of equations describing the competitive equilibrium, and sketch what we conjecture should be the corresponding shape of stems.

The final section contains some concluding remarks. In particular, we discuss the issue of phototropism, i.e. the tendency of plant stems to bend in the direction of the light source. Devising a mathematical model, which demonstrates phototropism as an advantageous trait, remains a challenging open problem. For a biological perspective on plant growth we refer to [9]. A recent mathematical study of the stabilization problem for growing stems can be found in [1].

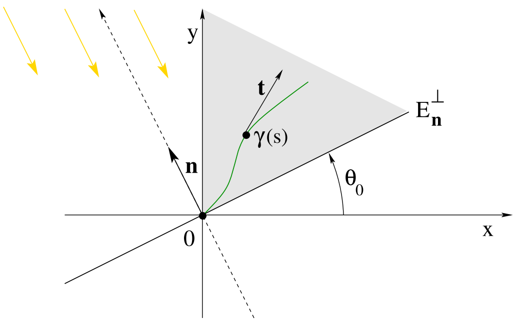

Fig. 1. By a reflection argument, it is not restrictive to assume that the tangent vector $\mathbf{t}(s)$ to the stem satisfies (2.4), i.e., it lies in the shaded cone.

2. Optimization problems for a single stem

We shall consider plant stems in the x-y plane, where y is the vertical coordinate. We assume that sunlight comes from the direction of the unit vector

$$\mathbf{n} = (n_1, n_2), \qquad n_2 < 0 < n_1.$$

As in Fig. 1, we denote by $\theta_0 \in]0, \pi/2[$ the angle such that

$$(-n_2, n_1) = (\cos\theta_0, \sin\theta_0). \tag{2.1}$$

Moreover, we assume that the light intensity $I(y) \in [0, 1]$ is a non-decreasing function of the height y. This is due to the presence of competing vegetation: close to the ground, less light can get through.

Model 1 (a stem with fixed length and constant thickness). We begin by studying a simple model, where each stem has a fixed length ℓ. Let $s \mapsto \gamma(s) = (x(s), y(s))$, $s \in [0, \ell]$, be an arc-length parameterization of the stem. As a first approximation, we assume that the leaves are uniformly distributed along the stem, with density κ. The total distribution of leaves in space is thus by a measure μ, with

$$\mu(A) = \kappa \cdot \text{meas}\Big(\big\{s \in [0, \ell]; \;\; \gamma(s) \in A\big\}\Big) \tag{2.2}$$

for every Borel set $A \subseteq \mathbb{R}^2$.

Among all stems with given length ℓ, we seek the shape which will collect the most sunlight. This can be formulated as an optimal control problem. Since γ is parameterized by arc-length, the map $s \mapsto \gamma(s)$ is Lipschitz continuous with constant 1. Hence the tangent vector

$$\mathbf{t}(s) = \dot\gamma(s) = (\cos\theta(s), \sin\theta(s))$$

is well defined for a.e. $s \in [0, \ell]$. The map $s \mapsto \theta(s)$ will be regarded as a control function.

According to the model in [5], calling $\Phi(\cdot)$ the density of the projection of μ on the space $E_{\mathbf{n}}^\perp$ orthogonal to $\mathbf{n}$, the total sunlight captured by the stem is

$$\mathcal{S}(\gamma) = \int \Big(1 - \exp\{-\Phi(z)\}\Big)\, dz$$

$$= \int_0^\ell I(y(s)) \cdot \left(1 - \exp\left\{\frac{-\kappa}{\cos(\theta(s) - \theta_0)}\right\}\right) \cos(\theta(s) - \theta_0)\, ds. \tag{2.3}$$

In order to maximize (2.3), we claim that it is not restrictive to assume that the angle satisfies

$$\theta_0 \le \theta(s) \le \frac{\pi}{2} \qquad \text{for all } s \in [0, \ell]. \tag{2.4}$$

Indeed, for any measurable map $s \mapsto \theta(s) \in]-\pi, \pi]$, we can define a modified angle function $\theta^\sharp(\cdot)$ by setting

$$\theta^\sharp(s) = \begin{cases} \theta(s) & \text{if } \theta(s) \in]0, \theta_0 + \pi/2], \\ -\theta(s) & \text{if } \theta(s) \in]-\pi,\, \theta_0 - \pi/2], \\ 2\theta_0 + \pi - \theta(s) & \text{if } \theta(s) \in]\theta_0 + \pi/2, \pi], \\ 2\theta_0 - \theta(s) & \text{if } \theta(s) \in]\theta_0 - \pi/2, 0]. \end{cases} \tag{2.5}$$

Calling $\gamma^\sharp : [0, \ell] \mapsto \mathbb{R}^2$ the curve whose tangent vector is $\dot\gamma^\sharp(s) = (\cos\theta^\sharp(s), \sin\theta^\sharp(s))$, since the light intensity function $y \mapsto I(y)$ is nondecreasing, we have $\mathcal{S}(\gamma^\sharp) \ge \mathcal{S}(\gamma)$.

By this first step, without loss of generality we can now assume $\theta(s) \in]0,\, \theta_0 + \pi/2]$. To proceed further, consider the piecewise affine map

$$\varphi(\theta) = \begin{cases} \theta & \text{if } \theta \in]\theta_0, \pi/2], \\ \pi - \theta & \text{if } \theta \in [\pi/2,\, \theta_0 + \pi/2], \\ 2\theta_0 - \theta & \text{if } \theta \in [0, \theta_0]. \end{cases} \tag{2.6}$$

Call γ^φ the curve whose tangent vector is $\dot\gamma^\varphi(s) = \Big(\cos(\varphi(\theta(s))),\, \sin(\varphi(\theta(s)))\Big)$. Since $I(\cdot)$ is nondecreasing, we again have $\mathcal{S}(\gamma^\varphi) \ge \mathcal{S}(\gamma)$. We now observe that, since $0 < \theta_0 < \pi/2$, there exists an integer $m \ge 1$ such that the m-fold composition $\varphi^m \doteq \varphi \circ \cdots \circ \varphi$ maps $[0, \theta_0 + \pi/2]$ into $[\theta_0, \pi/2]$. An inductive argument now yields $\mathcal{S}(\gamma^{\varphi^m}) \ge \mathcal{S}(\gamma)$, completing the proof of our claim.

As shown in Fig. 2, left, we call z the coordinate along the space $E_{\mathbf{n}}^\perp$ perpendicular to $\mathbf{n}$, and let y be the vertical coordinate. Hence

$$dz(s) = \cos(\theta(s) - \theta_0)\, ds, \qquad dy(s) = \sin(\theta(s))\, ds. \tag{2.7}$$

In view of (2.4), one can express both γ and θ as functions of the variable y. Introducing the function

$$g(\theta) \doteq \left(1 - \exp\left\{\frac{-\kappa}{\cos(\theta - \theta_0)}\right\}\right) \frac{\cos(\theta - \theta_0)}{\sin\theta}, \tag{2.8}$$

the problem can be equivalently formulated as follows.

(OP1) *Given a length $\ell > 0$, find $h > 0$ and a control function $y \mapsto \theta(y) \in [\theta_0, \pi/2]$ which maximizes the integral*

$$\int_0^h I(y)\, g(\theta(y))\, dy \tag{2.9}$$

subject to

$$\int_0^h \frac{1}{\sin\theta(y)}\, dy \;=\; \ell. \tag{2.10}$$

Model 2 (stems with variable length and thickness). Here we still assume that the plant consists of a single stem, parameterized by arc-length: $s \mapsto \gamma(s)$, $s \in [0, \ell]$. However, now we give no constraint on the length ℓ of the stem, and we allow the density of leaves to be variable along the stem.

Call $u(s)$ the density of leaves at the point $\gamma(s)$. In other words, μ is now the measure which is absolutely continuous w.r.t. arc-length measure on γ, with density u. Instead of (2.2) we thus have

$$\mu(A) \;=\; \int_{\{s\,;\ \gamma(s)\in A\}} u(s)\, ds\,. \tag{2.11}$$

Calling $I(y)$ the intensity of light at height y, the total sunlight gathered by the stem is now computed by

$$\mathcal{S}(\mu) \;=\; \int_0^\ell I(y(s))\cdot\left(1 - \exp\left\{\frac{-u(s)}{\cos(\theta(s)-\theta_0)}\right\}\right)\cos(\theta(s)-\theta_0)\, ds. \tag{2.12}$$

As in [5], we consider a cost for transporting water and nutrients from the root to the leaves. This is measured by

$$\mathcal{I}^\alpha(\mu) \;=\; \int_0^\ell \left(\int_s^\ell u(t)\, dt\right)^\alpha ds, \tag{2.13}$$

for some $0 < \alpha < 1$. Notice that, in Model 1, this cost was the same for all stems and hence it did not play a role in the optimization.

For a given constant $c > 0$, we now consider a second optimization problem:

$$\text{maximize:} \quad \mathcal{S}(\mu) - c\,\mathcal{I}^\alpha(\mu), \tag{2.14}$$

subject to:

$$y(0) \;=\; 0, \qquad\qquad \dot{y}(s) \;=\; \sin\theta(s). \tag{2.15}$$

The maximum is sought over all controls $\theta : \mathbb{R}_+ \mapsto [0, \pi]$ and $u : \mathbb{R}_+ \mapsto \mathbb{R}_+$. Calling

$$z(t) \doteq \int_t^{+\infty} u(s)\, ds, \tag{2.16}$$

$$G(\theta, u) \doteq \left(1 - \exp\left\{\frac{-u}{\cos(\theta - \theta_0)}\right\}\right)\cos(\theta - \theta_0), \tag{2.17}$$

this leads to an optimal control problem in a more standard form.

(OP2) *Given a sunlight intensity function $I(y)$, and constants $0 < \alpha < 1$, $c > 0$, find controls $\theta : \mathbb{R}_+ \mapsto [\theta_0, \pi/2]$ and $u : \mathbb{R}_+ \mapsto \mathbb{R}_+$ which maximize the integral*

$$\int_0^{+\infty} \Big[I(y)\, G(\theta, u) - c\, z^\alpha\Big]\, dt, \tag{2.18}$$

subject to

$$\begin{cases} \dot y(t) = \sin\theta, \\ \dot z(t) = -u, \end{cases} \qquad \begin{cases} y(0) = 0, \\ z(+\infty) = 0. \end{cases} \tag{2.19}$$

3. Optimal stems with fixed length and thickness

3.1. Existence of an optimal solution

Let $I(y)$ be the light intensity, which we assume is a non-decreasing function of the vertical component y. For a given $\kappa > 0$ (the thickness of the stem), we seek a curve $s \mapsto \gamma(s)$, starting at the origin and with a fixed length ℓ, which maximizes the sunlight functional defined at (2.9).

Theorem 3.1. *For any non-decreasing function $y \mapsto I(y) \in [0, 1]$ and any constants $\ell, \kappa > 0$ and $\theta_0 \in]0, \pi/2[$, the optimization problem* **(OP1)** *has at least one solution.*

Proof. 1. Let M be the supremum among all admissible payoffs in (2.9). By the analysis in [5] it follows that

$$0 \le M \le \kappa\, \mu(\mathbb{R}^2) = \kappa\, \ell.$$

Hence there exists a maximizing sequence of control functions $\theta_n : [0, h_n] \mapsto [\theta_0, \pi/2]$, so that

$$\int_0^{h_n} \frac{1}{\sin\theta_n(y)}\, dy = \ell \qquad \text{for all } n \ge 1, \tag{3.1}$$

$$\int_0^{h_n} I(y)g(\theta_n(y))\,dy \;\to\; M. \tag{3.2}$$

2. For each n, let $\theta_n^\sharp$ be the non-increasing rearrangement of the function θ_n. Namely, $\theta_n^\sharp$ is the unique (up to a set of zero measure) non-increasing function such that, for every $c \in \mathbb{R}$

$$\text{meas}\Big(\{s\,;\;\; \theta_n^\sharp(s) < c\}\Big) \;=\; \text{meas}\Big(\{s\,;\;\; \theta_n(s) < c\}\Big). \tag{3.3}$$

This can be explicitly defined as

$$\theta_n^\sharp(y) \;=\; \sup\Big\{\xi\,;\;\; \text{meas}\Big(\{\sigma \in [0, h_n]\,;\;\; \theta_n(\sigma) \geq \xi\}\Big) > y\Big\}.$$

For every $n \geq 1$ we claim that

$$\int_0^{h_n} \frac{1}{\sin\theta_n^\sharp(y)}\,dy \;=\; \int_0^{h_n} \frac{1}{\sin\theta_n(y)}\,dy \;=\; \ell, \tag{3.4}$$

$$\int_0^{h_n} I(y)g(\theta_n^\sharp(y))\,dy \;\geq\; \int_0^{h_n} I(y)g(\theta_n(y))\,dy. \tag{3.5}$$

Indeed, to prove the first identity we observe that, by (3.3), there exists a measure-preserving map $y \mapsto \zeta(y)$ from $[0, h_n]$ into itself such that $\theta_n^\sharp(y) = \theta_n(\zeta(y))$. Using ζ as new variable of integration, one immediately obtains (3.4).

To prove (3.5) we observe that the function g introduced at (2.8) is smooth and satisfies

$$g'(\theta) \;\leq\; 0 \qquad \text{for all } \theta \in [\theta_0,\, \pi/2]. \tag{3.6}$$

Therefore, the map $y \mapsto g(\theta_n^\sharp(y))$ coincides with the non-decreasing rearrangement of $y \mapsto g(\theta_n(y))$. On the other hand, since $I(\cdot)$ is non-decreasing, it trivially coincides with the non-decreasing rearrangement of itself. Therefore, (3.5) is an immediate consequence of the Hardy-Littlewood inequality [10].

3. Since all functions $\theta_n^\sharp$ are non-increasing, they have bounded variation. Using Helly's compactness theorem, by possibly extracting a subsequence, we can find $h > 0$ and a non-increasing function $\theta^* : [0, h] \mapsto [\theta_0, \pi/2]$ such that

$$\lim_{n\to\infty} h_n \;=\; h\,, \qquad \lim_{n\to\infty} \theta_n^\sharp(y) \;=\; \theta^*(y) \qquad \text{for a.e. } y \in [0, h]. \tag{3.7}$$

This implies

$$\int_0^{h} \frac{1}{\sin\theta^*(y)}\,dy \;=\; \ell, \qquad \int_0^{h} I(y)g(\theta^*(y))\,dy \;=\; M,$$

proving the optimality of θ^*. $\square$

3.2. Necessary conditions for optimality

Let $y \mapsto \theta^*(y)$ be an optimal solution. By the previous analysis we already know that the function $\theta^*(\cdot)$ is non-increasing. Otherwise, its non-increasing rearrangement achieves a better payoff. In particular, this implies that the left limit at the terminal point $y = h$ is well defined:

$$\theta^*(h) \;=\; \lim_{y\to h-} \theta^*(y). \tag{3.8}$$

Consider an arbitrary perturbation

$$\theta_\epsilon \;=\; \theta^* + \epsilon\Theta, \qquad\qquad h_\epsilon \;=\; h + \epsilon\eta.$$

The constraint (2.10) implies

$$\int_0^{h+\epsilon\eta} \frac{1}{\sin\theta_\epsilon(y)}\,dy \;=\; \ell. \tag{3.9}$$

Differentiating (3.9) w.r.t. ϵ one obtains

$$\frac{1}{\sin\theta^*(h)}\,\eta - \int_0^h \frac{\cos\theta^*(y)}{\sin^2\theta^*(y)}\,\Theta(y)\,dy \;=\; 0. \tag{3.10}$$

Next, calling

$$J_\epsilon \;\doteq\; \int_0^{h_\epsilon} I(y)g(\theta_\epsilon(y))dy$$

and assuming that $I(\cdot)$ is continuous at least at $y = h$, by (3.10) we obtain

$$\begin{aligned} 0 \;=\; \frac{d}{d\epsilon}J_\epsilon\Big|_{\epsilon=0} \;=\; & \int_0^h I(y)g'(\theta^*(y))\Theta(y)\,dy \\ & + I(h)g(\theta^*(h))\cdot\sin\theta^*(h)\int_0^h \frac{\cos\theta^*(y)}{\sin^2\theta^*(y)}\,\Theta(y)\,dy. \end{aligned} \tag{3.11}$$

Since (3.11) holds for arbitrary perturbations $\Theta(\cdot)$, the optimal control $\theta^*(\cdot)$ should satisfy the identity

$$I(y)g'\big(\theta^*(y)\big) + \lambda\cdot\frac{\cos\theta^*(y)}{\sin^2\theta^*(y)} \;=\; 0, \qquad\qquad \text{for a.e. } y\in[0,h], \tag{3.12}$$

where

$$\lambda \;=\; I(h)g(\theta^*(h))\cdot\sin\theta^*(h). \tag{3.13}$$

It will be convenient to write

$$g(\theta) \;=\; \frac{G(\theta)}{\sin\theta}, \qquad\qquad G(\theta) \;\doteq\; \left(1-\exp\left\{\frac{-\kappa}{\cos(\theta-\theta_0)}\right\}\right)\cos(\theta-\theta_0). \tag{3.14}$$

Inserting (3.14) in (3.12) one obtains the pointwise identities

$$I(y)\Big(G'(\theta^*(y))\sin\theta^*(y) - G(\theta^*(y))\cos\theta^*(y)\Big) + \lambda\cdot\cos\theta^*(y) \;=\; 0. \tag{3.15}$$

At $y=h$, the identities (3.13) and (3.15) yield

$$G'(\theta^*(h))\,\tan\theta^*(h) - G(\theta^*(h)) \;=\; -\,\frac{I(h)G(\theta^*(h))}{I(h)}\,.$$

Hence

$$G'(\theta^*(h))\,\tan\theta^*(h) \;=\; 0,$$

which implies

$$\theta^*(h) \;=\; \theta_0\,, \qquad\qquad \lambda \;=\; I(h)g(\theta_0)\sin\theta_0 \;=\; \big(1-e^{-\kappa}\big)\,I(h)\,. \tag{3.16}$$

Notice that (3.15) corresponds to

$$\theta^*(y) \;=\; \arg\max_{\theta\in[0,\pi]}\left\{I(y)\frac{G(\theta)}{\sin\theta} - \frac{\lambda}{\sin\theta}\right\}. \tag{3.17}$$

Equivalently, $\theta=\theta^*(y)$ is the solution to

$$G'(\theta)\tan\theta - G(\theta) \;=\; -\,\frac{\lambda}{I(y)}\,, \tag{3.18}$$

where G is the function at (3.14).

Lemma 3.2. *Let G be the function at (3.14). Then for every $z\in\,]-\infty,\ e^{-\kappa}-1]$ the equation*

$$F(\theta) \;\doteq\; G'(\theta)\tan\theta - G(\theta) \;=\; z \tag{3.19}$$

has a unique solution $\theta=\varphi(z)\in[\theta_0,\pi/2[$.

Proof. Observing that

$$\begin{cases} G(\theta_0) \;=\; 1-e^{-\kappa}, \\ G'(\theta_0) \;=\; 0, \end{cases} \qquad\qquad \begin{cases} G'(\theta) \;<\; 0 \\ G''(\theta) \;<\; 0 \end{cases} \qquad \text{for } \theta\in\,]\theta_0,\pi/2[\,, \tag{3.20}$$

we obtain $F(\theta_0) = e^{-\kappa} - 1$ and

$$F'(\theta) = G''(\theta)\tan\theta + G'(\theta)\tan^2\theta \; < \; 0 \qquad \text{for } \theta \in [\theta_0, \pi/2[\,.$$

Therefore, for $\theta \in [\theta_0, \pi/2[$, the left hand side of (3.19) is monotonically decreasing from $e^{-\kappa} - 1$ to $-\infty$. We conclude that (3.19) has a unique solution $\theta = \varphi(z)$ for any $z \in]-\infty, e^{-\kappa} - 1]$. □

The optimal control $\theta^*(\cdot)$ determined by the necessary condition (3.18) is thus recovered by

$$\theta^*(y) \;=\; \varphi\left(\frac{-\lambda}{I(y)}\right) \;=\; \varphi\left(\frac{(e^{-\kappa}-1)I(h)}{I(y)}\right). \tag{3.21}$$

Next, we need to determine h so that the constraint

$$L(h) \;\doteq\; \int_0^h \frac{1}{\sin(\theta^*(y))}\, dy \;=\; \ell \tag{3.22}$$

is satisfied. As shown by Example 3.4 below, the solution of (3.21)-(3.22) may not be unique.

In the following, we seek a condition on I which implies that L is monotone, i.e.,

$$L'(h) \;=\; \frac{1}{\sin(\theta_0)} + \int_0^h \frac{\cos\theta^*(y)}{\sin^2\theta^*(y)}\, \frac{1}{F'(\theta^*(y))}\, \frac{I'(h)}{I(y)} G(\theta_0)\, dy \;>\; 0\,. \tag{3.23}$$

This will guarantee that (3.22) has a unique solution. To get an upper bound for $F'(\theta)$, observe that, for $\theta \in [\theta_0, \pi/2[$,

$$\begin{aligned} F'(\theta) \;&\leq\; \tan(\theta) G''(\theta) \\ &=\; -\tan(\theta)\Bigg[\cos(\theta-\theta_0)\left(1 - \left(\frac{\kappa}{\cos(\theta-\theta_0)} + 1\right)\exp\left\{\frac{-\kappa}{\cos(\theta-\theta_0)}\right\}\right) \\ &\quad + \tfrac{\tan^2(\theta-\theta_0)}{\cos(\theta-\theta_0)}\kappa^2 \exp\left\{\tfrac{-\kappa}{\cos(\theta-\theta_0)}\right\}\Bigg] \\ &=\; -\tan(\theta)\, \cos(\pi/2 - \theta_0)\big(1 - (\kappa+1)e^{-\kappa}\big). \end{aligned}$$

Since $\theta^*(y) \in [\theta_0, \pi/2]$ and $G(\theta_0) = 1 - e^{-\kappa}$, using the above inequality one obtains

$$\begin{aligned} &\int_0^h \frac{\cos\theta^*(y)}{\sin^2\theta^*(y)} \cdot \frac{1}{|F'(\theta^*(y))|}\, \frac{I'(h)}{I(y)} G(\theta_0)\, dy \\ &\quad \leq \frac{\cos^2\theta_0}{\sin^3\theta_0} \cdot \frac{1-e^{-\kappa}}{\cos(\pi/2-\theta_0)\big(1-(\kappa+1)e^{-\kappa}\big)} \int_0^h \frac{I'(h)}{I(y)}\, dy\,. \end{aligned}$$

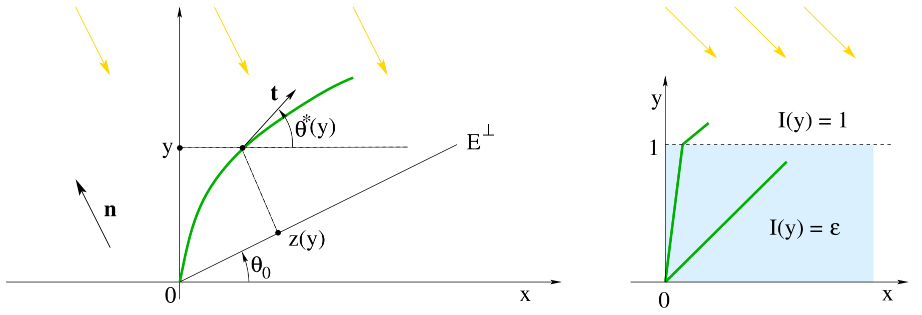

Fig. 2. Left: the optimal shape of a stem, as described in Theorem 3.3. Right: if the light intensity I changes abruptly as a function of the hight, the optimal shape may not be unique, as shown in Example 3.4.

Hence (3.23) is satisfied provided that

$$\int_0^h \frac{I'(h)}{I(y)}\,dy \;<\; \tan^2\theta_0 \cdot \frac{\cos(\pi/2-\theta_0)\big(1-(\kappa+1)e^{-\kappa}\big)}{1-e^{-\kappa}}\,. \tag{3.24}$$

From the above analysis, we conclude

Theorem 3.3. *Assume that the light intensity function I is Lipschitz continuous and satisfies the strict inequality (3.24) for a.e. $h \in [0, \ell]$. Then the optimization problem* **(OP1)** *has a unique optimal solution $\theta^* : [0, h^*] \mapsto [\theta_0, \pi/2]$. The function θ^* is non-increasing, and satisfies*

$$\theta^*(y) \;=\; \varphi\left((e^{-\kappa}-1)\frac{I(h^*)}{I(y)}\right), \tag{3.25}$$

where $z \mapsto \varphi(z) = \theta$ is the function implicitly defined by (3.19).

The following example shows that, without the bound (3.24) on the sunlight intensity function $I(\cdot)$, the conclusion of Theorem 3.3 can fail.

Example 3.4 *(non-uniqueness).* Choose $\mathbf{n} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\ell = 6/5 < \sqrt{2}$, $\kappa = 1$,

$$I(y) \;=\; \begin{cases} \varepsilon & \text{if} \quad y \in [0,1], \\ 1 & \text{if} \quad y > 1, \end{cases}$$

with $\varepsilon > 0$l.

By Theorem 3.1 at least one optimal solution exists. By the previous analysis, any optimal solution $\theta^* : [0, h^*] \mapsto [\theta_0, \pi/2]$ satisfies the necessary conditions (3.25). In this particular case, this implies that $\theta^*(y)$ is constant separately for $y < 1$ and for $y > 1$. As shown in Fig. 2, right, these necessary conditions can have two solutions.

Solution 1. If $h^* < 1$, then $I(y) = \varepsilon$ for all $y \in [0, h^*]$ and the necessary conditions (3.25) yield

$$\theta_1^*(y) \;=\; \varphi(e^{-1}-1) \;=\; \theta_0 \;=\; \pi/4 \qquad \text{for all } y\in[0,h^*].$$

The total sunlight collected is

$$\mathcal{S}_\varepsilon(\theta_1^*) \;=\; \frac{6}{5}(1-e^{-1})\,. \tag{3.26}$$

Solution 2. If $h^*>1$, then $I(h^*)=1$ and the necessary conditions (3.25) yield

$$\theta_2^*(y) \;=\; \varphi\left((e^{-1}-1)\frac{I(h^*)}{I(y)}\right) \;=\; \begin{cases} \varphi\left((e^{-1}-1)\varepsilon^{-1}\right) & \text{if} \quad y\in[0,1],\\ \pi/4 & \text{if} \quad y>1.\end{cases}$$

Calling $\alpha=\alpha(\varepsilon)\doteq\varphi\left((e^{-1}-1)\varepsilon^{-1}\right)$, the total sunlight collected in this case is

$$\mathcal{S}_\varepsilon(\theta_2^*) \;=\; \left(1-\exp\left\{-\frac{1}{\cos(\alpha-\pi/4)}\right\}\right)\cos(\alpha-\pi/4)\,\varepsilon+\left(\frac{6}{5}-\frac{1}{\sin\alpha}\right)\left(1-e^{-1}\right). \tag{3.27}$$

We claim that, for a suitable choice of $\varepsilon\in\,]0,1[$, the two quantities in (3.26) and (3.27) become equal. Indeed, as $\varepsilon\to0+$ we have

$$\alpha(\varepsilon) \;\doteq\; \varphi\left(\frac{e^{-1}-1}{\varepsilon}\right) \;\to\; \frac{\pi}{2}\,,$$

$$\mathcal{S}_\varepsilon(\theta_1^*) \;\to\; 0, \qquad\qquad \mathcal{S}_\varepsilon(\theta_2^*) \;\to\; \frac{1-e^{-1}}{5}\,. \tag{3.28}$$

On the other hand, as $\varepsilon\to1$ we have $\alpha(\varepsilon)\to\pi/4$. By continuity, there exists $\varepsilon_1\in\,]0,1[$ such that

$$\sin\alpha(\varepsilon_1) \;=\; \frac{5}{6}\,.$$

As $\varepsilon\to\varepsilon_1+$, we have

$$\mathcal{S}_\varepsilon(\theta_2^*) \;\to\; \left(1-\exp\left\{-\frac{1}{\cos(\alpha(\varepsilon_1)-\pi/4)}\right\}\right)\cos(\alpha(\varepsilon_1)-\pi/4)\,\varepsilon_1 \;<\; \mathcal{S}_{\varepsilon_1}(\theta_1^*). \tag{3.29}$$

Comparing (3.28) with (3.29), by continuity we conclude that there exists some $\widehat{\varepsilon}\in\,]0,\varepsilon_1[$ such that $\mathcal{S}_{\widehat{\varepsilon}}(\theta_1^*) \;=\; \mathcal{S}_{\widehat{\varepsilon}}(\theta_2^*)$. Hence for $\varepsilon=\widehat{\varepsilon}$ the optimization problem has two distinct solutions.

We remark that in this example the light intensity $I(y)$ is discontinuous at $y=1$. However, by a mollification one can still construct a similar example with two optimal configurations, also for $I(\cdot)$ smooth. Of course, in this case the derivative $I'(h)$ will be extremely large for $h\approx1$, so that the assumption (3.24) fails.

4. A competition model

In the previous analysis, the light intensity function $I(\cdot)$ was a priori given. We now consider a continuous distribution of stems, and determine the average sunlight $I(y)$ available at height y above ground, depending on the density of vegetation above y.

Let the constants $\ell, \kappa > 0$ be given, specifying the length and thickness of each stem. We now introduce another constant $\rho > 0$ describing the density of stems, i.e. how many stems grow per unit area. Assume that all stems have the same height and shape, described by the function $\theta : [0, h] \mapsto [\theta_0, \pi/2]$. For any $y \in [0, h]$, the total amount of vegetation at height $\geq y$, per unit length, is then measured by

$$\rho \cdot \int_y^h \frac{\kappa}{\sin\theta(y)}\, dy.$$

The corresponding light intensity function is defined as

$$I(y) \doteq \exp\left\{-\rho \cdot \int_y^h \frac{\kappa}{\sin\theta(y)}\, dy\right\} \qquad \text{for} \quad y \in [0, h], \tag{4.1}$$

while $I(y) = 1$ for $y \geq h$. We are interested in equilibrium configurations, where the shape of the stems is optimal for the light intensity $I(\cdot)$. We recall that θ_0 is the angle of incoming light rays, as in (2.1), while the constants $\ell, \kappa > 0$ denote the length and thickness of the stems.

Definition 4.1. Given an angle $\theta_0 \in]0, \pi/2]$ and constants $\ell, \kappa, \rho > 0$, we say that a light intensity function $I^* : \mathbb{R}_+ \mapsto [0, 1]$ and a stem shape function $\theta^* : [0, h^*] \mapsto [\theta_0, \pi/2]$ yield a **competitive equilibrium** if the following holds.

(i) The stem shape function $\theta^* : [0, h^*] \mapsto [\theta_0, \pi/2]$ provides an optimal solution to the optimization problem **(OP1)**, with light intensity function $I = I^*$.
(ii) For all $y \geq 0$, the light intensity at height y satisfies

$$I^*(y) = \exp\left\{-\rho \cdot \int_{\min\{y,h^*\}}^{h^*} \frac{\kappa}{\sin\theta^*(y)}\, dy\right\}. \tag{4.2}$$

If the density of vegetation is sufficiently small, we now show that an equilibrium configuration exists.

Theorem 4.2. *Let the light angle $\theta_0 \in]0, \pi/2]$ be given, together with the constants $\ell, \kappa > 0$ determining the common length and thickness of all the stems. Then there exists a constant $c_0 > 0$ such that, for all $0 < \rho \leq c_0$, an equilibrium configuration exists.*

Proof. 1. Consider the set of stem configurations

$$\mathcal{K} \doteq \Big\{\Theta : [0, \ell] \mapsto [\theta_0, \pi/2], \quad \Theta \text{ is nonincreasing}\Big\}, \tag{4.3}$$

and the set of light intensity functions

$$\mathcal{J} \doteq \Big\{ I : [0,+\infty[\mapsto [0,1]\,;\quad I \text{ is nondecreasing},\quad I(y)=1 \text{ for } y \geq \ell, \\ I \text{ is Lipschitz continuous with constant } \frac{\rho\kappa}{\sin\theta_0} \Big\}. \tag{4.4}$$

We observe that $\mathcal{K}$ is a compact, convex subset of $\mathbf{L}^1([0,\ell])$, while $\mathcal{J}$ is a compact, convex subset of $\mathcal{C}^0([0,+\infty[)$.

If $\Theta(\cdot) \in \mathcal{K}$ describes the common configuration of all stems, we denote by $I^{\Theta}(\cdot)$ the corresponding light intensity function. Moreover, for a given function $I(\cdot)$, we denote by $\Theta^*(I)$ the corresponding optimal configuration of plant stems.

In the following steps we shall prove that:

(i) The map $\Theta \mapsto I^{\Theta}$ is continuous from $\mathcal{K}$ into $\mathcal{J}$.
(ii) The map $I \mapsto \Theta^*(I)$ is continuous from $\mathcal{J}$ into $\mathcal{K}$.

As a consequence, the composed map $\Theta \mapsto \Theta^*(I^{\Theta})$ is continuous from $\mathcal{K}$ into itself. By Schauder's theorem, it has a fixed point, which provides an equilibrium solution.

2. Given $\Theta \in \mathcal{K}$, define the constant

$$\bar{h} \doteq \int_0^{\ell} \sin\Theta(t)\,dt\,. \tag{4.5}$$

More generally, for $s \in [0,\ell]$, set

$$y(s) \doteq \int_0^{s} \sin\Theta(t)\,dt \in [0,\bar{h}]. \tag{4.6}$$

We observe that, since $\Theta(t) \in [\theta_0, \pi/2]$, the inverse function $y \mapsto s(y)$ from $[0,\bar{h}]$ into $[0,\ell]$ is a strictly increasing bijection, with Lipschitz constant $L = \frac{1}{\sin\theta_0}$. The corresponding light intensity function is determined by

$$I^{\Theta}(y) = \begin{cases} \exp\big\{-\rho\kappa(\ell - s(y))\big\} & \text{if } y \in [0,\bar{h}], \\ 1 & \text{if } y > \ell. \end{cases} \tag{4.7}$$

From the above definitions it follows that $\Theta \mapsto I^{\Theta}$ is continuous from $\mathcal{K}$ into $\mathcal{J}$.

3. Next, let $I \in \mathcal{J}$. Given the constants ℓ, κ, by choosing $\rho > 0$ small enough, any Lipschitz continuous function $I : [0,\ell] \mapsto [0,1]$ with Lipschitz constant $L = \frac{\rho\kappa}{\sin\theta_0}$ will satisfy the inequality (3.24). Hence, by Theorem 3.3, the optimization problem **(OP1)** has a unique optimal solution $\theta^* : [0,h^*] \mapsto [\theta_0, \pi/2]$.

Notice that in Theorem 3.3 this solution is written in terms of the variable $y \in [0,h^*]$, and satisfies the optimality condition (3.25). In terms of the arc-length parameter $s \in [0,\ell]$, this corresponds to

$$\Theta^*(s) \;=\; \theta^*(h(s))$$

where the variable $y(s) \in [0, h^*]$ is implicitly defined by

$$\int_0^{y(s)} \frac{1}{\sin\theta^*(z)}\,dz \;=\; s.$$

In view of (2.3), given $I \in \mathcal{J}$ and $\Theta \in \mathcal{K}$, the total sunlight collected by the stem is computed by

$$\mathcal{S}(I,\Theta) \;=\; = \int_0^{\ell} I(y(s)) \cdot \left(1 - \exp\left\{\frac{-\kappa}{\cos(\Theta(s)-\theta_0)}\right\}\right)\cos(\Theta(s)-\theta_0)\,ds, \tag{4.8}$$

where

$$y(s) \;\doteq\; \int_0^{s} \sin\Theta(s)\,ds.$$

From the above formulas it follows that the map $(I,\Theta) \mapsto \mathcal{S}(I,\Theta)$ is continuous on the compact set $\mathcal{J} \times \mathcal{K}$. In particular, the function

$$I \;\mapsto\; \max_{\Theta\in\mathcal{K}} \mathcal{S}(I,\Theta) \tag{4.9}$$

is continuous on the compact set $\mathcal{J}$.

Given a light intensity function $I \in \mathcal{J}$, call $\Theta^*(I) \in \mathcal{K}$ the unique optimal stem shape. We claim that the map $I \mapsto \Theta^*(I)$ is continuous.

Indeed, this is a straightforward consequence of continuity and compactness. If continuity fails, there exists a convergent sequence $I_n \to I$ such that $\Theta(I_n)$ does not converge to $\Theta(I)$. By the compactness of $\mathcal{K}$, we can extract a subsequence such that

$$\Theta_{n_k} \;\to\; \Theta^\sharp \;\neq\; \Theta(I).$$

By continuity, one obtains

$$\begin{aligned}\mathcal{S}(I,\Theta(I)) = \sup_{\Theta\in\mathcal{K}} \mathcal{S}(I,\Theta) \;&=\; \lim_{k\to\infty}\,\sup_{\Theta\in\mathcal{K}} \mathcal{S}(I_{n_k},\Theta)\\ &= \lim_{k\to\infty} \mathcal{S}(I_{n_k},\Theta(I_{n_k}))) \;=\; \mathcal{S}(I,\Theta^\sharp).\end{aligned}$$

This contradicts the uniqueness of the optimal stem configuration, stated in Theorem 3.3. We thus conclude that the map $I \mapsto \Theta^*(I)$ is continuous, completing the proof. □

4.1. Uniqueness and representation of equilibrium solutions

By (3.21) and (4.2), this equilibrium configuration (h^*, θ^*) must satisfy the necessary condition

$$\theta^*(y) \;=\; \varphi\left((e^{-\kappa}-1)\exp\left\{\int_y^{h^*} \frac{\rho\kappa}{\sin\theta^*(y)}\,dy\right\}\right), \qquad y\in[0,h^*], \tag{4.10}$$

where φ is the function defined in Lemma 3.2. Here the constant h^* must be determined so that

$$\int_0^{h^*} \frac{1}{\sin\theta^*(y)}\,dy \;=\; \ell. \tag{4.11}$$

Based on (4.10), one obtains a simple representation of all equilibrium configurations, for any length $\ell > 0$. Indeed, for $t \in]-\infty, 0]$, let $t \mapsto \widehat{\zeta}(t)$ be the solution of the Cauchy problem

$$\zeta' \;=\; -\frac{\rho\kappa}{\sin\theta}, \qquad \text{where} \qquad \theta \;=\; \varphi\Big((e^{-\kappa}-1)\,e^{\zeta}\Big),$$

with terminal condition $\zeta(0) = 0$.

Notice that the corresponding function $t \mapsto \widehat{\theta}(t) = \varphi\Big((e^{-\kappa}-1)\,e^{\widehat{\zeta}(t)}\Big)$ satisfies

$$\widehat{\theta}(0) \;=\; \varphi(e^{-\kappa}-1) \;=\; \theta_0\,.$$

For any length ℓ of the stem, choose $h^* = h^*(\ell)$ so that

$$\int_{-h^*}^{0} \frac{1}{\sin\widehat{\theta}(t)}\,dt \;=\; \ell\,. \tag{4.12}$$

The shape of the stem that achieves the competitive equilibrium is then provided by

$$\theta^*(y) \;=\; \widehat{\theta}(y-h^*)\,, \qquad y\in[0,h^*]. \tag{4.13}$$

Since the backward Cauchy problem

$$\zeta' \;=\; -\frac{\rho\kappa}{\sin\Big(\varphi\big((e^{-\kappa}-1)\,e^{\zeta}\big)\Big)}\,, \qquad \zeta(0) = 0, \tag{4.14}$$

has a unique solution, we conclude that, if an equilibrium solution exists, by the representation (4.13) it must be unique. (See Fig. 3.)

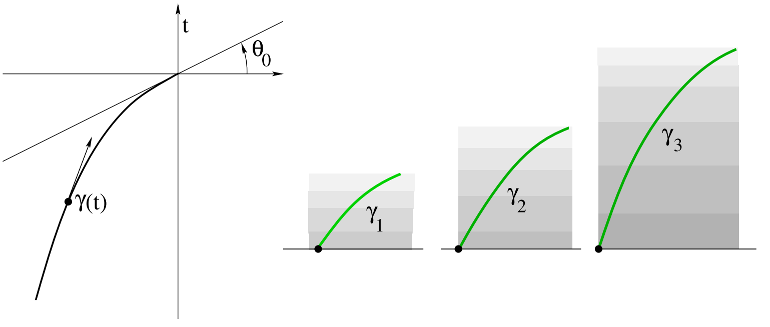

Fig. 3. Left: the curve γ, parameterized by the coordinate t. For $t < 0$, the tangent vector is $\frac{d\gamma}{dt} = (\tan\theta(t), 1)$, where $\theta(t)$ is obtained by solving the Cauchy problem (4.14). Right: for different lengths $0 < \ell_1 < \ell_2 < \ell_3$, the equilibrium configuration is obtained by taking the upper portion of the same curve γ, up to the length ℓ_i, $i = 1, 2, 3$.

5. Stems with variable length and thickness

We now consider the optimization problem **(OP2)**, allowing for stems of different lengths and with variable density of leaves.

5.1. Existence of an optimal solution

Theorem 5.1. *For any bounded, non-decreasing function $y \mapsto I(y) \in [0, 1]$ and any constants $0 < \alpha < 1$, $c > 0$ and $\theta_0 \in]0, \pi/2[$, the optimization problem* **(OP2)** *has at least one solution.*

Proof. 1. Consider a maximizing sequence of couples $(\theta_k, u_k) : \mathbb{R}_+ \mapsto [\theta_0, \pi/2] \times \mathbb{R}_+$. For $k \geq 1$, let

$$s \;\mapsto\; \gamma_k(s) \;=\; \left(\int_0^s \cos\theta_k(s)\,ds\,,\; \int_0^s \sin\theta_k(s)\,ds\right)$$

be the arc-length parameterization of the stem γ_k. Call μ_k the Radon measure on $\mathbb{R}^2$ describing the distribution of leaves along γ_k. For every Borel set $A \subseteq \mathbb{R}^n$, we thus have

$$\mu_k(A) \;=\; \int_{\{s\,;\;\gamma_k(s)\in A\}} u_k(s)\,ds. \tag{5.1}$$

For a given radius $\rho > 0$, we have the decomposition

$$\mu_k \;=\; \mu_k^\flat + \mu_k^\sharp\,,$$

where $\mu_k^\flat$ is the restriction of μ_k to the ball $B(0, \rho)$, while $\mu_k^\sharp$ is the restriction of μ_k to the complement $\mathbb{R}^2 \setminus B(0, \rho)$. By the same arguments used in steps **1-2** of the proof of Theorem 3.1 in [3], if the radius ρ is sufficiently large, then

$$\mathcal{S}(\mu_k^\flat) - c\mathcal{I}^\alpha(\mu_k^\flat) \;\geq\; \mathcal{S}(\mu_k) - c\mathcal{I}^\alpha(\mu_k) \tag{5.2}$$

for all $k \geq 1$. Here $\mathcal{S}$ and $\mathcal{I}^\alpha$ are the functionals defined at (2.12)-(2.13). According to (5.2), we can replace the measure μ_k with $\mu_k^\flat$ without decreasing the objective functional.

Without loss of generality we can thus choose $\ell > 0$ sufficiently large and assume that

$$u_k(s) = 0 \qquad \text{for all } s > \ell, \quad k \geq 1.$$

In turn, since $\mathcal{S}(\mu_k) - c\mathcal{I}^\alpha(\mu_k) \geq 0$, we obtain the uniform bound

$$\mathcal{I}^\alpha(\mu_k) \;\leq\; \kappa_1 \;\doteq\; \frac{1}{c}\mathcal{S}(\mu_k) \;\leq\; \frac{\ell}{c}. \tag{5.3}$$

2. In this step we show that the measures μ_k can be taken with uniformly bounded mass. Consider a measure μ_k for which (5.3) holds. By (2.13), for every $r \in [0, \ell]$ one has

$$\mathcal{I}^\alpha(\mu_k) \;\geq\; r \cdot \left(\int_r^\ell u_k(t)\,dt \right)^\alpha .$$

In view of (5.3), this implies

$$\int_r^\ell u_k(s)\,ds \;\leq\; \left(\frac{\kappa_1}{r}\right)^{1/\alpha}. \tag{5.4}$$

It thus remains to prove that, in our maximizing sequence, the functions u_k can be replaced with functions $\tilde{u}_k$ having a uniformly bounded integral over $[0, r]$, for some fixed $r > 0$.

Toward this goal we fix $0 < \varepsilon < \beta < 1$, and, for $j \geq 1$, we define $r_j = 2^{-j}$, and the interval $V_j =]r_{j+1}, r_j]$. Given $u = u_k$, if $\int_{V_j} u(s)\,ds > r_j^\varepsilon$, we introduce the functions

$$u_j(s) \;\doteq\; \chi_{V_j}(s) u(s), \qquad\qquad \tilde{u}_j(s) \;\doteq\; \min\{u_j(s), c_j\}, \tag{5.5}$$

choosing the constant $c_j \geq 2r_j^{\beta-1}$ so that

$$\int_{V_j} \tilde{u}_j(s)\,ds \;=\; r_j^\beta. \tag{5.6}$$

We then let $\mu_j = u_j\mu$ and $\tilde{\mu}_j = \tilde{u}_j\mu$ be the measures supported on V_j, corresponding to these densities.

For a fixed integer j^*, whose precise value will be chosen later, consider the set of indices

$$J \;\doteq\; \left\{ j \geq j^* \;\middle|\; \int_{V_j} u(s)\,ds > r_j^\varepsilon \right\} \tag{5.7}$$

and the modified density

$$\tilde{u}(s) \doteq u(s) + \sum_{j\in J} (\tilde{u}_j(s) - u_j(s)). \tag{5.8}$$

Moreover, call $\tilde{\mu}$ the measure obtained by replacing u with $\tilde{u}$ in (2.11). By (5.4) and (5.5) the total mass of $\tilde{\mu}$ is bounded. Indeed

$$\tilde{\mu}(\mathbb{R}^2) = \int_{r_{j^*}}^{\ell} \tilde{u}(s)\,ds + \int_0^{r_{j^*}} \tilde{u}(s)\,ds \le \left(\frac{\kappa_1}{r_{j^*}}\right)^{1/\alpha} + \sum_{j\ge j^*} r_j^{\varepsilon} \le \left(\frac{\kappa_1}{r_{j^*}}\right)^{1/\alpha} + \sum_{j\ge 1} 2^{-j\varepsilon} < +\infty. \tag{5.9}$$

We now claim that

$$\mathcal{S}(\tilde{\mu}) - c\mathcal{I}^{\alpha}(\tilde{\mu}) \ \ge\ \mathcal{S}(\mu) - c\mathcal{I}^{\alpha}(\mu). \tag{5.10}$$

Toward a proof of (5.10), we estimate

$$\begin{aligned} \mathcal{S}(\mu) - \mathcal{S}(\tilde{\mu}) \ &\le \sum_{j\in J} \Bigg(\int_{V_j} I(y(t)) \cos(\theta(t) - \theta_0)\,dt \\ &\qquad - \int_{V_j} I(y(t)) \left(1 - \exp\left\{ - \frac{\tilde{u}_j(t)}{\cos(\theta(t)-\theta_0)} \right\}\right) \cos(\theta(t) - \theta_0)\,dt \Bigg) \\ &\le \sum_{j\in J} \int_{r_{j+1}}^{r_j} \exp\big\{-\tilde{u}_j(t)\big\} dt \ \le\ \sum_{j\in J} r_{j+1} \exp\left\{-2 r_j^{\beta-1}\right\}. \end{aligned} \tag{5.11}$$

To estimate the difference in the irrigation cost, we first observe that the inequality

$$\left(\int_r^{\ell} u(t)\,dt\right)^{\alpha} \ \le\ \frac{1}{r}\mathcal{I}^{\alpha}(\mu) \ =\ \frac{\kappa_1}{r}$$

implies

$$\left(\int_r^{\ell} u(t)\,dt\right)^{\alpha-1} \ \ge\ \left(\frac{\kappa_1}{r}\right)^{\frac{\alpha-1}{\alpha}}. \tag{5.12}$$

Since $\tilde{u}(s) \le u(s)$ for every $s \in [0, \ell]$, using (5.12) we now obtain

$$\begin{aligned} \mathcal{I}^{\alpha}(\mu) - \mathcal{I}^{\alpha}(\tilde{\mu}) &= \int_0^1 \frac{d}{d\lambda} \mathcal{I}^{\alpha}\big(\lambda\mu + (1-\lambda)\tilde{\mu}\big)\,d\lambda \\ &= \int_0^1 \int_0^{\ell} \frac{d}{d\lambda} \left(\int_s^{\ell} [\lambda u(t) + (1-\lambda)\tilde{u}(t)]\,dt \right)^{\alpha} ds\,d\lambda \end{aligned}$$

$$\begin{aligned}
&= \int_0^1\int_0^\ell \left\{\alpha\left(\int_s^\ell [\lambda u(t)+(1-\lambda)\tilde u(t)]\,dt\right)^{\alpha-1}\int_s^\ell [u(t)-\tilde u(t)]\,dt\right\} ds\,d\lambda\\
&\geq \int_0^\ell \left\{\alpha\left(\int_s^\ell u(t)\,dt\right)^{\alpha-1}\int_s^\ell [u(t)-\tilde u(t)]\,dt\right\} ds\\
&\geq \sum_{j\in J}\int_{r_{j+2}}^{r_{j+1}}\left[\alpha\left(\int_s^\ell u(t)\,dt\right)^{\alpha-1}\int_{r_{j+1}}^{r_j}(u_j(t)-\tilde u_j(t))\,dt\right] ds\\
&\geq \sum_{j\in J}\alpha\left(\frac{\kappa_1}{r_{j+2}}\right)^{\frac{\alpha-1}{\alpha}}\cdot(r_j^\varepsilon-r_j^\beta)\cdot r_{j+2}\\
&= \sum_{j\in J}\kappa_2 r_j^{1/\alpha}(r_j^\varepsilon-r_j^\beta),
\end{aligned} \tag{5.13}$$

where $\kappa_2=\alpha(4\kappa_1)^{\frac{\alpha-1}{\alpha}}$. Combining (5.11) with (5.13) we obtain

$$c[\mathcal{I}^\alpha(\mu)-\mathcal{I}^\alpha(\tilde\mu)]-[\mathcal{S}(\mu)-\mathcal{S}(\tilde\mu)] \;\geq\; \sum_{j\in J}\left(c\kappa_2 r_j^{1/\alpha}(r_j^\varepsilon-r_j^\beta)-r_{j+1}\exp\left\{-2r_j^{\beta-1}\right\}\right). \tag{5.14}$$

By choosing the integer j^* large enough in (5.7), for $j\geq j^*$ all terms in the summation on the right hand side of (5.14) are ≥ 0. This implies (5.10).

3. By the two previous steps, w.l.o.g. we can assume that the measures μ_k have uniformly bounded support and uniformly bounded total mass. Otherwise, we can replace the sequence $(u_k)_{k\geq 1}$ with a new maximizing sequence $(\tilde u_k)_{k\geq 1}$ having these properties.

By taking a subsequence, we can thus assume the weak convergence $\mu_k \rightharpoonup \overline{\mu}$. The upper semicontinuity of the functional $\mathcal{S}$, proved in [5], yields

$$\mathcal{S}(\overline{\mu}) \;\geq\; \limsup_{k\to\infty}\;\mathcal{S}(\mu_k). \tag{5.15}$$

In addition, since all maps $s\mapsto\gamma_k(s)$ are 1-Lipschitz, by taking a further subsequence we can assume the convergence

$$\gamma_k(s) \;\to\; \overline{\gamma}(s) \tag{5.16}$$

for some limit function $\overline{\gamma}$, uniformly for $s\in[0,\ell]$.

Since each measure μ_k is supported on γ_k, the weak limit $\overline{\mu}$ is a measure supported on the curve $\overline{\gamma}$.

4. Since $\theta_k(s)\in[\theta_0,\pi/2]$, we can re-parameterize each stem γ_k in terms of the vertical variable

$$y_k(s) \;=\; \int_0^s \sin\theta_k(t)\,dt.$$

Calling $s = s_k(y)$ the inverse function, we thus obtain a maximizing sequence of couples

$$y \mapsto (\widehat{\theta}_k(y), \widehat{u}_k(y)) \doteq \Big(\theta_k(s_k(y)),\, u_k(s_k(y))\Big), \qquad y \in [0, h_k].$$

Moreover, the stem γ_k can be described as the graph of the Lipschitz function

$$x = x_k(y) = \int_0^{s_k(y)} \cos\theta_k(s)\,ds.$$

Since all functions $x_k(\cdot)$ satisfy $x_k(0) = 0$ and are non-decreasing, uniformly continuous with Lipschitz constant $L = \cos\theta_0 / \sin\theta_0$, by possibly extracting a further subsequence, we obtain the convergence $h_k \to \bar{h}$ and $x_k(\cdot) \to \bar{x}(\cdot)$. Here $\bar{x} : [0, \bar{h}] \mapsto \mathbb{R}$ is a nondecreasing continuous function with Lipschitz constant L, such that $\bar{x}(0) = 0$. More precisely, the convergence $x_k \to \bar{x}$ is uniform on every compact subinterval $[0, h]$ with $h < \bar{h}$.

5. We claim that the irrigation cost of $\overline{\mu}$ is no greater that the lim-inf of the irrigation costs for μ_k. Let $\sigma \mapsto \gamma(\sigma)$ be an arc-length parameterization of $\overline{\gamma}$. Since $s \mapsto \overline{\gamma}(s)$ is 1-Lipschitz, one has $d\sigma/ds \le 1$. We now compute

$$\begin{aligned}\mathcal{I}^\alpha(\overline{\mu}) &= \int_0^{\sigma(\ell)} \left(\int_\sigma^{\sigma(\ell)} \overline{u}(t)\,dt \right)^\alpha d\sigma = \int_0^{\sigma(\ell)} \left(\lim_{k\to\infty} \int_s^\ell u_k(t)\,dt \right)^\alpha d\sigma(s) \\ &\le \lim_{k\to\infty} \int_0^\ell \left(\int_s^\ell u_k(t)\,dt \right)^\alpha ds = \lim_{k\to\infty} \mathcal{I}^\alpha(\mu_k).\end{aligned} \tag{5.17}$$

6. Combining (5.15) with (5.17) we conclude that the measure $\overline{\mu}$, supported on the stem $\overline{\gamma}$, is optimal.

Let $\bar{u}$ be the density of the absolutely continuous part of $\overline{\mu}$ w.r.t. the arc-length measure on $\bar{\gamma}$, and call μ^* the measure that has density $\bar{u}$ w.r.t. arc-length measure. Since $\mathcal{S}(\mu^*) = \mathcal{S}(\overline{\mu})$, it follows that $\mu^* = \overline{\mu}$. Otherwise $\mathcal{I}^\alpha(\mu^*) < \mathcal{I}^\alpha(\overline{\mu})$ and $\overline{\mu}$ is not optimal. This argument shows that the optimal measure $\overline{\mu}$ is absolutely continuous w.r.t. the arc-length measure on $\overline{\gamma}$.

Calling $\sigma \mapsto \gamma(\sigma)$ the arc-length parameterization of $\overline{\gamma}$, the optimal solution to **(OP2)** is now provided by $\sigma \mapsto (\overline{\theta}(\sigma), \bar{u}(\sigma))$, where $\overline{\theta}$ is the orientation of the tangent vector:

$$\frac{d}{d\sigma}\overline{\gamma}(\sigma) = \big(\cos\overline{\theta}(\sigma),\, \sin\overline{\theta}(\sigma)\big). \quad \square$$

5.2. Necessary conditions for optimality

Let $t \mapsto (\theta^*(t), u^*(t))$ be an optimal solution to the problem **(OP2)**. The necessary conditions for optimality [4,6,7] yield the existence of dual variables p, q satisfying

$$\begin{cases} \dot{p} = -I'(y)\,G(\theta, u), \\ \dot{q} = c\alpha\, z^{\alpha-1}, \end{cases} \qquad \begin{cases} p(+\infty) = 0, \\ q(0) = 0, \end{cases} \tag{5.18}$$

and such that the maximality condition

$$(\theta^*(t), u^*(t)) = \arg\max_{\theta\in[0,\pi],\ u\geq 0} \Big\{p(t)\,\sin\theta - q(t)u + I(y(t))\,G(\theta,u) - cz^\alpha\Big\}. \tag{5.19}$$

We recall that $G(\theta, u)$ is the function defined at (2.17). An intuitive interpretation of the quantities on the right-hand side of (5.19) goes as follows:

- $p(t)$ is the rate of increase in the gathered sunlight, if the upper portion of stem $\{\gamma(s)\,;\ \ s > t\}$ is raised higher.
- $q(t)$ is the rate at which the irrigation cost increases, adding mass at the point $\gamma(t)$.
- $I(y(t))\,G(\theta, u)$ is the sunlight captured by the leaves at the point $\gamma(t)$.

6. Uniqueness of the optimal stem configuration

Aim of this section is to show that, if the light intensity $I(y)$ remains sufficiently close to 1 for all $y \geq 0$, then the shape of the optimal stem is uniquely determined. This models a case where the density of external vegetation is small.

Theorem 6.1. *Let $h \mapsto I(h) \in [0, 1]$ be a non-decreasing, absolutely continuous function which satisfies*

$$I'(y) \leq Cy^{-\beta} \qquad \text{for a.e. } y > 0, \tag{6.1}$$

for some constants $C > 0$ and $0 < \beta < 1$. If

$$I(0) \geq 1 - \delta \tag{6.2}$$

for some $\delta > 0$ sufficiently small, then the optimal solution to **(OP2)** *is unique.*

Proof. We will show that the necessary conditions for optimality have a unique solution. This will be achieved in several steps. **1.** Given I, p, q, define the functions Θ, U by setting

$$\Big(\Theta(I, p, q),\ U(I, p, q)\Big) \doteq \arg\max_{\theta\in[0,\pi],\ u\geq 0} \Big\{p\cdot\sin\theta - q\,u + I\cdot G(\theta,u) - cz^\alpha\Big\}. \tag{6.3}$$

We recall that G is the function defined at (2.17). Notice that one can write

$$G(\theta, u) = u\,\widetilde{G}\left(\frac{\cos(\theta-\theta_0)}{u}\right)$$

with

$$\widetilde{G}(x) \doteq \left(1 - \exp\left\{-\frac{1}{x}\right\}\right)x > 0, \qquad \widetilde{G}'(x) \leq 1, \qquad \widetilde{G}''(x) \leq 0, \qquad \text{for all } x > 0. \tag{6.4}$$

Denote by

$$\mathcal{H}(\theta, u) \doteq p\cdot\sin\theta - q\,u + I(y)\,G(\theta,u) - cz^\alpha \tag{6.5}$$

the quantity to be maximized in (6.3). Differentiating $\mathcal{H}$ w.r.t. θ and imposing that the derivative is zero, we obtain

$$\begin{aligned} \frac{p}{I} &= -\frac{G_\theta(\theta,u)}{\cos\theta} \\ &= \frac{\sin(\theta-\theta_0)}{\cos\theta}\left[1-\exp\left\{-\frac{u}{\cos(\theta-\theta_0)}\right\}-\frac{u}{\cos(\theta-\theta_0)}\exp\left\{-\frac{u}{\cos(\theta-\theta_0)}\right\}\right]. \end{aligned} \tag{6.6}$$

Similarly, differentiating $\mathcal{H}$ w.r.t. u, we find

$$-q+IG_u(\theta,u) \;=\; -q+I\exp\left\{-\frac{u}{\cos(\theta-\theta_0)}\right\} \;=\; 0.$$

This yields

$$u \;=\; -\ln\left(\frac{q}{I}\right)\cos(\theta-\theta_0). \tag{6.7}$$

A lengthy but elementary computation shows that the Hessian matrix of second derivatives of $\mathcal{H}$ w.r.t. θ, u is negative definite, and the critical point is indeed the point where the global maximum is attained. By (6.7) it follows

$$U(I,p,q) \;=\; -\ln\left(\frac{q}{I}\right)\cos\big(\Theta(I,p,q)-\theta_0\big). \tag{6.8}$$

Inserting (6.8) in (6.6) and using the identity

$$\frac{\sin(\theta-\theta_0)}{\cos\theta} \;=\; \cos\theta_0\tan\theta-\sin\theta_0$$

we obtain

$$\Theta(I,p,q) \;=\; \arctan\left(\tan\theta_0+\frac{\frac{1}{\cos\theta_0}\,\frac{p}{I}}{1-\frac{q}{I}+\frac{q}{I}\ln\left(\frac{q}{I}\right)}\right). \tag{6.9}$$

Introducing the function

$$w(I,p,q) \;\doteq\; \frac{p/I}{1-\frac{q}{I}+\frac{q}{I}\ln\left(\frac{q}{I}\right)}, \tag{6.10}$$

by (6.9) one has the identities

$$\begin{cases} \sin\big(\Theta(I,p,q)\big) \;=\; \dfrac{\sin\theta_0+w}{\sqrt{\cos^2\theta_0+(w+\sin\theta_0)^2}}, \\ \cos\big(\Theta(I,p,q)-\theta_0\big) \;=\; \dfrac{1+w\sin\theta_0}{\sqrt{\cos^2\theta_0+(w+\sin\theta_0)^2}}. \end{cases} \tag{6.11}$$

Note that $w\geq 0$, because $p,q,I\geq 0$. In turn, from (6.11) it follows

$$\begin{cases} \cos\big(\Theta(I,p,q)\big) \;=\; \dfrac{\cos\theta_0}{\sqrt{\cos^2\theta_0+(w+\sin\theta_0)^2}}, \\ \sin\left(\Theta(I,p,q)-\theta_0\right) \;=\; \dfrac{w\cos\theta_0}{\sqrt{\cos^2\theta_0+(w+\sin\theta_0)^2}}. \end{cases} \tag{6.12}$$

2. The necessary conditions for the optimality of a solution to **(OP2)** yield the boundary value problem

$$\begin{cases} \dot y(t) \;=\; \sin\Theta, \\ \dot z(t) \;=\; -U, \\ \dot p(t) \;=\; -I'(y)G\big(\Theta,U\big), \\ \dot q(t) \;=\; c\alpha z^{\alpha-1}, \end{cases} \qquad \begin{cases} y(0) \;=\; 0, \\ z(T) \;=\; 0, \\ p(T) \;=\; 0, \\ q(T) \;=\; I(y(T)), \\ q(0) \;=\; 0. \end{cases} \tag{6.13}$$

Here $[0,T[$ is the interval where $u>0$, while

$$\Theta \;=\; \Theta(I(y),p,q), \qquad U \;=\; U(I(y),p,q) \tag{6.14}$$

are the functions introduced at (6.3), or more explicitly at (6.8)-(6.9). Notice that the length T of the stem is a quantity to be determined, using the boundary conditions in (6.13).

3. Since the control system (2.19) and the running cost (2.18) do not depend explicitly on time, the Hamiltonian function

$$H(y,z,p,q) \;\doteq\; \max_{\theta\in[0,\pi],\; u\geq 0}\Big\{p\cdot\sin\theta-q\,u+I(y)\,G(\theta,u)-cz^\alpha\Big\} \tag{6.15}$$

is constant along trajectories of (6.13). Observing that the terminal conditions in (6.13) imply $H(y(T),z(T),p(T),q(T))=0$, one has the first integral

$$H(y(t),z(t),p(t),q(t)) \;=\; 0 \qquad\qquad \text{for all } t\in[0,T]. \tag{6.16}$$

This yields

$$\begin{aligned} 0 \;&=\; p\sin\Theta+\left[I(y)-q+q\ln\left(\frac{q}{I(y)}\right)\right]\cos\left(\Theta-\theta_0\right)-cz^\alpha \\ &=\; \frac{p\,[\sin\theta_0+w]+\left[I(y)-q+q\ln\left(\frac{q}{I(y)}\right)\right][1+w\sin\theta_0]}{\sqrt{\cos^2\theta_0+(w+\sin\theta_0)^2}}-cz^\alpha \\ &=\; I(y)\left[1-\frac{q}{I(y)}+\frac{q}{I(y)}\ln\left(\frac{q}{I(y)}\right)\right]\sqrt{\cos^2\theta_0+(w+\sin\theta_0)^2}-cz^\alpha. \end{aligned}$$

We can use this identity to express z as a function of the other variables:

$$z\big(I(y),p,q\big) = \left\{\frac{I(y)}{c}\left[1-\frac{q}{I(y)}+\frac{q}{I(y)}\ln\left(\frac{q}{I(y)}\right)\right]\sqrt{\cos^2\theta_0+(w+\sin\theta_0)^2}\right\}^{1/\alpha}$$
$$= c^{-1/\alpha}\left\{\left(\left[I(y)-q+q\ln\Big(\frac{q}{I(y)}\Big)\right]\cos\theta_0\right)^2 + \left(p+\left[I(y)-q+q\ln\Big(\frac{q}{I(y)}\Big)\right]\sin\theta_0\right)^2\right\}^{1/2\alpha}. \tag{6.17}$$

4. Since I is given as a function of the height y, it is convenient to rewrite the equations (6.13) using y as an independent variable. Using the identity (6.17), we obtain a system of two equations for the variables p, q:

$$\begin{aligned}\frac{d}{dy}p(y) &= -I'(y)\left[1-\frac{q(y)}{I(y)}\right]\frac{\cos\big(\Theta\big(I(y),p(y),q(y)\big)-\theta_0\big)}{\sin\Theta\big(I(y),p(y),q(y)\big)}\\ &= -I'(y)\left[1-\frac{q(y)}{I(y)}\right]\frac{1+w\sin\theta_0}{w+\sin\theta_0}\\ &\doteq -I'(y)\,f_1\big(I(y),p(y),q(y)\big),\end{aligned} \tag{6.18}$$

$$\begin{aligned}\frac{d}{dy}q(y) &= \frac{c\alpha\big[z\big(I(y),p(y),q(y)\big)\big]^{\alpha-1}}{\sin\Theta\big(I(y),p(y),q(y)\big)}\\ &= \frac{\alpha c^{1/\alpha}}{w+\sin\theta_0}\left[\cos^2\theta_0+(\sin\theta_0+w)^2\right]^{1-\frac{1}{2\alpha}}\\ &\quad\times\left[I(y)\left(1-\frac{q}{I(y)}+\frac{q}{I(y)}\ln\left(\frac{q}{I(y)}\right)\right)\right]^{1-\frac{1}{\alpha}}\\ &\doteq f_2\big(I(y),p(y),q(y)\big),\end{aligned} \tag{6.19}$$

where $w = w(I,p,q)$ is the function introduced at (6.10). Note that under our assumptions, f_1 remains bounded, while f_2 diverges as $q(y) \to I(y)$. The system (6.13) can now be equivalently formulated as

$$\begin{cases} p'(y) = -I'(y)\,f_1\big(I(y),p,q\big),\\ q'(y) = f_2\big(I(y),p,q\big),\end{cases}\qquad \begin{cases} p(h) = 0,\\ q(h) = I(h),\end{cases}\qquad q(0)=0. \tag{6.20}$$

5. To prove uniqueness of the solution to the boundary value problem (6.13), it thus suffices to prove the following (see Fig. 4, right).

(U) *Call*

$$y \mapsto \big(p(y,h),\ q(y,h)\big) \tag{6.21}$$

the solution to the system (6.20), with the two terminal conditions given at $y = h$. Then there is a unique choice of $h > 0$ which satisfies also the third boundary condition

$$q(0,h) = 0. \tag{6.22}$$

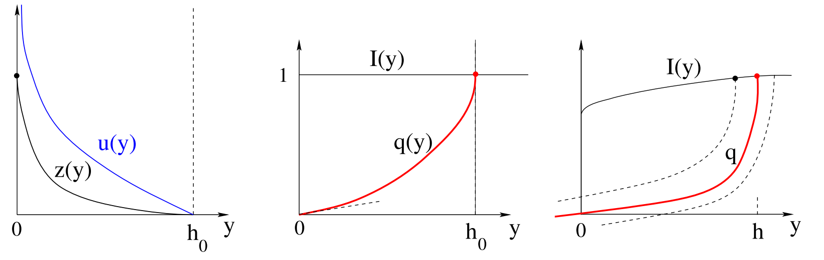

Fig. 4. Left and center: sketch of the solution of the system (5.18) in the case where $I(y) \equiv 1$. Left: the graphs of the functions z in (6.25) and $u = -\ln q$. Center: the graph of the function q at (6.26). The figure on the right shows the case where $I(\cdot)$ is not constant. As before, h must be determined so that $q(0, h) = 0$.

To make the argument more clear, the uniqueness property **(U)** will be proved in two steps.

(i) When $I(y) \equiv 1$, the map

$$h \mapsto q(0, h) \tag{6.23}$$

is strictly decreasing, hence it vanishes at a unique point h_0.

(ii) For all functions $I(\cdot)$ sufficiently close to the constant map $\equiv 1$, the map (6.23) is strictly decreasing in a neighborhood of h_0.

In the case $I(y) \equiv 1$, recalling (6.9) we obtain (see Fig. 4)

$$I'(y) = 0, \qquad p(y, h) = 0, \qquad \Theta(I, 0, q) = \theta_0, \qquad G(\theta_0, U) = 1 - e^{-U},$$
$$U(1, 0, q) = \operatorname*{argmax}_u \left\{-qu + G(\theta_0, U)\right\} = \operatorname*{argmax}_u \{-qu + 1 - e^{-u}\} = -\ln q,$$

The system (6.13) can now be written as

$$\begin{cases} p'(y) = 0, \\ q'(y) = \dfrac{c\alpha z^{\alpha-1}}{\sin\theta_0}, \\ z'(y) = \dfrac{\ln q}{\sin\theta_0}, \end{cases} \qquad \begin{cases} p(h) = 0, \\ q(h) = 1, \\ z(h) = 0, \end{cases} \qquad q(0) = 0. \tag{6.24}$$

From (6.24) it follows $p(y) \equiv 0$, while

$$\frac{dz}{dq} = \frac{\ln q}{c\alpha z^{\alpha-1}}.$$

Integrating the above ODE with terminal conditions $q = 1$, $z = 0$, one obtains

$$z = c^{-1/\alpha}\Big[1 + q\ln q - q\Big]^{1/\alpha}. \tag{6.25}$$

The second equation in (6.24) thus becomes

$$q'(y) \;=\; \frac{\alpha c^{1/\alpha}}{\sin\theta_0}\Big[1+q\ln|q|-q\Big]^{\frac{\alpha-1}{\alpha}}. \tag{6.26}$$

Notice that here the right hand side is strictly positive for all $q\in\,]-1,1[$. Of course, only positive values of q are relevant for the optimization problem, but for the analysis it is convenient to extend the definition also to negative values of q. The solution of (6.26) with terminal condition $q(h)=1$ is implicitly determined by

$$h-y \;=\; \frac{\sin\theta_0}{\alpha c^{1/\alpha}}\int_{q(y)}^{1}\Big[1+s\ln|s|-s\Big]^{\frac{1-\alpha}{\alpha}}\,ds\,. \tag{6.27}$$

The map $h\mapsto q(0,h)$ thus vanishes at the unique point

$$h_0 \;=\; \frac{\sin\theta_0}{\alpha c^{1/\alpha}}\int_{0}^{1}\Big[1+s\ln|s|-s\Big]^{\frac{1-\alpha}{\alpha}}\,ds. \tag{6.28}$$

As expected, the height h_0 of the optimal stem decreases as we increase the constant c in the transportation cost. A straightforward computation yields

$$\frac{\partial}{\partial h}q(0,h) \;=\; -\frac{\alpha c^{1/\alpha}}{\sin\theta_0}\Big[1+q(0,h)\ln|q(0,h)|-q(0,h)\Big]^{\frac{1-\alpha}{\alpha}}. \tag{6.29}$$

In particular, at $h=h_0$ we have $q^{(h_0)}(0)=0$ and hence

$$\frac{d}{dh}q(0,h)\bigg|_{h=h_0} \;=\; -\frac{\alpha c^{1/\alpha}}{\sin\theta_0} \;<\; 0. \tag{6.30}$$

6. We will show that a strict inequality as in (6.30) remains valid for a more general function $I(\cdot)$, provided that the assumptions (6.1)-(6.2) hold.

Toward this goal, we need to determine how p and q vary w.r.t. the parameter h. Denoting by

$$P(y)\;\doteq\;\frac{\partial p(y,h)}{\partial h},\qquad Q(y)\;\doteq\;\frac{\partial q(y,h)}{\partial h} \tag{6.31}$$

their partial derivatives, by (6.20) one obtains the linear system

$$\begin{pmatrix}P(y)\\Q(y)\end{pmatrix}' \;=\; \begin{pmatrix}-I'(y)f_{1,p} & -I'(y)f_{1,q}\\ f_{2,p} & f_{2,q}\end{pmatrix}\begin{pmatrix}P(y)\\Q(y)\end{pmatrix}. \tag{6.32}$$

The boundary conditions at $y=h$ require some careful consideration. As $y\to h-$, we expect $f_2(I(y),p(y),q(y))\to+\infty$ and $Q(y)\to-\infty$. To cope with this singularity we introduce the new variable

$$\widetilde{Q}(y)\;\doteq\;\frac{Q(y)}{f_2\big(I(y),p(y),q(y)\big)}\,. \tag{6.33}$$

The system (6.32), together with the new boundary conditions for P, $\widetilde{Q}$, can now be written as

$$\begin{cases} P'(y) \;=\; -I'(y)\big[f_{1,p}P + f_{1,q}f_2\widetilde{Q}\big], \\ \widetilde{Q}'(y) \;=\; \dfrac{f_{2,p}}{f_2}P - \dfrac{I'(y)[f_{2,I} - f_{2,p}f_1]}{f_2}\widetilde{Q}, \end{cases} \qquad \begin{cases} P(h) \;=\; 0, \\ \widetilde{Q}(h) \;=\; -1. \end{cases} \tag{6.34}$$

To analyze this system we must compute the partial derivatives of f_1 and f_2. From the definition (6.10) it follows

$$\frac{\partial w}{\partial I} \;=\; \frac{w^2}{p}\Big[1-\frac{q}{I}\Big], \qquad \frac{\partial w}{\partial p} \;=\; \frac{w}{p}, \qquad \frac{\partial w}{\partial q} \;=\; -\frac{w^2}{p}\ln\Big(\frac{q}{I}\Big). \tag{6.35}$$

Using (6.35), from (6.18), (6.19) we obtain

$$\begin{cases} f_{1,p}\big(I(y),p,q\big) \;=\; \dfrac{1-\frac{q}{I(y)}}{I(y)\tan^2\Theta\Big[1-\frac{q}{I(y)}+\frac{q}{I(y)}\ln\Big(\frac{q}{I(y)}\Big)\Big]}, \\ f_{1,q}\big(I(y),p,q\big) \;=\; \dfrac{1}{I(y)}\dfrac{\cos(\Theta-\theta_0)}{\sin\Theta} - \dfrac{\sin(\Theta-\theta_0)\cos\Theta\Big[1-\frac{q}{I(y)}\Big]\ln\big(\frac{q}{I}\big)}{I(y)\sin^2\Theta\Big[1-\frac{q}{I(y)}+\frac{q}{I(y)}\ln\Big(\frac{q}{I(y)}\Big)\Big]}, \\ f_{2,p}\big(I(y),p,q\big) \;=\; -\Big[1+\dfrac{\alpha}{\sin^2\Theta}-2\alpha\Big]\dfrac{1}{z\big(I(y),p,q\big)}, \\ f_{2,q}\big(I(y),p,q)\big) \;=\; -\Big[\dfrac{(1-\alpha)\sin\theta_0}{\sin^2\Theta} - \dfrac{\sin(\Theta-\theta_0)}{\cos\Theta}\Big(1+\dfrac{\alpha}{\sin^2\Theta}-2\alpha\Big)\Big]\dfrac{\ln\Big(\frac{q}{I(y)}\Big)}{z\big(I(y),p,q\big)}, \\ f_{2,I}\big(I(y),p,q\big) \;=\; -\Big[\dfrac{(1-\alpha)\sin\theta_0}{\sin^2\Theta} + \dfrac{\sin(\Theta-\theta_0)}{\cos\Theta}\Big(1+\dfrac{\alpha}{\sin^2\Theta}-2\alpha\Big)\Big]\dfrac{1-\frac{q}{I(y)}}{z\big(I(y),p,q\big)}. \end{cases} \tag{6.36}$$

At this stage, the strategy of the proof is straightforward. When $I'(y)\equiv 0$, the solution to (6.34) is trivially given by $P(y)\equiv 0$, $\widetilde{Q}(y)\equiv -1$. This implies

$$\frac{\partial}{\partial h}q(0,h) \;=\; \widetilde{Q}(0)\cdot f_2(I(0),p(0),q(0)) \;<\; 0.$$

We need to show that the same strict inequality holds when $\delta > 0$ in (6.2) is small enough. Notice that, if the right hand sides of the equations in (6.34) were bounded, letting $\|I'\|_{\mathbf{L}^\infty}\to 0$ a continuity argument would imply the uniform convergence $P(y)\to 0$ and $\widetilde{Q}(y)\to -1$. The same conclusion can be achieved provided that the right hand sides in (6.34) are uniformly integrable. This is precisely what will be proved in the next two steps, relying on the identities (6.36).

7. In this step we prove an inequality of the form

$$0 \;<\; \theta_0 \;\le\; \Theta(I,p,q) \;\le\; \theta^+ \;<\; \frac{\pi}{2}. \tag{6.37}$$

As a consequence, this implies that all terms in (6.36) involving $\sin\Theta$ or $\cos\Theta$ remain uniformly positive.

The lower bound $\Theta \geq \theta_0$ is an immediate consequence of (6.9). To obtain an upper bound on Θ, we set

$$q^\sharp \doteq \frac{q(y)}{I(y)}.$$

By (6.13), a differentiation yields

$$\dot q^\sharp = \frac{c\alpha z^{\alpha-1} - q^\sharp I' \sin(\Theta)}{I}.$$

Next, we observe that, by (6.13), one has

$$\frac{dz}{dq^\sharp} = \ln q^\sharp \cdot \cos(\Theta-\theta_0) \cdot \frac{I}{c\alpha z^{\alpha-1} - q^\sharp I' \sin(\Theta)} = \varphi_1(q^\sharp) \cdot \ln q^\sharp \cdot \alpha z^{\alpha-1}, \qquad \begin{cases} z(h) = 0, \\ q^\sharp(h) = 1. \end{cases}$$

In (6.2) we can now choose $\delta \leq c\alpha M^{\alpha-1}$, where $M \geq z(0)$ is an a priori bound on the mass of the stem, derived in Section 5. This ensures that φ_1 is a bounded, uniformly positive function for y close enough to h, say

$$0 < c^- \leq \varphi_1 \leq c^+,$$

for some constants c^-, c^+. Integrating, we obtain

$$z^\alpha = \int_0^z \alpha \zeta^{\alpha-1}\, d\zeta = -\int_{q^\sharp}^1 \varphi_1(s) \ln s\, ds = -\varphi_2(q^\sharp) \int_{q^\sharp}^1 \ln s\, ds = \varphi_3(q^\sharp) \cdot (1-q^\sharp)^2, \quad (6.38)$$

and

$$\frac{dq^\sharp}{dy} = \frac{c\alpha}{\sin\Theta} \left(-\int_{q^\sharp}^1 \varphi_1(s) \ln s\, ds \right)^{\frac{\alpha-1}{\alpha}} = \varphi_4(q^\sharp) \cdot \left(-\int_{q^\sharp}^1 \ln s\, ds \right)^{\frac{\alpha-1}{\alpha}}$$
$$= \varphi_5(q^\sharp) \cdot (1-q^\sharp)^{\frac{2(\alpha-1)}{\alpha}}. \quad (6.39)$$

Here the φ_k are uniformly positive, bounded functions. Integrating (6.39) we obtain

$$\int_{q^\sharp}^1 \frac{1}{\varphi_5(s)} (1-s)^{\frac{2(1-\alpha)}{\alpha}}\, ds = h - y. \quad (6.40)$$

To fix the ideas, assume

$$0 < c_3 \leq \varphi_5(s) \leq C_3.$$

Then

$$\frac{1}{c_3}\int_{q^\sharp}^{1}(1-s)^{\frac{2(1-\alpha)}{\alpha}}\,ds \;=\; \frac{\alpha}{(2-\alpha)c_3}(1-q^\sharp)^{\frac{2-\alpha}{\alpha}}\,ds \;\geq\; h-y.$$

$$1-q^\sharp(y)\;\geq\;\left(\frac{(2-\alpha)c_3}{\alpha}\right)^{\frac{\alpha}{2-\alpha}}(h-y)^{\frac{\alpha}{2-\alpha}}. \tag{6.41}$$

A similar argument yields

$$1-q^\sharp(y)\;\leq\;\left(\frac{(2-\alpha)C_3}{\alpha}\right)^{\frac{\alpha}{2-\alpha}}(h-y)^{\frac{\alpha}{2-\alpha}}. \tag{6.42}$$

Using (6.1) and (6.42) in the equation (6.18) we obtain a bound of the form

$$-p'(y)\;\leq\;C_1(1-q(y))\;\leq\;C_2(h-y)^{\frac{\alpha}{2-\alpha}} \tag{6.43}$$

for y in a left neighborhood of h, which yields

$$p(y)\;\leq\;\frac{C_2}{\alpha+1}(h-y)^{\frac{2}{2-\alpha}}. \tag{6.44}$$

Since $\alpha<1$, using (6.41) and (6.44) in (6.9) we obtain the limit $\Theta(y)\to\theta_0$ as $y\to h-$.

On the other hand, when y is bounded away from h, the denominator in (6.10) is strictly positive and the quantity $w=w(I,p,q)$ remains uniformly bounded. By (6.9), we obtain the upper bound $\Theta\leq\theta^+$, for some $\theta^+<\pi/2$.

8. Relying on (6.36), in this step we prove that all terms on the right hand sides of the ODEs in (6.34) are uniformly integrable.

(i) We first consider the terms appearing in the ODE for $P(y)$. Concerning $f_{1,p}$, as $y\to h-$ one has

$$f_{1,p}\;=\;\mathcal{O}(1)\cdot\left(1-\frac{q}{I}\right)^{-1}\;=\;\mathcal{O}(1)\cdot(h-y)^{\frac{-\alpha}{2-\alpha}}, \tag{6.45}$$

because of (6.41). Since $\alpha<1$, this implies that $f_{1,p}$ is an integrable function of y.

(ii) By the second equation in (6.36), as $y\to h-$ one has

$$f_{1,q}\;=\;\mathcal{O}(1)\cdot\frac{(1-q^\sharp)\ln(q^\sharp)}{1-q^\sharp+q^\sharp\ln(q^\sharp)}\;=\;\mathcal{O}(1). \tag{6.46}$$

(iii) The term f_2 blows up as $y\to h-$, due to the factor $z^{\alpha-1}$. However, this factor is integrable in y because, by (6.38), (6.41) and (6.42)

$$z^\alpha\big(I(y),p(y),q(y)\big)\;=\;\mathcal{O}(1)\cdot(h-y)^{\frac{2\alpha}{2-\alpha}}. \tag{6.47}$$

This implies

$$f_2\big(I(y),p(y),q(y)\big) = \mathcal{O}(1)\cdot z^{\alpha-1}\big(I(y),p(y),q(y)\big) \\ = \mathcal{O}(1)\cdot (h-y)^{-1+\frac{\alpha}{2-\alpha}}, \tag{6.48}$$

showing that f_2 is integrable, because $\alpha>0$.

(iv) We now solve the linear ODE for P in (6.34) with terminal condition $P(h)=0$. By the estimates (6.45)-(6.46) and (6.48) one obtains a bound of the form

$$P(y) \;=\; \mathcal{O}(1)\cdot (h-y)^{\frac{\alpha}{2-\alpha}}, \tag{6.49}$$

valid in a left neighborhood of $y=h$.

(v) In a neighborhood of the origin, the function $f_{1,q}$ contains a logarithm which blows up as $y\to 0+$. However, this is integrable because, for $y\approx 0$, we have

$$\frac{q(y)}{I(y)} \;\approx\; \left(\frac{d}{dy}\frac{q(y)}{I(y)}\right)\Bigg|_{y=0}\cdot y \;=\; \frac{c\alpha}{(z(0))^{1-\alpha}I(0)\sin(\Theta(0))}\,y,$$

and $\ln y$ is integrable in y. Recalling (6.1), as y ranges in a right neighborhood of the origin, i.e. for $y>0$, we conclude

$$\begin{cases} I'(y)\cdot f_{1,q}f_2 \;=\; \mathcal{O}(1)\cdot I'(y)f_{1,q} \;=\; \mathcal{O}(1)\cdot y^{-\beta}\ln y, \\ I'(y)\cdot f_{1,p} \;=\; \mathcal{O}(1)\cdot I'(y) \;=\; \mathcal{O}(1)\cdot y^{-\beta}. \end{cases} \tag{6.50}$$

This shows that, in (6.34), the coefficients in first equation are uniformly integrable in a right neighborhood of the origin.

(vi) It remains to consider the terms appearing in the ODE for $\widetilde{Q}(y)$. We first observe that

$$\frac{f_{2,p}}{f_2} \;=\; -\,\frac{\sin\Theta}{c\alpha}\left[1+\frac{\alpha}{\sin^2\Theta}-2\alpha\right]z^{-\alpha}\big(I(y),p(y),q(y)\big).$$

As $y\to h-$, by (6.47) and (6.49) this implies

$$\frac{f_{2,p}}{f_2}\cdot P \;=\; \mathcal{O}(1)\cdot (h-y)^{\frac{-2\alpha}{2-\alpha}}\cdot (h-y)^{\frac{\alpha}{2-\alpha}}, \tag{6.51}$$

which is integrable for $\alpha<1$.

(vii) Finally, as $y\to h-$, we consider

$$\frac{f_{2,I}}{f_2} \;=\; -\,\frac{\sin\Theta}{c\alpha}\left[\frac{(1-\alpha)\sin\theta_0}{\sin^2\Theta}+\frac{\sin(\Theta-\theta_0)}{\cos\Theta}\left(1+\frac{\alpha}{\sin^2\Theta}-2\alpha\right)\right] \\ \times\frac{1-\frac{q}{I(y)}}{z^{\alpha}\big(I(y),p(y),q(y)\big)} \tag{6.52}$$

$$=\;\mathcal{O}(1)\cdot(1-q^\sharp)z^{-\alpha}\big(I(y),p(y),q(y)\big) \;=\; \mathcal{O}(1)\cdot(h-y)^{\frac{\alpha}{2-\alpha}}\cdot(h-y)^{\frac{-2\alpha}{2-\alpha}},$$

which is integrable in y since $\alpha < 1$. Similarly, by (6.51), (6.18), and (6.42), it follows

$$\frac{f_{2,p}}{f_2} \cdot f_1 \;=\; \mathcal{O}(1) \cdot (h-y)^{\frac{-2\alpha}{2-\alpha}} \cdot (h-y)^{\frac{\alpha}{2-\alpha}}, \tag{6.53}$$

which is again integrable.

9. The proof can now be accomplished by a contradiction argument. If the conclusion of the theorem were not true, one could find a sequence of absolutely continuous, non-decreasing functions $I_n : \mathbb{R}_+ \mapsto [0,1]$, all satisfying (6.1), with $I_n(0) \to 1$, and such that, for each $n \geq 1$, the optimization problem **(OP2)** has two distinct solutions, say $(\check{\theta}_n, \check{u}_n)$ and $(\hat{\theta}_n, \hat{u}_n)$. As a consequence, for each $n \geq 1$ the system (6.13) has two solutions. To fix the ideas, let the first solution be defined on $[0, \check{h}_n]$ and the second on $[0, \hat{h}_n]$, with $\check{h}_n < \hat{h}_n$. These two solutions will be denoted by $(\check{p}_n, \check{q}_n, \check{z}_n)$ and $(\hat{p}_n, \hat{q}_n, \hat{z}_n)$. They both satisfy the boundary conditions

$$\check{p}_n(\check{h}_n) = \hat{p}_n(\hat{h}_n) = 0, \qquad \check{q}_n(\check{h}_n) = I(\check{h}_n), \qquad \hat{q}_n(\hat{h}_n) = I(\hat{h}_n), \qquad \check{q}_n(0) = \hat{q}_n(0) = 0. \tag{6.54}$$

As a preliminary, we observe that, for $\delta > 0$ small, the heights $\hat{h}$, $\check{h}$ of optimal stems must remain uniformly positive. Indeed, by (2.3) the sunlight gathered by a stem γ of length ℓ is bounded by

$$\mathcal{S}(\gamma) \;\leq\; \ell.$$

Hence, for a sequence of stems γ_n with heights $\hat{h}_n \to 0$, the total sunlight satisfies

$$\mathcal{S}(\gamma_n) \;\leq\; \ell_n \;\leq\; \frac{\hat{h}_n}{\sin\theta_0} \;\to\; 0.$$

Therefore, for n large, none of these stems can be optimal.

Thanks to the last identity in (6.54), by the mean value theorem there exists some intermediate point $k_n \in [\check{h}_n, \hat{h}_n]$ such that, with the notation introduced at (6.21),

$$\frac{\partial q_n}{\partial h}(0, k_n) \;=\; 0. \tag{6.55}$$

For each $n \geq 1$ consider the corresponding system

$$\begin{cases} P_n'(y) \;=\; -I_n'(y)\big[f_{1,p}P_n + f_{1,q}f_2\widetilde{Q}_n\big], \\[2mm] \widetilde{Q}'(y) \;=\; \dfrac{f_{2,p}}{f_2}P_n - \dfrac{I_n'(y)[f_{2,I} - f_{2,p}f_1]}{f_2}\widetilde{Q}_n, \end{cases} \qquad \begin{cases} P_n(k_n) \;=\; 0, \\[2mm] \widetilde{Q}_n(k_n) \;=\; -1. \end{cases} \tag{6.56}$$

Since $f_2\big(I_n(0), p_n(0,k_n), 0\big) > 0$, by (6.55) it follows

$$\widetilde{Q}_n(0) \;=\; \frac{1}{f_2\big(I_n(0), p_n(0,k_n), 0\big)} \cdot \frac{\partial q_n}{\partial h}(0, k_n) \;=\; 0. \tag{6.57}$$

Let

$$P_n(y) \doteq \frac{\partial p(y,k_n)}{\partial h}, \qquad \widetilde{Q}_n(y) \doteq \frac{1}{f_2\big(I_n(y), p_n(y,k_n), q_n(y,k_n)\big)} \cdot \frac{\partial q(y,k_n)}{\partial h},$$

be the solutions to (6.56). By the previous steps, their derivatives $\big(P_n', \widetilde{Q}_n'\big)_{n\geq 1}$ form a sequence of uniformly integrable functions defined on the intervals $[0,k_n]$. Note that the existence of an upper bound $\sup_n k_n \doteq h^+ < +\infty$ follows from the existence proof.

Thanks to the uniform integrability, by possibly taking a subsequence, we can assume the convergence $k_n \to \overline{h} \in [0,h^+]$, the weak convergence of derivatives $P_n' \rightharpoonup P'$, $\widetilde{Q}_n' \rightharpoonup \widetilde{Q}'$ in $\mathbf{L}^1$, and the convergence

$$P_n \to P, \qquad \widetilde{Q}_n \to \widetilde{Q},$$

uniformly on every subinterval $[0,h]$ with $h < \bar{h}$.

Recalling that every I_n' satisfies the uniform bounds (6.1), since $I_n(y) \to I(y) \equiv 1$ uniformly for all $y \geq 0$, we conclude that $(P, \widetilde{Q})$ provides a solution to the linear system (6.34) on $[0,\bar{h}]$, corresponding to the constant function $I(y) \equiv 1$. We now observe that, when $I(y) \equiv 1$, the solution to (6.34) is $P(y) \equiv 0$ and $\widetilde{Q}(y) \equiv -1$. On the other hand, our construction yields

$$\widetilde{Q}(0) = \lim_{n\to\infty} \widetilde{Q}_n(0) = 0.$$

This contradiction achieves the proof of Theorem 6.1. □

7. Existence of an equilibrium solution

Given a nondecreasing light intensity function $I : \mathbb{R}_+ \mapsto [0,1]$, in the previous section we proved the existence of an optimal solution (θ^*, u^*) for the maximization problem **(OP2)**.

Conversely, let $\rho_0 > 0$ be the constant density of stems, i.e. the number of stems growing per unit area. If all stems have the same configuration, described by the couple of functions $y \mapsto (\theta(y), u(y))$ as in (2.18), then the corresponding intensity of light at height y above ground is computed as

$$I^{(\theta,u)}(y) \doteq \exp\left\{ -\frac{\rho_0}{\cos\theta_0} \int_y^{+\infty} \frac{u(\zeta)}{\sin\theta(\zeta)}\, d\zeta \right\}. \tag{7.1}$$

The main goal of this section is to find a competitive equilibrium, i.e. a fixed point of the composition of the two maps $I \mapsto (\theta^*, u^*)$ and $(\theta,u) \mapsto I^{(\theta,u)}$.

Definition 7.1. Given an angle $\theta_0 \in]0, \pi/2[$ and a constant $\rho_0 > 0$, we say that the light intensity function $I^* : \mathbb{R}_+ \mapsto [0,1]$ and the stem configuration $(\theta^*, u^*) : \mathbb{R}_+ \mapsto [\theta_0, \pi/2] \times \mathbb{R}_+$ yield a **competitive equilibrium** if the following holds.

(i) The couple (θ^*, u^*) provides an optimal solution to the optimization problem **(OP2)**, with light intensity function $I = I^*$.
(ii) The identity $I^* = I^{(\theta^*,u^*)}$ holds.

The main result of this section provides the existence of a competitive equilibrium, assuming that the density ρ_0 of stems is sufficiently small.

Theorem 7.2. *Let an angle $\theta_0 \in]0, \pi/2[$ be given. Then, for all $\rho_0 > 0$ sufficiently small, a unique competitive equilibrium (I^*, θ^*, u^*) exists.*

Proof. 1. Setting $C = 1$ and $\beta = 1/2$ in (6.1), we define the family of functions

$$\mathcal{F} \doteq \Big\{ I : \mathbb{R}_+ \mapsto [1-\delta,\, 1]; \quad I \text{ is absolutely continuous}, \quad I'(y) \in \big[0,\, y^{-1/2}\big] \quad \text{for a.e. } y > 0 \Big\}, \tag{7.2}$$

where $\delta > 0$ is chosen small enough so that the conclusion of Theorem 6.1 holds.

2. For each $I \in \mathcal{F}$, let $(\theta^{(I)}, u^{(I)})$ describe the corresponding optimal stem. Calling

$$h^{(I)} = \sup \big\{ y \geq 0; \; u^{(I)}(y) > 0 \big\}$$

the height of this stem, by the a priori bounds proved in Section 6 we have a uniform bound

$$h^{(I)} \leq h^+$$

for all $I \in \mathcal{F}$. Let $p^{(I)}, q^{(I)} : [0, h^{(I)}] \mapsto \mathbb{R}_+$ be the corresponding solutions of (6.20). For convenience, we extend all these functions to the larger interval $[0, h^+]$ by setting

$$p^{(I)}(y) \doteq p^{(I)}\big(h^{(I)}\big), \qquad q^{(I)}(y) \doteq q^{(I)}\big(h^{(I)}\big), \qquad \text{for all } y \in [h^{(I)}, h^+].$$

3. By the analysis in Section 6, for any $I \in \mathcal{F}$, the solution to the system of optimality conditions (6.13) satisfies

$$\theta_0 \leq \Theta(I(y), p(y), q(y)) \leq \theta^+, \qquad c_0\, y \leq \frac{q(y)}{I(y)} \leq 1, \tag{7.3}$$

for some $\theta^+ < \pi/2$ and $c_0 > 0$ sufficiently small. In view of (6.8), this implies

$$U(I(y), p(y), q(y)) \doteq -\ln\left(\frac{q(I)}{I(y)}\right) \cos\big(\Theta(I(y), p(y), q(y)) - \theta_0\big) \leq -\ln(c_0 y). \tag{7.4}$$

Note that $\Theta(I(y), p^{(I)}(y), q^{(I)}(y)) = \theta^{(I)}(y)$ and $U(I(y), p^{(I)}(y), q^{(I)}(y)) = u^{(I)}(y)$. Thus, if we choose $\rho_0 > 0$ small enough, it follows that the corresponding light intensity function $I^{(\theta,u)}$ at (7.1) is again in $\mathcal{F}$. A competitive equilibrium will be obtained by constructing a fixed point of the composition of the two maps

$$\Lambda_1 : I \mapsto \big(\theta^{(I)},\, u^{(I)}\big), \qquad \Lambda_2 : (\theta, u) \mapsto I^{(\theta,u)}. \tag{7.5}$$

In order to use Schauder's theorem, we need to check the continuity of these maps, in a suitable topology.

We start by observing that $\mathcal{F} \subset \mathcal{C}^0([0, h^+])$ is a compact, convex set. Again by the analysis in Section 6, as I varies within the domain $\mathcal{F}$, the corresponding functions $\theta^{(I)}$ are uniformly bounded in $\mathbf{L}^\infty([0, h^+])$, while $u^{(I)}$ is uniformly bounded in $\mathbf{L}^1([0, h^+])$.

From the estimate (6.43) it follows that the functions $p^{(I)}$ are equicontinuous on $[0, h^+]$. Recalling that $q = q^\sharp \cdot I$, by (6.39) we conclude that the functions $q^{(I)}$ are equicontinuous as well.

4. Motivated by (7.3)-(7.4), we consider the set of functions

$$\mathcal{U} \doteq \Big\{(\theta, u) \in \mathbf{L}^1\big([0, h^+];\ \mathbb{R}^2\big), \quad \theta(y) \in [\theta_0, \theta^+],\ \ 0 \leq u(y) \leq -\ln(c_0 y)\Big\}. \tag{7.6}$$

Thanks to the uniform bounds imposed on θ and u in the definition (7.6), the continuity of the map $\Lambda_2 : \mathcal{U} \mapsto \mathcal{C}^0$, defined at (7.1) is now straightforward.

5. To prove the continuity of the map Λ_1, consider a sequence of functions $I_n \in \mathcal{F}$, with $I_n \to I$ uniformly on $[0, h^+]$. Let $(\theta_n, u_n) : [0, h^+] \mapsto \mathbb{R}^2$ be the corresponding unique optimal solutions.

We claim that $(\theta_n, u_n) \to (\theta, u)$ in $\mathbf{L}^1([0, h^+])$, where (θ, u) is the unique optimal solution, given the light intensity I.

To prove the claim, let (p_n, q_n) be the corresponding solutions of the system (6.20). By the estimates on p', q' proved in Section 6, the functions (p_n, q_n) are equicontinuous. From any subsequence we can thus extract a further subsequence and obtain the convergence

$$p_{n_j} \to \widehat{p}, \qquad q_{n_j} \to \widehat{q}, \qquad I_{n_j} \to I, \tag{7.7}$$

for some functions $\widehat{p}, \widehat{q}$, uniformly on $[0, h^+]$.

For every $j \geq 1$ we now have

$$\theta_{n_j}(y) = \Theta\big(I_{n_j}(y), p_{n_j}(y), q_{n_j}(y)\big), \qquad\qquad u_{n_j}(y) = U\big(I_{n_j}(y), p_{n_j}(y), q_{n_j}(y)\big),$$

where U and Θ are the functions in (6.8)-(6.9). By the dominated convergence theorem, the convergence (7.7) together with the uniform integrability of θ_{n_j} and u_{n_j} yields the $\mathbf{L}^1$ convergence

$$\|\theta_{n_j} - \widehat{\theta}\|_{\mathbf{L}^1} \to 0, \qquad\qquad \|u_{n_j} - \widehat{u}\|_{\mathbf{L}^1} \to 0. \tag{7.8}$$

In turn this implies that $(\widehat{p}, \widehat{q})$ provide a solution to the problem (6.20), in connection with the light intensity I. By uniqueness, $\widehat{p} = p$ and $\widehat{q} = q$. Therefore, $\widehat{\theta} = \theta$ and $\widehat{u} = u$ as well.

The above argument shows that, from any subsequence, one can extract a further subsequence so that the $\mathbf{L}^1$-convergence (7.8) holds. Therefore, the entire sequence $(\theta_n, u_n)_{n \geq 1}$ converges to (θ, u) in $\mathbf{L}^1([0, h^+])$. This establishes the continuity of the map Λ_1.

6. The map $\Lambda_2 \circ \Lambda_1$ is now a continuous map of the compact, convex domain $\mathcal{F} \subset \mathcal{C}^0([0, h^+])$ into itself. By Schauder's theorem it admits a fixed point $I^*(\cdot)$. By construction, the optimal stem configuration $\big(\theta^{(I^*)}, u^{(I^*)}\big)$ yields a competitive equilibrium, in the sense of Definition 7.1.

7. To prove uniqueness, we derive a set of necessary conditions satisfied by the equilibrium solution, and show that this system has a unique solution.

Using (6.8) and (6.11), we can rewrite the light intensity function (7.1) as

$$I(y) \;=\; \exp\left\{\frac{\rho_0}{\cos\theta_0}\int_y^\infty \ln\left(\frac{q}{I}\right)\frac{1+w\sin\theta_0}{\sin\theta_0+w}\,d\zeta\right\},$$

where $w = w(I, p, q)$ is the function introduced at (6.10). Differentiating w.r.t. y one obtains

$$I'(y) \;=\; -\frac{\rho_0}{\cos\theta_0}\ln\left(\frac{q}{I}\right)\frac{1+w\sin\theta_0}{\sin\theta_0+w}\cdot I \;\doteq\; f_3(I,p,q). \tag{7.9}$$

Combining (7.9) with (6.20), we conclude that the competitive equilibrium satisfies the system of equations and boundary conditions

$$\begin{cases} p'(y) = -f_1\big(I(y),p(y),q(y)\big)\cdot f_3(I(y),p(y),q(y)),\\ q'(y) \;=\; f_2\big(I(y),p(y),q(y)\big),\\ I'(y) \;=\; f_3(I(y),p(y),q(y)),\end{cases}\qquad \begin{cases} p(h) \;=\; 0,\\ q(h) \;=\; 1,\\ I(h) \;=\; 1,\end{cases} \tag{7.10}$$

together with

$$q(0) \;=\; 0. \tag{7.11}$$

Here the common height of the stems $h > 0$ is a constant to be determined.

8. The uniqueness of solutions to (7.10) will be achieved by a contradiction argument. Since this is very similar to the one used in the proof of Theorem 6.1, we only sketch the main steps.

In analogy with (6.31), (6.33), denote by $p(y,h), q(y,h), I(y,h)$ the unique solution to the Cauchy problem (7.10), with terminal conditions given at $y = h$. Consider the functions

$$P(y) \doteq \frac{\partial p(y,h)}{\partial h},\qquad \widetilde{Q}(y) \doteq \frac{1}{f_2(I,p,q)}\frac{\partial q(y,h)}{\partial h},\qquad J(y) \doteq \frac{\partial I(y,h)}{\partial h}.$$

By (7.10), these functions satisfy

$$\begin{cases} P'(y) \;=\; -\big[f_{3,I}f_1 + f_3 f_{1,I}\big]J - \big[f_{3,p}f_1 + f_3 f_{1,p}\big]P - \big[f_{3,q}f_1 + f_3 f_{1,q}\big]f_2\widetilde{Q},\\ \widetilde{Q}'(y) \;=\; \dfrac{f_{2,I}}{f_2}J + \dfrac{f_{2,p}}{f_2}P - \dfrac{f_3}{f_2}\big[f_{2,I} - f_{2,p}f_1\big]\widetilde{Q},\\ J'(y) \;=\; f_{3,I}J + f_{3,p}P + f_{3,q}f_2\widetilde{Q},\end{cases} \tag{7.12}$$

with boundary conditions

$$P(h) = 0,\qquad \widetilde{Q}(h) = -1,\qquad J(h) = 0.$$

Set $d_0 = \frac{\rho_0}{\cos\theta_0}$. Several of the partial derivatives on the right-hand side of (7.12) were computed in (6.36). The remaining ones are

$$f_{1,I}(I,p,q) = \frac{q}{I^2}\cdot\frac{1+w\sin\theta_0}{\sin\theta_0+w} - \frac{\cos^2\theta_0}{(\sin\theta_0+w)^2}\frac{w^2}{p}\Big[1-\frac{q}{I}\Big],$$

$$f_{3,I}(I,p,q) = -d_0\bigg[\Big(\ln\Big(\frac{q}{I}\Big)-1\Big)\frac{1+w\sin\theta_0}{\sin\theta_0+w} - I\ln\Big(\frac{q}{I}\Big)\frac{\cos^2\theta_0}{(\sin\theta_0+w)^2}\frac{w^2}{p}\Big(1-\frac{q}{I}\Big)\bigg],$$

$$f_{3,p}(I,p,q) = d_0 I\ln\Big(\frac{q}{I}\Big)\frac{\cos^2\theta_0}{(\sin\theta_0+w)^2}\frac{w}{p},$$

$$f_{3,q}(I,p,q) = -d_0 I\bigg[\frac{1}{q}\cdot\frac{1+w\sin\theta_0}{\sin\theta_0+w} + \Big[\ln\Big(\frac{q}{I}\Big)\Big]^2\frac{\cos^2\theta_0}{(\sin\theta_0+w)^2}\frac{w^2}{p}\bigg].$$

By the same arguments used in step **8** of the proof of Theorem 6.1, we conclude that the right-hand side of (7.12) is uniformly integrable.

9. Let a density $\rho_0 > 0$ be given. Assume that the problem (7.10)-(7.11) has two distinct solutions $(\hat p, \hat q, \hat I)$ and $(\check p, \check q, \check I)$, defined on $[0, \hat h]$ and $[0, \check h]$ say with $\hat h < \check h$. Since $\hat q(0) = \check q(0) = 0$, by the mean value theorem there exists $k \in [\hat h, \check h]$ such that $\frac{\partial q}{\partial h}(0, k) = 0$.

Next, if multiple solutions exist for arbitrarily small values of the density ρ_0, we can find a decreasing sequence $\rho_{0,n} \downarrow 0$ and corresponding solutions P_n, Q_n, I_n of (7.12), defined for $y \in [0, k_n]$, such that

$$P_n(k_n) = 0, \qquad \widetilde Q_n(k_n) = -1, \qquad J_n(k_n) = 0, \qquad \widetilde Q_n(0) = 0. \tag{7.13}$$

Thanks to the uniform integrability of the right hand sides of (7.12), by possibly extracting a subsequence we can achieve the convergence $k_n \to \bar h \in [0, h^+]$, the weak convergence $P_n' \rightharpoonup P'$, $\widetilde Q_n' \rightharpoonup \widetilde Q'$, $J_n' \rightharpoonup J'$ in $\mathbf{L}^1$, and the strong convergence

$$P_n \to P, \qquad \widetilde Q_n \to \widetilde Q, \qquad J_n \to J,$$

uniformly on every subinterval $[0, h]$ with $h < \bar h$.

To reach a contradiction, we observe that

$$J_n(y) = -\int_y^{k_n} J_n'(z)\,dz$$

and the right-hand side of J_n' in (7.12) consists of uniformly integrable terms which are multiplied by $\rho_{0,n}$. This implies $J(y) \equiv 0$. This corresponds to the case of an intensity function $I(y) \equiv 1$. But in this case we know that $\widetilde Q(y) \equiv -1$, contradicting the fact that, by (7.13),

$$\widetilde Q(0) = \lim_{n\to\infty} \widetilde Q_n(0) = 0. \qquad \square$$

8. Stem competition on a domain with boundary

We consider here the same model introduced in Section 2, where all stems have fixed length ℓ and constant thickness κ. But we now allow the sunlight intensity $I = I(x, y)$ to vary w.r.t. both variables x, y. As shown in Fig. 5, left, we denote by

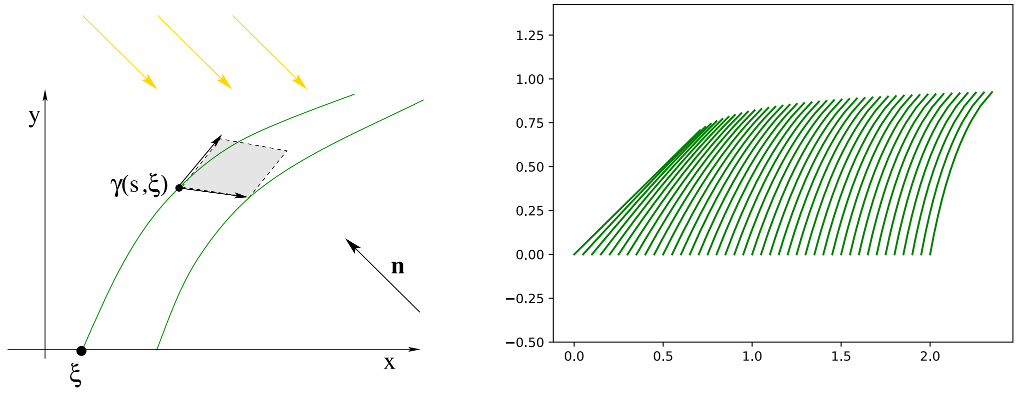

Fig. 5. Left: to leading order, the amount of vegetation in the shaded region is proportional to $\kappa\,\bar\rho(\xi)d\xi ds$. Since the area is computed in terms of the cross product $\frac{\partial\gamma}{\partial\xi}\times\frac{\partial\gamma}{\partial s}$, this motivates the formula (8.4). Right: a possible competitive equilibrium, where the light rays come from the direction $\mathbf{n}=(\frac{-1}{\sqrt2},\frac{1}{\sqrt2})$ and stems are distributed along the positive half line, with density as in (8.9). In this case, stems originating from points close to the origin have no incentive to grow upward, because they already receive a nearly maximum light intensity. Hence they bend to the right, almost perpendicularly to the light rays.

$$s \mapsto \gamma(s,\xi) = (x(s), y(s)), \qquad s\in[0,\ell], \tag{8.1}$$

the arc-length parameterization of the stem whose root is located at $(\xi,0)$, and write g for the function introduced at (2.8). This leads to the optimization problem

(OP3) *Given a light intensity function $I=I(x,y)$, find a control $s\mapsto\theta(s)\in[0,\pi]$ which maximizes the integral*

$$\int_0^\ell I(x(s),y(s))\,g(\theta(s))\,ds \tag{8.2}$$

subject to

$$\frac{d}{ds}(x(s),y(s)) = (\cos\theta(s),\sin\theta(s)), \qquad (x(0),y(0)) = (\xi,0). \tag{8.3}$$

Next, consider a function $\bar\rho(\xi)\geq 0$ describing the density of stems which grow near $\xi\in\mathbb{R}$. At any point in space reached by a stem, i.e. such that

$$(x,y)=\gamma(s,\xi) \qquad \text{for some } \xi\in\mathbb{R},\ \ s\in[0,\ell],$$

the density of vegetation is

$$\rho(x,y) = \rho(\gamma(s,\xi)) = \kappa\,\bar\rho(\xi)\cdot\left[\frac{\partial\gamma}{\partial\xi}\times\frac{\partial\gamma}{\partial s}\right]^{-1}. \tag{8.4}$$

The light intensity at a point $P = (x, y) \in \mathbb{R}^2$ is now given by

$$I(P) \;=\; \exp\left\{ -\int_0^{+\infty} \rho(P + t\mathbf{n})\,dt \right\}. \tag{8.5}$$

Definition 8.1. Given the constants ℓ, κ and the density $\bar\rho \in \mathbf{L}^\infty(\mathbb{R})$, we say that the maps $\gamma : [0, \ell] \times \mathbb{R}$ and $I : \mathbb{R} \times \mathbb{R}_+ \mapsto [0, 1]$ yield a **competitive equilibrium** if the following holds:

(i) For each $\xi \in \mathbb{R}$, the stem $\gamma(\cdot, \xi)$ provides an optimal solution to **(OP3)**.
(ii) The function $I(\cdot)$ coincides with the light intensity determined by (8.4)-(8.5).

We shall not analyze the existence or uniqueness of the competitive equilibrium, in the case where the distribution of stem roots is not uniform. We only observe that, if the stem $\gamma(\cdot, \xi)$ in (8.1) is optimal, the necessary conditions yield the existence of a dual vector $s \mapsto \mathbf{p}(s)$ satisfying

$$\dot{\mathbf{p}}(s) \;=\; -\nabla I\big(x(s), y(s)\big)\, g(\theta(s)), \qquad\qquad \mathbf{p}(\ell) \;=\; (0, 0), \tag{8.6}$$

and such that, for a.e. $s \in [0, \ell]$, the optimal angle $\theta^*(s)$ satisfies

$$\theta^*(s) \;=\; \operatorname*{argmax}_{\theta} \Big\{ \mathbf{p}(s) \cdot (\cos\theta, \sin\theta) + I(x(s), y(s)) g(\theta) \Big\}. \tag{8.7}$$

Differentiating the expression on the right hand side of (8.7) one obtains an implicit equation for $\theta^*(s)$, namely

$$I\big(x(s), y(s)\big)\big) g'(\theta^*(s)) + \mathbf{p}(s) \cdot \mathbf{n}(s) \;=\; 0 \tag{8.8}$$

for a.e. $s \in [0, \ell]$. Here $\mathbf{n}(s) \;\doteq\; \big(-\sin\theta(s), \cos\theta(s)\big)$ is the unit vector perpendicular to the stem. Moreover, by (8.6) one has

$$\mathbf{p}(s) \;=\; \int_s^\ell \nabla I\big(x(\sigma), y(\sigma)\big)\, g(\theta^*(\sigma)\big)\, d\sigma.$$

An interesting case is where stems grow only on the half line $\{\xi \geq 0\}$. For example, one can take

$$\bar\rho(\xi) \;=\; \begin{cases} 0 & \text{if } \xi < 0, \\ b^{-1}\xi & \text{if } \xi \in [0, b], \\ 1 & \text{if } \xi > b. \end{cases} \tag{8.9}$$

In this case, we conjecture that the competitive equilibrium has the form illustrated in Fig. 5, right.

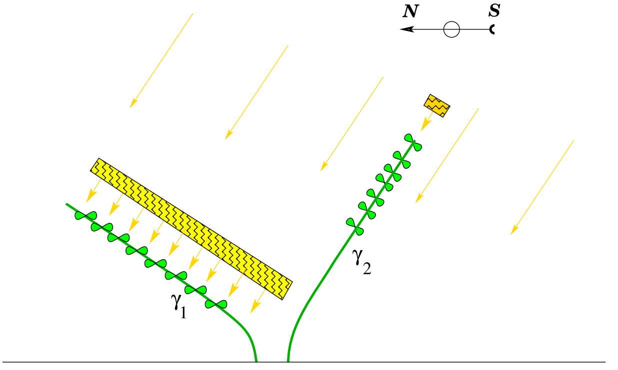

Fig. 6. The stem γ_1, oriented perpendicularly to the sun rays, collects much more sunlight than γ_2. Indeed, γ_1 would give the best orientation for solar panels. Notice that γ_2 minimizes the sunlight gathered because the upper leaves put the lower ones in shade.

9. Concluding remarks

A motivation for the present study was to understand whether competition for sunlight could explain phototropism, i.e. the tendency of plant stems to bend toward the light source. A naive approach may suggest that, if a stem bends in the direction of the light rays, the leaves will be closer to the sun and hence gather more light. However, since the average distance of the earth from the sun is approximately 90 million miles, getting a few inches closer cannot make a difference.

As shown in Fig. 6, if a single stem were present, to maximize the collected sunlight it should be perpendicular to the light rays, not parallel. In the presence of competition among several plant stems, our analysis shows that the best configuration is no longer perpendicular to light rays: the lower part of the stems should grow in a nearly vertical direction, while the upper part bends away from the sun.

Still, our competition models do not predict the tilting of stems in the direction of the sun rays. This may be due to the fact that these models are “static”, i.e., they do not describe how plants grow in time. This leaves open the possibility of introducing further models that can explain phototropism in a time-dependent framework. As suggested in [12], the preemptive conquering of space, in the direction of the light rays, can be an advantageous strategy. We leave these issues for future investigation.

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