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RANDOM WALK ON UNIPOTENT MATRIX GROUPS

BY PERSI DIACONIS AND ROBERT HOUGH

ABSTRACT. — We introduce a new method for proving central limit theorems for random walk on nilpotent groups. The method is illustrated in a local central limit theorem on the Heisenberg group, weakening the necessary conditions on the driving measure. As a second illustration, the method is used to study walks on the $n \times n$ uni-upper triangular group with entries taken modulo p . The method allows sharp answers to the behavior of individual coordinates: coordinates immediately above the diagonal require order p^2 steps for randomness, coordinates on the second diagonal require order p steps; coordinates on the k th diagonal require order $p^{\frac{2}{k}}$ steps.

RÉSUMÉ. — Nous introduisons une nouvelle méthode pour prouver les théorèmes limites centraux pour la marche aléatoire sur groupes nilpotents. La méthode est illustrée dans un théorème de la limite centrale locale sur le groupe Heisenberg, affaiblissant les conditions nécessaires sur la mesure sous-jacente. Comme deuxième illustration, la méthode est utilisée pour étudier les marches aléatoires sur le groupe triangulaire des matrices uni-supérieures $n \times n$ avec des entrées prises modulo p . La méthode permet des réponses précises sur le comportement des coordonnées individuelles: les coordonnées immédiatement au-dessus de la diagonale nécessitent un ordre p^2 pour devenir aléatoire, les coordonnées sur la deuxième diagonale nécessitent un ordre de p pas pour converger; les coordonnées sur la k -ième diagonale nécessitent un ordre de magnitude de $p^{\frac{2}{k}}$ pas.

1. Introduction

Let $\mathbb{H}(\mathbb{R})$ denote the real Heisenberg group

$$(1) \quad \mathbb{H}(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

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Abbreviate $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ with $[x, y, z]$, identified with a vector in \mathbb{R}^3 . Consider a simple random walk on $G = \mathbb{H}(\mathbb{R})$ driven by Borel probability measure μ . For $N \geq 1$, the law of this walk is the convolution power μ^{*N} where, for Borel measures μ, ν on G , and for $f \in C_c(G)$,

$$(2) \quad \langle f, \mu * \nu \rangle = \int_{g, h \in G} f(gh) d\mu(g) d\nu(h).$$

Say that measure μ is non-lattice (aperiodic) if its support is not contained in a proper closed subgroup of G . For general non-lattice μ of compact support Breuillard [6] uses the representation theory of G to prove a local limit theorem for the law of μ^{*N} , asymptotically evaluating its density in translates of bounded Borel sets. However, in evaluating μ^{*N} on Borel sets translated on both the left and the right he makes a decay assumption on the Fourier transform of the abelianization of the measure μ , and raises the question of whether this is needed. We show that this condition is unnecessary. In doing so we give an alternative approach to the local limit theorem on G treating it as an extension of the classical local limit theorem on \mathbb{R}^n . We also obtain the best possible rate. The method of argument is analogous to (though simpler than) the analysis of quantitative equidistribution of polynomial orbits on G from [14].

Recall that the abelianization $G_{\text{ab}} = G/[G, G]$ of G is isomorphic to \mathbb{R}^2 with projection $p : G \rightarrow G_{\text{ab}}$ given by $p([x, y, z]) = [x, y]$. Assume that the probability measure μ satisfies the following conditions.

i. *Compact support.*

ii. *Centered.* The projection p satisfies

$$(3) \quad \int_G p(g) d\mu(g) = 0.$$

iii. *Full dimension.* Let $\Gamma = \overline{\langle \text{supp } \mu \rangle}$ be the closure of the subgroup of G generated by the support of μ . The quotient G/Γ is compact.

Section 2 gives a characterization of closed subgroups Γ of G of full dimension.

Under the above conditions, the central limit theorem for μ is known. Let $(d_t)_{t>0}$ denote the semigroup of dilations given by

$$(4) \quad d_t([x, y, z]) = [tx, ty, t^2z]$$

and denote the Gaussian semigroup $(\nu_t)_{t>0}$ defined by its generator (see [6], [26])

$$(5) \quad \begin{aligned} \mathcal{A}f &= \frac{d}{dt} \Big|_{t=0} \int_{g \in G} f(g) d\nu_t(g) \\ &= \bar{z} \partial_z f(\text{id}) + \bar{x}\bar{y} \partial_{xy}^2 f(\text{id}) + \frac{1}{2} \bar{x}^2 \partial_x^2 f(\text{id}) + \frac{1}{2} \bar{y}^2 \partial_y^2 f(\text{id}) \end{aligned}$$

where $\sigma_x^2 = \bar{x}^2 = \int_{g=[x,y,z] \in G} x^2 d\mu(g)$ and similarly $\sigma_y^2 = \bar{y}^2$, $\sigma_{xy}^2 = \bar{x}\bar{y}$, \bar{z} . With $\nu = \nu_1$, the central limit theorem for μ states that for $f \in C_c(G)$,

$$(6) \quad \left\langle f, d_{\frac{1}{\sqrt{N}}} \mu^{*N} \right\rangle \rightarrow \langle f, \nu \rangle.$$

For $g \in G$ define the left and right translation operators $L_g, R_g : L^2(G) \rightarrow L^2(G)$,

$$(7) \quad L_g f(h) = f(gh), \quad R_g f(h) = f(hg).$$

Our local limit theorem in the non-lattice case is as follows.

THEOREM 1. – *Let μ be a Borel probability measure of compact support on $G = \mathbb{H}(\mathbb{R})$, which is centered and full dimension. Assume that the projection to the abelianization μ_{ab} is non-lattice. Let ν be the limiting Gaussian measure of $d_{\frac{1}{\sqrt{N}}} \mu^{*N}$. For $f \in C_c(G)$, uniformly for $g, h \in G$, as $N \rightarrow \infty$,*

$$(8) \quad \langle L_g R_h f, \mu^{*N} \rangle = \langle L_g R_h f, d_{\sqrt{N}} \nu \rangle + o_{\mu, f}(N^{-2}).$$

If the Cramér condition holds:

$$(9) \quad \sup_{\lambda \in \widehat{\mathbb{R}^2}, |\lambda| > 1} \left| \int_{g=[x,y,z] \in G} e^{-i\lambda \cdot (x,y)} d\mu(g) \right| < 1,$$

then uniformly for $g, h \in G$ and Lipschitz $f \in C_c(G)$, as $N \rightarrow \infty$

$$(10) \quad \langle L_g R_h f, \mu^{*N} \rangle = \langle L_g R_h f, d_{\sqrt{N}} \nu \rangle + O_{\mu, f}(N^{-\frac{5}{2}}).$$

REMARK. – The rate is best possible as may be seen by projecting to the abelianization. A variety of other statements of the local theorem are also derived, see eqn. (74) in Section 3.

REMARK. – For non-lattice μ , [6] obtains (8) with $h = \text{id}$ and for general h subject to Cramér's condition. A condition somewhat weaker than Cramér's would suffice to obtain (10).

REMARK. – In the case that μ is supported on a closed discrete subgroup or has a density with respect to Haar measure, [1, 2] obtains an error of $O(N^{-\frac{5}{2}})$ in approximating $\mu^{*N}(g)$, $g \in \Gamma$.

Our proof of Theorem 1 applies also in the case when μ_{ab} has a lattice component, and gives a treatment which is more explicit than the argument in [1]. To illustrate this, we determine the leading constant in the probability of return to 0 in simple random walk on $\mathbb{H}(\mathbb{Z})$, giving an alternative proof of a result of [19]. Our proof applies equally well to determine the return probability to 0 of lattice random walks with arbitrary finitely supported driving measures.

THEOREM 2. – *Let μ_0 be the measure on $\mathbb{H}(\mathbb{Z})$ which assigns equal probability $\frac{1}{5}$ to each element of the generating set*

$$(11) \quad \left\{ \text{id}, \begin{pmatrix} 1 & \pm 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \pm 1 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

$$\text{As } N \rightarrow \infty, \mu_0^{*N}(\text{id}) = \frac{25}{16N^2} + O(N^{-\frac{5}{2}}).$$

The basic idea which drives the proof is that permuting segments of generators in a typical word of the walk generates smoothness in the central coordinate of the product, while leaving the abelianized coordinates unchanged. This observation permits passing from a limit theorem to a local limit theorem by smoothing at a decreasing sequence of scales. When studying μ^{*N} near the scale of its distribution, we use a Lindeberg replacement scheme in which one copy at a time of μ is replaced with a Gaussian measure in the abelianization. To handle uniformity in the translation in Theorem 1 in the case where the Cramér condition is not assumed we are forced to treat frequencies α which are unbounded, and thus must consider the large spectrum

$$(12) \quad \text{Spec}_{\vartheta}(\mu_{\text{ab}}) = \{\alpha \in \mathbb{R}^2 : |\hat{\mu}_{\text{ab}}(\alpha)| > 1 - \vartheta\},$$

where $\vartheta \rightarrow 0$ as a function of N . In treating this, we use an approximate lattice structure of $\text{Spec}_{\vartheta}(\mu_{\text{ab}})$, see Section 2.1.

As a further application of the word rearrangement technique, answering a question of [9] we determine the mixing time of the central coordinate in a natural class of random walks on the group $N_n(\mathbb{Z}/p\mathbb{Z})$ of $n \times n$ uni-upper triangular matrices with entries in $\mathbb{Z}/p\mathbb{Z}$.

THEOREM 3. – *Let $n \geq 2$ and let μ be a probability measure on \mathbb{Z}^{n-1} which satisfies the following conditions.*

- i. Bounded support.
- ii. Full support. $\langle \text{supp } \mu \rangle = \mathbb{Z}^{n-1}$
- iii. Lazy. $\mu(0) > 0$
- iv. Mean zero. $\sum_{x \in \mathbb{Z}^{n-1}} x \mu(x) = 0$
- v. Trivial covariance.

$$(13) \quad \left(\sum_{x \in \mathbb{Z}^{n-1}} x^{(i)} x^{(j)} \mu(x) \right)_{i,j=1}^{n-1} = I_{n-1}.$$

Push forward μ to a probability measure $\tilde{\mu}$ on $N_n(\mathbb{Z})$ via, for all $x \in \mathbb{Z}^{n-1}$,

$$(14) \quad \tilde{\mu} \left(\begin{pmatrix} 1 & x^{(1)} & 0 & \cdots & 0 \\ 0 & 1 & x^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ & & 0 & 1 & x^{(n-1)} \\ 0 & \cdots & & 0 & 1 \end{pmatrix} \right) = \mu(x).$$

Write $Z : N_n(\mathbb{Z}) \rightarrow \mathbb{Z}$ for the upper right corner entry of a matrix of $N_n(\mathbb{Z})$. There exists $C > 0$ such that, for all primes p , for $N \geq 1$,

$$(15) \quad \sum_{x \bmod p} \left| \tilde{\mu}^{*N}(Z \equiv x \bmod p) - \frac{1}{p} \right| \ll \exp \left(-C \frac{N}{p^{\frac{2}{n-1}}} \right).$$

REMARK. – Informally, the top right corner entry mixes in time $O\left(p^{\frac{2}{n-1}}\right)$. This is tight, since archimedean considerations show that the L^1 distance to uniform is $\gg 1$ if the number of steps of the walk is $\ll p^{\frac{2}{n-1}}$.

REMARK. – Although we have considered only the top right corner entry in $U_n(\mathbb{Z}/p\mathbb{Z})$, this result determines the mixing time of each entry above the diagonal by iteratively projecting to the subgroups determined by the top left or bottom right $m \times m$ sub-matrices.

REMARK. – Our argument permits treating measures not supported on the first super-diagonal essentially without change, since entries above the first diagonal introduce a lower degree tensor which is annihilated by the application of the Gowers-Cauchy-Schwarz inequality. We treat the simplified case stated in order to ease the notation.

After completion of this work, the second named author has extended the methods developed here to treat the local limit theorem on an arbitrary connected, simply connected nilpotent Lie group [15].

History

Random walk on groups is a mature subject with myriad projections into probability, analysis and applications. Useful overviews with extensive references are in [5], [22]. Central limit theorems for random walk on Lie groups were first proved by [27] with [25] carrying out the details for the Heisenberg group. Best possible results under a second moment condition for nilpotent Lie groups are in [21].

A general local limit theorem for the Heisenberg group appears in [6], which contains a useful historical review. There similar conditions to those of our Theorem 1 are made, but the argument treats only the non-lattice case and needs a stronger condition on the characteristic function of the measure projected to the abelianization. Remarkable local limit theorems are in [1, 2]. The setting is groups of polynomial growth, and so “essentially” nilpotent Lie groups via Gromov’s Theorem. The first paper gives quite complete results assuming that the generating measure has a density. The second paper gives results for measures supported on a lattice. The arguments in [2] have been adapted in [4] to give a local limit theorem for non-lattice measures supported on finitely many points.

Just as for the classical abelian case, many variations have been studied. Central limit theorems for walks satisfying a Lindeberg condition on general Lie groups are proved in [23], see also references therein. Large deviations for walks on nilpotent groups are proved in [3]. Central limit theorems on covering graphs with nilpotent automorphism groups are treated in [17, 18]. This allows walks on Cayley graphs with some edges and vertices added and deleted. Brownian motion and heat kernel estimates are also relevant, see [16, 12].

Random walk on finite nilpotent groups are a more recent object of study. Diaconis and Saloff-Coste [11, 10, 9] show that for simple symmetric random walk on $\mathbb{Z}/n\mathbb{Z}$, order n^2 steps are necessary and sufficient for convergence to uniform. The first paper uses Nash inequalities, the second lifts to random walk on the free nilpotent group and applies central limit theorems of Hebisch, Saloff-Coste and finally Harnack inequalities to transfer back to the finite setting. The third paper uses geometric ideas of moderate growth to show that

for groups of polynomial growth, diameter-squared steps are necessary and sufficient to reach uniformity. This paper raises the question of the behavior of the individual coordinates on $U_n(\mathbb{Z}/p\mathbb{Z})$ which is finally answered in Theorem 3. A direct non-commuting Fourier approach to $\mathbb{H}(\mathbb{Z}/p\mathbb{Z})$ is carried out in [7], where it is shown that order $p \log p$ steps suffice to make the central coordinate random, improved here to order p steps, which is best possible. For a review of the $\mathbb{H}(\mathbb{Z})$ results, see [8]. Finally there have been quite a number of papers studying the walk on $U_n(\mathbb{Z}/p\mathbb{Z})$ when both p and n grow. We refer to [20], which contains a careful review and definitive results.

Notation and conventions

Vectors from \mathbb{R}^d , $d \geq 1$ are written in plain text w , their coordinates with superscripts $w^{(i)}$, and sequences of vectors with an underline \underline{w} . The sum of a sequence of vectors \underline{w} is indicated \overline{w} . w^t denotes the transpose of w . We frequently identify matrix elements in the group U_n with vectors from Euclidean space, and have attempted to indicate the way in which the vectors should be interpreted. As a rule of thumb, when the group law is written multiplicatively, the product is in the group U_n , and when additively, in Euclidean space.

The arguments presented use permutation group actions on sequences of vectors. Given integer $N \geq 1$, denote \mathfrak{S}_N the symmetric group on $[N] = \mathbb{Z} \cap [1, N]$, which acts on length N sequence of vectors by permuting the indices:

$$(16) \quad \mathfrak{S}_N \ni \sigma : (w_1, \dots, w_N) \mapsto (w_{\sigma(1)}, \dots, w_{\sigma(N)}).$$

C_2 is the two-element group. For $d \geq 1$, identify C_2^d with the d -dimensional hypercube $\{0, 1\}^d$. $\mathbf{1}_d$ is the element of C_2^d corresponding to the sequence of all 1's on the hypercube. C_2^d acts on sequences of vectors of length 2^d with the j th factor determining the relative order of the first and second blocks of 2^{j-1} elements. To illustrate the action of C_2^2 on $\underline{x} = x_1 x_2 x_3 x_4$:

$$(17) \quad \begin{aligned} (0, 0) \cdot \underline{x} &= x_1 x_2 x_3 x_4 \\ (1, 0) \cdot \underline{x} &= x_2 x_1 x_3 x_4 \\ (0, 1) \cdot \underline{x} &= x_3 x_4 x_1 x_2 \\ (1, 1) \cdot \underline{x} &= x_3 x_4 x_2 x_1. \end{aligned}$$

The 2-norm on \mathbb{R}^d is indicated $\|\cdot\|$ and $\|\cdot\|_{(\mathbb{R}/\mathbb{Z})^d}$ denotes distance to the nearest integer lattice point. Given $\xi \in \mathbb{R}^d$, $e_\xi(\cdot)$ denotes the character of \mathbb{R}^d , $e_\xi(x) = e^{2\pi i \xi \cdot x}$.

Use δ_x to indicate the Dirac delta measure at $x \in \mathbb{R}^d$. Given $f \in C_c(\mathbb{R}^d)$ and measure μ , $\langle f, \mu \rangle$ denotes the bilinear pairing

$$(18) \quad \langle f, \mu \rangle = \int_{\mathbb{R}^d} f(x) d\mu(x).$$

Denote the Fourier transform of function f , resp. the characteristic function of measure μ by, for $\xi \in \mathbb{R}^d$,

$$(19) \quad \hat{f}(\xi) = \int_{\mathbb{R}^d} e_{-\xi}(x) f(x) dx, \quad \hat{\mu}(\xi) = \int_{\mathbb{R}^d} e_{-\xi}(x) d\mu(x).$$

For $x \in \mathbb{R}^d$, $T_x f$ denotes function f translated by x ,

$$(20) \quad T_x f(y) = f(y - x), \quad \widehat{T_x f}(\xi) = e_{-\xi}(x) \hat{f}(\xi)$$

and for real $t > 0$, f_t denotes f dilated by t ,

$$(21) \quad f_t(x) = t^d f(tx), \quad \widehat{f_t}(\xi) = \hat{f}\left(\frac{\xi}{t}\right).$$

For smooth f ,

$$(22) \quad f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e_{\xi}(x) d\xi.$$

By a bump function ρ on \mathbb{R}^n we mean a smooth non-negative function of compact support and integral 1. The Fourier transform of ρ satisfies, for each $A > 0$ there is a constant $C(A, \rho) > 0$ such that

$$(23) \quad |\hat{\rho}(\xi)| \leq \frac{C(A, \rho)}{(1 + \|\xi\|)^A}.$$

This follows from integration by parts.

For $r \in \mathbb{R}$ and $\sigma > 0$, $\eta(r, \sigma)$ denotes the one-dimensional Gaussian distribution with mean r and variance σ^2 , with density and characteristic function

$$(24) \quad \eta(r, \sigma)(x) = \frac{\exp\left(-\frac{(x-r)^2}{2\sigma^2}\right)}{\sqrt{2\pi}\sigma}, \quad \widehat{\eta(r, \sigma)}(\xi) = e_{-\xi}(r) \exp(-2\pi^2 \sigma^2 \xi^2).$$

A centered (mean zero) normal distribution η in dimension d is specified by its covariance matrix

$$(25) \quad \sigma^2 = \left(\int_{\mathbb{R}^d} x^{(m)} x^{(n)} \eta(x) \right)_{m,n=1}^d$$

and has density and characteristic function

$$(26) \quad \eta(0, \sigma)(x) = \frac{\exp\left(-\frac{x^t(\sigma^2)^{-1}x}{2}\right)}{(2\pi)^{\frac{d}{2}} (\det \sigma^2)^{\frac{1}{2}}}, \quad \widehat{\eta(0, \sigma)}(\xi) = \exp(-2\pi^2 \xi^t \sigma^2 \xi).$$

All of our arguments concern the repeated convolution μ^{*N} of a fixed measure μ on the upper triangular matrices. The product measure $\mu^{\otimes N}$ is abbreviated \mathbb{U}_N . Asymptotic statements are with respect to N as the large parameter. The Vinogradov notation $A \ll B$, resp. $A \gg B$, means $A = O(B)$, resp. $B = O(A)$. $A \asymp B$ means $A \ll B$ and $B \ll A$.

2. Background to Theorems 1 and 2

This section collects together several background statements regarding the Heisenberg group, its Gaussian semigroups of probability measures and statements of elementary probability which are needed in the course of the argument.

Write $A = [1, 0, 0]$, $B = [0, 1, 0]$, $C = [0, 0, 1]$. The following commutators are useful,

$$\begin{aligned}
 [A, B] &= ABA^{-1}B^{-1} = [0, 0, 1] = C, \\
 (27) \quad [A^{-1}, B^{-1}] &= A^{-1}B^{-1}AB = [0, 0, 1] = C, \\
 [A, B^{-1}] &= AB^{-1}A^{-1}B = [0, 0, -1] = C^{-1}, \\
 [A^{-1}, B] &= A^{-1}BAB^{-1} = [0, 0, -1] = C^{-1}.
 \end{aligned}$$

A convenient representation for $[x, y, z] \in \mathbb{H}(\mathbb{R})$ is $C^z B^y A^x$. Using the commutator rules above, the multiplication rule for $\underline{w} \in \mathbb{H}(\mathbb{R})^N$ becomes

$$(28) \quad \prod_{i=1}^N [w_i^{(1)}, w_i^{(2)}, w_i^{(3)}] = [\underline{w}^{(1)}, \underline{w}^{(2)}, \underline{w}^{(3)} + H(\underline{w})],$$

where $\bar{\cdot}$ and H act on sequences of vectors from \mathbb{R}^3 via

$$(29) \quad \underline{w} = \sum_i w_i \quad H(\underline{w}) = \sum_{i < j} w_i^{(1)} w_j^{(2)}.$$

It is also convenient to define

$$\begin{aligned}
 (30) \quad H^*(\underline{w}) &= H(\underline{w}) - \frac{1}{2} \underline{w}^{(1)} \underline{w}^{(2)} + \frac{1}{2} \sum_{i=1}^N w_i^{(1)} w_i^{(2)} \\
 &= \frac{1}{2} \sum_{1 \leq i < j \leq N} (w_i^{(1)} w_j^{(2)} - w_i^{(2)} w_j^{(1)}),
 \end{aligned}$$

and for $w = [x, y, z]$, $\tilde{w} = [x, y, z - \frac{1}{2}xy]$, so that the multiplication rule may be written

$$(31) \quad \prod_{i=1}^N w_i = \tilde{w} + \left[0, 0, \frac{1}{2} \underline{w}^{(1)} \underline{w}^{(2)} + H^*(\underline{w}) \right].$$

Let $S = \text{supp } \mu$. Recall that $\Gamma = \overline{\langle S \rangle}$ is the closure of the group generated by S . Its abelianization, $\Gamma_{\text{ab}} = \Gamma/[\Gamma, \Gamma]$ is equal to $p(\Gamma)$ where p is the projection $p : G \rightarrow G_{\text{ab}}$. Let Γ_0 be the semigroup generated by S . We record the following descriptions of Γ and Γ_0 .

PROPOSITION 4. – *Let $\Gamma \leq \mathbb{H}(\mathbb{R})$ be a closed subgroup of full dimension. The structure of the abelianization $\Gamma_{\text{ab}} = \Gamma/[\Gamma, \Gamma]$ and of Γ falls into one of the following cases.*

i.

$$(32) \quad \Gamma_{\text{ab}} = \mathbb{R}^2, \quad \Gamma = \{[\gamma, r] : \gamma \in \Gamma_{\text{ab}}, r \in \mathbb{R}\}$$

ii. *There exist non-zero orthogonal $v_1, v_2 \in \mathbb{R}^2$, such that*

$$\begin{aligned}
 (33) \quad \Gamma_{\text{ab}} &= \{nv_1 + rv_2 : n \in \mathbb{Z}, r \in \mathbb{R}\}, \\
 \Gamma &= \{[\gamma, r] : \gamma \in \Gamma_{\text{ab}}, r \in \mathbb{R}\}
 \end{aligned}$$

iii. *There exist non-zero $v_1, v_2 \in \mathbb{R}^2$, linearly independent over \mathbb{R} , such that*

$$(34) \quad \Gamma_{\text{ab}} = \{n_1 v_1 + n_2 v_2 : n_1, n_2 \in \mathbb{Z}\}.$$

In this case, Γ falls into one of two further cases

iv. $\Gamma = \{[\gamma, r] : \gamma \in \Gamma_{\text{ab}}, r \in \mathbb{R}\}$

v. There exists $\lambda \in \mathbb{R}_{>0}$ and $f : \Gamma_{\text{ab}} \rightarrow \mathbb{R}$ such that

$$(35) \quad \Gamma = \{[\gamma, \lambda(f(\gamma) + n)] : \gamma \in \Gamma_{\text{ab}}, n \in \mathbb{Z}\}.$$

Proof of Proposition 4. – The full dimension condition guarantees that Γ_{ab} is a two dimensional closed subgroup of \mathbb{R}^2 , and the three possibilities given are all such closed subgroups. Likewise, the center of Γ is a non-trivial subgroup of \mathbb{R} , hence either \mathbb{R} or $\lambda \cdot \mathbb{Z}$ for some real $\lambda > 0$. It follows that the fiber over $\gamma \in \Gamma_{\text{ab}}$ is a translate of the center. Let v_1, v_2 be two linearly independent elements of the abelianization, and choose $g_1 = [v_1, z_1]$, $g_2 = [v_2, z_2]$ in Γ . The commutator $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$ is bilinear in v_1, v_2 , is non-zero, and lies in the center. It follows that if one of v_1, v_2 may be scaled by a continuous parameter in the abelianization then the center is \mathbb{R} . \square

LEMMA 5. – The closure of the semigroup Γ_0 is $\overline{\Gamma_0} = \Gamma$.

Proof. – Write $\Gamma_{0,\text{ab}} = p(\Gamma_0)$ where p denotes projection to the abelianization G_{ab} . That $\overline{\Gamma_{0,\text{ab}}} = \Gamma_{\text{ab}}$ follows from the local limit theorem on \mathbb{R}^2 . To treat the central fiber, in the case $\Gamma_{\text{ab}} = \mathbb{R}^2$ let $0 < \epsilon < \frac{1}{4}$ be a fixed small parameter and choose x, x', y, y' in Γ_0 such that

$$(36) \quad p(x), p(x'), p(y), p(y') \approx e_1, -e_1, e_2, -e_2$$

where the approximation means within distance ϵ . Take a word \underline{w} in $T = \{\text{id}, x, x', y, y'\}$ of length $4n$ with product approximating the identity in Γ_{ab} within ϵ , which is such that each of x, x', y, y' appear $> (1 - O(\epsilon))n$ times in \underline{w} . The abelianization of the product is independent of the ordering of \underline{w} , but if the letters in \underline{w} appear in order y, x, y', x' then the central element is $< -(1 + O(\epsilon))n^2$, while if they appear in order y', x, y, x' then the central element is $> (1 + O(\epsilon))n^2$. Moving from an ordering of the first type to an ordering of the second by swapping generators one at a time changes the central element by $O(1)$ at each step. Let $\epsilon \downarrow 0$ to deduce that $\overline{\Gamma_0}$ contains positive and negative central elements, and hence that $\overline{\Gamma_0}$ is a group, equal to Γ . In the case Γ_{ab} has a one or two dimensional lattice component, replace either e_1 or both e_1, e_2 above with a basis for the lattice component and repeat the argument. \square

More quantitative structural statements are as follows.

LEMMA 6. – Let μ be a measure on $\mathbb{H}(\mathbb{R})$, with abelianization μ_{ab} not supported on a lattice of \mathbb{R}^2 . If the Cramér condition holds for the measure μ_{ab} then it holds also for the measure μ' on \mathbb{R} obtained by pushing forward $\mu_{\text{ab}} \otimes \mu_{\text{ab}}$ by $H^*(w_1, w_2)$.

Proof. – Let $\xi \in \mathbb{R}$, $|\xi| \geq 1$ and fix $w_2 \in \text{supp}(\mu_{\text{ab}})$, bounded away from 0. Write $H^*(w_1, w_2) = \frac{w_1 \wedge w_2}{2} = \frac{1}{2} w_1 \cdot \hat{w}_2$. The claim follows since $\left| \int e_{-\xi} (H^*(w_1, w_2)) d\mu_{\text{ab}}(w_1) \right|$ is bounded away from 1 uniformly in ξ and w_2 . \square

LEMMA 7. – Let μ be a measure on \mathbb{R}^2 of compact support, with support generating a subgroup of \mathbb{R}^2 of full dimension. If μ is lattice supported, assume that the co-volume of the lattice is at least 1. There is a constant $c = c(\mu) > 0$ such that, uniformly in $0 < \xi \leq \frac{1}{2}$, for $N = N(\xi) = \left\lfloor \frac{1}{2\xi} \right\rfloor$,

$$(37) \quad \left| \int_{\mathbb{R}^2 \times \mathbb{R}^2} e_{-\xi} (H^*(w_1, w_2)) d\mu^{*N}(w_1) d\mu^{*N}(w_2) \right| \leq 1 - c(\mu).$$

Proof. – When μ is lattice with lattice of covolume V , the measure

$$H^*(w_1, w_2) d\mu(w_1) d\mu(w_2)$$

is lattice distributed with step size V . Hence the bound on $|\xi|$ suffices to guarantee the claim for N bounded.

For N growing, a standard application of the functional central limit theorem implies that $\frac{1}{N} H^*(w_1, w_2) d\mu^{*N}(w_1) d\mu^{*N}(w_2)$ converges to a non-zero density on \mathbb{R} as $N \rightarrow \infty$. \square

Normalize Haar measure on $\mathbb{H}(\mathbb{R})$ to be given in coordinates by $dg = dx dy dz$. The density of a Gaussian measure ν on $\mathbb{H}(\mathbb{R})$ can be understood as the rescaled limit of the density of a random walk with independent Gaussian inputs in the abelianization. Consider the distribution on the Heisenberg group given by $\nu_{2,\sigma} = [\eta(0, \sigma), 0]$, which has projection to the abelianization given by a two dimensional normal distribution of covariance σ , and with trivial central fiber. Write $\nu_2 = \nu_{2,I_2}$ for the measure in which σ is the two dimensional identity matrix. The rescaled distribution $d_{\frac{1}{\sqrt{N}}} \nu_2^{*N}$ converges to a Gaussian measure ν_0 on $\mathbb{H}(\mathbb{R})$ as $N \rightarrow \infty$. Note that we have not included a covariance term, which can be accommodated with a linear change of coordinates. Also, we do not consider randomness in the central coordinate as it would scale only as \sqrt{N} , whereas the central coordinate has distribution at scale N .

Let $\alpha \in \mathbb{R}^2$ and $\xi \in \mathbb{R}$. Write the modified characteristic function of ν_0 as (recall $\tilde{z} = z - \frac{xy}{2}$)

$$(38) \quad I(\alpha, \xi) = \int_{g=[x,y,z] \in G} e_{-\alpha}(g_{ab}) e_{-\xi}(\tilde{z}) d\nu_0(g)$$

and

$$(39) \quad I(\alpha, \xi; N) = \int_{(\mathbb{R}^2)^N} e_{-\alpha} \left(\frac{\bar{x}}{\sqrt{N}} \right) e_{-\xi} \left(\frac{H^*(\underline{x})}{N} \right) d\nu_{2,ab}^{\otimes N}(\underline{x}).$$

LEMMA 8. – Let $\alpha \in \mathbb{R}^2, \xi \in \mathbb{R}$ and let σ^2 be the covariance matrix of a non-degenerate two dimensional normal distribution of determinant $\delta^2 = \det \sigma^2, \delta > 0$. Then

$$(40) \quad \int_{(\mathbb{R}^2)^N} e_{-\alpha} \left(\frac{\bar{x}}{\sqrt{N}} \right) e_{-\xi} \left(\frac{H^*(\underline{x})}{N} \right) d\eta(0, \sigma)^{\otimes N}(\underline{x}) = I(\sigma\alpha, \delta\xi; N).$$

Proof. – Making the change of variables, for each i , $\sigma y_i = x_i$ in the density $\frac{1}{2\pi\delta} \exp\left(-\frac{x_i^T \sigma^{-2} x_i}{2}\right)$ changes $\bar{x} = \sigma \bar{y}$ and $H^*(\underline{x}) = \det \sigma \cdot H^*(\underline{y})$. \square

In view of the multiplication rule (31), for $\|\alpha\|, |\xi| = O(1)$

$$(41) \quad \lim_{N \rightarrow \infty} I(\alpha, \xi; N) \rightarrow I(\alpha, \xi).$$

The following rate of convergence is given in the appendix.

THEOREM 9. – For all $\alpha \in \mathbb{R}^2$, $\xi \in \mathbb{R}$ such that $(1 + \|\alpha\|^2)(1 + \xi^2) < N$,

$$(42) \quad I(\alpha, \xi; N) = \frac{1 + O\left(\frac{(1 + \|\alpha\|^2)(1 + \xi^2)}{N}\right)}{\exp\left(\frac{2\pi\|\alpha\|^2}{\xi \coth \pi \xi}\right) \cosh \pi \xi}.$$

In particular,

$$(43) \quad I(\alpha, \xi) = \frac{\exp\left(-\frac{2\pi\|\alpha\|^2}{\xi \coth \pi \xi}\right)}{\cosh \pi \xi}.$$

REMARK. – While $I(\alpha, \xi)$ characterizes the Gaussian measure, it does not behave well under convolution.

Along with the above characteristic function calculation the following moment calculation is used.

LEMMA 10. – Let η be a two dimensional Gaussian with identity covariance. For each $k \geq 1$, and $N \geq 2$,

$$(44) \quad \mathbf{E}_{\eta^{\otimes N}} \left[H^*(w)^{2k} \right] \leq \mu_{2k}^2 \frac{N^{2k}}{2^{2k}},$$

where $\mu_{2k} = \frac{(2k)!}{2^k k!}$ is the $2k$ th moment of a standard one dimensional Gaussian.

For any compactly supported probability measure μ of mean zero on \mathbb{R}^2 , for any $k \geq 1$, as $N \rightarrow \infty$,

$$(45) \quad \mathbf{E}_{\mu^{\otimes N}} \left[H^*(w)^{2k} \right] \leq O_{k,\mu} \left(N^{2k} \right).$$

Proof. – Write

$$(46) \quad H^*(w) = \frac{1}{2} \sum_{1 \leq i \neq j \leq N} (-1)^{\mathbf{1}(i > j)} w_i^{(1)} w_j^{(2)}$$

and expand the moment to find

$$(47) \quad \begin{aligned} \mathbf{E}_{\eta^{\otimes N}} \left[H^*(w)^{2k} \right] &\leq \frac{1}{2^{2k}} \mathbf{E}_{\eta^{\otimes N}} \left[\sum_{1 \leq m_1, \dots, m_{2k}, n_1, \dots, n_{2k} \leq N} w_{m_1}^{(1)} \dots w_{m_{2k}}^{(1)} w_{n_1}^{(2)} \dots w_{n_{2k}}^{(2)} \right] \\ &= \frac{1}{2^{2k}} \left[\mathbf{E}_{\eta^{\otimes N}} \left[\sum_{1 \leq m_1, \dots, m_{2k} \leq N} w_{m_1}^{(1)} \dots w_{m_{2k}}^{(1)} \right] \right]^2 \\ &= \mu_{2k}^2 \frac{N^{2k}}{2^{2k}}. \end{aligned}$$

When treating general μ of compact support,

$$(48) \quad \mathbf{E}_{\mu^{\otimes N}} \left[H^*(w)^{2k} \right] = \frac{1}{2^{2k}} \mathbf{E}_{\mu^{\otimes N}} \left[\sum_{1 \leq m_1, \dots, m_{2k}, n_1, \dots, n_{2k} \leq N} \varepsilon_{\underline{m}, \underline{n}} w_{m_1}^{(1)} \dots w_{m_{2k}}^{(1)} w_{n_1}^{(2)} \dots w_{n_{2k}}^{(2)} \right],$$

with $\varepsilon_{\underline{m}, \underline{n}} \in \{-1, 0, 1\}$. The expectation vanishes unless each index in $[N]$ which appears among $m_1, \dots, m_{2k}, n_1, \dots, n_{2k}$ appears at least twice. There are $O(N^{2k})$ ways to choose

which indices appear and $O_k(1)$ ways to assign m_1, \dots, n_{2k} to the indices which appear. For those assignments which don't vanish, the expectation is $O_{k,\mu}(1)$ by the compact support. \square

We make the following convention regarding rare events. Say that a sequence of measurable events $\{A_N\}_{N \geq 1}$ such that $A_N \subset S^N$ occurs *with high probability* (w.h.p.) if the complements satisfy the decay estimate,

$$(49) \quad \forall C \geq 0, \quad \mu^{\otimes N}(A_N^c) = O_C(N^{-C})$$

as $N \rightarrow \infty$. The sequence of complements is said to be *negligible*. A sequence of events $\{A_N\}$ which is negligible for $\mu^{\otimes N}$ is also negligible when $\mu^{\otimes N}$ is conditioned on a non-negligible sequence of events $\{B_N\}$.

2.1. The large spectrum

Let μ be a mean 0, compactly supported probability measure on \mathbb{R}^2 . For $0 < \vartheta < 1$, define the large spectrum of $\hat{\mu}$ to be

$$(50) \quad \text{Spec}_\vartheta(\mu) = \{\alpha \in \mathbb{R}^2 : |\hat{\mu}(\alpha)| > 1 - \vartheta\}$$

and let

$$(51) \quad \mathcal{M}_\vartheta(\mu) = \{\alpha \in \text{Spec}_\vartheta(\mu) : |\hat{\mu}(\alpha)| \text{ is a local maximum}\}.$$

Let

$$(52) \quad \check{\mu}(A) = \mu(\{x^{-1} : x \in A\})$$

and set $\mu_2 = \mu * \check{\mu}$. The measure μ_2 is still mean 0, compactly supported and satisfies

$$(53) \quad \hat{\mu}_2(\alpha) = \int_{\mathbb{R}^2} \cos(2\pi\alpha \cdot x) d\mu_2(x) = |\hat{\mu}(\alpha)|^2,$$

so $\text{Spec}_\vartheta(\mu) = \text{Spec}_{2\vartheta-\vartheta^2}(\mu_2)$ and $\mathcal{M}_\vartheta(\mu) = \mathcal{M}_{2\vartheta-\vartheta^2}(\mu_2)$.

For a differential operator $D_\beta = D_{i_1} D_{i_2} \cdots D_{i_\ell}$, set $|\beta| = \ell$.

LEMMA 11. – Let $0 \leq \vartheta \leq 1$, let $\alpha \in \text{Spec}_\vartheta(\mu_2)$ and let D_β be a differential operator. Then

$$(54) \quad D_\beta \hat{\mu}_2(\alpha) = \begin{cases} O_\beta(\vartheta^{\frac{1}{2}}) & |\beta| \text{ odd} \\ D_\beta \hat{\mu}_2(0) + O_\beta(\vartheta) & |\beta| \text{ even} \end{cases}.$$

Proof. – Let $D_\beta = D_{i_1} \cdots D_{i_\ell}$. Differentiating under the integral, if ℓ is odd then

$$(55) \quad D_\beta \hat{\mu}_2(\alpha) = (-1)^{\mathbf{1}(\ell \equiv 1 \pmod{4})} (2\pi)^\ell \int_{\mathbb{R}^2} x_{i_1} \cdots x_{i_\ell} \sin(2\pi\alpha \cdot x) d\mu_2(x)$$

so that, using the compact support of μ_2 and then Cauchy-Schwarz,

$$(56) \quad \begin{aligned} |D_\beta \hat{\mu}_2(\alpha)| &\ll_\ell \int_{\mathbb{R}^2} |\sin(2\pi\alpha \cdot x)| d\mu_2(x) \\ &= \int_{\mathbb{R}^2} \sqrt{1 - \cos^2(2\pi\alpha \cdot x)} d\mu_2(x) \\ &\leq \left(\int_{\mathbb{R}^2} 1 - \cos^2(2\pi\alpha \cdot x) d\mu_2(x) \right)^{\frac{1}{2}} \leq (2\vartheta)^{\frac{1}{2}}. \end{aligned}$$

If ℓ is even, then

$$(57) \quad \begin{aligned} D_\beta \hat{\mu}_2(\alpha) &= (-1)^{\frac{\ell}{2}} (2\pi)^\ell \int_{\mathbb{R}^2} x_{i_1} \cdots x_{i_\ell} \cos(2\pi \alpha \cdot x) d\mu_2(x) \\ &= D_\beta \hat{\mu}_2(0) - (-1)^{\frac{\ell}{2}} (2\pi)^\ell \int_{\mathbb{R}^2} x_{i_1} \cdots x_{i_\ell} (1 - \cos(2\pi \alpha \cdot x)) d\mu_2(x). \end{aligned}$$

Again using the compact support, the integral in the last line is $O(\vartheta)$. \square

The previous lemma has the following consequences.

LEMMA 12. – *There is a constant $C_1 = C_1(\mu)$, $0 < C_1 < 1$ such that if $0 < \vartheta < C_1$ then the following hold:*

- i. *There are constants $C_2(\mu), C_3(\mu) > 0$ such that if $\alpha_0 \in \mathcal{M}_\vartheta(\mu_2)$ and $\|\alpha - \alpha_0\| < C_2$ then*

$$(58) \quad \hat{\mu}_2(\alpha) \leq \hat{\mu}_2(\alpha_0) - C_3 \|\alpha - \alpha_0\|^2.$$

- ii. *There is a constant $C_4(\mu) > 0$ such that if $\alpha \in \text{Spec}_\vartheta(\mu_2)$ then there exists $\alpha_0 \in \mathcal{M}_\vartheta(\mu_2)$ with*

$$(59) \quad \|\alpha - \alpha_0\| \leq C_4 \sqrt{\vartheta}.$$

Furthermore, if μ does not have a lattice component, then there is a growth function $\mathcal{F}(\vartheta)$ tending to infinity as $\vartheta \downarrow 0$ such that, if $\alpha_0 \neq \alpha_1$ are distinct elements of $\mathcal{M}_\vartheta(\mu_2)$ then

$$(60) \quad \|\alpha_0 - \alpha_1\| > \mathcal{F}(\vartheta).$$

Proof. – To prove i., Taylor expands about α_0 using that the first derivatives vanish and that the third derivatives of $\hat{\mu}_2$ are uniformly bounded. The term from the second degree Taylor expansion may be replaced with the corresponding term at $\alpha_0 = 0$, making an error which is $O(\vartheta)$. This may be absorbed into the constant C_3 by making C_1 sufficiently small.

To prove ii., first reduce $C_1(\mu)$ to guarantee that there is a ball $B_\delta(\alpha)$, $0 < \delta < 1$ a fixed constant, such that the maximum of $\hat{\mu}_2$ does not occur on the boundary of the ball. This may be achieved by Taylor expanding about α , which now includes a linear term, which is $O(\vartheta^{\frac{1}{2}})$. Let α_0 be the global maximum in the interior, and now apply part i. and $\hat{\mu}_2(\alpha_0) - \hat{\mu}_2(\alpha) \leq \vartheta$ to conclude that $\|\alpha - \alpha_0\| \ll \sqrt{\vartheta}$.

To prove the final statement, note that for $0 \leq \vartheta \leq \frac{1}{4}$, if $\alpha_0, \alpha_1 \in \text{Spec}_\vartheta(\mu_2)$ then $\alpha_0 - \alpha_1 \in \text{Spec}_{4\vartheta}(\mu_2)$, see [24], p. 183. An easier proof is possible here since the spectrum is positive, indeed,

$$(61) \quad \begin{aligned} 1 - \hat{\mu}_2(\alpha_0 - \alpha_1) &= \int_{\mathbb{R}^2} 1 - \cos(2\pi \alpha_0 \cdot x) \cos(2\pi \alpha_1 \cdot x) d\mu_2 \\ &\quad - \int_{\mathbb{R}^2} \sin(2\pi \alpha_0 \cdot x) \sin(2\pi \alpha_1 \cdot x) d\mu_2. \end{aligned}$$

Bound the first integral by

$$(62) \quad \int_{\mathbb{R}^2} 1 - \cos(2\pi \alpha_0 \cdot x) d\mu_2 + \int_{\mathbb{R}^2} \cos(2\pi \alpha_0 \cdot x) (1 - \cos(2\pi \alpha_1 \cdot x)) d\mu_2 \leq 2\vartheta.$$

By Cauchy-Schwarz, the second integral is bounded in size by

$$(63) \quad \left(\int_{\mathbb{R}^2} 1 - \cos^2(2\pi\alpha_0 \cdot x) d\mu_2 \int_{\mathbb{R}^2} 1 - \cos^2(2\pi\alpha_1 \cdot x) d\mu_2 \right)^{\frac{1}{2}} \leq 2\vartheta.$$

The claim now follows on considering $\hat{\mu}_2(\alpha)$ in growing balls about 0. \square

The following lemma gives information about variation of the phase of $\hat{\mu}(\alpha)$ in the large spectrum.

LEMMA 13. – *Let μ be a measure of mean 0 and compact support on \mathbb{R}^2 , let $0 \leq \vartheta \leq \frac{1}{2}$ and let $\alpha_0 \in \mathcal{M}_{\vartheta}(\mu)$. The following hold.*

i. $\text{Im } D_i \log \hat{\mu}(\alpha_0) = O_{\mu}(\vartheta)$.

ii. $\text{Im } D_i D_j \log \hat{\mu}(\alpha_0) = O_{\mu}(\sqrt{\vartheta})$.

iii. For all $\alpha \in \text{Spec}_{\frac{1}{2}}(\mu)$,

$$(64) \quad \text{Im } D_{i_1} D_{i_2} D_{i_3} \log \hat{\mu}(\alpha) = O(1).$$

Proof. – Let $\hat{\mu}(\alpha_0) = e_{\alpha_0}(\phi_0)|\hat{\mu}(\alpha_0)|$.

For i.

$$(65) \quad D_i \log \hat{\mu}(\alpha_0) = \frac{D_i \hat{\mu}(\alpha_0)}{\hat{\mu}(\alpha_0)} = \frac{2\pi i}{|\hat{\mu}(\alpha_0)|} \int_{\mathbb{R}^2} x_i e_{\alpha_0}(x - \phi_0) d\mu(x).$$

Since μ is mean 0,

$$(66) \quad \text{Im } D_i \log \hat{\mu}(\alpha_0) = \frac{2\pi}{|\hat{\mu}(\alpha_0)|} \int_{\mathbb{R}^2} x_i (\cos(2\pi\alpha_0 \cdot (x - \phi_0)) - 1) d\mu(x).$$

By the compact support,

$$(67) \quad |\text{Im } D_i \log \hat{\mu}(\alpha_0)| \ll \int_{\mathbb{R}^2} 1 - \cos(2\pi\alpha_0 \cdot (x - \phi_0)) d\mu(x) \\ = 1 - |\hat{\mu}(\alpha_0)| \leq \vartheta.$$

For ii., write

$$(68) \quad D_i D_j \log \hat{\mu}(\alpha_0) = \frac{D_i D_j \hat{\mu}(\alpha_0)}{\hat{\mu}(\alpha_0)} - \frac{D_i \hat{\mu}(\alpha_0) D_j \hat{\mu}(\alpha_0)}{\hat{\mu}(\alpha_0)^2}.$$

The subtracted term is real since $\frac{D_i \hat{\mu}(\alpha_0)}{\hat{\mu}(\alpha_0)}$ is imaginary (α_0 is a maximum for $|\hat{\mu}(\alpha_0)|$). Hence, again using the compact support and Cauchy-Schwarz,

$$(69) \quad \text{Im } D_i D_j \log \hat{\mu}(\alpha_0) = \frac{-4\pi^2}{|\hat{\mu}(\alpha_0)|} \int_{\mathbb{R}^2} x_i x_j \sin(2\pi\alpha_0 \cdot (x - \phi_0)) d\mu(x) \\ \ll \int_{\mathbb{R}^2} \sqrt{1 - \cos^2(2\pi\alpha_0 \cdot (x - \phi_0))} d\mu(x) \\ \ll \left(\int_{\mathbb{R}^2} 1 - \cos(2\pi\alpha_0 \cdot (x - \phi_0)) d\mu(x) \right)^{\frac{1}{2}} \leq \vartheta^{\frac{1}{2}}.$$

To obtain iii., note that the first three derivatives of $\hat{\mu}$ are bounded due to the compact support. \square

The results of this section are collected into the following lemma which permits approximating $\hat{\mu}(\alpha)$ in neighborhoods of a local maximum for $|\hat{\mu}(\alpha)|$.

LEMMA 14. – *Let μ be a probability measure on \mathbb{R}^n of covariance matrix σ^2 . There is a constant $C = C(\mu) > 0$ such that for all $0 \leq \vartheta \leq C$ and for all $\alpha_0 \in \mathcal{M}_\vartheta(\mu)$ we have*

$$(70) \quad \hat{\mu}(\alpha_0 + \alpha) = \hat{\mu}(\alpha_0) \mathbf{E}[e_\alpha(X)] + O(\vartheta \|\alpha\| + \|\alpha\|^3)$$

with X distributed as $\eta(0, \sigma)$.

Proof. – Taylor expands $\log \frac{\hat{\mu}(\alpha_0 + \alpha)}{\hat{\mu}(\alpha_0)}$ in a ball of constant radius about $\alpha = 0$ to find

$$(71) \quad \log \frac{\hat{\mu}(\alpha_0 + \alpha)}{\hat{\mu}(\alpha_0)} = \frac{1}{2} \alpha^t H_0 \alpha + O\left(\vartheta \|\alpha\| + \vartheta^{\frac{1}{2}} \|\alpha\|^2 + \|\alpha\|^3\right),$$

with H_0 the Hessian of $\log \hat{\mu}(\alpha)$ at 0. In making this expansion, we've used the estimates for derivatives of $\hat{\mu}_2(\alpha_0 + \alpha)$ in Lemma 11 together with

$$(72) \quad \operatorname{Re} \log \frac{\hat{\mu}(\alpha_0 + \alpha)}{\hat{\mu}(\alpha_0)} = \frac{1}{2} \log \frac{\hat{\mu}_2(\alpha_0 + \alpha)}{\hat{\mu}_2(\alpha_0)}$$

and the estimates for derivatives of $\operatorname{Im} \log \hat{\mu}(\alpha_0 + \alpha)$ in Lemma 13.

Then (we've absorbed the $\vartheta^{\frac{1}{2}} \|\alpha\|^2$ error term into the others)

$$(73) \quad \hat{\mu}(\alpha_0 + \alpha) = \hat{\mu}(\alpha_0) \mathbf{E}[e_\alpha(X)] + O(\vartheta \|\alpha\| + \|\alpha\|^3).$$

Since $\hat{\mu}$ is bounded, this formula holds for all α by adjusting the constants appropriately. \square

3. Proof of Theorem 2

We first treat Theorem 2 which is illustrative of the main idea, before proving Theorem 1. Identify $\underline{n} = (n_1, n_2, n_3)^t \in \mathbb{Z}^3$, with $\underline{g}_n = [n_1, n_2, n_3] \in \mathbb{H}(\mathbb{Z})$ and let ν be the limiting Gaussian measure under convolution by μ .

PROPOSITION 15. – *For each $\underline{n} = (n_1, n_2, n_3)^t \in \mathbb{Z}^3$,*

$$(74) \quad P_N(n_1, n_2, n_3) := \mu^{*N}(\{g_n\}) = \frac{1}{N^2} \cdot \frac{d\nu}{dg} \left(d_{\frac{1}{\sqrt{N}}} \underline{g}_n \right) + O\left(N^{-\frac{5}{2}}\right).$$

Recalling the multiplication rule

$$(75) \quad \prod_{i=1}^N [w_i^{(1)}, w_i^{(2)}, 0] = \left[\underline{w}^{(1)}, \underline{w}^{(2)}, \frac{1}{2} \underline{w}^{(1)} \underline{w}^{(2)} + H^*(\underline{w}) \right],$$

which is valid for $w_i = [\pm 1, 0, 0]$ or $[0, \pm 1, 0]$, it suffices to calculate, with \mathbb{U}_N standing for the product measure $\mu^{\otimes N}$ and expectation with respect to \mathbb{U}_N ,

$$(76) \quad \mathbb{U}_N \left(\underline{w}_{\text{ab}} = (n_1, n_2)^t, H^*(\underline{w}) = n_3 - \frac{1}{2} n_1 n_2 \right) = \int_{(\mathbb{R}/\mathbb{Z})^3} e_\alpha((n_1, n_2)^t) e_\xi \left(n_3 - \frac{n_1 n_2}{2} \right) \mathbf{E} \left[\overline{e_\alpha(\underline{w}_{\text{ab}})} e_\xi(H^*(\underline{w})) \right] d\xi d\alpha.$$

3.1. Reduction to a central limit theorem

The following two lemmas reduce to a quantitative central limit theorem by truncating frequency space to the scale of the distribution.

LEMMA 16. – For any $A > 0$ there is $C = C(A) > 0$ such that if $\|\xi\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{C \log N}{N}$, for all $\alpha \in \mathbb{R}^2$,

$$(77) \quad |\mathbf{E}_{\mathbb{U}_N} [e_\alpha(\overline{w}_{\text{ab}}) e_\xi(H^*(\underline{w}))]| \leq N^{-A}.$$

Proof. – Choose $k = k(\xi)$ according to the rule

$$(78) \quad k(\xi) = \begin{cases} 1, & |\xi| > \frac{1}{10}, \\ \left\lfloor \frac{1}{2|\xi|} \right\rfloor, & |\xi| \leq \frac{1}{10}. \end{cases}$$

Let $N' = \lfloor \frac{N}{2k} \rfloor$. The group $G_k = C_2^{N'}$ acts on strings of length N with j th factor exchanging the order of the substrings of length k ending at $(2j-1)k$ and $2jk$.

Given string \underline{w} , write $\hat{\underline{w}}$ for the string of length $2N'$ with j th entry given by

$$(79) \quad \hat{w}_j = \sum_{i=1}^k w_{(j-1)k+i}.$$

Write

$$(80) \quad H^*(\underline{w}) = H_k^1(\underline{w}) + H_k^2(\underline{w}), \quad H_k^2(\underline{w}) = \sum_{j=1}^{N'} H^*(\hat{w}_{2j-1}, \hat{w}_{2j}).$$

Both \overline{w}_{ab} and H_k^1 are invariant under G_k . Exchanging the order of the expectations, which is justified because the group action is finite,

$$(81) \quad \begin{aligned} \mathbf{E}_{\mathbb{U}_N} [e_\alpha(\overline{w}_{\text{ab}}) e_\xi(H^*(\underline{w}))] &= \mathbf{E}_{\underline{\tau} \in G_k} [\mathbf{E}_{\mathbb{U}_N} [e_\alpha(\overline{w}_{\text{ab}}) e_\xi(H^*(\underline{\tau} \cdot \underline{w}))]] \\ &= \mathbf{E}_{\mathbb{U}_N} [e_\alpha(\overline{w}_{\text{ab}}) e_\xi(H_k^1(\underline{w})) \mathbf{E}_{\underline{\tau} \in G_k} [e_\xi(H_k^2(\underline{\tau} \cdot \underline{w}))]] \end{aligned}$$

and, therefore,

$$(82) \quad |\mathbf{E}_{\mathbb{U}_N} [e_\alpha(\overline{w}_{\text{ab}}) e_\xi(H^*(\underline{w}))]| \leq \mathbf{E}_{\mathbb{U}_N} [|\mathbf{E}_{\underline{\tau} \in G_k} [e_\xi(H_k^2(\underline{\tau} \cdot \underline{w}))]|].$$

By Cauchy-Schwarz,

$$(83) \quad \mathbf{E}_{\mathbb{U}_N} [|\mathbf{E}_{\underline{\tau} \in G_k} [e_\xi(H_k^2(\underline{\tau} \cdot \underline{w}))]|]^2 \leq \mathbf{E}_{\mathbb{U}_N} [|\mathbf{E}_{\underline{\tau} \in G_k} [e_\xi(H_k^2(\underline{\tau} \cdot \underline{w}))]|^2].$$

One checks, using the product group structure,

$$(84) \quad |\mathbf{E}_{\underline{\tau} \in G_k} [e_\xi(H_k^2(\underline{\tau} \cdot \underline{w}))]|^2 = \prod_{j=1}^{N'} \left(\frac{1 + \cos(2\pi \xi H^*(\hat{w}_{2j-1}, \hat{w}_{2j}))}{2} \right),$$

and hence, since the coordinates in \underline{w} are i.i.d.,

$$(85) \quad \mathbf{E}_{\mathbb{U}_N} [|\mathbf{E}_{\underline{\tau} \in G_k} [e_\xi(H_k^2(\underline{\tau} \cdot \underline{w}))]|^2] = \left(\frac{1 + \mathbf{E}_{\mathbb{U}_N} [\cos(2\pi \xi H^*(\hat{w}_1, \hat{w}_2))]}{2} \right)^{N'}.$$

By Lemma 7 the expectation in the cosine is uniformly bounded in size by $1 - c(\mu)$ for some $c(\mu) > 0$. The claim is completed by using the estimate $(1 - x)^{N'} \leq e^{-N'x}$, which is valid for $0 \leq x \leq 1$. \square

The following lemma obtains cancelation in α .

LEMMA 17. – Let $A, \epsilon > 0$ and $0 \leq \|\xi\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{C \log N}{N}$ where C is as in Lemma 16. For all N sufficiently large, if $\|\alpha\|_{\mathbb{R}^2/\mathbb{Z}^2} \geq N^{\epsilon-\frac{1}{2}}$, then

$$(86) \quad |\mathbf{E}_{\mathbb{U}_N} [e_\alpha(\overline{w}_{\text{ab}}) e_\xi(H^*(w))]| \leq N^{-A}.$$

Proof. – Let $N' = \lfloor N^{1-\epsilon} \rfloor$. Let \underline{w}_0 be \underline{w} truncated at N' and let \underline{w}_t be the remainder of \underline{w} so that \underline{w} is the concatenation $\underline{w}_0 \oplus \underline{w}_t$. Write

$$(87) \quad H^*(w) = H^*(\underline{w}_0) + H^*(\overline{w}_0, \overline{w}_t) + H^*(\underline{w}_t)$$

to bound

$$(88) \quad |\mathbf{E}_{\mathbb{U}_N} [e_\alpha(\overline{w}_{\text{ab}}) e_\xi(H^*(w))]| \\ \leq \mathbf{E}_{\underline{w}_t \sim \mu^{\otimes(N-N')}} \left[\left| \mathbf{E}_{\underline{w}_0 \sim \mu^{\otimes N'}} [e_\alpha(\overline{w}_{0,\text{ab}}) e_\xi(H^*(\underline{w}_0) + H^*(\overline{w}_0, \overline{w}_t))] \right| \right].$$

Truncate the outer integral to $\|\overline{w}_t\| \leq \sqrt{N} \log N$, which holds w.h.p. Let $E_k(x)$ denote the degree k Taylor expansion of $e_1(x)$ about 0, and recall that the integral form of Taylor's theorem gives

$$(89) \quad |e_1(x) - E_k(x)| = \left| (2\pi x)^{k+1} \int_0^1 \frac{(1-t)^k}{k!} e_1(xt) dt \right| \leq \frac{(2\pi|x|)^{k+1}}{(k+1)!}.$$

Use Lemma 10 to choose $k = k(A, \epsilon)$ odd and sufficiently large so that

$$(90) \quad \mathbf{E}_{\underline{w}_0 \sim \mu^{\otimes N'}} [|E_k(\xi H^*(\underline{w}_0)) - e_\xi(H^*(\underline{w}_0))|] \\ \leq \frac{(2\pi|\xi|)^{k+1}}{(k+1)!} \mathbf{E}_{\underline{w}_0 \sim \mu^{\otimes N'}} [|H^*(\underline{w}_0)|^{k+1}] \\ \leq O_{k,\mu}(|\xi|N)^{k+1} \leq \frac{1}{2N^A}.$$

It thus suffices to estimate

$$(91) \quad \mathbf{E}_{\mathbb{U}_{N'}} \left[e_\alpha(\overline{w}_{0,\text{ab}}) e_\xi(H^*(\overline{w}_0, \overline{w}_t)) E_k(\xi H^*(\underline{w}_0)) \right].$$

Expand E_k into $\text{Poly}(N)$ terms, each depending on boundedly many indices from \underline{w}_0 . Expectation over the remaining terms factors as a product which is exponentially small in a power of N , hence negligible. \square

3.2. Quantitative Gaussian approximation

In the range $\|\alpha\| \leq N^{\epsilon-\frac{1}{2}}$, $|\xi| \ll \frac{\log N}{N}$, expectation with respect to μ is replaced with expectation taken over a measure with projection to the abelianization given by a Gaussian of the same covariance matrix as μ_{ab} . The modified characteristic function in the Gaussian case is evaluated precisely in Theorem 9, which finishes the proof.

Let σ^2 be the covariance matrix of μ_{ab} and let $\eta(0, \sigma)$ be a centered Gaussian of the same covariance. Set $\delta = \det \sigma$. Taylor expands $\log \hat{\mu}(\beta)$ about $\beta = 0$ to find a cubic map $T(\beta)$ such that

$$(92) \quad \hat{\mu}_{\text{ab}}(\beta) = \hat{\eta}(\beta) (1 + T(\beta)) + O(\|\beta\|^4).$$

In the phase $e_\alpha(\underline{w}_{ab}) e_\xi(H^*(\underline{w}))$ let

$$(93) \quad \alpha_j(\underline{w}) = \alpha + \frac{\xi}{2} \left[\sum_{i \neq j} (-1)^{\mathbf{1}(i < j)} w_i^{(2)}, \sum_{i \neq j} (-1)^{\mathbf{1}(i > j)} w_i^{(1)} \right]^t,$$

so that $\alpha_j(\underline{w}) \cdot w_j$ is the part which depends on w_j . The Gaussian replacement scheme is as follows.

LEMMA 18. – Let $0 < \epsilon < \frac{1}{2}$ and $C > 0$ be constants. For $\|\alpha\| \leq N^{\epsilon - \frac{1}{2}}$ and $|\xi| \leq \frac{C \log N}{N}$,

$$(94) \quad \mathbf{E}_{\mathbb{U}_N} [e_\alpha(\underline{w}_{ab}) e_\xi(H^*(\underline{w}))] = I(\sqrt{N} \sigma \cdot \alpha, N \delta \xi) + O(N^{-1+O(\epsilon)})$$

$$+ \mathbf{E}_{\eta^{\otimes N}} \left[e_\alpha(\underline{w}_{ab}) e_\xi(H^*(\underline{w})) \left(\sum_j T(\alpha_j(\underline{w})) \right) \right].$$

Proof. – Since $\mathbf{E}_{\eta^{\otimes N}} [e_\alpha(\underline{w}_{ab}) e_\xi(H^*(\underline{w}))] = I(N^{\frac{1}{2}} \sigma \alpha, N \delta \xi; N)$, and since in the stated range of α, ξ ,

$$(95) \quad I(N^{\frac{1}{2}} \sigma \alpha, N \delta \xi; N) = I(N^{\frac{1}{2}} \sigma \alpha, N \delta \xi) + O(N^{-1+O(\epsilon)})$$

by Theorem 9, it suffices to prove

$$(96) \quad \mathbf{E}_{\mathbb{U}_N} [e_\alpha(\underline{w}_{ab}) e_\xi(H^*(\underline{w}))] + O(N^{-1+O(\epsilon)})$$

$$= \mathbf{E}_{\eta^{\otimes N}} \left[e_\alpha(\underline{w}_{ab}) e_\xi(H^*(\underline{w})) \left(1 + \sum_j T(\alpha_j(\underline{w})) \right) \right].$$

For convenience, write

$$(97) \quad T_j(\alpha, \xi, \underline{w}) = T(\alpha_j(\underline{w}))$$

and, for $k \neq j$,

$$(98) \quad T_j(\alpha, \xi, \underline{w}) = T_j^k(\alpha, \xi, \underline{w}) + \hat{T}_j^k(\alpha, \xi, \underline{w}),$$

in which T_j^k collects monomials in T_j which depend on w_k , and \hat{T}_j^k collects monomials which don't depend on w_k .

Since the expectation does not depend upon the third coordinate, write $\mu_{ab}^{\otimes N}$ in place of \mathbb{U}_N . For $0 \leq j \leq N$ consider the measure $\mu_j = \mu_{ab}^{\otimes j} \otimes \eta^{\otimes (N-j)}$ in which the first j coordinates are i.i.d. with measure μ_{ab} and last $N - j$ coordinates are independent of the first j and are i.i.d.

We prove (96) iteratively by showing that, for each $k \geq 1$,

$$(99) \quad S_k := \mathbf{E}_{\mu_k} \left[e_\alpha(\underline{w}_{ab}) e_\xi(H^*(\underline{w})) \left(1 + \sum_{j > k} T_j(\alpha, \xi, \underline{w}) \right) \right]$$

$$= O(N^{-2+O(\epsilon)}) + \mathbf{E}_{\mu_{k-1}} \left[e_\alpha(\underline{w}_{ab}) e_\xi(H^*(\underline{w})) \left(1 + \sum_{j > k-1} T_j(\alpha, \xi, \underline{w}) \right) \right]$$

$$= O(N^{-2+O(\epsilon)}) + S_{k-1},$$

which suffices since (96) may be written as $|S_N - S_0| = O(N^{-1+O(\epsilon)})$. By the triangle inequality, and setting apart expectation in the k th variable as the inner integral,

(100)

$$\begin{aligned} |S_k - S_{k-1}| &\leq \mathbf{E}_{\mu_{ab}^{\otimes(k-1)} \otimes \eta^{\otimes(N-k)}} \left| \int_{w_k} e_{\alpha_k(\underline{w})}(w_k) (d\mu_{ab} - (1 + T_k(\alpha, \xi, \underline{w})) d\eta) \right| \\ &\quad + \mathbf{E}_{\mu_{ab}^{\otimes(k-1)} \otimes \eta^{\otimes(N-k)}} \left| \int_{w_k} e_{\alpha_k(\underline{w})}(w_k) \left(\sum_{j>k} T_j(\alpha, \xi, \underline{w}) \right) (d\mu_{ab} - d\eta) \right|. \end{aligned}$$

In the first line of the right hand side, note that $T_k(\alpha, \xi, \underline{w})$ does not depend on w_k , so that Taylor expanding the exponential obtains a bound of $O(\|\alpha_k(\underline{w})\|^4)$, which suffices since

$$(101) \quad \mathbf{E}_{\mu_{ab}^{\otimes(k-1)} \otimes \eta^{\otimes(N-k)}} [\|\alpha_k(\underline{w})\|^4] = O(N^{-2+4\epsilon}).$$

In the second line, write $T_j = T_j^k + \hat{T}_j^k$. Since \hat{T}_j^k does not depend on w_k , matching the first two moments of μ_{ab} and η gives

$$\begin{aligned} (102) \quad &\mathbf{E}_{\mu_{ab}^{\otimes(k-1)} \otimes \eta^{\otimes(N-k)}} \left| \int_{w_k} e(\alpha_k(\underline{w}) \cdot w_k) \left(\sum_{j>k} \hat{T}_j^k(\alpha, \xi, \underline{w}) \right) (d\mu_{ab} - d\eta) \right| \\ &\ll \mathbf{E}_{\mu_{ab}^{\otimes(k-1)} \otimes \eta^{\otimes(N-k)}} \left[\|\alpha_k(\underline{w})\|^3 \left| \sum_{j>k} \hat{T}_j^k(\alpha, \xi, \underline{w}) \right| \right] \ll N^{-2+6\epsilon}. \end{aligned}$$

Finally, to bound the terms from T_j^k , Taylor expands $e(\alpha_k(\underline{w}) \cdot w_k)$ to degree 2 to bound

$$\begin{aligned} (103) \quad &\mathbf{E}_{\mu_{ab}^{\otimes(k-1)} \otimes \eta^{\otimes(N-k)}} \left| \int_{w_k} e(\alpha_k(\underline{w}) \cdot w_k) \left(\sum_{j>k} T_j^k(\alpha, \xi, \underline{w}) \right) (d\mu_{ab} - d\eta) \right| \\ &\ll \mathbf{E}_{\mu_{ab}^{\otimes(k-1)} \otimes \eta^{\otimes(N-k)}} \left| \int_{w_k} (1 + 2\pi i \alpha_k(\underline{w}) \cdot w_k) \sum_{j>k} T_j^k(\alpha, \xi, \underline{w}) (d\mu_{ab} - d\eta) \right| \\ &\quad + \mathbf{E}_{\mu_{ab}^{\otimes(k-1)} \otimes \eta^{\otimes(N-k)}} \left| \int_{w_k} \|\alpha_k(\underline{w})\|^2 |w_k|^2 \sum_{j>k} T_j^k(\alpha, \xi, \underline{w}) (d\mu_{ab} + d\eta) \right|. \end{aligned}$$

Since the first two moments of μ_{ab} and η match, the only terms which survive the first line here are degree 3 in w_k , and these contribute $O(N^{-2+3\epsilon})$. In the second line here, keeping in mind that T_j^k contains only monomials that have a factor of $\xi w_k^{(1)}$ or $\xi w_k^{(2)}$, one obtains a bound of $O(N^{-2+5\epsilon})$ by applying Cauchy-Schwarz to separate the integrands. This completes the iterative estimate (99). \square

We give two estimates for the error term

$$(104) \quad \mathcal{T} = \mathbf{E}_{\eta^{\otimes N}} \left[e_{\alpha}(\underline{w}_{ab}) e_{\xi}(H^*(\underline{w})) \left(\sum_j T(\alpha_j(\underline{w})) \right) \right],$$

depending on the relative sizes of α and ξ . In this part of the argument we assume that $\eta(0, \sigma)$ has identity covariance, which may be achieved by rescaling α and ξ by constant factors.

LEMMA 19. – *There exists $c > 0$ such that, for $\|\alpha\| \leq N^{\epsilon - \frac{1}{2}}$ and $|\xi| \ll \frac{\log N}{N}$,*

$$(105) \quad \mathcal{T} = O\left(N\|\alpha\|^3 + N^{\frac{5}{2}}|\xi|^3\right) \exp(-cN|\xi|).$$

Proof. – Bound each term

$$(106) \quad \mathbf{E}_{\eta \otimes N} [e_{\alpha}(\underline{w}_{\text{ab}}) e_{\xi}(H^*(\underline{w})) T(\alpha_j(\underline{w}))]$$

individually by setting

$$(107) \quad k = \begin{cases} \left\lfloor \frac{1}{2|\xi|} \right\rfloor, & |\xi|N > 1 \\ \left\lfloor \frac{N}{2} \right\rfloor, & \text{otherwise} \end{cases}$$

and allowing G_k to act as in Lemma 16. Let G'_k be the subgroup omitting the factor that moves w_j . Then G'_k leaves $T(\alpha_j(\underline{w}))$ invariant, so that

$$(106) = \mathbf{E}_{\eta \otimes N} [e_{\alpha}(\underline{w}_{\text{ab}}) \mathbf{E}_{\tau \in G'_k} [e_{\xi}(H^*(\tau \cdot \underline{w}))] T(\alpha_j(\underline{w}))].$$

By Cauchy-Schwarz,

$$|(106)|^2 \leq \mathbf{E}_{\eta \otimes N} [\mathbf{E}_{\tau \in G'_k} [e_{\xi}(H^*(\tau \cdot \underline{w}))]^2] \mathbf{E}_{\eta \otimes N} [|T(\alpha_j(\underline{w}))|^2].$$

Arguing as in Lemma 16 now obtains the estimate, for some $c > 0$,

$$(108) \quad |(104)| \ll \left(N\|\alpha\|^3 + N^{\frac{5}{2}}|\xi|^3\right) \exp(-c|\xi|N). \quad \square$$

To obtain decay in $\|\alpha\|$ instead of $|\xi|$, consider the degree 3 polynomial $\sum_j T(\alpha_j(\underline{w}))$ which consists of monomials of which

- i. Those constant in \underline{w} and cubic in α have absolute sum of coefficients $O(N)$.
- ii. Those linear in $\xi \underline{w}$ and quadratic in α have absolute sum of coefficients $O(N^2)$.
- iii. Those quadratic in $\xi \underline{w}$ and linear in α have absolute sum of coefficients $O(N^3)$. Of these, those with a repeated factor from \underline{w} have absolute sum of coefficients $O(N^2)$.
- iv. Those that are cubic in $\xi \underline{w}$ have absolute sum of coefficients $O(N^4)$. Of these, those with a repeated factor from \underline{w} have absolute sum of coefficients $O(N^3)$.

Write M for the typical monic monomial, so that M is of form

$$(109) \quad 1, w_{i_1}^{(\epsilon_1)}, w_{i_1}^{(\epsilon_1)} w_{i_2}^{(\epsilon_2)}, w_{i_1}^{(\epsilon_1)} w_{i_2}^{(\epsilon_2)} w_{i_3}^{(\epsilon_3)},$$

with $\epsilon_j \in \{1, 2\}$, according as the case is i., ii., iii. or iv.. Given a typical monomial M of T , write $\omega(M)$ for the number of variables from \underline{w} which are odd degree in M .

LEMMA 20. – *There is a constant $c > 0$ such that we have, for $0 < |\xi| \ll \frac{\log N}{N}$ and $\sqrt{|\xi|} \leq \|\alpha\| \leq N^{\epsilon - \frac{1}{2}}$,*

$$(110) \quad \mathcal{T} = \left[O\left(\|\alpha\|(1 + N\|\alpha\|^2)(1 + N^3|\xi|^3)\right) \times \exp\left(-c\left(\|\alpha\|^2 \min\left(N, \frac{1}{N|\xi|^2}\right)\right)\right) \right].$$

Proof. – Consider the expectation

$$(111) \quad E_M = \mathbf{E}_{\eta \otimes N} [Me_\alpha(\underline{w}_{ab}) e_\xi(H^*(\underline{w}))].$$

We show that, for some $c > 0$,

$$(112) \quad E_M \ll \|\alpha\|^{\omega(M)} \exp\left(-c \left(\|\alpha\|^2 \min\left(N, \frac{1}{N|\xi|^2}\right)\right)\right),$$

which suffices on summing over the monomials described in i. through iv. above.

Let $c_1 > 0$ be a small constant, and let

$$(113) \quad N' = \begin{cases} \left\lfloor \frac{c_1}{|\xi|} \right\rfloor, & N|\xi| > c_1 \\ N, & \text{otherwise} \end{cases}.$$

Let \underline{w}_0 be the initial string of \underline{w} of length N' and assume that this includes any variables from M ; the general case may be handled by a straightforward modification. Write $\underline{w} = \underline{w}_0 \oplus \underline{w}_t$ so that \underline{w}_t contains the remaining variables. Write

$$(114) \quad H^*(\underline{w}) = H^*(\underline{w}_0) + H^*(\underline{w}_0, \underline{w}_t) + H^*(\underline{w}_t).$$

Write $\hat{\alpha} = \alpha + \frac{\xi}{2} [\underline{w}_t^{(2)}, -\underline{w}_t^{(1)}]^t$. Bound

$$(115) \quad |E_M| \leq \mathbf{E}_{\underline{w}_t \sim \eta \otimes (N-N')} \left[\left| \mathbf{E}_{\underline{w}_0 \sim \eta \otimes N'} [Me_{\hat{\alpha}}(\underline{w}_0) e_\xi(H^*(\underline{w}_0))] \right| \right].$$

Expand $e_\xi(H^*(\underline{w}_0))$ in Taylor series to degree $L := \lfloor N^{2\epsilon} \rfloor$. The error in doing so is bounded by

$$(116) \quad \frac{(2\pi|\xi|)^{L+1}}{(L+1)!} \mathbf{E}_{\underline{w}_0 \sim \eta \otimes N'} [|M| |H^*(\underline{w}_0)|^{L+1}].$$

Apply Cauchy-Schwarz to remove the monomial M , then insert the moment bound of Lemma 10 to estimate this by

$$(117) \quad \begin{aligned} &\ll \frac{(2\pi|\xi|)^{L+1}}{(L+1)!} \mathbf{E}_{\underline{w}_0 \sim \eta \otimes N'} [H^*(\underline{w}_0)^{2L+2}]^{\frac{1}{2}} \\ &\leq \frac{(2\pi|\xi|)^{L+1}}{(L+1)!} \frac{(2L+2)!}{(L+1)! 2^{2L+2}} (N')^{L+1} \\ &\leq (2\pi|\xi|N')^{L+1} \leq (2\pi c_1)^{L+1}. \end{aligned}$$

If $c_1 < \frac{1}{2\pi}$ then this is bounded by, for some $c > 0$, $\exp(-cN^{2\epsilon})$.

In the Taylor expansion, expectation over \underline{w}_0 is bounded by

$$(118) \quad \begin{aligned} &\sum_{\ell=0}^L \frac{(2\pi|\xi|)^\ell}{\ell!} \left| \mathbf{E}_{\eta \otimes N'} [Me_{\hat{\alpha}}(\underline{w}_0) H^*(\underline{w}_0)^\ell] \right| \\ &\leq \sum_{\ell=0}^L \frac{(2\pi|\xi|)^\ell}{2^\ell \ell!} \sum_{\underline{m}, \underline{n} \in [N']^\ell} \left| \mathbf{E}_{\eta \otimes N'} [Me_{\hat{\alpha}}(\underline{w}_0) w_{m_1}^{(1)} \cdots w_{m_\ell}^{(1)} w_{n_1}^{(2)} \cdots w_{n_\ell}^{(2)}] \right|. \end{aligned}$$

The expectation factors as a product. Those indices of $[N']$ which do not have a monomial factor contribute, for some $c_2 > 0$,

$$(119) \quad \leq \exp(-2\pi^2(N' - 2\ell - 3)\|\hat{\alpha}\|^2) \leq \exp(-c_2 N' \|\hat{\alpha}\|^2).$$

Let \mathcal{E} (resp. \mathcal{O}) be those indices in $[N']$ which appear a positive even (resp. odd) number of times among the factors in M and $m_1, \dots, m_\ell, n_1, \dots, n_\ell$.

For indices which appear a positive even h_j number of times, bound $|e_{\hat{\alpha}}(w_j)| \leq 1$, so that the expectation is bounded by the h_j th moment of a 1-dimensional Gaussian,

$$(120) \quad \mu_{h_j} = \frac{h_j!}{2^{\frac{h_j}{2}} \left(\frac{h_j}{2}\right)!}.$$

At indices which appear an odd h_j number of times, set $e_{\hat{\alpha}}(w_j) = 1 + O(\|\hat{\alpha}\| \|w_j\|)$. Expectation against 1 vanishes. The remaining expectation is bounded by

$$(121) \quad \ll \|\hat{\alpha}\| \mu_{h_j+1}.$$

The configurations in which no index outside M appears with multiplicity greater than 2, and no more than one of the m_j, n_j fall on an odd degree index of M and none fall on an even degree index of M , make a dominant contribution. Call these *base configurations*. The *type* of a base configuration is described by a triple $(\underline{p}, \ell_1, \ell_2)$ where \underline{p} indicates whether an index from $\underline{m}, \underline{n}$ falls on each odd degree index present in M , where ℓ_1 counts the number of indices which appear once, and ℓ_2 counts the number of indices which appear twice. Let $|\underline{p}|$ be the number of indices which fall on M . Thus

$$(122) \quad 2\ell = |\underline{p}| + \ell_1 + 2\ell_2.$$

Let $\mathcal{N}(\underline{p}, \ell_1, \ell_2)$ be the number of base configurations that have a given type. There are

$$(123) \quad \frac{(2\ell)!}{|\underline{p}|! \ell_1! \ell_2! 2^{\ell_2}}$$

ways to allot the 2ℓ indices of $\underline{m}, \underline{n}$ to belong to \underline{p} , the ℓ_1 singletons or ℓ_2 doubles, and $\ll (N')^{\ell_1+\ell_2}$ ways to place the indices in $[N']$ once they have been so arranged, so that

$$(124) \quad \mathcal{N}(\underline{p}, \ell_1, \ell_2) \ll \frac{(2\ell)!}{|\underline{p}|! \ell_1! \ell_2! 2^{\ell_2}} (N')^{\ell_1+\ell_2}.$$

Given $\underline{m}, \underline{n}$ of type $(\underline{p}, \ell_1, \ell_2)$,

$$(125) \quad \mathbf{E}_{\eta \otimes N'} \left[M e_{\hat{\alpha}}(\overline{w}_0) w_{m_1}^{(1)} \cdots w_{m_\ell}^{(1)} w_{n_1}^{(2)} \cdots w_{n_\ell}^{(2)} \right] \\ \leq \exp(-c_2 N' \|\hat{\alpha}\|^2) O(1)^{\ell_1+\ell_2} \|\hat{\alpha}\|^{\omega(M)-|\underline{p}|+\ell_1}.$$

Indicating restriction of $\underline{m}, \underline{n}$ to base configurations with a ι ,

$$(126) \quad \sum_{\ell=0}^L \frac{(2\pi|\xi|)^\ell}{2^\ell \ell!} \sum_{\substack{\underline{m}, \underline{n} \in [N']^\ell \\ \iota}} \left| \mathbf{E}_{\eta \otimes N'} \left[M e_{\hat{\alpha}}(\overline{w}_0) w_{m_1}^{(1)} \cdots w_{m_\ell}^{(1)} w_{n_1}^{(2)} \cdots w_{n_\ell}^{(2)} \right] \right| \\ \ll \exp(-c_2 N' \|\hat{\alpha}\|^2) \\ \times \sum_{\substack{\underline{p} \\ \ell_1, \ell_2=0 \\ |\underline{p}|+\ell_1 \text{ even}}} \sum_{\ell_1+\ell_2=0}^{\infty} \frac{(\pi|\xi|)^{\frac{|\underline{p}|+\ell_1+2\ell_2}{2}} \|\hat{\alpha}\|^{\omega(M)-|\underline{p}|+\ell_1} (|\underline{p}| + \ell_1 + 2\ell_2)!}{\left(\frac{|\underline{p}|+\ell_1+2\ell_2}{2}\right)! \ell_1! \ell_2! 2^{\ell_2}} O(N')^{\ell_1+\ell_2}.$$

Bound

$$(127) \quad \frac{(|p| + \ell_1 + 2\ell_2)!}{|p|! \left(\frac{|p| + \ell_1 + 2\ell_2}{2}\right)! \ell_1! \ell_2!} \leq 4^{|p| + \ell_1 + 2\ell_2} \frac{\ell_2!}{\left(\frac{|p| + \ell_1 + 2\ell_2}{2}\right)!} \leq \frac{4^{|p| + \ell_1 + 2\ell_2}}{\left(\frac{|p| + \ell_1}{2}\right)!}.$$

If the constant c_1 in (113) is chosen sufficiently small, then the sum over ℓ_2 converges to a bounded quantity and the sum over ℓ_1 is bounded by

$$(128) \quad \ll \exp(O(1)\|\hat{\alpha}\|^2|\xi|(N')^2).$$

Since $|\xi|N' \leq c_1$, if c_1 is sufficiently small this obtains a bound,

$$(129) \quad \ll \exp\left(-\frac{c_2}{2}N'\|\hat{\alpha}\|^2\right) \left(\|\hat{\alpha}\|^{\omega(m)} + \|\hat{\alpha}\|^{\omega(m)-1}\sqrt{|\xi|} + \cdots + |\xi|^{\frac{\omega(m)}{2}}\right).$$

This bound with α in place of $\hat{\alpha}$ obtains (112) for the dominant terms, and hence bounds the dominant terms unless $|\xi| \gg \frac{1}{N}$. In the remaining case, the bound is acceptable unless $\|\hat{\alpha}\| < c_3\|\alpha\|$ for a small constant $c_3 > 0$. In this case one obtains $\|\xi\bar{w}_t\| \gg \|\alpha\|$. Since \bar{w}_t is a Gaussian with variance of order $N - N' < N$, the event $\|\xi\bar{w}_t\| \gg \|\alpha\|$ occurs with w_t -probability, for some $c_4 > 0$, $\ll \exp\left(-c_4\frac{\|\alpha\|^2}{N\xi^2}\right)$, which is again satisfactory.

To obtain a bound for all configurations from the bound for base ones, configurations with $|\mathcal{O}| = \ell_1, |\mathcal{E}| = \ell_2$ may be enumerated by adding a number k of double indices to an existing base configuration. There are $O(L)^k$ ways of choosing the indices where the new doubles will be added, $O(L)^{2k}$ ways of inserting the indices into the list $\underline{m}, \underline{n}$, and the new indices make a gain in the calculated moments of $O(L)^k$. Meanwhile, a factor of $|\xi|^k$ is saved in the outer sum. Recall $L \leq N^{2\epsilon}$. If $\epsilon < \frac{1}{8}$ then the sum over k is $O(1)$ for all N sufficiently large. \square

Proof of Theorem 2. – Combining Lemmas 16 and 17 obtains, for any $A > 0, 0 < \epsilon < \frac{1}{4}$, for some $c > 0$

$$(130) \quad P_N(n_1, n_2, n_3) + O_A(N^{-A}) = \iint_{\substack{\|\alpha\| \leq N^{\epsilon-\frac{1}{2}} \\ |\xi| \leq \frac{\log N}{N}}} e_\alpha((n_1, n_2)^t) e_\xi\left(n_3 - \frac{n_1 n_2}{2}\right) \\ \times \left[I\left(\sqrt{N}\sigma\alpha, N\delta\xi\right) + O(E) \right] d\alpha d\xi,$$

where the error term E satisfies the estimates of Lemmas 18, 19 and 20. Over the range of integration the error integrates to $O\left(N^{-\frac{5}{2}}\right)$.

Making a change of variables and extending the integral to \mathbb{R}^3 obtains

$$(131) \quad P_N(n_1, n_2, n_3) + O\left(N^{-\frac{5}{2}}\right) = \frac{1}{\delta^2 N^2} \int_{\mathbb{R}^3} e_\alpha\left(\sigma^{-1} \left[\frac{(n_1, n_2)^t}{\sqrt{N}} \right]\right) \\ \times e_\xi\left(\frac{1}{\delta} \left(\frac{n_3 - \frac{n_1 n_2}{2}}{N} \right)\right) I(\alpha, \xi) d\alpha d\xi.$$

The right hand side is the Gaussian density of the limit theorem. To obtain the return probability to 0, use $\delta^2 = \frac{4}{25}$ and $\int_{\mathbb{R}^3} I(\alpha, \xi) d\alpha d\xi = \frac{1}{4}$ in

$$(132) \quad P_N(0, 0, 0) = \frac{1}{\delta^2 N^2} \int_{\mathbb{R}^3} I(\alpha, \xi) d\alpha d\xi + O\left(N^{-\frac{5}{2}}\right) \\ = \frac{25}{16N^2} + O\left(N^{-\frac{5}{2}}\right). \quad \square$$

4. Proof of Theorem 1, Cramér case

Theorem 1 treats measures for which the abelianized walk is non-lattice. In this case the fibered distribution is also dense in \mathbb{R} , and when the abelianized distribution satisfies the Cramér condition, the fibered distribution does, also. We assume the Cramér condition in this section and treat the general case in the next section. In this case, after making an arbitrary translation on the left and right, the test functions may be taken to be of the form

$$(133) \quad f([x, y, z]) = F\left(x - x_0, y - y_0, z - \frac{xy}{2} - Ax - By - z_0\right),$$

where F is a Lipschitz function of compact support and x_0, y_0, z_0, A, B are real parameters.

Let ρ be a smooth, compactly supported bump function on \mathbb{R}^3 , for $t > 0$, $\rho_t(x) = t^3 \rho(tx)$ and $F_t = F * \rho_t$ the convolution on \mathbb{R}^3 . Since F is Lipschitz, $\|F - F_t\|_\infty \ll \frac{1}{t}$ as $t \rightarrow \infty$. Set

$$(134) \quad f_t([x, y, z]) = F_t\left(x - x_0, y - y_0, z - \frac{xy}{2} - Ax - By - z_0\right).$$

Choosing $t = t(N) = N^{\frac{5}{2}}$,

$$(135) \quad \langle f, \mu^{*N} \rangle = O\left(N^{-\frac{5}{2}}\right) + \int_{(\alpha, \xi) \in \mathbb{R}^3} \hat{F}_t(\alpha, \xi) \\ \times \left\langle e_\alpha\left((x - x_0, y - y_0)^t\right) e_\xi\left(z - \frac{xy}{2} - Ax - By - z_0\right), \mu^{*N} \right\rangle d\alpha d\xi.$$

Using the decay of the Fourier transform of the bump function ρ_t , truncate the integral to $\|\alpha\|, |\xi| = O(N^{O(1)})$ with admissible error.

Apply the multiplication rule (31) to write the central coordinate of a product of group elements \underline{w} as

$$(136) \quad \tilde{z} = z - \frac{xy}{2} = H^*(\underline{w}) + \tilde{\underline{w}}^{(3)}.$$

The mean of $\tilde{\underline{w}}^{(3)}$ is $N\tilde{z}$. Let $\tilde{\underline{w}}_0^{(3)} = \tilde{\underline{w}}^{(3)} - N\tilde{z}$. Let $\tilde{\alpha} = \alpha - \xi \cdot (A, B)^t$. Thus

$$(137) \quad \left\langle e_\alpha\left((x - x_0, y - y_0)^t\right) e_\xi\left(z - \frac{xy}{2} - Ax - By - z_0\right), \mu^{*N} \right\rangle \\ = e_\alpha\left(-(x_0, y_0)^t\right) e_\xi\left(N\tilde{z} - z_0\right) \int_{\mathbb{H}(\mathbb{R})^N} e_{\tilde{\alpha}}(\underline{w}_{ab}) e_\xi\left(H^*(\underline{w}) + \tilde{\underline{w}}_0^{(3)}\right) d\mu^{\otimes N}.$$

The argument of Lemma 16 applies as before to truncate to $|\xi| = O\left(\frac{\log N}{N}\right)$. This uses Lemma 7 in the case $|\xi| = O(1)$ and Lemma 6 in the case $|\xi| \gg 1$. The argument of Lemma 17 applies as before to truncate to $\|\tilde{\alpha}\| \ll N^{-\frac{1}{2} + \epsilon}$.

A small modification is needed to the application of Lemma 18 which we now describe. Here one can now include in the measure μ_{ab} a third dimension corresponding the $\tilde{z} - \tilde{z}$, and make η a 3 dimensional Gaussian with the same covariance matrix. The Gaussian replacement scheme goes through essentially unchanged, the addition of the third coordinate evaluated at the small frequency ξ making a negligible change; these terms do not need to be included in $T_j(\tilde{\alpha}, \xi, \underline{w})$. The main term becomes

$$(138) \quad \mathbf{E}_{\eta^{\otimes N}} \left[e_{\tilde{\alpha}}(\underline{w}_{ab}) e_\xi(H^*(\underline{w})) e_\xi\left(\tilde{\underline{w}}_0^{(3)}\right) \right].$$

After a linear change of coordinates, the third coordinate is independent of the first two and $\tilde{\alpha}$ is mapped to $\alpha' = \tilde{\alpha} + O(\xi)$. Hence

$$(139) \quad \mathbf{E}_{\eta^{\otimes N}} \left[e_{\tilde{\alpha}}(\overline{w}_{ab}) e_{\xi}(H^*(\underline{w})) e_{\xi}(\overline{w}_0^{(3)}) \right] = (1 + O(\xi^2 N)) \mathbf{E}_{\eta^{\otimes N}} \left[e_{\alpha'}(\overline{w}_{ab}) e_{\xi}(H^*(\underline{w})) \right].$$

Note that, since $\|\alpha'\|^2 = \|\tilde{\alpha}\|^2 + O(\|\tilde{\alpha}\| |\xi|) + O(|\xi|^2)$,

$$(140) \quad I \left(N^{\frac{1}{2}} \sigma \alpha', N \delta \xi \right) = \frac{\exp \left(-\frac{2\pi \|\alpha'\|^2}{\delta \xi \coth N \delta \pi \xi} \right)}{\cosh N \delta \pi \xi} \\ = I \left(N^{\frac{1}{2}} \sigma \tilde{\alpha}, N \delta \xi \right) (1 + O(\|\tilde{\alpha}\| + |\xi|)).$$

In the error term,

$$(141) \quad \mathbf{E}_{\eta^{\otimes N}} \left[e_{\tilde{\alpha}}(\overline{w}_{ab}) e_{\xi}(H^*(\underline{w})) e_{\xi}(\overline{w}_0^{(3)}) \sum_j T_j(\tilde{\alpha}, \xi, \underline{w}) \right],$$

the factor of $e_{\xi}(\overline{w}_0^{(3)})$ may be removed by Taylor expanding to degree 1, so that this part of the argument is unchanged.

To complete the argument, integrate as before

$$(142) \quad \langle f, \mu^{*N} \rangle = \int_{\substack{\|\tilde{\alpha}\| \ll N^{-\frac{1}{2}+\epsilon} \\ |\xi| \ll \frac{\log N}{N}}} \hat{F}_t(\alpha, \xi) e_{\alpha}(-(x_0, y_0)^t) e_{\xi}(N \bar{z} - z_0) \\ \times \left[I \left(N^{\frac{1}{2}} \sigma \alpha', N \delta \xi \right) + O(E) \right] d\alpha d\xi + O \left(N^{-\frac{5}{2}} \right).$$

The argument is now completed essentially as before, to find

$$(143) \quad \langle f, \mu^{*N} \rangle = \langle f_t, d_{\sqrt{N}} v \rangle + O \left(N^{-\frac{5}{2}} \right) = \langle f, d_{\sqrt{N}} v \rangle + O \left(N^{-\frac{5}{2}} \right).$$

5. Proof of Theorem 1, general case

We now consider the case in which μ_{ab} does not necessarily satisfy a Cramér condition. In this section the test functions take the form

$$(144) \quad f([x, y, z]) = F \left(x - x_0, y - y_0, z - \frac{xy}{2} - Ax - By - z_0 \right),$$

with F continuous and of compact support. Since we ask only for an asymptotic, it suffices by Selberg's theory of band-limited majorants and minorants [13] to assume that F takes the form

$$(145) \quad F([x, y, z]) = \phi_{ab}(x, y) \phi_3(z)$$

with ϕ_{ab} and ϕ_3 functions with Fourier transform of compact support. In this case, writing $\tilde{\alpha} = \alpha - \xi \cdot (A, B)^t$,

$$(146) \quad \langle f, \mu^{*N} \rangle = \int_{\|\alpha\|, |\xi| = O(1)} \hat{\phi}_{ab}(\alpha) \hat{\phi}_3(\xi) e_{\alpha}(-(x_0, y_0)^t) e_{\xi}(N \bar{z} - z_0) \\ \times \int_{\mathbb{H}(\mathbb{R})^N} e_{\tilde{\alpha}}(\overline{w}_{ab}) e_{\xi} \left(H^*(\underline{w}) + \overline{w}_0^{(3)} \right) d\mu^{\otimes N}.$$

Argue as in Lemma 16 to truncate to $|\xi| \ll \frac{\log N}{N}$. Since A and B are unconstrained, a further difficulty is encountered in applying Lemma 17 to truncate α . Let $\epsilon > 0$. For $|\xi| \ll \frac{\log N}{N}$ and $\tilde{\alpha} \notin \text{Spec}_{N^{-1+\epsilon}}(\mu_{\text{ab}})$, Lemma 17 demonstrates that the integral over $\mathbb{H}(\mathbb{R})^N$ is, for any $A > 0$, $O_A(N^{-A})$. The following modification of Lemma 18 permits an asymptotic evaluation of

$$(147) \quad \mathbf{E}_{\mathbb{U}_N} \left[e_{\tilde{\alpha}}(\underline{w}_{\text{ab}}) e_{\xi} \left(H^*(\underline{w}) + \underline{\bar{w}}_0^{(3)} \right) \right]$$

at points $\tilde{\alpha}$ in the large spectrum of μ_{ab} , $\text{Spec}_{N^{-1+\epsilon}}(\mu_{\text{ab}})$.

LEMMA 21. – *Let μ_{ab} have covariance matrix σ^2 and set $\delta = \det(\sigma)$. Let $0 < \epsilon < \frac{1}{4}$, $\vartheta = N^{-1+\epsilon}$, let $\tilde{\alpha} \in \text{Spec}_{N^{-1+\epsilon}}(\mu_{\text{ab}})$, $|\xi| \ll \frac{\log N}{N}$, and let $\alpha_0 \in \mathcal{M}_{\vartheta}(\mu_{\text{ab}})$ satisfy $\|\tilde{\alpha} - \alpha_0\| \ll \sqrt{\vartheta}$. Then*

$$(148) \quad \mathbf{E}_{\mathbb{U}_N} \left[e_{\tilde{\alpha}}(\underline{w}_{\text{ab}}) e_{\xi} \left(H^*(\underline{w}) + \underline{\bar{w}}_0^{(3)} \right) \right] = O \left(N^{-\frac{1}{2}+O(\epsilon)} \right) + \hat{\mu}_{\text{ab}}(\alpha_0)^N I \left(N^{\frac{1}{2}}\sigma(\tilde{\alpha} - \alpha_0), N\delta\xi \right).$$

Proof. – Write $e_{\xi}(\underline{\bar{w}}_0^{(3)}) = 1 + O(|\xi| |\underline{\bar{w}}_0^{(3)}|)$. Since

$$(149) \quad \mathbf{E}_{\mathbb{U}_N} \left[|\xi| |\underline{\bar{w}}_0^{(3)}| \right] = O \left(N^{-\frac{1}{2}+\epsilon} \right),$$

it suffices to prove that

$$(150) \quad \mathbf{E}_{\mathbb{U}_N} \left[e_{\tilde{\alpha}}(\underline{w}_{\text{ab}}) e_{\xi} (H^*(\underline{w})) \right] = \hat{\mu}_{\text{ab}}(\alpha_0)^N I \left(N^{\frac{1}{2}}\sigma(\tilde{\alpha} - \alpha_0), N\delta\xi \right) + O \left(N^{-\frac{1}{2}+O(\epsilon)} \right).$$

Let $\eta = \eta(0, \sigma)$ be a centered two dimensional Gaussian with covariance equal to that of μ_{ab} . Let X be distributed according to η . Set $\alpha = \tilde{\alpha} - \alpha_0$. By Lemma 14,

$$(151) \quad \hat{\mu}_{\text{ab}}(\alpha_0 + \alpha) = \hat{\mu}_{\text{ab}}(\alpha_0) \mathbf{E}_{\eta} [e_{\alpha}(X)] + O(\vartheta \|\alpha\| + \|\alpha\|^3).$$

In analogy with Lemma 18, define

$$(152) \quad \alpha_j(\underline{w}) = \alpha + \frac{\xi}{2} \left[\sum_{i \neq j} (-1)^{\delta(i < j)} w_i^{(2)}, \sum_{i \neq j} (-1)^{\delta(i > j)} w_i^{(1)} \right]^t.$$

Set, for $1 \leq k \leq N$, $\mu_k = \mu_{\text{ab}}^{\otimes k} \otimes \eta^{\otimes(N-k)}$. Also set $\underline{\bar{w}}_{\text{ab},k} = \sum_{j=1}^k w_{j,\text{ab}}$ and

$$(153) \quad S_k = \mathbf{E}_{\mu_k} \left[e_{\alpha}(\underline{w}_{\text{ab}}) e_{\alpha_0}(\underline{\bar{w}}_{\text{ab},k}) e_{\xi}(H^*(\underline{w})) \right],$$

so that $S_N = \mathbf{E}_{\mathbb{U}_N} [e_{\tilde{\alpha}}(\underline{w}_{\text{ab}}) e_{\xi}(H^*(\underline{w}))]$ and

$$(154) \quad S_0 = I \left(\sqrt{N}\sigma\alpha, N\delta\xi; N \right) = I \left(\sqrt{N}\sigma\alpha, N\delta\xi \right) + O \left(N^{-1+O(\epsilon)} \right).$$

Holding all but the k th variable fixed obtains

$$(155) \quad \begin{aligned} |S_k - \hat{\mu}_{\text{ab}}(\alpha_0) S_{k-1}| &\ll \mathbf{E}_{\mu_{\text{ab}}^{\otimes(k-1)} \otimes \eta^{\otimes(N-k)}} \left[\vartheta \|\alpha_k(\underline{w})\| + \|\alpha_k(\underline{w})\|^3 \right] \\ &= O \left(N^{-\frac{3}{2}+O(\epsilon)} \right). \end{aligned}$$

The proof is now complete, since

$$(156) \quad \left| S_N - \hat{\mu}_{ab}(\alpha_0)^N S_0 \right| \leq \sum_{k=0}^{N-1} \left| \hat{\mu}_{ab}(\alpha_0)^k S_{N-k} - \hat{\mu}_{ab}(\alpha_0)^{k+1} S_{N-k-1} \right| \\ \leq \sum_{k=0}^{N-1} |S_{N-k} - \hat{\mu}_{ab}(\alpha_0) S_{N-k-1}| \ll N^{-\frac{1}{2}+O(\epsilon)}. \quad \square$$

Proof of Theorem 1, general case. – Let $0 < \epsilon < \frac{1}{4}$ and let $\vartheta = N^{-1+\epsilon}$. By Lemma 12 there is a constant $c_1 = c_1(\mu)$ such that

$$\text{Spec}_{\vartheta}(\mu_{ab}) \subset \bigcup_{\alpha_0 \in \mathcal{M}_{\vartheta}} B_{c_1 \vartheta^{\frac{1}{2}}}(\alpha_0).$$

Let $\text{supp } \hat{\phi}_{ab} \subset B_R(0)$. Define the set

$$(157) \quad \Xi_{\text{good}} = \left\{ \xi : |\xi| \ll \frac{\log N}{N}, -\xi \cdot (A, B)^t \in \bigcup_{\alpha_0 \in \mathcal{M}_{\vartheta}} B_{R+c_1 \vartheta^{\frac{1}{2}}}(\alpha_0) \right\}.$$

Assume that N is sufficiently large so that if α_0, α_1 are distinct points of \mathcal{M}_{ϑ} then $\|\alpha_0 - \alpha_1\| \geq 2(R + c_1 \vartheta^{\frac{1}{2}})$. Given $\xi \in \Xi_{\text{good}}$ let $\alpha_0(\xi)$ be the nearest point to $-\xi \cdot (A, B)^t$ in \mathcal{M}_{ϑ} . Also, define

$$(158) \quad A_{\xi} = \{\alpha \in B_R(0) : \tilde{\alpha} = \alpha - \xi \cdot (A, B)^t \in \text{Spec}_{\vartheta}(\mu_{ab})\}.$$

By Lemma 12, $|A_{\xi}| \ll N^{-1+\epsilon}$.

In the evaluation from above,

$$(159) \quad \langle f, \mu^{*N} \rangle + O_A(N^{-A}) = \int_{\xi \in \Xi_{\text{good}}} \int_{\alpha: \tilde{\alpha} \in \text{Spec}_{\vartheta}(\mu_{ab})} \hat{\phi}_{ab}(\alpha) \hat{\phi}_3(\xi) e_{\alpha}(-(x_0, y_0)^t) e_{\xi}(N\bar{z} - z_0) \\ \times \mathbf{E}_{\mu \otimes N} [e_{\tilde{\alpha}}(\bar{w}_{ab}) e_{\xi}(H^*(\underline{w}))] d\alpha d\xi$$

insert the asymptotic formula for the expectation from Lemma 21. The error term here is bounded by

$$(160) \quad \int_{\xi: |\xi| \ll \frac{\log N}{N}} \int_{\substack{\alpha \in B_R(0), \\ \tilde{\alpha} \in \text{Spec}_{\vartheta}(\mu_{ab})}} N^{-\frac{1}{2}+O(\epsilon)} d\alpha d\xi \\ \leq \int_{\xi \in \Xi_{\text{good}}} N^{-\frac{1}{2}+O(\epsilon)} |A_{\xi}| d\xi = O(N^{-\frac{5}{2}+O(\epsilon)}).$$

This leaves the main term

$$(161) \quad \int_{\xi \in \Xi_{\text{good}}} \int_{\alpha: \tilde{\alpha} \in \text{Spec}_{\vartheta}(\mu_{ab})} \hat{\phi}_{ab}(\alpha) \hat{\phi}_3(\xi) e_{\alpha}(-(x_0, y_0)^t) e_{\xi}(N\bar{z} - z_0) \\ \times I(\sqrt{N}\sigma(\tilde{\alpha} - \alpha_0(\xi)), N\delta\xi) d\alpha d\xi.$$

The contribution from the part of this integral where $\alpha_0(\xi) = 0$ contributes $\langle f, d_{\sqrt{N}}\nu \rangle + O(N^{-A})$, by extending the integral with the same integrand to all of \mathbb{R}^3 , so it remains to bound the contribution from ξ for which $\alpha_0(\xi) \neq 0$.

For a fixed $\xi \in \Xi_{\text{good}}$, the formula

$$(162) \quad I(\alpha, \xi) = \frac{\exp\left(-\frac{2\pi\|\alpha\|^2}{\xi \coth \xi}\right)}{\cosh \pi \xi}$$

gives that integration in α is bounded absolutely by

$$(163) \quad \int_{\alpha} I\left(\sqrt{N}\sigma(\tilde{\alpha} - \alpha_0(\xi)), N\delta\xi\right) d\alpha \ll \frac{\max\left(\frac{1}{N}, |\xi|\right)}{\cosh N\pi\delta\xi}.$$

The contribution of $\xi \in \Xi_{\text{good}}$ for which $\alpha_0(\xi) \neq 0$ is bounded by

$$(164) \quad \int_{\xi \in \Xi_{\text{good}}, \alpha_0(\xi) \neq 0} \frac{\max\left(\frac{1}{N}, |\xi|\right)}{\cosh N\pi\delta\xi} d\xi.$$

Since the $\alpha_0(\xi)$ are $\mathcal{F}(\vartheta)$ spaced, this is bounded by

$$(165) \quad \ll \frac{1}{\mathcal{F}(\vartheta)} \int_0^\infty \frac{\max\left(\frac{1}{N}, |\xi|\right)}{\cosh N\pi\delta\xi} d\xi = o\left(\frac{1}{N^2}\right). \quad \square$$

6. Random walk on $N_n(\mathbb{Z})$, proof of Theorem 3

The case $n = 2$ is classical so assume $n \geq 3$.

Let $M : \mathbb{Z}^{n-1} \rightarrow N_n(\mathbb{Z})$ be the map

$$(166) \quad M : \mathbb{Z}^{n-1} \ni v = \begin{pmatrix} v^{(1)} \\ v^{(2)} \\ \vdots \\ v^{(n-1)} \end{pmatrix} \mapsto \begin{pmatrix} 1 & v^{(1)} & 0 & \cdots & 0 \\ 0 & 1 & v^{(2)} & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 1 & v^{(n-1)} & \\ 0 & \cdots & 0 & 1 & \end{pmatrix}.$$

Recall that, given $m \in N_n$ we write $Z(m)$ for the upper right corner. Given sequence of vectors $\underline{v} = \{v_i\}_{i=1}^N \in (\mathbb{Z}^{n-1})^N$ the central coordinate satisfies the product rule

$$(167) \quad Z\left(\prod_{i=1}^N M(v_i)\right) = \sum_{1 \leq i_1 < i_2 < \dots < i_{n-1} \leq N} v_{i_1}^{(1)} v_{i_2}^{(2)} \cdots v_{i_{n-1}}^{(n-1)}.$$

Write

$$(168) \quad Z_n^N = \sum_{1 \leq i_1 < i_2 < \dots < i_{n-1} \leq N} e_{i_1}^{(1)} \otimes \cdots \otimes e_{i_{n-1}}^{(n-1)}$$

for the corresponding tensor. $Z_{n,\mu}^N$ denotes the measure on \mathbb{Z} obtained by pushing forward measure μ on \mathbb{Z}^{n-1} via M to measure $\tilde{\mu}$ on $N_n(\mathbb{Z})$, then obtaining $\langle Z, \tilde{\mu}^{*N} \rangle$. Equivalently, $Z_{n,\mu}^N$ is the distribution of Z_n^N evaluated on N vectors v_i drawn i.i.d. from μ .

Given a probability measure ν on \mathbb{Z} and prime p , Cauchy-Schwarz and Plancherel give

$$(169) \quad \sum_{x \bmod p} \left| \nu(x \bmod p) - \frac{1}{p} \right| \leq \left(\sum_{0 \neq \xi \bmod p} \left| \hat{\nu}\left(\frac{\xi}{p}\right) \right|^2 \right)^{\frac{1}{2}},$$

where

$$(170) \quad \hat{v}(\alpha) = \sum_{n \in \mathbb{Z}} e_{-\alpha}(n) v(n).$$

Theorem 3 thus reduces to the following estimate on the characteristic function of $Z_{n,\mu}^N$.

PROPOSITION 22. – *Let $n \geq 3$ and let μ be a measure on \mathbb{Z}^{n-1} satisfying the same conditions as in Theorem 3. There exists constant $C > 0$ such that for all $N > 0$ and all $0 < |\xi| \leq \frac{1}{2}$,*

$$(171) \quad \left| \hat{Z}_{n,\mu}^N(\xi) \right| \ll \exp\left(-CN|\xi|^{\frac{2}{n-1}}\right).$$

Deduction of Theorem 3. – Recall $N = cp^{\frac{2}{n-1}}$ and let $c \geq 1$. Apply the upper bound of Proposition 22. By (169),

$$(172) \quad \left(\sum_{x \bmod p} \left| Z_{n,\mu}^N(x) - \frac{1}{p} \right| \right)^2 \leq \sum_{\substack{\xi \in \mathbb{Z} \\ 0 < |\xi| < \frac{p}{2}}} \left| \hat{Z}_{n,\mu}^N\left(\frac{\xi}{p}\right) \right|^2 \\ \ll \sum_{0 < |\xi| < \frac{p}{2}} \exp\left(-Cc|\xi|^{\frac{2}{n-1}}\right) \\ \ll \exp(-Cc). \quad \square$$

6.1. Proof of Proposition 22

Let C_2^{n-2} act on blocks of vectors of length $k2^{n-2}$ with the j th factor from C_2^{n-2} , $j \geq 1$ switching the relative order of the first $k2^{j-1}$ and second $k2^{j-1}$ indices. Thus, for instance, in case $n = 5$, if each of x_1, \dots, x_8 represents a block of k consecutive indices and $\underline{x} = x_1x_2x_3x_4x_5x_6x_7x_8$,

$$(173) \quad \begin{aligned} \tau_2 \underline{x} &= x_3x_4x_1x_2x_5x_6x_7x_8 \\ \tau_1 \tau_3 \underline{x} &= \tau_3 \tau_1 \underline{x} = x_5x_6x_7x_8x_2x_1x_3x_4 \\ \tau_1 \tau_2 \tau_3 \underline{x} &= x_5x_6x_7x_8x_3x_4x_2x_1. \end{aligned}$$

For $k \geq 1$ set $N' = \left\lfloor \frac{N}{k2^{n-2}} \right\rfloor$ and let $G_k = (C_2^{n-2})^{N'}$. G_k acts on sequences of length N with, for $j \geq 1$, the j th factor of G_k acting on the contiguous subsequence of indices of length $k2^{n-2}$ ending at $jk2^{n-2}$. For fixed k and fixed $\underline{w} \in W_N = (\text{supp } \mu)^N$, let

$$(174) \quad Z_k(\underline{w}) = \mathbf{E}_{\underline{\tau} \in G_k} \left[\delta_{Z_n^N(\underline{\tau} \cdot \underline{w})} \right].$$

Continue to abbreviate $\mathbb{U}_N = \mu^{\otimes N}$. For any k ,

$$(175) \quad Z_{n,\mu}^N = \mathbf{E}_{\mathbb{U}_N} [Z_k(\underline{w})].$$

We introduce a second, dual action of G_k on a linear dual space. Let

$$(176) \quad \mathcal{J}_N = \{\underline{i} = (i_1, i_2, \dots, i_{n-1}) : 1 \leq i_1 < i_2 < \dots < i_{n-1} \leq N\}.$$

Given $\underline{i} \in \mathcal{J}_N$ and $k \geq 1$, let $\mathfrak{S}_{\underline{i},k} \subset \mathfrak{S}_{n-1}$ be the subset of permutations $\mathfrak{S}_{\underline{i},k} = \{\sigma_{\underline{\tau},\underline{i}} : \underline{\tau} \in G_k\}$, where

$$(177) \quad \forall 1 \leq j \leq n-1, \quad \sigma_{\underline{\tau},\underline{i}}(j) = \#\{1 \leq k \leq n-1 : \underline{\tau}(i_k) \leq \underline{\tau}(i_j)\}.$$

That is, $\sigma_{\underline{\tau}, \underline{i}}(j)$ is the relative position of $\underline{\tau} \cdot i_j$ when $\underline{\tau} \cdot \underline{i}$ is sorted to be in increasing order. Put another way, suppose $\underline{\tau}$ maps $i_1 < \dots < i_{n-1}$ to $j_1 < \dots < j_{n-1}$ in some order (and vice versa, $\underline{\tau}$ is an involution) and calculate

$$\begin{aligned} e_{j_1}^{(1)} \otimes \dots \otimes e_{j_{n-1}}^{(n-1)}(\underline{\tau} \cdot \underline{w}) &= e_{\underline{\tau} j_1}^{(1)} \otimes \dots \otimes e_{\underline{\tau} j_{n-1}}^{(n-1)}(\underline{w}) \\ (178) \qquad \qquad \qquad &= e_{i_{\sigma^{-1}(1)}}^{(1)} \otimes \dots \otimes e_{i_{\sigma^{-1}(n-1)}}^{(n-1)}(\underline{w}) \\ &= e_{i_1}^{(\sigma(1))} \otimes \dots \otimes e_{i_{n-1}}^{(\sigma(n-1))}(\underline{w}), \end{aligned}$$

where $\sigma_{\underline{\tau}, \underline{i}}$ is abbreviated σ .

Let

$$(179) \qquad X_{N,k} = \left\{ e_{i_1}^{(\sigma(1))} \otimes \dots \otimes e_{i_{n-1}}^{(\sigma(n-1))} : \underline{i} \in \mathcal{I}_N, \sigma \in \mathfrak{S}_{\underline{i},k} \right\}.$$

The action of $\underline{\tau} \in G_k$ is defined on a representative set within $X_{N,k}$ by, for each $\underline{i} \in \mathcal{I}_N$,

$$(180) \qquad \underline{\tau} \cdot \left(e_{i_1}^{(1)} \otimes \dots \otimes e_{i_{n-1}}^{(n-1)} \right) = e_{i_1}^{(\sigma_{\underline{\tau}, \underline{i}}(1))} \otimes \dots \otimes e_{i_{n-1}}^{(\sigma_{\underline{\tau}, \underline{i}}(n-1))}.$$

The following lemma justifies that this definition extends to a unique group action of G_k on all of $X_{N,k}$.

LEMMA 23. – *Let $\underline{\tau}, \underline{\tau}' \in G_k$ and $\underline{i} \in \mathcal{I}_N$ satisfy $\sigma_{\underline{\tau}, \underline{i}} = \sigma_{\underline{\tau}', \underline{i}}$. Then for any $\underline{\tau}'' \in G_k$, $\sigma_{\underline{\tau} + \underline{\tau}'', \underline{i}} = \sigma_{\underline{\tau}' + \underline{\tau}'', \underline{i}}$. In particular, (180) extends to a unique group action on $X_{N,k}$.*

Proof. – This follows, since, for any $1 \leq i < j \leq N$ there is at most one factor of $G = C_2^{n-2}$ in G_k , and one index ℓ , $1 \leq \ell \leq n-2$ of G which exchanges the order of i and j . To define the group action in general, given $\underline{\tau} \in G_k$, $\underline{i} \in \mathcal{I}_N$ and $\sigma \in \mathfrak{S}_{\underline{i},k}$ choose any $\underline{\tau}_0$ such that $\sigma = \sigma_{\underline{\tau}_0, \underline{i}}$. Let $\sigma' = \sigma_{\underline{\tau}_0 + \underline{\tau}, \underline{i}}$. Then

$$(181) \qquad \underline{\tau} \cdot e_{i_1}^{(\sigma(1))} \otimes \dots \otimes e_{i_{n-1}}^{(\sigma(n-1))} = e_{i_1}^{(\sigma'(1))} \otimes \dots \otimes e_{i_{n-1}}^{(\sigma'(n-1))}.$$

The definition is clearly unique, since (180) surjects on $X_{N,k}$. \square

The actions of G_k on W_N and on $X_{N,k}$, although not adjoint, are compatible on Z_n^N , in the sense that for any $\underline{\tau}$,

$$(182) \qquad (\underline{\tau} \cdot Z_n^N)(\underline{w}) = Z_n^N(\underline{\tau} \cdot \underline{w})$$

so that

$$(183) \qquad Z_k(\underline{w}) = \mathbf{E}_{\underline{\tau} \in G_k} \left[\delta_{(\underline{\tau} \cdot Z_n^N)(\underline{w})} \right].$$

Note that in general, although G_k is a product group, the separate factors τ_i do not act independently in $\underline{\tau} \cdot Z_n^N$. For instance, when $n = 5$ and $k = 1$, the change in a tensor of type $e_1^{(1)} \otimes e_2^{(2)} \otimes e_{2^{n-2}+1}^{(3)} \otimes e_{2^{n-2}+2}^{(4)}$ under the first factor of G_k depends upon whether the second factor has been applied. Thus the characteristic function

$$(184) \qquad \chi_k(\xi, \underline{w}) = \mathbf{E}_{\underline{\tau} \in G_k} \left[e^{-\xi} (Z_n^N(\underline{\tau} \cdot \underline{w})) \right]$$

needs not factor as a product.

A pleasant feature of the general case is that this difficulty is rectified by estimating instead of $\chi_k(\xi, \underline{w})$, a function $F_k(\xi, \underline{w})$ which is the result of applying the Gowers-Cauchy-Schwarz inequality to $\chi_k(\xi, \underline{w})$. To describe this, write $G_k = (C_2^{n-2})^{N'} = (C_2^{N'})^{n-2}$, and thus

$$(185) \quad \mathbf{E}_{\underline{\tau} \in G_k} [f(\underline{\tau})] = \mathbf{E}_{\underline{\tau}_1 \in C_2^{N'}} \cdots \mathbf{E}_{\underline{\tau}_{n-2} \in C_2^{N'}} [f(\underline{\tau}_1, \dots, \underline{\tau}_{n-2})].$$

Then, setting apart one expectation at a time and applying Cauchy-Schwarz,

$$(186) \quad |\chi_k(\xi, \underline{w})|^{2^{n-2}} \leq \mathbf{E}_{\underline{\tau}_1, \underline{\tau}'_1 \in C_2^{N'}} \cdots \mathbf{E}_{\underline{\tau}_{n-2}, \underline{\tau}'_{n-2} \in C_2^{N'}} \left[e_{-\xi} \left(\sum_{S \subset [n-2]} (-1)^{n-2-|S|} \underline{\tau}_S \cdot \underline{w} \right) \right] \\ = \mathbf{E}_{\underline{\tau}, \underline{\tau}' \in G_k} \left[e_{-\xi} \left(\sum_{S \subset [n-2]} (-1)^{n-2-|S|} \underline{\tau}_S \cdot \underline{w} \right) \right] \\ =: F_k(\xi, \underline{w}),$$

where

$$(187) \quad \underline{\tau}_S = (\tau_{S,1}, \dots, \tau_{S,n-2}), \quad \tau_{S,i} = \begin{cases} \tau_i & i \in S, \\ \tau'_i & i \notin S. \end{cases}$$

LEMMA 24. – $F_k(\xi, \underline{w})$ factors as the product

$$(188) \quad F_k(\xi, \underline{w}) = \prod_{j=1}^{N'} \left(1 - \frac{1}{2^{n-2}} + \frac{F_{k,j}(\xi, \underline{w})}{2^{n-2}} \right),$$

where $F_{k,j}(\xi, \underline{w})$ is a function of $\underline{w}_{j,k} = (\omega_1, \dots, \omega_{2^{n-2}})$ with the

$$(189) \quad \omega_i = \sum_{(2^{n-2}(j-1)+i-1)k < \ell \leq (2^{n-2}(j-1)+i)k} w_\ell$$

the sum of consecutive blocks of length k in \underline{w} . Identify C_2^{n-2} with $\{0, 1\}^{n-2}$ and write $|\tau| = \sum_{i=1}^{n-2} \mathbf{1}(\tau_i \neq 0)$. Then

$$(190) \quad F_{k,j}(\xi, \underline{w}) = \mathbf{E}_{\tau \in C_2^{n-2}} \left[e_{-\xi} \left(\sum_{\tau' \in C_2^{n-2}} (-1)^{|\tau'|} Z_n^{2^{n-2}}((\tau + \tau') \cdot \underline{w}_{j,k}) \right) \right]$$

with the action of C_2^{n-2} on blocks of size 1 in $\underline{w}_{j,k}$.

Proof. – Consider for fixed $\underline{\tau}, \underline{\tau}' \in G_k$ the sum

$$(191) \quad Z_n^N(\underline{\tau}, \underline{\tau}')(\underline{w}) = \sum_{S \subset [n-2]} (-1)^{n-2-|S|} \underline{\tau}_S \cdot Z_n^N(\underline{w}).$$

After replacing \underline{w} with $\underline{\tau}' \underline{w}$ and $\underline{\tau}$ with $\underline{\tau} + \underline{\tau}'$ it suffices to consider $\underline{\tau}' = \text{id}$.

Consider the action of

$$(192) \quad \hat{\tau} = \sum_{S \subset [n-2]} (-1)^{n-2-|S|} \tau_S$$

on a tensor

$$(193) \quad \mathbf{e} = e_{i_1}^{(1)} \otimes e_{i_2}^{(2)} \otimes \cdots \otimes e_{i_{n-1}}^{(n-1)}, \quad 1 \leq i_1 < i_2 < \cdots < i_{n-1} \leq N$$

appearing in Z_n^N . Let $G = C_2^{n-2}$ identified with subsets S of $[n-2]$, let $G^0 = \text{stab}(\underline{i}) \leq C_2^{n-2}$ be the subgroup consisting of S for which $\underline{\tau}_S \cdot \mathbf{e} = \mathbf{e}$, and let $G^1 = C_2^{n-2}/G^0$. By the group action property, for all $x \in G^0$, for all $y \in G^1$, $\underline{\tau}_{x+y} \mathbf{e} = \underline{\tau}_y \mathbf{e}$ so that when $G^0 \neq \{1\}$, $\hat{\tau} \cdot \mathbf{e} = 0$.

A necessary and sufficient condition for $G^0 = \{1\}$ is that, for some $1 \leq j \leq N'$,

$$(194) \quad \begin{aligned} 2^{n-2}(j-1)k < i_1 &\leq 2^{n-2}(j-1)k + k \\ \forall 1 < \ell \leq n-1 \quad 2^{n-2}(j-1)k + 2^{\ell-2}k < i_\ell &\leq 2^{n-2}(j-1)k + 2^{\ell-1}k, \end{aligned}$$

and $\tau_j = \mathbf{1}_{n-2} \in C_2^{n-2}$. In words, the indices must all belong to a common block of length $2^{n-2}k$ acted on by a single factor from G_k , within this block, the first $2^{\ell-1}k$ elements must contain i_ℓ and the second $2^{\ell-1}k$ must contain $i_{\ell+1}$ for $\ell = 1, 2, \dots, n-2$, and the factor τ_j acting on the block must be the element $\mathbf{1}_{n-2}$ of the hypercube C_2^{n-2} .

The product formula given summarizes this condition. \square

Proof of Proposition 22. – Assume without loss that $|\xi| \geq N^{-\frac{n-1}{2}}$.

Let $k = \max\left(\left\lfloor |\xi|^{-\frac{2}{n-1}} \right\rfloor, M(\mu)\right)$, where $M(\mu)$ is a constant depending upon μ . By the triangle inequality and Hölder's inequality,

$$(195) \quad \begin{aligned} \left| \hat{Z}_{n,\mu}^N(\xi) \right| &\leq \left| \mathbf{E}_{\mathbb{U}_N} [\chi_k(\xi, \underline{w})] \right| \\ &\leq \mathbf{E}_{\mathbb{U}_N} [|\chi_k(\xi, \underline{w})|] \\ &\leq \mathbf{E}_{\mathbb{U}_N} \left[|\chi_k(\xi, \underline{w})|^{2^{n-2}} \right]^{\frac{1}{2^{n-2}}} \\ &\leq \mathbf{E}_{\mathbb{U}_N} [F_k(\xi, \underline{w})]^{\frac{1}{2^{n-2}}}. \end{aligned}$$

Since disjoint blocks are i.i.d., Lemma 24 implies that the expectation of $F_k(\xi, \underline{w})$ factors as a product

$$(196) \quad \mathbf{E}_{\mathbb{U}_N} [F_k(\xi, \underline{w})] = \left(1 - \frac{1}{2^{n-2}} + \frac{\mathbf{E}[F_{k,1}(\xi, \underline{w})]}{2^{n-2}} \right)^{N'}.$$

In the limit as $|\xi| \downarrow 0$,

$$(197) \quad \xi \sum_{\tau' \in C_2^{n-2}} (-1)^{|\tau'|} Z_n^{2^{n-2}}((\tau + \tau') \cdot \underline{w}_{j,k})$$

has a continuous limiting distribution, which is a polynomial of degree $n-1$ in independent normal random variables, and hence the characteristic function $\mathbf{E}[F_{k,1}(\xi, \underline{w})]$ has size bounded uniformly away from 1 for all $|\xi|$ smaller than a fixed constant $\epsilon(\mu)$. It follows that

$$(198) \quad \left| \hat{Z}_{n,\mu}^N(\xi) \right| \leq \exp(-C'N') \leq \exp\left(-C|\xi|^{\frac{2}{n-1}}N\right).$$

To handle the remaining range $\epsilon(\mu) \leq |\xi| \leq \frac{1}{2}$, choose N_1, N_2, \dots, N_{n-1} minimal such that μ^{*N_i} gives positive mass to the i th standard basis vector in \mathbb{R}^{n-1} . Set $k = N_1 + \dots + N_{n-1}$ and recall that μ assigns positive probability to 0. Then with $\mu^{\otimes 2^{n-2}k}$ -positive probability, for each $1 \leq j \leq n-1$, ω_{2^j-1} is the j th standard basis vector and all other ω_i are 0. For this configuration, $Z_n^{2^{n-2}}(\tau \cdot \underline{w}_{1,k}) = 1$ if τ is the identity, and 0 otherwise. Again, this gives that

the characteristic function is uniformly bounded away from 1. We thus conclude, as before, that

$$(199) \quad \left| \hat{Z}_{n,\mu}^N(\xi) \right| \leq \left| \hat{Z}_{n,\mu}^N(\xi) \right| \leq \exp(-CN). \quad \square$$

Appendix

The characteristic function of a Gaussian measure on the Heisenberg group

This section proves Theorem 9, which gives a rate of convergence to the characteristic function of a Gaussian measure on the Heisenberg group when the steps in the walk are normally distributed in the abelianization.

Recall that

$$(200) \quad I(\alpha, \xi; N) = \int_{(\mathbb{R}^2)^N} e_{-\alpha} \left(\frac{\bar{x}}{\sqrt{N}} \right) e_{-\xi} \left(\frac{H^*(x)}{N} \right) dv_2^{\otimes N}(x),$$

where $v_2(x) = \frac{1}{2\pi} \exp\left(-\frac{\|x\|^2}{2}\right)$.

First consider the case $\alpha = 0$. Integrate away $\underline{x}^{(1)}$ to obtain,

$$(201) \quad I(0, \xi; N) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} \exp\left(-\frac{1}{2} \underline{y}^t \left((1 - \xi_0^2) I_N + \xi_0^2 H\right) \underline{y}\right) d\underline{y},$$

where

$$(202) \quad \xi_0 = \frac{\pi \xi}{N}, \quad H_{i,j} = N - 2|i - j|;$$

this follows from $H^*(x) = \frac{1}{2} \sum_{i < j} (-1)^{\delta(j < i)} x_i^{(1)} x_j^{(2)}$, $\hat{\eta}(\xi) = e^{-2\pi^2 \xi^2}$ for a standard one dimensional Gaussian, and

$$(203) \quad \underline{y}^t (H - I_N) \underline{y} = \sum_{i=1}^N \left(\sum_{j \neq i} (-1)^{\delta(j < i)} y_j \right)^2.$$

Thus,

$$(204) \quad I(0, \xi; N) = \frac{1}{\sqrt{\det((1 - \xi_0^2) I_N + \xi_0^2 H)}},$$

as may be seen by using an orthogonal matrix to diagonalize the quadratic form.

We perform elementary row operations to simplify the computation of the determinant.

Let

$$(205) \quad U_- = I_N - \sum_{i=1}^{N-1} e_i \otimes e_{i+1}.$$

Thus

$$(206) \quad (U_-^t U_-)_{i,j} = \begin{cases} 1, & i = j = 1 \\ 2, & i = j > 1 \\ -1, & |i - j| = 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$(207) \quad (U_-^t H U_-)_{i,j} = \begin{cases} N, & i = j = 1 \\ -2, & i = 1, j > 1 \text{ or } j = 1, i > 1 \\ 4, & i = j > 1 \\ 0, & \text{otherwise} \end{cases},$$

so that

$$(208) \quad U_-^t ((1 - \xi_0^2) I_N + \xi_0^2 H) U_- = (1 + \xi_0^2) \left[2I_N - \frac{1 - \xi_0^2}{1 + \xi_0^2} \sum_{i=1}^{N-1} (e_i \otimes e_{i+1} + e_{i+1} \otimes e_i) \right. \\ \left. - \frac{2\xi_0^2}{1 + \xi_0^2} \sum_{i=1}^N (e_1 \otimes e_i + e_i \otimes e_1) + \frac{(N+1)\xi_0^2 - 1}{1 + \xi_0^2} e_1 \otimes e_1 \right].$$

Set $\zeta = \frac{1 - \xi_0^2}{1 + \xi_0^2}$. We diagonalize the tridiagonal matrix with 2's on the diagonal and $-\zeta$ on the first sub and super diagonal by working from the lower right corner and adding up and to the left, and treat the remainder of the matrix as a rank 2 perturbation.

Define sequences

$$(209) \quad \begin{aligned} \varepsilon_1 &= 2, & \forall i \geq 1, \varepsilon_{i+1} &= 2 - \frac{\zeta^2}{\varepsilon_i} \\ \pi_0 &= 1, & \forall i \geq 1, \pi_i &= \prod_{j=1}^i \varepsilon_j \\ \delta_1 &= 1, & \forall i \geq 1, \delta_{i+1} &= 1 + \frac{\zeta \delta_i}{\varepsilon_i}. \end{aligned}$$

These parameters have the following behavior with proof postponed until the end of this section.

LEMMA 25. – For $\xi \in (0, N^{\frac{1}{2}}]$ the following asymptotics hold

$$(210) \quad \begin{aligned} \pi_N &= N \frac{\sinh(2\pi\xi)}{2\pi\xi} \left(1 + O\left(\frac{1 + \xi^2}{N}\right) \right) \\ \varepsilon_N &= 1 + \frac{2\pi\xi}{N} \coth(2\pi\xi) \left(1 + O\left(\frac{1 + \xi^2}{N}\right) \right) \\ \delta_N &= \frac{N \tanh \pi\xi}{2\pi\xi} \left(1 + O\left(\frac{1 + \xi^2}{N}\right) \right) \\ \sum_{j=1}^{N-1} \frac{\delta_j^2}{\varepsilon_j} &= \frac{N^3}{8\pi^3 \xi^3} [2\pi\xi - 2 \tanh \pi\xi] \left(1 + O\left(\frac{1 + \xi^2}{N}\right) \right). \end{aligned}$$

Set

$$(211) \quad L_\varepsilon = I_N + \zeta \sum_{i=1}^{N-1} \frac{e_{i+1} \otimes e_i}{\varepsilon_{N-i+1}}, \quad D_\varepsilon = \frac{1}{(1 + \xi_0^2)} \sum_{i=1}^N \frac{e_i \otimes e_i}{\varepsilon_{N+1-i}}.$$

The diagonalization process is summarized in the following matrix equation, in which $D_\varepsilon^{\frac{1}{2}}$ is multiplied on left and right to obtain 1's on the diagonal

$$(212) \quad D_\varepsilon^{\frac{1}{2}} L_\varepsilon^t U_-^t \left((1 - \xi_0^2) I_N + \xi_0^2 H \right) U_- L_\varepsilon D_\varepsilon^{\frac{1}{2}} = I_N + P$$

and where P is the rank two symmetric matrix which results from applying the diagonalization operators to the second line on the right hand side of (208),

$$(213) \quad P = \frac{-2\xi_0^2}{1 + \xi_0^2} \sum_{i=1}^N \frac{\delta_{N+1-i}}{\sqrt{\varepsilon_N \varepsilon_{N-i+1}}} (e_1 \otimes e_i + e_i \otimes e_1) + \frac{(N+1)\xi_0^2 - 1}{\varepsilon_N(1 + \xi_0^2)} e_1 \otimes e_1.$$

Then, for some orthogonal matrix O , and $\lambda_+ \geq \lambda_-$,

$$(214) \quad O^t (I_N + P) O = (\lambda_+ e_1 \otimes e_1 + \lambda_- e_2 \otimes e_2) \oplus I_{N-2}.$$

By direct calculation, expanding by the top row, $\det(I_N + P)$ is equal to the $e_1 \otimes e_1$ entry plus the sum of the squares of the $e_1 \otimes e_i$ entries, $1 < i \leq N$,

$$(215) \quad \begin{aligned} \det(I_N + P) &= \lambda_+ \lambda_- = \left(1 - \frac{1}{\varepsilon_N(1 + \xi_0^2)} \right) + \frac{(N+1)\xi_0^2}{\varepsilon_N(1 + \xi_0^2)} \\ &\quad - \frac{4\xi_0^2}{1 + \xi_0^2} \frac{\delta_N}{\varepsilon_N} - \frac{4\xi_0^4}{(1 + \xi_0^2)^2 \varepsilon_N} \sum_{j=1}^{N-1} \frac{\delta_j^2}{\varepsilon_j} \\ &= \frac{\pi \xi \coth \pi \xi}{N} \left(1 + O\left(\frac{1 + \xi^2}{N}\right) \right). \end{aligned}$$

Since

$$(216) \quad \det(D_\varepsilon)^{-1} = (1 + \xi_0^2)^N \pi_N = \left(1 + O\left(\frac{1 + \xi^2}{N}\right) \right) \frac{N \sinh 2\pi \xi}{2\pi \xi},$$

$$(217) \quad \det((1 - \xi_0^2) I_N + \xi_0^2 H) = (\cosh \pi \xi)^2 \left(1 + O\left(\frac{1 + \xi^2}{N}\right) \right).$$

Now consider the general case in which $\alpha \neq 0$. Treat \underline{x} as N vectors in \mathbb{R}^2 . When $\text{SO}_2(\mathbb{R})$ acts diagonally on $(\mathbb{R}^2)^N$ rotating each x_i simultaneously, H^* and the Gaussian density are preserved. Thus, $I(\alpha, \xi; N) = I((0, \|\alpha\|)^t, \xi; N)$. Calculate

$$(218) \quad [1, 1, \dots, 1] U_- L_\varepsilon D_\varepsilon^{\frac{1}{2}} = \frac{e_1}{\sqrt{\varepsilon_N(1 + \xi_0^2)}}.$$

It follows that after making the change of coordinates $\underline{y}' =: U_1 L_\varepsilon D_\varepsilon^{\frac{1}{2}} \underline{y}$ the phase has magnitude $\frac{2\pi \|\alpha\|}{\sqrt{N \varepsilon_N(1 + \xi_0^2)}}$ and is now in the e_1 direction. Let v_+, v_- be unit vectors generating the eigenspaces λ_+, λ_- respectively. Since e_1 lies in the span of v_+, v_- it follows that

$$(219) \quad I(\alpha, \xi; N) = \exp \left(\frac{-2\pi^2 \|\alpha\|^2}{N \varepsilon_N(1 + \xi_0^2)} \left(\frac{\langle v_+, e_1 \rangle^2}{\lambda_+} + \frac{\langle v_-, e_1 \rangle^2}{\lambda_-} \right) \right) I(0, \xi; N).$$

Calculate

$$(220) \quad T = \lambda_+ + \lambda_- = 2 + e_1^t P e_1 = 1 + O\left(\frac{\xi^2}{N}\right),$$

so that

$$(221) \quad (\lambda_+, \lambda_-) = \left(1 + O\left(\frac{1 + \xi^2}{N}\right)\right) \left(1, \frac{\pi \xi \coth \pi \xi}{N}\right).$$

Also,

$$(222) \quad \langle v_+, e_1 \rangle^2 + \langle v_-, e_1 \rangle^2 = 1$$

$$\lambda_+ \langle v_+, e_1 \rangle^2 + \lambda_- \langle v_-, e_1 \rangle^2 = 1 + e_1^t P e_1 = O\left(\frac{1 + \xi^2}{N}\right),$$

so that

$$(223) \quad \langle v_+, e_1 \rangle^2 = O\left(\frac{1 + \xi^2}{N}\right), \quad \langle v_-, e_1 \rangle^2 = 1 + O\left(\frac{1 + \xi^2}{N}\right).$$

It follows that

$$(224) \quad \frac{\langle v_+, e_1 \rangle^2}{\lambda_+} + \frac{\langle v_-, e_1 \rangle^2}{\lambda_-} = \frac{N}{\pi \xi \coth \pi \xi} \left(1 + O\left(\frac{1 + \xi^2}{N}\right)\right).$$

In particular

$$(225) \quad I(\alpha, \xi; N) = \frac{\exp\left(\frac{-2\pi \|\alpha\|^2}{\xi \coth \pi \xi}\right)}{\cosh \pi \xi} \left(1 + O\left(\frac{(1 + \|\alpha\|^2)(1 + \xi^2)}{N}\right)\right).$$

Proof of Lemma 25. – Recall $\zeta = \frac{1 - \xi_0^2}{1 + \xi_0^2} \cdot \pi_n$ satisfies the recurrence

$$(226) \quad \pi_n = 2\pi_{n-1} - \zeta^2 \pi_{n-2}, \quad \pi_0 = 1, \pi_1 = 2.$$

The following closed forms hold,

$$(227) \quad \pi_n = \frac{(1 + \xi_0)^{2n+2} - (1 - \xi_0)^{2n+2}}{4\xi_0(1 + \xi_0^2)^n}$$

$$\varepsilon_n = 1 + \frac{2\xi_0}{1 + \xi_0^2} \frac{(1 + \xi_0)^{2n} + (1 - \xi_0)^{2n}}{(1 + \xi_0)^{2n} - (1 - \xi_0)^{2n}}$$

$$\delta_n = \frac{1}{2\xi_0} \frac{\left(\frac{1 + \xi_0}{1 - \xi_0}\right)^n + \left(\frac{1 - \xi_0}{1 + \xi_0}\right)^n - 2}{\left(\frac{1 + \xi_0}{1 - \xi_0}\right)^n - \left(\frac{1 - \xi_0}{1 + \xi_0}\right)^n} + \frac{1}{2}.$$

The formula for π_n is immediate from the recurrence relation, since

$$(228) \quad \frac{(1 + \xi_0)^2}{1 + \xi_0^2}, \quad \frac{(1 - \xi_0)^2}{1 + \xi_0^2}$$

are the two roots of $x^2 - 2x + \zeta^2 = 0$. The formula for ε_n follows from $\varepsilon_n = \frac{\pi_n}{\pi_{n-1}}$. The formula for δ_n is obtained on summing the geometric series

$$(229) \quad \delta_n = \frac{\zeta^{n-1}}{\pi_{n-1}} \sum_{j=0}^{n-1} \frac{\pi_j}{\zeta^j},$$

and use

$$(230) \quad \begin{aligned} \frac{\pi_n}{\xi^n} &= \frac{(1 + \xi_0)^{2n+2} - (1 - \xi_0)^{2n+2}}{4\xi_0(1 - \xi_0^2)^n} \\ &= \frac{1 - \xi_0^2}{4\xi_0} \left[\left(\frac{1 + \xi_0}{1 - \xi_0} \right)^{n+1} - \left(\frac{1 - \xi_0}{1 + \xi_0} \right)^{n+1} \right]. \end{aligned}$$

The claimed asymptotics for π , ε , δ are straightforward. For instance, to obtain the correct relative error in π_n , write

$$(231) \quad \frac{(1 + \xi_0)^{2n+2} - (1 - \xi_0)^{2n+2}}{4\xi_0} = \frac{(1 + \xi_0)^{2n+2} - (1 - \xi_0)^{2n+2}}{2((1 + \xi_0) - (1 - \xi_0))}$$

as a geometric series of positive terms. In each term of the series, approximate the power with an exponential with acceptable relative error, then sum the sequence of exponentials. The correct relative error may be obtained in the other cases similarly.

Using the exact formulae for ε and δ yields

$$(232) \quad \frac{\delta_j^2}{\varepsilon_j} = \left(1 + O\left(\frac{1}{j} + \frac{\xi^2}{N}\right) \right) \frac{N^2}{4\pi^2\xi^2} \tanh\left(\frac{j\pi\xi}{N}\right)^2.$$

Approximating with a Riemann sum,

$$(233) \quad \sum_{j=1}^{N-1} \frac{\delta_j^2}{\varepsilon_j} = \left(1 + O\left(\frac{1}{N} + \frac{\xi^2}{N}\right) \right) \frac{N^3}{4\pi^2\xi^2} \int_0^1 \tanh(t\pi\xi)^2 dt,$$

which gives the claimed estimate. □

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