Impact of Confirmation Bias on Competitive Information Spread in Social Networks

Yanbing Mao, Emrah Akyol, and Naira Hovakimyan

***Abstract*—This article investigates the impact of confirmation bias on competitive information spread in the social network that comprises individuals in a social network and competitive information sources at a cyber layer. We formulate the problem of information spread as a zero-sum game, which admits a unique Nash equilibrium in pure strategies. We characterize the dependence of pure Nash equilibrium on the public's innate opinions, the social network topology, as well as the parameters of confirmation bias. We uncover that confirmation bias moves the equilibrium toward the center only when the innate opinions are not neutral, and this move does not occur for the competitive information sources simultaneously. Numerical examples in the context of well-known Krackhardt's advice network are provided to demonstrate the correctness of theoretical results.**

***Index Terms*—Competitive information spread, confirmation bias, innate opinion, Nash equilibrium, social network topology, zero-sum game.**

I. Introduction

Mathematical models for the opinion formation in networks have been an important research subject for decades, see e.g., [1], [2]. A few well-known models include the DeGroot model [3] (whose roots go back to [2] and [4]) that considers opinion evolution within a network in terms of the weighted average of individuals' connections, where weights are determined by influences. The Friedkin–Johnsen model [5] incorporates individual innate opinions, thereby making the model more suitable to several real-life scenarios, as well as real applications, e.g., optimal investment for competing camps [6] and debiasing social influence [7]. In [8], a bounded confidence model is presented where individuals are influenced by their neighbors that are not too far from their opinion. The majority of the recent works are variations of these models, with a few exceptions. An overview of opinion dynamic models can be found in relevant tutorial papers, see, e.g., [9], [10], and the references therein.

While opinion evolution models have always been an active research area, recently with the wide use of social media [11], in conjunction with automated news generation with the help of artificial intelligence technologies [12], they have gained vital importance in studying misinformation spread and polarization. In this regard, confirmation bias (CB) plays a key role. CB broadly refers to cognitive bias toward favoring information sources that affirm existing opinion [13]. It is well understood that CB helps create "echo chambers" within networks, in which misinformation and polarization thrive, see, e.g., [14] and [15].

In this article, we study competitive information spread in social networks, with a particular focus on the impact of CB on the results. Competitive information spread has been studied extensively in recent years. Building on the DeGroot model [3], Zhao *et al.* in [16] investigated how to enhance a competitor's competitiveness through adding new communication links to normal agents to maximize the number of supporters or the supporting degree toward a competitor. Rusinowska and Taalaibekova in [17] proposed a model of competitive opinion formation with three persuaders, who, respectively, hold extreme, opposite, and centrist opinions, while Grabisch *et al.* in [18] investigated the model of influence with a set of nonstrategic agents and two strategic agents that have fixed but opposed opinions. Dhamal *et al.* in [6] incorporated opponent stubborn agents into the Friedkin–Johnsen model [5]. Employing diffusion dynamics, Eshghi *et al.* in [19] studied optimal allocating of a finite budget across several advertising channels. Meanwhile, Proskurnikov *et al.* in [20] studied the opinion dynamics with negative weights, which models antagonistic or competitive interactions, with its origins dating back to the seminal work of Altafini [21]. We note, however, that these prior works on competitive camps and competitive/antagonistic interactions do not consider CB in their analysis.

Among the aforementioned opinion evolution models, CB can be modeled within the context of bounded confidence models [11] such as the Hegselmann–Krause model [8] and its recent variations [22]. However, these models involve a discontinuity in the influence impact: An individual is either influenced by an information source (or her neighbors) fully or not at all, depending on the opinion differences. This binary influence effect renders the analysis of the steady-state point difficult in general. As a remedy, in [23], a new opinion dynamics model is proposed as a variation of the Friedkin–Johnsen model [5] with a continuous bias model.

In this article, building on preliminary analysis in [24], we analyze the information spread over a network with two competitive information sources, where the only control variables are the

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opinions of information sources. We adopt the opinion dynamics in [23] with two information sources and a (state-dependent) piecewise linear CB model. We formulate the problem as a zero-sum game and show that this game admits a unique Nash equilibrium which is in pure strategies. We particularly study the impact of CB on the Nash equilibrium. We analyze how the equilibrium achieving strategies depend on the public's innate opinions, the network topology, as well as the CB parameters.

This article is organized as follows. In Section II, we present preliminaries. The problem is formulated in Section III. In Section IV, we investigate the Nash equilibrium and the impact of CB on Nash equilibrium. We present numerical simulations in Section V. Finally, Section VI concludes this article.

II. Preliminaries

A. Notation

Let $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the set of n-dimensional real vectors and the set of $m\times n$-dimensional real matrices, respectively. $\mathbb{N}$ represents the set of the positive integers, and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. We define I as the identity matrix with proper dimension. We let $\mathbf{1}$ denote the vector of all ones. The superscript "$\top$" stands for the matrix transposition. For a vector $x\in\mathbb{R}^n$, $\|x\|$ stands for its l_1 norm, i.e., $\|x\|=\sum_{i=1}^n|x_i|$. For $W=[w_{ij}]\in\mathbb{R}^{n\times n}$, we use $\|W\|_1$ and $\|W\|_\infty$ to denote $\max_{j=1,\dots,n}\{\sum_{i=1}^n|w_{ij}|\}$ and $\max_{i=1,\dots,n}\{\sum_{j=1}^n|w_{ij}|\}$, respectively.

The social network considered in this article is composed of n individuals. The interaction among the individuals is modeled by a digraph $\mathfrak{G}=(\mathbb{V},\mathbb{E})$, where $\mathbb{V}=\{\mathrm{v}_1,\dots,\mathrm{v}_\mathrm{n}\}$ is a set of vertices representing the individuals and $\mathbb{E}\subseteq\mathbb{V}\times\mathbb{V}$ is a set of edges representing the influence structure. We take the network to have no self-loops, i.e., for any $\mathrm{v_i}\in\mathbb{V}$, we assume that $(\mathrm{v_i},\mathrm{v_i})\notin\mathbb{E}$. A node v_j is said to be reachable from node v_i if there is a directed path from v_i to v_j in digraph $\mathfrak{G}$. A digraph $\mathfrak{G}$ is strongly connected if every two nodes are mutually reachable; $\mathfrak{G}$ is quasi-strongly connected if for every two nodes v_i and v_j, there is a node v_u from which both v_i and v_j are reachable.

B. Opinion Dynamics

In this article, we adopt the opinion evolution model in the presence of two competitive information sources in [23]

$$x_i(k+1)=\alpha_i(x_i(k))s_i+\sum_{j\in\mathbb{V}}w_{ij}x_j(k)+\overline{w}(x_i(k))h + \underline{w}(x_i(k))g,\quad i\in\mathbb{V}. \tag{1}$$

In the following, we describe the elements of this model.

1) $x_i(k)\in[0,1]$ is individual $\mathrm{v_i}$'s opinion at time k. This opinion evolves in time as described in (1).
2) $s_i\in[0,1]$ is individual $\mathrm{v_i}$'s innate opinion which is fixed in time. We define the extremal innate opinions as follows:
$$\overline{s}\triangleq\max_{i\in\mathbb{V}}\{s_i\},\qquad \underline{s}\triangleq\min_{i\in\mathbb{V}}\{s_i\}.$$
3) w_{ij} represents the influence of individual $\mathrm{v_j}$ on $\mathrm{v_i}$,
$$w_{ij}=\begin{cases}>0, & \text{if } (\mathrm{v_i},\mathrm{v_j})\in\mathbb{E}\\ =0, & \text{otherwise.}\end{cases}$$
This is a standard model parameter, with its origins dating back to the seminal work of Friedkin and Johnsen [5]. In this article, we do not consider antagonistic interactions, as studied in [20], which would imply negative values for w_{ij}.
4) h and g are the opinions of competitive information sources (or stubborn individuals), Hank and Georgia, respectively. Their objectives are to move the public opinion to two extremes they represent. We assume that the values of g,h satisfy the following:
$$1\geq h\geq\overline{s}\geq\underline{s}\geq g\geq 0. \tag{2}$$
This assumption states that the information sources are more extreme than the most extreme innate opinion of the public, prior to any external influence.
5) $\overline{w}(x_i(k))$ and $\underline{w}(x_i(k))$ are the state-dependent influence weights of information sources Hank and Georgia on individual $\mathrm{v_i}$. These weights model "symmetric" confirmation bias as
$$\overline{w}(x_i(k))=\beta-\gamma\,|x_i(k)-h| \tag{3a}$$
$$\underline{w}(x_i(k))=\beta-\gamma\,|x_i(k)-g| \tag{3b}$$
where $\beta\in\mathbb{R}$ and $\gamma\in\mathbb{R}$ are bias parameters. The model indicates that every individual in social networks has access to both information sources, which is due to information overload in the modern information era [11], [25]. Throughout this article, we make the following assumption on the bias parameters and influence weights.
Assumption 1: Given $W\in\mathbb{R}^{n\times n}$, $\beta\in\mathbb{R}$, and $\gamma\in\mathbb{R}$
$$\beta\geq\gamma\geq 0 \tag{4a}$$
$$1-\max\{\|W\|_\infty,\|W\|_1\}\geq\max\{2\beta,4\gamma\} \tag{4b}$$
$$W \text{ has a positive eigenvector.} \tag{4c}$$
Here, the CB model (3) is assumed to be piecewise linear. We note that in the original model used in [23], this bias function is taken in general possibly nonlinear and decreasing.
6) $\alpha_i(x_i(k))$ is the "resistance parameter" of individual $\mathrm{v_i}$ and is determined in such a way that it satisfies
$$\alpha_i(x_i(k))+\sum_{j\in\mathbb{V}}w_{ij}+\overline{w}(x_i(k))+\underline{w}(x_i(k))=1 \tag{5}$$
$\forall i\in\mathbb{V}$ and $\forall k\in\mathbb{N}_0$. This is a standard assumption common in all classical opinion dynamics models, see, e.g., [5]–[7]. The entire model essentially represents that individuals form opinions by taking weighted averages over a convex polytope of different contributing factors.

Remark 1: Confirmation bias refers to the tendency to acquire or process new information in a way that confirms one's preconceptions and avoids contradiction with prior belief [13]. We note that function (3) is more like state-dependent social influence weights used to model homophily [26], [27], which is a remedied version of bounded confidence models. While following [11] wherein a bounded confidence model is leveraged to describe confirmation bias in the era of information overload, the model (3) is also used in this article to describe the confirmation bias to some extent, which is motivated by following observations.

1) Both polarization and homogeneity are the results of the conjugate effect of confirmation bias and social influence [28], [29].
2) Confirmation bias happens when a person gives more weight to evidence that confirms their beliefs and undervalues evidence that could disprove it [30].

Remark 2: We assume that individual–individual influence weights w_{ij}s are trust-based and fixed, whose motivations are as follows.

1) Social influence among individuals is based on the trust (e.g., cognition-based trust and knowledge-based trust) which tends to vary little over a long period of time, with exception being the swift trust which is required in the environments wherein there is little or no time to develop trust among persons [31].
2) Cognitive factors that can influence trust decisions are founded on a deeper knowledge of the other person and the stability of the other's behavior across time and contexts [31].

We note that the fixed influence weights w_{ij}s can also describe the conformity behavior in social network to some extent, i.e., without information sources and innate opinions, an individual updates her opinion as an average of her neighbors [3].

Remark 3: The model (3) describes the symmetric bias for the social problems, e.g., Senate or House Member evolving ideology [32], whose implicit (e.g., binary or triple) mapping can be ignored. To capture the asymmetric bias, e.g., in US President Election, model (3) updates as

$$\overline{w}(x_i(k)) = \beta - \gamma\,|x_i(k) - h| + \upsilon\mathfrak{h}((h - x_i(k))(x_i(k) - 0.5))$$
$$\underline{w}(x_i(k)) = \beta - \gamma\,|x_i(k) - g| + \upsilon\mathfrak{h}((g - x_i(k))(x_i(k) - 0.5))$$

where the added terms correspond to the implicit triple mapping, $\upsilon > 0$ and $\mathfrak{h}(\cdot)$ denotes the Heaviside step function. The investigation of competitive information spread under asymmetric confirmation bias is beyond the scope of our article, which constitutes a part of our future research directions.

Remark 4: We obtain from (3) and (5) that

$$\begin{aligned}\alpha_i(x_i(k)) &= 1 - \sum_{j\in\mathbb{V}} w_{ij} - \overline{w}(x_i(k)) - \underline{w}(x_i(k)) \\ &= 1 - \sum_{j\in\mathbb{V}} w_{ij} - 2\beta + (|x_i(k) - h| + |x_i(k) - g|)\,\gamma \\ &\geq 1 - \sum_{j\in\mathbb{V}} w_{ij} - 2\beta\end{aligned}$$

which indicates that to guarantee the nonnegativeness of $\alpha_i(x_i(k))$, we require $1 - \sum_{j\in\mathbb{V}} w_{ij} \geq 2\beta$ for any $i \in \mathbb{V}$, or

$$1 - \|W\|_\infty \geq 2\beta \tag{6}$$

which holds under Assumption 1. We note that Assumption 1, in conjunction with (5), also guarantees the nonnegativeness of state-dependent influence weights (3) and the convergence of dynamics in (1), whose detailed proofs are included in the proof of Theorem 1 (see Appendix B).

We next express (1) in the vector form

$$x(k+1) = \mathcal{A}(x(k))s + Wx(k) + \overline{\mathcal{W}}(x(k))h + \underline{\mathcal{W}}(x(k))g \tag{7}$$

where we define

$$s \triangleq [s_1, \ldots, s_n]^\top \in \mathbb{R}^n \tag{8a}$$
$$x(k) \triangleq [x_1(k), \ldots, x_n(k)]^\top \in \mathbb{R}^n \tag{8b}$$
$$W \triangleq [w_{ij}] \in \mathbb{R}^{n\times n} \tag{8c}$$
$$\mathcal{A}(x(k)) \triangleq \text{diag}\{\alpha_1(x_1(k)), \ldots, \alpha_n(x_n(k))\} \in \mathbb{R}^{n\times n} \tag{8d}$$
$$\overline{\mathcal{W}}(x(k)) \triangleq [\overline{w}(x_1(k)), \ldots, \overline{w}(x_n(k))]^\top \in \mathbb{R}^n \tag{8e}$$
$$\underline{\mathcal{W}}(x(k)) \triangleq [\underline{w}(x_1(k)), \ldots, \underline{w}(x_n(k))]^\top \in \mathbb{R}^n. \tag{8f}$$

In [23], it is shown that similar dynamics converge to a unique steady state, independent of the initial opinions, for more general bias functions and information sources. Here, we show that the derived convergence condition (4) is more relaxed for this more specific model. Moreover, we analytically analyze the steady-state point achieved by the opinion dynamics. We reiterate that the primary advantage of the model described in (1), in contrast with the classical bounded confidence models such as the Hegselmann–Krause model [8], is that (1) allows us to examine the steady-state point analytically. This is because the state-dependent weights in the classical models can be equal to zero when the opinion distance is larger than the confidence bound, which renders analysis difficult, while state-dependent weights in (1) are nonzero for almost all scenarios. For an analytical expression, similar settings are imposed on state-dependent susceptibility of polar opinion dynamics [33], [34]. Before presenting our results formally, we define the following matrices:

$$E \triangleq I - W + (g - h)\,\gamma I \in \mathbb{R}^{n\times n} \tag{9}$$

$$D \triangleq \text{diag}\left\{\sum_{j\in\mathbb{V}} w_{1j}, \sum_{j\in\mathbb{V}} w_{2j}, \ldots, \sum_{j\in\mathbb{V}} w_{nj}\right\} \in \mathbb{R}^{n\times n}. \tag{10}$$

With these definitions at hand, we present our convergence result, whose proof appears in Appendix B.

Theorem 1: For any $\mathbf{x}(0)$, the dynamics in (1) converge to

$$\begin{aligned}x^*(g,h) &= E^{-1}\left(I - D - 2\beta I + (h - g)\gamma I\right)s \\ &\quad + E^{-1}\left((h+g)\beta\mathbf{1} + (g^2 - h^2)\gamma\mathbf{1}\right).\end{aligned} \tag{11}$$

Remark 5: In light of the Gershgorin circle theorem, we straightforwardly verify from (4) and (9) that all of the eigenvalues of E are nonzero, thus, E is invertible.

III. Problem Formulation

In this work, we analyze the values of information sources (or stubborn individuals as referred in some prior work, e.g., [6]) Hank and Georgia would provide in a setting, where they strive to move the steady-state opinion (whose exact expression is provided in Theorem 1) of the network to the two binary extremes. This problem constitutes an unconstrained zero-sum game between Hank and Georgia, with continuous strategy spaces $g \in [0, \underline{s}]$ and $h \in [\overline{s}, 1]$, for which Nash equilibria are sought.

At first glance, it might be tempting to conclude that the trivial choice of $g = 0$ and $h = 1$ are the equilibrium achieving strategies for Hank and Georgia. Indeed, we formally show that (in Section IV-B, Corollary 2) these strategies are equilibrium-achieving, in the absence of CB. However, this is exactly the aspect in which CB renders this problem a formidable research challenge. The strategic considerations incentivize Hank and/or Georgia to move toward the center (from extremal positions of $h = 1$ and $g = 0$) to increase their influence over the public opinion. More broadly, we explore the following questions in this article.

Q1: What are the properties of Nash equilibrium? Is it unique? Does it exist in pure or mixed strategies?

Q2: How does CB impact equilibrium achieving strategies?

Q3: Does CB affect both Hank and Georgia symmetrically in the sense that they move to the center at equal amounts at the equilibrium? Do they move simultaneously, or only one of them moves?

Let us now recall a technical lemma regarding the eigenvector and eigenvalue of network adjacency matrix.

Lemma 1: [35] Let $\mathfrak{G} = (\mathbb{V}, \mathbb{E})$ be a connected weighted graph. Assume there is a positive vector $\bar{c}$, such that the adjacency matrix A satisfies $A\bar{c} = \bar{\lambda}\bar{c}$. Then, $\bar{\lambda} = \max_{i\in\mathbb{V}}\{|\lambda_i(A)|\}$ and the eigenvalue has multiplicity 1.

Remark 6: Lemma 1 is a consequence of the Perron–Frobenius theorem on nonnegative matrices [36]. We note that the Perron–Frobenius theorem requires the adjacency matrix A to be irreducible, i.e., the implicit digraph $\mathfrak{G}$ must be strongly connected, while Lemma 1 removes this strict requirement, such that the digraph $\mathfrak{G}$ can be quasi-strongly connected with the strong component being aperiodic. This is the motivation behind (4c) in Assumption 1.

We formulate the problem as a zero-sum game, whose cost function in light of Lemma 1 is

$$f(g, h) = c^\top x^*(g, h) \tag{12}$$

where $x^*(g, h)$ is computed via (11), and $c = [c_1, c_2, \ldots, c_n]^\top \in \mathbb{R}^n$ is the eigenvector associated with the largest eigenvalue of $W^\top$

$$W^\top c = \lambda c, \qquad \lambda = \max_{i\in\mathbb{V}}\{|\lambda_i(W)|\}. \tag{13}$$

Remark 7: The cost function $f(g, h) = \mathbf{1}^\top x^*(g, h)$ in [6] indicates that the decision maker treats individuals' opinions equally, which, however, does not hold in many real social examples. For example, in a company, the CEO usually has larger decision-making power than managers. Motivated by this observation, we assign relative scores, i.e, the entries of vector c, to all individuals in a network based on the concept that the high-scoring individual contributes more influence to the decision-making than the low-scoring individual. According to (13), the score vector c in the cost function (12) is referred to the vector of out-eigenvector centralities that measures the importance of an individual in influencing other individuals' opinions [37].

Here, Hank's objective is to maximize $f(g, h)$, while Georgia's objective is to minimize $f(g, h)$. We next define two different notions

$$\widehat{s} \triangleq \sum_{i\in\mathbb{V}} \widehat{c}_i s_i, \quad \chi \triangleq \sum_{i\in\mathbb{V}} \widehat{c}_i \sum_{j\in\mathbb{V}} s_i w_{ij}, \quad \widehat{c}_i \triangleq \frac{c_i}{\sum_{j\in\mathbb{V}} c_j}. \tag{14}$$

We note that $\widehat{s}$ represents the eigencentrality weighted average of innate opinions over the network. In the special case of neutral public opinions, i.e, $s_i = 1/2$ for all i, it follows from (13) that $\chi = \frac{\sum_{i\in\mathbb{V}}\sum_{j\in\mathbb{V}} c_i w_{ij}}{2\sum_{l\in\mathbb{V}} c_l} = \frac{c^\top W\mathbf{1}}{2c^\top\mathbf{1}} = \frac{\lambda c^\top \mathbf{1}}{2c^\top\mathbf{1}} = \frac{\lambda}{2}$, which holds regardless of the remaining network parameters.

We express the cost function as a function of g and h in the following corollary whose proof appears in Appendix C.

Corollary 1: The cost function (12) can be expressed as

$$f(g, h) = \frac{((1-2\beta+(h-g)\gamma)\widehat{s}-\chi+(h+g)\beta+(g^2-h^2)\gamma}{1-\lambda+(g-h)\gamma}c^\top\mathbf{1}. \tag{15}$$

This game can be viewed from two different perspectives, each of which provides a lower/upper bound for the value of the cost function $f(g, h)$ defining the game. The first one is a max-min optimization problem for Hank $\max_h \min_g\{f(g, h)\}$, where Hank expresses her opinion h to maximize $f(g, h)$, anticipating the rational best response of Georgia $g^*(h)$, as formally stated below

$$g^*(h) \triangleq \operatorname*{argmin}_{g\in[0,\underline{s}]}\{f(g, h), \text{ for all } h \in [\overline{s}, 1]\} \tag{16a}$$

$$h^* \triangleq \operatorname*{argmax}_{h\in[\overline{s},1]}\{f(g^*(h), h)\}. \tag{16b}$$

Similarly, a min-max optimization for Georgia would be $\min_g \max_h\{f(g, h)\}$. In this scenario, Georgia acts as the leader, with the objective to minimize $f(g, h)$ while taking the best response of Hank into account. The strategies are referred to the pair (h^*, g^*), such that

$$h^*(g) \triangleq \operatorname*{argmax}_{h\in[\overline{s},1]}\{f(g, h), \text{ for } g \in [0, \underline{s}]\} \tag{17a}$$

$$g^* \triangleq \operatorname*{argmin}_{g\in[0,\underline{s}]}\{f(g, h^*(g))\}. \tag{17b}$$

In the next section, we formally show that this game indeed admits a unique, pure-strategy Nash equilibrium, and hence, solving one of these optimization problems would be sufficient to derive the equilibrium-achieving strategies g^* and h^*.

IV. Nash Equilibrium

We first recall the definition of strategic form game, which will be used to investigate the properties of Nash equilibrium.

Definition 1: [38] A strategic form game is a triplet $\langle\mathbb{I}, (\mathbb{A}_i)_{i\in\mathbb{I}}, (u_i)_{i\in\mathbb{I}}\rangle$, where

1) $\mathbb{I}$ is a finite set of players;
2) $\mathbb{A}_i$ is a nonempty set of available actions for player i;
3) $u_i : \mathbb{A} \to \mathbb{R}$ is the cost function of player i, where $\mathbb{A} = \prod_{i\in\mathbb{I}} \mathbb{A}_i$.

We then transform the zero-sum games (16) and (17) to a strategic form game: $\langle\mathbb{I}, (\mathbb{A}_i)_{i\in\mathbb{I}}, (u_i)_{i\in\mathbb{I}}\rangle$, where

$$\mathbb{I} = \{\text{Hank}, \text{Georgia}\}, \ \mathbb{A}_{\text{Georgia}} = [0, \underline{s}], \ \mathbb{A}_{\text{Hank}} = [\overline{s}, 1] \tag{18a}$$

$$u_{\text{Georgia}}(a_{\text{Georgia}}, a_{\text{Hank}}) = -f(g, h) \tag{18b}$$

$$u_{\text{Hank}}(a_{\text{Hank}}, a_{\text{Georgia}}) = f(g, h). \tag{18c}$$

A. Existence and Uniqueness of Nash Equilibrium

We start with the properties of Nash equilibrium, which are formally stated in the following theorem whose proof is presented in Appendix D.

Theorem 2: The Nash equilibrium (g^*, h^*) is unique and it is in pure strategies.

We proceed with the analysis of the aforementioned Nash equilibrium in the absence, and in the presence of CB.

B. Nash Equilibrium Without CB: $\gamma = 0$

We start with the case of no CB for which we have the intuitive solution, $(g^*, h^*) = (0, 1)$, regardless of the remaining problem parameters. We note that in our model setting $\gamma = 0$ in (3) yields "the no CB scenario." Our result is stated formally in the following theorem whose proof is given in Appendix E.

Corollary 2: When no individual holds CB toward the opinions of information sources Hank and Georgia, the pure Nash equilibrium of the competitive information spread problems (16) and (17) is $(g^*, h^*) = (0, 1)$.

C. Nash Equilibrium With CB ($\gamma \neq 0$)

Before stating our main result, we define two simple functions that are used in the description of the equilibrium

$$q(g, h) \triangleq (\beta + \gamma\widehat{s} - 2\gamma h)a_1 + b_1\gamma + \gamma^2 h^2 \tag{19}$$

$$m(g, h) \triangleq (\beta - \gamma\widehat{s} + 2\gamma g)a_2 + g^2\gamma^2 - b_2\gamma \tag{20}$$

with

$$a_1 = 1 - \lambda + \gamma g \tag{21}$$

$$b_1 = (1 - 2\beta - g\gamma)\widehat{s} - \chi + g\beta + g^2\gamma \tag{22}$$

$$a_2 = 1 - \lambda - h\gamma \tag{23}$$

$$b_2 = (1 - 2\beta + h\gamma)\,\widehat{s} + h\beta - h^2\gamma - \chi. \tag{24}$$

These functions are related to the partial derivatives of the cost function $f(g, h)$ as follows:

$$\frac{\partial f(g, h)}{\partial h} = \frac{q(g, h)c^\top \mathbf{1}}{(a_1 - \gamma h)^2} \tag{25}$$

$$\frac{\partial f(g, h)}{\partial g} = \frac{m(g, h)c^\top \mathbf{1}}{(a_2 + g\gamma)^2}. \tag{26}$$

We next define the following auxiliary functions:

$$r(g) \triangleq -\frac{\sqrt{(1-\lambda)(1-\lambda-\beta+2g\gamma)-(2-\lambda-2\beta)\gamma\widehat{s}-2g\beta\gamma+\gamma\chi}}{\gamma} + \frac{1-\lambda}{\gamma} + g \tag{27}$$

$$w(h) \triangleq \frac{\sqrt{(1-\lambda)(1-\lambda-2h\gamma-\beta)+(2-\lambda-2\beta)\gamma\widehat{s}+2h\beta\gamma-\chi\gamma}}{\gamma} - \frac{1-\lambda}{\gamma} + h. \tag{28}$$

With these definitions at hand, we present the Nash equilibrium in the following theorem, and its proof in Appendix F.

Theorem 3: The Nash equilibrium (g^*, h^*) for the competitive information spread problems (16) and (17) is as follows.

1) If $m(0, 1) \geq 0$

$$(g^*, h^*) = \begin{cases} (0, 1), & \text{if } q(0, 1) \geq 0 \\ (0, \overline{s}), & \text{if } q(0, \overline{s}) \leq 0 \\ (0, r(0)), & \text{otherwise.} \end{cases} \tag{29}$$

2) If $m(\underline{s}, 1) \leq 0$

$$(g^*, h^*) = (\underline{s}, 1). \tag{30}$$

3) Otherwise

$$(g^*, h^*) = (w(1), 1). \tag{31}$$

Remark 8: Through comparing (29) with (30) and (31), we conclude that if $m(0, 1) \geq 0$, Georgia's pure strategy is fixed as $g^* = 0$; otherwise,Hank's strategy is fixed as $h^* = 1$.

Remark 9: In the scenario of neutral innate opinions, i.e., $s_1 = s_2 = \ldots = s_n = \frac{1}{2}$, we have $\widehat{s} = \frac{1}{2}$ and $\chi = \frac{\lambda}{2}$, substituting, which into $m(0, 1)$ and $q(0, 1)$ yields $m(0, 1) = q(0, 1) \geq 0$. In light of Theorem 3, the Nash equilibrium is $(g^*, h^*) = (0, 1)$, which indicates that CB does not move equilibrium when the public's innate opinions are neutral.

Remark 10: A rather interesting observation of Nash equilibrium here is the following: CB can influence only Hank's or Georgia's pure strategy, while it cannot influence both of them simultaneously. In other words, either Hank or Georgia moves to the center (or neither does so), but under no condition both Hank and Georgia move toward the center at equilibrium.

Remark 11: The studied problem of competitive information spread assumes that the social dynamics (1) and the cost function (12) are known to information sources. Theorem 3 indicates that the information sources need to explore the inference algorithms of social network topology, innate opinions, and confirmation bias parameters included in model (1) and cost function (12) for optimal information spread strategies. These observations first inspired our proposed inference mechanism of network topology and confirmation bias [39]. Cost-function inference is a much more challenging and deeper problem than learning dynamics and it has deep roots to inverse optimality, which was first investigated by Kalman in his seminal paper "When is a linear control optimal" in [40] and then later studied by the machine learning community in the context of inverse reinforcement learning. To address this problem, we will propose a method inspired by the work in [41], which can be thought of as an extension of inverse optimality for nonlinear systems based on Hamilton–Jacobi–Bellman theory.

Remark 12: Theorem 3 indicates that the studied problem in this article can address a security concern in social networks, where the attacker influences public opinions through dispersing misinformation, while the defender counters the influence of misinformation on public opinions by spreading truthful information. The dependencies of Nash equilibrium presented in Theorem 3 also indicate that the defender can hinder the attacker's optimal information spread strategy through preserving the privacy of cooperative individuals' communication topology and their associated innate opinions and confirmation bias parameters from inference by attacker [39].

D. Impact of CB

Substituting $\gamma = 0$ into $q(0, 1)$ and $m(0, 1)$ straightforwardly yields $q(0, 1) \geq 0$ and $m(0, 1) \geq 0$, which, in conjunction with Theorem 2 as well as the the conditions of (29)–(31), indicate

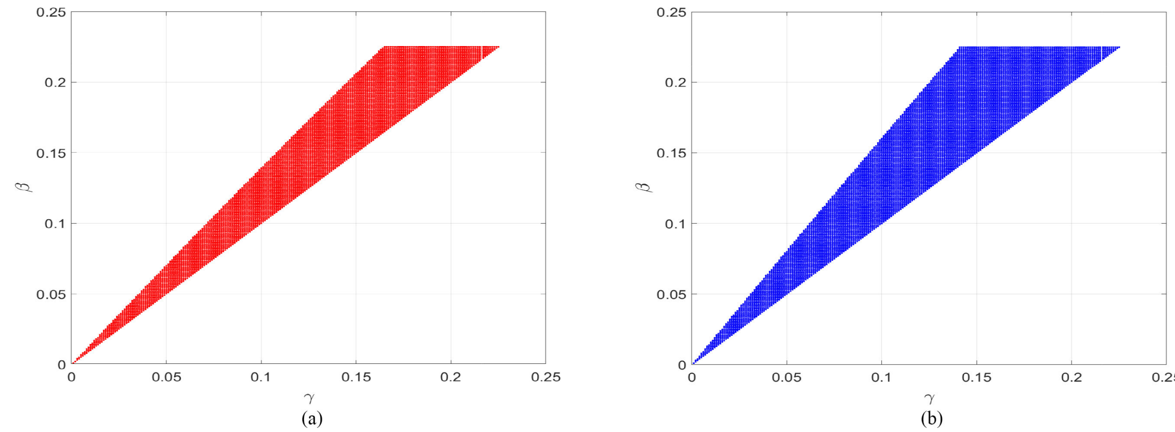

Fig. 1. Under fixed $\xi = 0.14$. (a) Feasible areas of $m(0,1) < 0$ under $(\widehat{s}, \lambda) = (0.75, 0.3)$ and $(\widehat{s}, \lambda) = (0.75, 0.1)$. (b) Feasible areas of $q(0,1) < 0$ under $(\widehat{s}, \lambda) = (0.65, 0.05)$ and $(\widehat{s}, \lambda) = (0.65, 0.1)$.

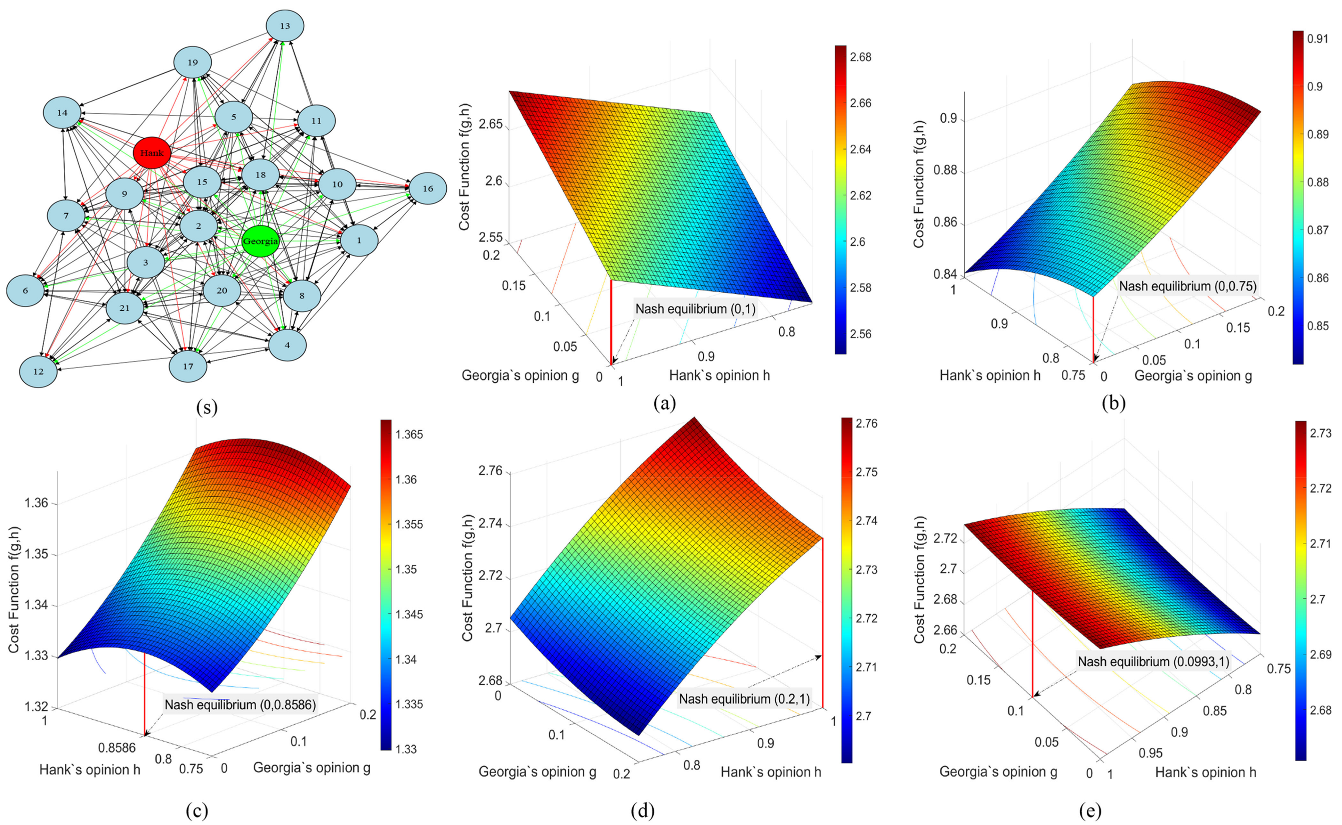

Fig. 2. (s) Krackhardt's advice network [42] in the presence of competitive information sources Hank and Georgia. Surface plots of cost functions under different conditions. (a) No CB ($m(0,1)\geq 0$ and $q(0,1)\geq 0$). (b) $m(0,1)\geq 0$ and $q(0,\overline{s}) \leq 0$. (c) $m(0,1) \geq 0$ and $q(0,1) < 0$ and $q(0,\overline{s}) > 0$. (d) $m(\underline{s},1) \leq 0$. (e) $m(0,1) < 0$ and $m(\underline{s},1) > 0$.

that the CB can influence the pure Nash equilibrium only when its parameters cause $m(0,1) < 0$, or $q(0,1) < 0$. The feasible areas of $m(0,1) < 0$ and $q(0,1) < 0$ in each subfigure of Fig. 1 show that under fixed innate opinions and social network structure, there exists a range of β and γ in influencing the pure Nash equilibrium. The characterization of the conditions for which CB influences the Nash equilibrium is stated formally in the following theorem, whose proof is presented in Appendix G.

Theorem 4: CB changes the equilibrium-achieving strategy of Georgia, i.e., $g^* \neq 0$, if and only if

$$0 < \frac{1 - \lambda - 2\gamma + 2\widehat{s}\gamma}{2\widehat{s} - \lambda\widehat{s} - \gamma - \chi} < \frac{\gamma}{\beta} \leq 1 \tag{32}$$

and CB alters Hank's strategy, i.e., $h^* \neq 1$, if and only if

$$0 < \frac{1 - \lambda}{2 - 2\lambda - 2\widehat{s} + \widehat{s}\lambda + 2\beta\widehat{s} + \chi - \gamma} < \frac{\gamma}{\beta} \leq 1. \tag{33}$$

V. Numerical Results

In this section, we numerically demonstrate our results in the well-known Krackhardt's advice network [42] with 21 individuals. The network topology is shown in Fig. 2(s), where Hank and Georgia represent two competitive information sources. For the weight matrix W, if individual v_i asks for advice from her neighbor v_j, then $w_{ij} = \frac{1}{25+\Gamma_i^{\text{in}}}$ for all the individuals v_j that influence individual v_i, where Γ_i^{in} denotes in-degree of individual v_i. The largest eigenvalue of the adjacency matrix W that describes the structure of Krackhardt's advice network is computed as $\lambda = 0.2369$. We demonstrate the five different Nash equilibriums.

a) No CB: We let the innate opinions of individuals v_1 and v_2 be the same as 0.2, while others are uniformly set as 0.75. For CB, we set $\beta = 0.06$ and $\gamma = 0$, which indicates that no individual holds CB toward the opinions of Hank and Georgia. We verify that $q(0,1)\geq 0$ and $m(0,1)\geq 0$. By Corollary 2 or Theorem 3, we theoretically expect the Nash equilibrium to be $(g^*, h^*) = (0, 1)$, which is demonstrated by Fig. 2(a).

b) $m(0,1)\geq 0$ and $q(0,\overline{s}) \leq 0$: We set individual v_{21}'s innate opinion as 0.2, while others are uniformly set as 0.75. For CB, we let $\beta = \gamma = 0.06$. Under this setting, we have

$$\underline{s} = 0.2,\ \overline{s} = 0.75,\ \widehat{s} = 0.2283,\ \chi = 0.0580$$

by which, we verify from (19)–(24) that $m(0,1)\geq 0$ and $q(0,\overline{s}) \leq 0$. Hence, from Theorem 3 we expect the Nash equilibrium to be $(g^*, h^*) = (0, \overline{s}) = (0, 0.75)$, which is demonstrated by Fig. 2(b).

c) $m(0,1) \geq 0$ and $q(0,1) < 0$ and $q(0,\overline{s}) > 0$: We let the innate opinions of individuals $\mathrm{v}_{18} - \mathrm{v}_{21}$ be the same as 0.75, while others are uniformly set as 0.2. For CB, we choose $\beta = \gamma = 0.06$. Under this setting, we have

$$\underline{s} = 0.2,\ \overline{s} = 0.75,\ \widehat{s} = 0.3637,\ \chi = 0.0945$$

by which, we verify from (19)–(24) that $m(0,1) \geq 0$ and $q(0,1) < 0$ and $q(0,\overline{s}) > 0$. Moreover, we obtain from (27) that $r(0) = 0.8586$. Therefore, from Theorem 3 we expect the Nash equilibrium to be $(g^*, h^*) = (0, r(0)) = (0, 0.8586)$, which is demonstrated by Fig. 2(c).

d) $m(\underline{s}, 1) \leq 0$: We set the innate opinions of individuals v_1 and v_2 as the same as 0.2, others are uniformly set as 0.75. For CB, we choose $\beta = \gamma = 0.06$. Under this setting, we have

$$\underline{s} = 0.2,\ \overline{s} = 0.75,\ \widehat{s} = 0.7265,\ \chi = 0.1693$$

by which, we verify from (19)–(24) that $m(\underline{s}, 1) \leq 0$. From Theorem 3 we expect Nash equilibrium to be $(g^*, h^*) = (\underline{s}, 1) = (0.2, 1)$, which is demonstrated by Fig. 2(d).

e) $m(0,1) < 0$ and $m(\underline{s}, 1) > 0$: In this case, we choose the same setting of innate opinions in case D, but for CB, we let $\beta = 0.06$ and $\gamma = 0.048$. We verify from (19)–(24) that $m(0,1) < 0$ and $m(\underline{s}, 1) > 0$. Moreover, by (28) we have $w(1) = 0.0993$. Hence, from Theorem 3 we expect the Nash equilibrium to be $(g^*, h^*) = (w(1), 1) = (0.0993, 1)$, which is demonstrated by Fig. 2(e).

VI. Conclusion

In this article, we have studied the competitive information spread with CB over social networks, which is formulated as a zero-sum game and have investigated the pure Nash equilibrium point. We have analyzed the impact of CB and innate opinions of the Nash equilibrium, particularly the following tradeoff for information sources: A move to the extremal opinions to maximally change public opinion, and another move, due to the existence CB, to the center to maximize the influence. Our analysis has uncovered a few rather surprising results: CB moves the Nash equilibrium toward the center only when the innate opinions are not neutral, and this move occurs for only one of the information sources. Theoretical results are verified by numerical examples.

Appendix A
Auxiliary Results

This section presents the auxiliary results for the proofs of main results.

Lemma 2: The matrix E defined in (9) satisfies

$$\frac{c^\top E}{1 - \lambda + (g-h)\gamma} = c^\top. \tag{34}$$

Proof: We note that W is an adjacency matrix and its transposition does not change its eigenvalues; by Lemma 1 we have (13). It follows from (9) and (13) that

$$c^\top E = c^\top - c^\top W + (g-h)c^\top\gamma = (1 - \lambda + (g-h)\gamma)c^\top$$

from which (34) is obtained. ■

Lemma 3: With $\gamma \neq 0$, $q(g,h)$ and $m(g,h)$ in (19) and (20) satisfy $m(g,h) + q(g,h) > 0$.

Proof: The partial derivative of (20) w.r.t. g is

$$\frac{\partial m(g,h)}{\partial g} = 2\gamma a_2 + 2\gamma^2 g. \tag{35}$$

Noticing the eigenvalue λ given in (13), the condition (4), in conjunction with Ger š gorin disk theorem [43], imply that

$$1 - \lambda \geq 1 - \max\left\{\|W\|_1, \|W\|_\infty\right\} > \max\left\{2\beta, 4\gamma\right\} \tag{36}$$

which together with (23) and the fact $0 < h < 1$ imply that $a_2 > 0$. Thus, we conclude from (35) that

$$\frac{\partial m(g,h)}{\partial g} \geq 0, \quad \text{for } g, h \in [0,1]. \tag{37}$$

The partial derivative of (20) w.r.t. h satisfies

$$\frac{\partial m(g,h)}{\partial h} = -2(\beta - \gamma(h-g))\gamma \leq 0, \text{for } g, h \in [0,1]. \tag{38}$$

Applying the same analysis to (19), we have

$$\frac{\partial q(g,h)}{\partial g} = 2(\beta + (g-h)\gamma)\gamma \geq 0, \text{for } g, h \in [0,1] \tag{39}$$

$$\frac{\partial q(g,h)}{\partial h} = -2(1-\lambda+(g-h)\gamma)\gamma \leq 0, \text{for } g, h \in [0,1]. \tag{40}$$

From (37)–(38) and (39)–(40), we have

$$m(g,h) \geq m(0,h) \geq m(0,1) \tag{41}$$

$$q(g,h) \geq q(0,h) \geq q(0,1). \tag{42}$$

Combining (41) and (42) yields

$$m(g,h) + q(g,h) \geq m(0,1) + q(0,1). \tag{43}$$

Substituting the values of $m(0,1)$ and $q(0,1)$ into (43), we have

$$\begin{aligned} m(g,h)+q(g,h) &\geq 2(1-\frac{\lambda}{2}-\gamma-\gamma\lambda)\beta+2\gamma^2+2\gamma\lambda \\ &\geq 2(1-\frac{\lambda}{2}-\gamma-\gamma\lambda)\gamma+2\gamma^2+2\gamma\lambda \\ &= 2\gamma+\lambda\gamma-2\gamma^2\lambda = 2\gamma(1-\gamma\lambda)+\lambda\gamma \\ &> 0 \end{aligned}$$

where the inequalities follow from $\gamma \neq 0$, (4a), and (36). ■

Lemma 4: Consider $f(g,h)$, $q(g,h)$, and $m(g,h)$ given by (15), (19), and (20), respectively. If $q(g,h) \geq 0$, $m(\tilde{g},\tilde{h}) \geq 0$, $1 \geq g \geq \tilde{g} \geq 0$, and $1 \geq h \geq \tilde{h} \geq 0$, then $f(g,h) \geq f(\tilde{g},\tilde{h})$.

Proof: It follows from (40) that $q(g,h) \geq 0$ implies $q(g,\breve{h}) \geq 0$ for $\breve{h} \in [\tilde{h},h] \subseteq [0,1]$, which, in conjunction with (25), results in

$$f(g,h) \geq f(g,\tilde{h}). \tag{44}$$

Meanwhile, following (39), $m(\tilde{g},\tilde{h}) \geq 0$ implies $m(\breve{g},\tilde{h}) \geq 0$ for $\breve{g} \in [\tilde{g},g] \subseteq [0,1]$, which, in conjunction with (26), results in $f(g,\tilde{h}) \geq f(\tilde{g},\tilde{h})$, which along with (44) leads to $f(g,h) \geq f(g,\tilde{h}) \geq f(\tilde{g},\tilde{h})$. ■

Appendix B
Proof of Theorem 1

Noting that $s_i, x_i(0) \in [0,1]\ \forall i \in \mathbb{V}$, in conjunction with (5), we obtain $x_i(k) \in [0,1]$. Thus, the nonnegativeness of state-dependent influence weights (3) directly follows from (4a). We denote the mapping executed by the dynamics in (1) from time k to $k+1$ as Ψ, i.e., $x_i(k+1) \triangleq \Psi_i(x_i(k))$. For two vectors x and y, we have

$$\begin{aligned} &\Psi_i(x_i)-\Psi_i(y_i) \\ &= (\alpha_i(x_i)-\alpha_i(y_i))s_i + \sum_{j\in\mathbb{V}} w_{ij}(x_j-y_j) \\ &\quad + (\overline{w}(x_i)-\overline{w}(y_i))h + (\underline{w}(x_i)-\underline{w}(y_i))g,\ i \in \mathbb{V}. \end{aligned} \tag{45}$$

Also noting that

$$\begin{aligned} |\alpha_i(x_i)-\alpha_i(y_i)| &= \gamma\left(|x_i-h|-|y_i-h|+|x_i-g|-|y_i-g|\right) \\ &\leq 2\gamma\,|x_i-y_i|\,. \end{aligned} \tag{46}$$

Moreover, from (3) we have

$$|\overline{w}(x_i)-\overline{w}(y_i)| = \gamma\,||y_i-h|-|x_i-h|| \leq \gamma\,|x_i-y_i| \tag{47a}$$

$$|\underline{w}(x_i)-\underline{w}(y_i)| = \gamma\,||y_i-g|-|x_i-g|| \leq \gamma\,|x_i-y_i|\,. \tag{47b}$$

Combining (45) with (46) and (47) yields

$$\begin{aligned} \|\Psi(x)-\Psi(y)\| &= \sum_{i\in\mathbb{V}} |\Psi_i(x_i)-\Psi_i(y_i)| \\ &\leq 4\gamma\sum_{i\in\mathbb{V}}|x_i-y_i| + \sum_{i\in\mathbb{V}}\sum_{j\in\mathbb{V}} w_{ij}\,|x_j-y_j| \\ &= 4\gamma\sum_{i\in\mathbb{V}}|x_i-y_i| + \sum_{i\in\mathbb{V}}|x_i-y_i|\sum_{j\in\mathbb{V}} w_{ji} \\ &\leq 4\gamma\sum_{i\in\mathbb{V}}|x_i-y_i| + \sum_{i\in\mathbb{V}}|x_i-y_i|\max_{i\in\mathbb{V}}\left\{\sum_{j\in\mathbb{V}} w_{ji}\right\} \\ &= \left(4\gamma+\|W\|_1\right)\|x-y\|\,. \end{aligned} \tag{48}$$

Here we note that if $1-\|W\|_1 > 4\gamma$, due to Banach fixed-point theorem (following the steps in the proof of [21, Th. 1], the dynamics in (1) converge to a unique point, regardless of the initial state. This condition, in conjunction with (6) yields (4b).

To solve for $x^*(g,h)$, we set $h\mathbf{1} \geq x(0) \geq g\mathbf{1}$ (since $x^*(g,h)$ is independent of the initial state, we can set arbitrary initial condition). Using (2) and (5), we have $h\mathbf{1} \geq x(k) \geq g\mathbf{1}$ for $\forall k \in \mathbb{N}_0$, and hence $h\mathbf{1} \geq x^*(g,h) \geq g\mathbf{1}$. This implies, by reexpressing (3) for $x^*(g,h)$, that

$$\overline{w}(x_i^*(g,h)) = \beta - h\gamma + \gamma x_i^*(g,h) \tag{49a}$$

$$\underline{w}(x_i^*(g,h)) = \beta + g\gamma - \gamma x_i^*(g,h) \tag{49b}$$

and also from (5)

$$\alpha_i(x_i^*(g,h)) = 1 - \sum_{j\in\mathbb{V}} w_{ij} - 2\beta + (h-g)\,\gamma,\ i \in \mathbb{V}. \tag{50}$$

Plugging (49) and (50) in (7), we obtain (11).

Appendix C
Proof of Corollary 1

Plugging (11) and (34) into (12) yields, after some algebra

$$\begin{aligned} f(g,h) = \frac{\sum_{i\in\mathbb{V}} c_i}{1-\lambda+(g-h)\gamma}&\left(\sum_{i\in\mathbb{V}}((h+g)\beta+(g^2-h^2)\gamma)\frac{c_i}{\sum_{i\in\mathbb{V}} c_i}\right. \\ &\left.+\sum_{i\in\mathbb{V}}\left(1-\sum_{j\in\mathbb{V}} w_{ij}-2\beta+(h-g)\gamma\right)\frac{c_i s_i}{\sum_{i\in\mathbb{V}} c_i}\right) \end{aligned}$$

which is equivalent to (15), considering (14).

Appendix D
Proof of Theorem 2

Let us consider the transformed strategic form game $\left\langle \mathbb{I}, (\mathbb{A}_i)_{i\in\mathbb{I}}, (u_i)_{i\in\mathbb{I}} \right\rangle$ with elements given in (18). We obtain from (26) with (20), (23), (24), and (18) that

$$\frac{\partial^2 u_{\text{Georgia}}(a_{\text{Georgia}}, a_{\text{Hank}})}{\partial a_{\text{Georgia}}^2} = -\frac{2\gamma c^\top \mathbf{1}\bar{\mathrm{m}}(g,h)}{(a_2+g\gamma)^3} \tag{51}$$

where

$$\begin{aligned} \bar{\mathrm{m}}(g,h) = &\ (1-\lambda-h\gamma)\,(1-\lambda-h\gamma-\beta+\gamma\widehat{s}) \\ &+ \left((1-2\beta+h\gamma)\,\widehat{s}+h\beta-h^2\gamma-\chi\right)\gamma. \end{aligned} \tag{52}$$

Noticing (14), we have $\chi \leq \widehat{s}$ (since $0 \leq \sum_{j\in\mathbb{V}} w_{ij} \leq 1$), moreover, considering (2) we have $h \geq \widehat{s}$. We then obtain from (52) that

$$\begin{aligned} \bar{\mathrm{m}}(g,h) \geq &\ (1-\lambda-h\gamma)\,(1-\lambda-h\gamma-\beta+\gamma\widehat{s}) \\ &+ \left((1-2\beta+\widehat{s}\gamma)\,\widehat{s}+\widehat{s}\beta-\widehat{s}^2\gamma-\widehat{s}\right)\gamma \\ = &\ (1-\lambda-h\gamma)\,(1-\lambda-h\gamma-\beta+\gamma\widehat{s})-\beta\gamma\widehat{s} \end{aligned}$$

which, in conjunction with (4), further implies

$$\bar{\mathrm{m}}(g,h) \geq (1-\lambda-h\gamma)(1-\lambda-h\gamma-\beta+\gamma\hat{s})-(1-\lambda-h\gamma)\gamma\hat{s}$$
$$= (1-\lambda-h\gamma)(1-\lambda-h\gamma-\beta) \geq 0. \quad (53)$$

Substituting (53) into (51) yields $\frac{\partial^2 u_{\text{Georgia}}(a_{\text{Georgia}}, a_{\text{Hank}})}{\partial a^2_{\text{Georgia}}} \leq 0$. Following the same method, we also obtain $\frac{\partial^2 u_{\text{Hank}}(a_{\text{Georgia}}, a_{\text{Hank}})}{\partial a^2_{\text{Hank}}} \leq 0$. Thus, $\langle \mathbb{I}, (\mathbb{A}_i)_{i\in\mathbb{I}}, (u_i)_{i\in\mathbb{I}} \rangle$ is a concave game.

Following the proof of Lemma 4 in Appendix A, we obtain $u_{\text{Georgia}}(a_{\text{Georgia}}, a_{\text{Hank}}) \geq u_{\text{Georgia}}(\bar{a}_{\text{Georgia}}, a_{\text{Hank}})$ and $u_{\text{Hank}}(a_{\text{Georgia}}, a_{\text{Hank}}) \geq u_{\text{Hank}}(\bar{a}_{\text{Georgia}}, \bar{a}_{\text{Hank}})$, for any $a_{\text{Georgia}} < \bar{a}_{\text{Georgia}} \in \mathbb{A}_{\text{Georgia}}$ and $a_{\text{Hank}} > \bar{a}_{\text{Hank}} \in \mathbb{A}_{\text{Hank}}$. Here, we conclude that the concave game $\langle \mathbb{I}, (\mathbb{A}_i)_{i\in\mathbb{I}}, (u_i)_{i\in\mathbb{I}} \rangle$ satisfies the dominance solvability condition in [44]. Hence, it satisfies Rosen's well-known conditions for existence and uniqueness of a pure strategy Nash equilibrium [45].

Appendix E
Proof of Corollary 2

Substituting $\gamma = 0$, $h = 1$, and $g = 0$ into (20) with (23) and (24) yields $q(0,1) \geq 0$ and $m(0,1) \geq 0$. It follows from (41) that $m(g,h) \geq 0$ for $g, h \in [0,1]$. Then, from (26) we obtain the optimal solution of (16a) as $g^*(h) = 0$. By (42), we obtain from $q(0,1) \geq 0$ that $q(0,h) \geq 0$, and from (25) we have $\frac{\partial f(0,h)}{\partial h} \geq 0$. Thus, the optimal solution of (16b) is $h^* = 1$. Consequently, the max-min strategy of (16) is $(g^*, h^*) = (0,1)$, which is also the pure Nash equilibrium via considering Theorem 2.

Appendix F
Proof of Theorem 3

As Theorem 2 states, the pure Nash equilibrium exists for the games (16) and (17). Therefore, to derive it, we only need to study min-max or max-min strategy. In this proof, we study the max-min problem (16).

Based on (37) and (38), the rest of the proof considers five different cases with the following auxiliary function (assuming $\gamma \neq 0$):

$$\delta(g) \triangleq -\frac{\sqrt{(\beta+2g\gamma)(\beta+\lambda-1)-(\lambda+2\beta-2)\hat{s}\gamma-\chi\gamma}}{\gamma} - \frac{1-\lambda}{\gamma} + h. \quad (54)$$

Case A. $m(0,1) \geq 0$**:** Due to (41), $m(0,1) \geq 0$, in conjunction with (26), indicate that given any $h \in [\overline{s}, 1]$, we have

$$\frac{\partial f(g,h)}{\partial g} \geq 0, \text{ for any } g \in [0, \underline{s}] \quad (55)$$

which implies that $f(g,h)$ is nondecreasing with respect to g. Thus, from (16a) we have

$$g^*(h) = 0. \quad (56)$$

We next insert (56) into (15) and take its derivative w.r.t. h

$$\frac{\mathrm{d}f(g^*(h),h)}{\mathrm{d}h} = \frac{q(0,h)c^\top \mathbf{1}}{(1-\lambda-\gamma h)^2} \quad (57)$$

where $q(0,h)$ is given in (19). We note that (40) indicates that $q(0,h)$ is nonincreasing w.r.t. h. Thus, if $q(0,1) \geq 0$, $q(0,h) \geq 0$ for any $h \in [\overline{s}, 1]$. We conclude from (57) that $f(g^*(h),h)$ is nondecreasing w.r.t. h. Hence via (16b), $h^* = 1$. If $q(0,\overline{s}) \leq 0$, we have $q(0,h) \leq 0$ for any $h \in [\overline{s}, 1]$. Thus, $f(g^*(h),h)$ is nonincreasing w.r.t. h, and hence $h^* = \overline{s}$. If $q(0,\overline{s}) > 0$ and $q(0,1) < 0$, it can be verified from (27) and (19) that $q(0, r(0)) = 0$. Then, from (57) we have $\frac{\mathrm{d}f(g^*(h),h)}{\mathrm{d}h} \geq 0$ for $h \in [\overline{s}, r(0)]$ and $\frac{\mathrm{d}f(g^*(h),h)}{\mathrm{d}h} < 0$ for $h \in [r(0), 1]$, which implies that $h^* = r(0)$. The equilibrium point in this case is summarized in (29).

Case B. $m(\underline{s},\overline{s}) \leq 0$ **:** Due to (37) and (38), $m(\underline{s},\overline{s}) \leq 0$ implies that given any $h \in [\overline{s}, 1]$, $\frac{\partial f(g,h)}{\partial g} \leq 0$ for $g \in (0, \underline{s}]$, from which and (16a), we have $g^*(h) = \underline{s}$. We now plug $g^*(h)$ into (15) and take its derivative w.r.t. h

$$\frac{\mathrm{d}f(g^*(h),h)}{\mathrm{d}h} = \frac{q(\underline{s},h)c^\top \mathbf{1}}{(1-\lambda-\gamma h+\gamma)^2}. \quad (58)$$

Due to (40), if $q(\underline{s},1) \geq 0$, $q(\underline{s},h) \geq 0$ for any $h \in [\overline{s}, 1]$. We conclude from (58) that $f(\underline{s},h)$ is nondecreasing w.r.t h; thus, $h^* = 1$. If $q(\underline{s},\overline{s}) \leq 0$, then $q(\underline{s},h) \leq 0$ for any $h \in [\overline{s}, 1]$. Thus, $f(g^*(h),h)$ is nonincreasing w.r.t h, and hence $h^* = \overline{s}$. If $q(\underline{s},\overline{s}) > 0$ and $q(\underline{s},1) < 0$, it can be verified from (27) and (19) that $q(\underline{s}, r(\underline{s})) = 0$. Then, via (58), we have $\frac{\mathrm{d}f(\underline{s},h)}{\mathrm{d}h} \geq 0$ for $h \in [\overline{s}, r(\underline{s})]$ and $\frac{\mathrm{d}f(\underline{s},h)}{\mathrm{d}h} < 0$ for $h \in [r(\underline{s}), 1]$, which implies that $h^* = r(\underline{s})$. The equilibrium is summarized as

$$\text{If } m(\underline{s},\overline{s}) \leq 0, \quad (g^*,h^*) = \begin{cases} (\underline{s}, 1), & \text{if } q(\underline{s},1) \geq 0 \\ (\underline{s}, \overline{s}), & \text{if } q(\underline{s},\overline{s}) \leq 0 \\ (\underline{s}, r(\underline{s})), & \text{otherwise.} \end{cases} \quad (59)$$

By Lemma 3, the condition $m(\underline{s},\overline{s}) \leq 0$ & $q(\underline{s},\overline{s}) \leq 0$ in (59) contradicts with $m(\underline{s},\overline{s}) + q(\underline{s},\overline{s}) > 0$. The "otherwise" in (59) represents $q(\underline{s},1) < 0$ & $q(\underline{s},\overline{s}) > 0$ & $m(\underline{s},\overline{s}) \leq 0$, which with (38) imply $q(\underline{s},1) + m(\underline{s},1) \leq q(\underline{s},1) + m(\underline{s},\overline{s}) < 0$. Note that this contradicts with $q(\underline{s},1) + m(\underline{s},1) \geq 0$, which is a consequence of Lemma 3. Thus, the "otherwise" condition in (59) does not hold as well. Therefore, we have

$$\text{If } m(\underline{s},\overline{s}) \leq 0, \quad (g^*,h^*) = (\underline{s}, 1). \quad (60)$$

Case C. $m(0,\overline{s}) \leq 0$ **&** $m(\underline{s},1) \geq 0$ **:** It follows from (38) that $m(0,\overline{s}) \leq 0$ and $m(\underline{s},1) \geq 0$ imply $m(0,h) \leq 0$ and $m(\underline{s},h) \geq 0$ for $h \in [\overline{s}, 1]$, which indicate that $\frac{\partial f(g,h)}{\partial g} \geq 0$ for any $g \in (w(h), \underline{s}]$, and $\frac{\partial f(g,h)}{\partial g} \leq 0$ for any $g \in [0, w(h)]$, where $w(h)$ is given by (28). The relation $g^*(h) = w(h)$ follows from (16a), whose derivate w.r.t. h is:

$$\frac{\mathrm{d}g^*(h)}{\mathrm{d}h} = \frac{\mathrm{d}w(h)}{\mathrm{d}h} = 1 - \frac{1-\lambda-\beta}{\Xi} \quad (61)$$

where

$$\Xi = \sqrt{(1-\lambda)(1-\lambda-2h\gamma-\beta)+(2-\lambda-2\beta)\gamma\hat{s}+2h\beta\gamma-\chi\gamma}.$$

Replacing g in (15) by $g^*(h)$ and taking its derivative w.r.t. h, we get

$$\frac{\mathrm{d}f(g^*(h),h)}{\mathrm{d}h} = \frac{(r_1r_3+r_2+(r_1r_4-r_2)\frac{\mathrm{d}g^*(h)}{\mathrm{d}h})c^\top \mathbf{1}}{(a_2+g\gamma)^2} \tag{62}$$

where we define

$$r_1 \triangleq 1-\lambda-h\gamma+\gamma g^*(h) \tag{63a}$$

$$\begin{aligned} r_2 \triangleq\ & (1-2\beta+h\gamma)\widehat{s}\gamma+h\beta\gamma-h^2\gamma^2-\chi\gamma \\ & +(\beta-\gamma\widehat{s}+\gamma g^*(h))\gamma g^*(h) \end{aligned} \tag{63b}$$

$$r_3 \triangleq \beta+\widehat{s}\gamma-2h\gamma \tag{63c}$$

$$r_4 \triangleq \beta-\widehat{s}\gamma+2\gamma g^*(h). \tag{63d}$$

Since $g^*(h)=w(h)\geq 0$ due to (28), we have

$$\Xi \geq 1-\lambda-h\gamma\geq 0. \tag{64}$$

Moreover, $1-\lambda-h\gamma-(1-\lambda-\beta)=\beta-h\gamma\geq 0$, which implies that $1-\lambda-h\gamma\geq 1-\lambda-\beta$. Then, noting (64) and (36), we obtain $\Xi\geq 1-\lambda-h\gamma\geq 1-\lambda-\beta\geq 0$, which, in conjunction with (61), yields

$$0\leq \frac{\mathrm{d}g^*(h)}{\mathrm{d}h}\leq 1. \tag{65}$$

If $r_1r_4-r_2\leq 0$, it follows from (63) and (65) that:

$$\begin{aligned} & r_1r_3+r_2+(r_1r_4-r_2)\frac{\mathrm{d}g^*(h)}{\mathrm{d}h} \\ & \geq r_1r_3+r_2+r_1r_4-r_2 \qquad (66)\\ & = 2\left(1-\lambda-h\gamma+\gamma g^*(h)\right)\left(\beta-h\gamma+\gamma g^*(h)\right)\geq 0 \qquad (67) \end{aligned}$$

where (66) follows from (36) and (4a).

From (14), we have

$$\chi\leq \frac{1}{\sum_{i\in\mathbb{V}} c_i}\sum_{i\in\mathbb{V}} c_is_iw_{\max}=w_{\max}\widehat{s}. \tag{68}$$

where we use $w_{\max}\triangleq \max_{i\in\mathbb{V}}\sum_{j\in\mathbb{V}} w_{ij}$. We obtain from (63) that

$$\begin{aligned} & (r_1r_3+r_2)-(r_1r_4-r_2) \\ & = 2\gamma(g^*(h))^2-2\gamma h^2+2h\gamma\widehat{s}-2\,g^*(h)\gamma\widehat{s}+h\beta+g^*(h)\beta \\ & \quad -\chi-2\beta\widehat{s}+h+g^*(h)+\lambda\widehat{s}-\lambda h-\lambda g^*(h) \qquad (69)\\ & \geq -2\gamma h^2+2h\gamma\widehat{s}+h\beta-\chi-2\beta\widehat{s}+h+\lambda\widehat{s}-\lambda h \qquad (70)\\ & \geq -\widehat{s}\beta-\chi+\widehat{s} \qquad (71)\\ & \geq (1-w_{\max}-\beta)\,\widehat{s}>0 \qquad (72) \end{aligned}$$

where (70) follows from (69) considering

$$(1+\beta-\lambda-2\gamma\widehat{s}+2\gamma g^*(h))g^*(h)\geq 0$$

[due to (36)], and (71) follows from (70) due to $-2\,h^2\gamma+(2\gamma\widehat{s}+1+\beta-\lambda)h\geq(1+\beta-\lambda)\widehat{s}$, since $\frac{1+\beta-\lambda+2\gamma\widehat{s}}{4\gamma}\geq 1$ and $h\in[\widehat{s},1]$. We note also that (72) follows from (71) due to (68).

If $r_1r_4-r_2>0$, from (72) we have

$$r_1r_3+r_2\geq r_1r_4-r_2>0.$$

From (65), we have

$$r_1r_3+r_2+(r_1r_4-r_2)\frac{\mathrm{d}g^*(h)}{\mathrm{d}h}\geq r_1r_3+r_2>0,$$

which, in conjunction with (67) and (62), result in $\frac{\mathrm{d}f(g^*(h),h)}{\mathrm{d}h}\geq 0$. We obtain here $h^*=1$, and consequently, $g^*=w(1)$. The equilibrium in this case is expressed as

$$\text{If } m(0,\overline{s})\leq 0\ \&\ m(\underline{s},1)\geq 0,\ \ (g^*,h^*)=(w(1),1). \tag{73}$$

Case D. $m(\underline{s},\overline{s})>0$ ***&*** $m(0,1)<0$ ***&*** $m(0,\overline{s})>0$***) :*** Due to (38), we obtain from $m(0,1)<0$ & $m(0,\overline{s})>0$

$$m(0,h)\leq 0, h\in[\delta(0),1] \tag{74a}$$

$$m(0,h)>0, h\in[\overline{s},\delta(0)). \tag{74b}$$

It follows from (37) and (74b) that $m(g,h)>0$ for $h\in[\overline{s},\delta(0))$ and $g\in[0,\underline{s}]$. Thus, we have $g^*=0$. Then, following the same analysis in Case A, we arrive at

$$\text{For } h\in[\overline{s},\delta(0)), (g^*,h^*)=\begin{cases}(0,\delta(0)), & \text{if } q(0,\delta(0))\geq 0\\ (0,\overline{s}), & \text{if } q(0,\overline{s})\leq 0\\ (0,r(0)), & \text{otherwise.}\end{cases} \tag{75}$$

We note that the "otherwise" in (75) is $q(0,\delta(0))<0$ & $q(0,\overline{s})>0$. Following (40), we have $q(0,1)\leq q(0,\overline{s})\leq 0$ or $q(0,1)\leq q(0,\delta(0))\leq 0$. Noticing $m(0,1)<0$, we obtain $q(0,1)+m(0,1)<0$ that contradicts with $q(0,1)+m(0,1)>0$ implied by Lemma 3. Thus, the conditions of the second and third items in (75) do not hold. Therefore, equation (75) in this case can be expressed as

$$\text{For } h\in[\overline{s},\delta(0)),\ \ (g^*,h^*)=(0,\delta(0)). \tag{76}$$

If $m(\underline{s},1)\geq 0$, we have $m(\underline{s},h)\geq 0$ for $h\in[\overline{s},1]$, which follows from (38). With the consideration of (74a), following the same analysis in Case C, we obtain

$$\text{For } h\in[\delta(0),1)\ \&\ m(\underline{s},1)\geq 0,\ (g^*,h^*)=(w(1),1). \tag{77}$$

Since (76) and (77) are, respectively, based on $m(0,\delta(0))=0$ and $q(w(1),1)=0$, due to Lemma 4 we have $f(w(1),1)\geq f(0,\delta(0))$. From (76) and (77), we obtain

$$\begin{aligned} & \text{If } m(\underline{s},\overline{s})>0\ \&\ m(0,1)<0\ \&\ m(0,\overline{s})>0\ \&\ m(\underline{s},1)\geq 0 \\ & \quad (g^*,h^*)=(w(1),1). \end{aligned} \tag{78}$$

If $m(\underline{s},1)<0$, we have $m(g,1)<0$ for $g\in[0,\underline{s}]$, which follows from (37). We note that $m(0,\overline{s})>0$ implies that $m(g,\overline{s})>0$ for $g\in[0,\underline{s}]$. Noting (38), we conclude

$$m(g,h)>0 \text{ for any } h\in[\overline{s},\delta(g)] \tag{79a}$$

$$m(g,h)\leq 0 \text{ for any } h\in(\delta(g),1] \tag{79b}$$

where $\delta(g)$ is given in (54). Taking its derivative w.r.t. g, we have

$$\begin{aligned} \frac{\mathrm{d}\delta(g)}{\mathrm{d}g} = & \frac{1-\lambda-\beta}{\sqrt{(\beta+2g\gamma)(\beta+\lambda-1)-(\lambda+2\beta-2)\widehat{s}\gamma-\bar{w}\gamma}} \\ & +1>0 \end{aligned} \tag{80}$$

where the inequality is obtained via considering (36). Since (80) implies that $\delta(g)$ is an increasing function, we have $\delta(\underline{s}) \geq \delta(0)$. It follows from (20) with (54) that $m(0, \delta(0)) = 0$ and $m(\underline{s}, \delta(\underline{s})) = 0$, which, respectively, imply $m(0, h) \leq 0$ and $m(\underline{s}, h) \geq 0$ for $h \in [\delta(0), \delta(\underline{s})]$ [that is due to (38)]. Following the analysis in Case C, we obtain

$$\text{For } h \in [\delta(0), \delta(\underline{s})] \ \& \ m(\underline{s}, 1) < 0$$
$$(g^*, h^*) = (w(\delta(\underline{s})), \delta(\underline{s})). \quad (81)$$

Noting (26), we obtain from (79b) that $g^* = \underline{s}$ for $h \in [\delta(\underline{s}), 1]$. Following the analysis in Case B, we arrive at

$$\text{For } h \in [\delta(\underline{s}), 1] \ \& \ m(\underline{s}, 1) < 0$$
$$(g^*, h^*) = \begin{cases} (\underline{s}, 1), & \text{if } q(\underline{s}, 1) \geq 0 \\ (\underline{s}, \delta(\underline{s})), & \text{if } q(\underline{s}, \delta(\underline{s})) \leq 0 \\ (\underline{s}, r(\underline{s})), & \text{otherwise.} \end{cases} \quad (82)$$

We note that "otherwise" condition in (82) is $q(\underline{s}, 1) < 0$ & $q(\underline{s}, \delta(\underline{s})) > 0$. Due to (39), we have $q(\underline{s}, 1) \leq q(\underline{s}, \delta(\underline{s})) \leq 0$, which, in conjunction with $m(\underline{s}, 1) < 0$, results in $q(\underline{s}, 1) + m(\underline{s}, 1) < 0$ that contradicts with $q(\underline{s}, 1) + m(\underline{s}, 1) > 0$ implied by Lemma 3. Thus, the conditions of the second and the third items in (82) do not hold. Thus, equation (82) is simplified as

$$\text{For } h \in [\delta(\underline{s}), 1] \ \& \ m(\underline{s}, 1) < 0 \ \& \ q(\underline{s}, 1) \geq 0$$
$$(g^*, h^*) = (\underline{s}, 1). \quad (83)$$

Noting $w(\delta(\underline{s})) \leq \underline{s}$, $w(1) \leq \underline{s}$, the condition $q(\underline{s}, 1) \geq 0$ in (83), the result in (81) is based on $m(w(\delta(\underline{s})), \delta(\underline{s})) = 0$, and the result in (76) is based on $m(0, \delta(0)) \geq 0$. By Lemma 4 we obtain $f(\underline{s}, 1) \geq f(w(\delta(\underline{s})), \delta(\underline{s}))$ and $f(\underline{s}, 1) \geq f(0, \delta(0))$, which, in conjunction with (76), (81), and (83), yields the equilibrium

$$\text{If } m(\underline{s}, \overline{s}) > 0 \ \& \ m(0, 1) < 0 \ \& \ m(0, \overline{s}) > 0 \ \& \ m(\underline{s}, 1) < 0$$
$$(g^*, h^*) = (\underline{s}, 1). \quad (84)$$

Case E.** $m(\underline{s}, \overline{s}) > 0$ **&** $m(0, 1) < 0$ **&** $m(\underline{s}, 1) < 0$**) : Noting (37), we obtain from $m(\underline{s}, \overline{s}) > 0$ & $m(\underline{s}, 1) < 0$ that $m(\underline{s}, \delta(\underline{s})) = 0$, where $\delta(\underline{s})$ is given by (54). Consequently

$$m(\underline{s}, h) \leq 0, h \in [\delta(\underline{s}), 1] \quad (85a)$$
$$m(\underline{s}, h) > 0, h \in [\overline{s}, \delta(\underline{s})). \quad (85b)$$

It follows from (37) and (85a) that $m(g, h) \leq 0$ for $h \in [\delta(\underline{s}), 1]$ and $g \in [0, \underline{s}]$. Thus, we have $g^* = \underline{s}$. Then, following the analysis in Case B, we arrive at

$$\text{For } h \in [\delta(\underline{s}), 1], (g^*, h^*) = \begin{cases} (\underline{s}, 1), & \text{if } q(\underline{s}, 1) \geq 0 \\ (\underline{s}, \delta(\underline{s})), & \text{if } q(\underline{s}, \delta(\underline{s})) \leq 0 \\ (\underline{s}, r(\underline{s})), & \text{otherwise.} \end{cases} \quad (86)$$

We note that "otherwise" in (86) includes the condition $q(\underline{s}, 1) < 0$. By (40), we have $q(\underline{s}, 1) \leq q(\underline{s}, \delta(\underline{s})) \leq 0$. Noticing $m(\underline{s}, 1) < 0$ in this case, we have $q(\underline{s}, 1) + m(\underline{s}, 1) < 0$, which contradicts with $q(\underline{s}, 1) + m(\underline{s}, 1) > 0$ implied by Lemma 3. Thus, the conditions of the second and third items in (86) do not hold. Therefore, equation (86) reduces to

$$\text{For } h \in [\delta(\underline{s}), 1] \ \& \ q(\underline{s}, 1) \geq 0, \quad (g^*, h^*) = (\underline{s}, 1). \quad (87)$$

If $m(0, \overline{s}) \leq 0$, from (38) we have $m(0, h) \leq 0$ for $h \in [\overline{s}, 1]$. With the consideration of (85b), following the analysis in Case C, we obtain

$$\text{For } h \in [\overline{s}, \delta(\underline{s})) \ \& \ m(0, \overline{s}) \leq 0$$
$$(g^*, h^*) = (w(\delta(\underline{s})), \delta(\underline{s})). \quad (88)$$

Noting $1 \geq \delta(\underline{s})$, $\underline{s} \geq w(\delta(\underline{s}))$, the result in (88) is based on $m(w(\delta(\underline{s})), \delta(\underline{s})) = 0$ and the condition $q(\underline{s}, 1) \geq 0$ in (87). From Lemma 4 we have $f(\underline{s}, 1) \geq f(w(\delta(\underline{s})), \delta(\underline{s}))$. Consequently, we get

$$\text{If } m(\underline{s}, \overline{s}) > 0 \ \& \ m(0, 1) < 0 \ \& \ m(\underline{s}, 1) < 0 \ \& \ m(0, \overline{s}) \leq 0$$
$$(g^*, h^*) = (\underline{s}, 1). \quad (89)$$

If $m(0, \overline{s}) > 0$, we have $m(g, \overline{s}) > 0$ for $g \in [0, \underline{s}]$, which is due to (37). We note that $m(\underline{s}, 1) < 0$ implies that $m(g, 1) < 0$ for $g \in [0, \underline{s}]$. Here, we conclude (79). Considering (26), we obtain from (79a) that $g^* = 0$ for $h \in [\overline{s}, \delta(0)]$. Following the analysis in Case A, we have

$$\text{For } h \in [\overline{s}, \delta(0)] \ \& \ m(0, \overline{s}) > 0$$
$$(g^*, h^*) = \begin{cases} (0, \delta(0)), & \text{if } q(0, \delta(0)) \geq 0 \\ (0, \overline{s}), & \text{if } q(0, \overline{s}) \leq 0 \\ (0, r(0)), & \text{otherwise.} \end{cases} \quad (90)$$

The "otherwise" in (90) includes the condition $q(0, \delta(0)) < 0$. Due to (40), we have $q(0, 1) \leq q(0, \overline{s}) \leq 0$ and $q(0, 1) \leq q(0, \delta(0)) < 0$. Noting $m(0, 1) < 0$ in this case, we have $m(0, 1) + q(0, 1) < 0$, which contradicts with $q(0, 1) + m(0, 1) > 0$ implied by Lemma 3. Thus, the conditions of the second the third items in (90) do not hold. Therefore, equation (90) reduces to

$$\text{For } h \in [\overline{s}, \delta(0)] \ \& \ m(0, \overline{s}) > 0 \ \& \ q(0, \delta(0)) \geq 0$$
$$(g^*, h^*) = (0, \delta(0)). \quad (91)$$

Following the steps used in the derivation of (81), we obtain

$$\text{For } h \in [\delta(0), \delta(\underline{s})] \ \& \ m(0, \overline{s}) > 0$$
$$(g^*, h^*) = (w(\delta(\underline{s})), \delta(\underline{s})). \quad (92)$$

We note that (90) [which leads to (91)] is based on the condition $m(g, h) > 0$ for $h \in [\overline{s}, \delta(0))$ and $g \in [0, \underline{s}]$, which implies $m(0, \delta(0)) > 0$. Moreover, for (92), we have $m(w(\delta(\underline{s})), \delta(\underline{s})) = 0$. Then, by Lemma 4, from (87), (91), and (92) we have $f(\underline{s}, 1) \geq f(0, \delta(0))$ and $f(\underline{s}, 1) \geq f(w(\delta(\underline{s})), \delta(\underline{s}))$, and combining the conditions in (87), (91), and (92), we arrive at

$$\text{If } m(\underline{s}, \overline{s}) > 0 \ \& \ m(0, 1) < 0 \ \& \ m(\underline{s}, 1) < 0 \ \& \ m(0, \overline{s}) > 0$$
$$(g^*, h^*) = (\underline{s}, 1). \quad (93)$$

Summary for Cases B–E: We note that (29) is obtained in Case A. Combining (78) and (73) yields (31), while combining (60), (84), (89), and (93) results in (30).

APPENDIX G
PROOF OF THEOREM 4

It follows from (20) with (23) and (24) that

$$m(0,1) = (1-\lambda-2\gamma+2\widehat{s}\gamma)\beta - (2\widehat{s}-\lambda\widehat{s}-\gamma-\chi)\gamma. \quad (94)$$

We note that (4) implies $1-\lambda-2\gamma+2\widehat{s}\gamma > 0$, which implies that if we require $m(0,1)<0$, we must have $2\widehat{s}-\lambda\widehat{s}-\gamma-\chi > 0$, which means the left hand of (32). Then noticing $0 \le \frac{\gamma}{\beta} \le 1$, but $\frac{\gamma}{\beta}=0$ implies $\gamma=0$, we straightforwardly verify from (94) that (32) is equivalent to $m(0,1)<0$. Following the same analysis, we conclude that (33) is equivalent to $q(0,1)<0$.

From (29)–(31) we conclude that the CB influences Georgia's strategy (i.e., $g^* \neq 0$) if and only if $m(0,1)<0$. Thus, the proof of (32) is established.

From (29) we conclude that the CB influences Hank's strategy (i.e., $h^* \neq 1$) if and only if $m(0,1) \ge 0$ and $q(0,1)<0$. We note that $q(0,1)<0$ and $m(0,1)<0$ simultaneously do not hold, since the condition $q(0,1)+m(0,1)>0$ implied by Lemma 3 is violated. Therefore, the sufficient and necessary condition is $q(0,1)<0$.

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