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On Conditions for Rate-induced Tipping in Multi-dimensional Dynamical Systems

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Abstract
The possibility of *rate-induced tipping* (R-tipping) away from an attracting fixed point has been thoroughly explored in 1-dimensional systems. In these systems, it is impossible to have R-tipping away from a path of quasi-stable equilibria that is *forward basin stable* (FBS), but R-tipping is guaranteed for paths that are non-FBS of a certain type. We will investigate whether these results carry over to multi-dimensional systems. In particular, we will show that the same conditions guaranteeing R-tipping in 1-dimension also guarantee R-tipping in higher dimensions; however, it is possible to have R-tipping away from a path that is FBS even in 2-dimensional systems. We will propose a different condition, *forward inflowing stability* (FIS), which we show is sufficient to prevent R-tipping in all dimensions. The condition, while natural, is difficult to verify in concrete examples. *Monotone systems* are a class for which FIS is implied by an easily verifiable condition. As a result, we see how the additional structure of these systems makes predicting the possibility of R-tipping straightforward in a fashion similar to 1-dimension. In particular, we will prove that the FBS and FIS conditions in monotone systems reduce to comparing the relative positions of equilibria over time. An example of a monotone system is given that demonstrates how these ideas are applied to determine exactly when R-tipping is possible.

Keywords Tipping · Loss of stability · Transitions · Time-dependent parameters

1 Introduction

Tipping can be described as a sudden, drastic, irreversible change in the behavior of a solution as a result of a small change to the system. In part, tipping is interesting because it can be observed in nature. A recent example in the literature concerns the rise of temperature in peatlands (see [12]). There are different reasons that tipping can happen in a system; in particular Ashwin et al. [2] describes three types of tipping: bifurcation-, noise-, and rate-induced. This paper will focus on the third kind of tipping, which results from a fast parameter

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change in a dynamical system. It is the *rate* at which the parameter changes that causes the tipping, not simply the amount that it changes. For a thorough introduction into rate-induced tipping, the reader is encouraged to look in Ashwin et al. [3], but we will give a summary here that is sufficient for the rest of the paper.

Suppose we have an autonomous dynamical system

$$\dot{x} = f(x, \lambda) \tag{1.1}$$

where $x \in U$ for $U \subset \mathbb{R}^n$ open, $\lambda \in \mathbb{R}^m$, and $f \in C^2(U \times \mathbb{R}^m, \mathbb{R}^n)$. If we want to explore the possibility of rate-induced tipping in this system, we must allow the parameter λ to change over time. Without loss of generality, we may assume that $\lambda \in \mathbb{R}$ because if not, we can parametrize each component of λ with a different one-dimensional parameter. We want the parameter change to be bounded and sufficiently differentiable, so we choose the *parameter shift* $\Lambda \in \mathcal{P}(\lambda_-, \lambda_+)$ for some $\lambda_- < \lambda_+$ where

$$\mathcal{P}(\lambda_-, \lambda_+) = \left\{ \Lambda \in C^2(\mathbb{R}, (\lambda_-, \lambda_+)) : \lim_{s \to \pm\infty} \Lambda(s) = \lambda_\pm \text{ and } \lim_{s \to \pm\infty} \frac{d\Lambda}{ds} = 0 \right\}$$

and obtain a corresponding non-autonomous system

$$\dot{x} = f(x, \Lambda(rt)) \tag{1.2}$$

for some $r > 0$. The value r can be thought of as the *rate* at which Λ changes. When r is small, the parameter change is gradual, and when r is large, the parameter change is very sudden. We are interested in comparing the behavior of system (1.2) for different values of r.

Since we prefer to work with autonomous systems, we introduce the variable $s = rt$ and augment system (1.2) as

$$\begin{aligned} \dot{x} &= f(x, \Lambda(s)) \\ \dot{s} &= r. \end{aligned} \tag{1.3}$$

Notice that if $r = 0$, then (1.3) reduces to (1.1) where $\lambda = \Lambda(s)$.

Suppose that for all $s \in \mathbb{R}$, $X(s)$ is an attracting equilibrium for the corresponding autonomous system (1.1) with $\lambda = \Lambda(s)$ that depends continuously on s and

$$X_\pm = \lim_{s \to \pm\infty} X(s)$$

are also attracting equilibria for $\lambda = \lambda_\pm$. Then we say $(X(s), \Lambda(s))$ is a *stable path.*

As shown in Theorem 2.2 of Ashwin et al. [3], there is a unique trajectory $x^r(t)$ of (1.3) such that $x^r(t) \to X_-$ as $t \to -\infty$, which is the *local pullback attractor* to X_-. If $\lim_{t\to\infty} x^r(t) = X_+$, then we say that $x^r(t)$ *endpoint tracks* the stable path $(X(s), \Lambda(s))$. (Often we will just say *tracks* for short.) By Lemma 2.3 of Ashwin et al. [3], $x^r(t)$ endpoint tracks $(X(s), \Lambda(s))$ for all sufficiently small $r > 0$. However, if $x^r(t) \not\to X_+$ as $t \to \infty$, then $x^r(t)$ does not endpoint track $(X(s), \Lambda(s))$, and we say that *rate-induced tipping* (or *R-tipping*) has occurred.

This kind of tipping is sometimes called *irreversible* rate-induced tipping because it depends on the end behavior of the pullback attractor. This is different from *transient* rate-induced tipping (not discussed in this paper), in which the pullback attractor for some $r > 0$ may leave a neighborhood of the the stable path $X(s)$ during intermediate time values but then approach X_+ as $t \to \infty$. (The "compost-bomb instability" in Wieczorek et al. [12] is an example of transient rate-induced tipping.)

Our interest is in showing what kinds of parameter changes Λ can lead to R-tipping for some $r > 0$. Some results are already known for 1-dimensional systems ($n = 1$), and we will give these in Sect. 2. These results are phrased in the language of *forward basin stability* or *forward basin stable* paths (FBS), so we will focus on ways that FBS (or lack thereof) relates to R-tipping in multi-dimensional systems ($n > 1$).

In Sect. 3, we will give a constructive proof showing that R-tipping will happen in certain cases of no FBS, namely, if the position of a stable path $(X(s), \Lambda(s))$ at time s_1 is contained in the basin of attraction of a different stable path $(Y(s), \Lambda(s))$ at a later time s_2. We will look at the Lorenz '63 system and show how varying a parameter in a way that satisfies this condition leads to R-tipping there. In Sect. 4 we will give an example of a 2-dimensional system in which a path is FBS but the pullback attractor does not track it. In particular, this will show that FBS is not sufficient to prevent R-tipping in multi-dimensional systems. We will define a different condition, *forward inflowing stability* (FIS), which is sufficient to prevent R-tipping away from a stable path.

In Sect. 5 we will focus on R-tipping in monotone systems. We will be able to use the results from Sects. 3 and 4 to give conditions for guaranteeing or preventing R-tipping that rely only on the relative positions of the equilibria in the system. For this reason, we will see that monotone systems are ideal systems for thinking about R-tipping. In Sect. 6, we will show how the methods described in this paper give a nearly complete characterization of the possibilities of R-tipping in a particular 2-dimensional monotone system. Finally in Sect. 7 we will have some discussion about how the method of FIS could apply to a broader range of examples than those explicitly covered here.

2 R-Tipping in 1-Dimensional Systems

We begin by giving the definition of forward basin stability and stating a result from Ashwin et al. [3] about R-tipping in 1-dimensional systems (when $n = 1$) that we will reference in later sections. Unless explicitly stated, we will continue to use the notation from Sect. 1.

Definition 1 For $s \in \mathbb{R}$, let $\mathbb{B}(X(s), \Lambda(s))$ be the basin of attraction of the stable equilibrium $X(s)$ for the autonomous system (1.1) with $\lambda = \Lambda(s)$. A stable path $(X(s), \Lambda(s))$ is *forward basin stable* (FBS) if

$$\overline{\{X(u) : u < s\}} \subset \mathbb{B}(X(s), \Lambda(s)) \text{ for all } s \in \mathbb{R}.$$

The following theorem follows from Theorem 3.2 of Ashwin et al. [3] and its proof:

Theorem 1 *Suppose we have a system of the form (1.3) for $n = 1$. Let $(X(s), \Lambda(s))$ be a stable path. Set $X_\pm = \lim_{s\to\pm\infty} X(s)$.*

1. *If $(X(s), \Lambda(s))$ is FBS, there can be no R-tipping away from X_- for this Λ.*
2. *If there is another stable path $(Y(s), \Lambda(s))$ with $Y_+ = \lim_{s\to\infty} Y(s)$ such that $Y_+ \neq X_+$ and there are $u < v$ such that*

$$X(u) \in \mathbb{B}(Y(v), \Lambda(v)),$$

then $(X(s), \Lambda(s))$ is not FBS and there is a parameter shift $\tilde{\Lambda} \in \mathcal{P}(\lambda_-, \lambda_+)$ such that there is R-tipping away from X_- for this $\tilde{\Lambda}$ for some $r > 0$.

3. *If there is a* $Y_+ \neq X_+$ *such that* Y_+ *is an attracting equilibrium of* (1.1) *for* $\lambda = \lambda_+$ *and*

$$X_- \in \mathbb{B}(Y_+, \lambda_+),$$

then $(X(s), \Lambda(s))$ *is not FBS and there is R-tipping away from* X_- *for this* Λ *for all sufficiently large* $r > 0$.

In Sects. 3 and 4 we will see how the three parts of Theorem 1 do or do not generalize to systems with $n > 1$.

3 Conditions to Guarantee R-Tipping

In this section we will prove that statements 2 and 3 of Theorem 1 generalize to multi-dimensional systems. First, we must establish some lemmas that will be useful later. The proofs are given in "Appendix". In what follows, we will assume $(X(s), \Lambda(s))$ is a stable path of (1.3) with $X_\pm = \lim_{s\to\pm\infty} X(s)$.

This first lemma deals with the initial behavior of the pullback attractor to X_-.

Lemma 1 *Let* $x^r(t)$ *be the pullback attractor to* X_- *in* (1.3). *Given* $\epsilon > 0$, *there exists an* $S > 0$ *such that* $x^r(t) \in B_\epsilon(X_-)$ *when* $rt < -S$.

Next, we discuss the end behavior of a trajectory of (1.3). The purpose of Lemmas 2–5 is to show that if X_+ is an attracting fixed point of (1.1) with $\lambda = \lambda_+$ and $\mathbb{B}(X_+, \lambda_+)$ is its basin of attraction, then any trajectory $x(t)$ of (1.3) that is in a compact subset of $\mathbb{B}(X_+, \lambda_+)$ for large enough t will converge to X_+. In what follows, it will be helpful to distinguish between the flow of the augmented system (1.3) and the flow of the reduced systems (1.1) for different values of λ. So, we will use the notation

$$x \cdot_{\lambda'} t$$

to denote a trajectory of (1.1) with $\lambda = \lambda'$, while $x(t)$ will denote a trajectory in (1.2) or (1.3).

In autonomous systems, the *omega limit set* of a point x is defined to be $\omega(x) = \{y : x \cdot t_n \to y \text{ for some } t_n \to \infty\}$. Omega limit sets have the property that if $z \in \omega(y)$ and $y \in \omega(x)$, then $z \in \omega(x)$. (See Section 4.1 of [10].) This next lemma states that, in a certain sense, this property holds in non-autonomous systems like (1.2).

Lemma 2 *Suppose* $y \cdot_{\lambda_+} s_n \to z$ *for some* $\{s_n\} \to \infty$. *If* $x(t)$ *is a trajectory of* (1.2) *such that* $x(t_n) \to y$ *for some* $\{t_n\} \to \infty$, *then there exist* $\{u_n\} \to \infty$ *for which* $x(u_n) \to z$.

If p is an attracting fixed point of an autonomous system, there are arbitrarily small forward invariant neighborhoods of p. (This is follows from the Stable Manifold Theorem; see Theorem 2.1 in Chapter 10 of [9].) This next lemma states that a similar statement is true for X_+ in (1.3), where the forward invariant neighborhoods around X_+ extend both in the x- and s-dimensions.

Lemma 3 *For all sufficiently small* $\epsilon > 0$, *there exists an* $S > 0$ *such that if* $x(T) \in B_\epsilon(X_+)$ *for* $rT > S$, *then* $x(t) \in B_\epsilon(X_+)$ *for all* $t \geq T$.

If p is an attracting fixed point of an autonomous system, then by the Stable Manifold Theorem there is a neighborhood V of p such that all trajectories with initial conditions in V converge to p. This next lemma shows that a similar thing is true for X_+ in (1.3), where the attracting neighborhood around X_+ extends both in the x- and s-dimensions.

Lemma 4 *There exists an* $\epsilon > 0$ *and an* $S > 0$ *such that if* $|x(t) - X_+| < \epsilon$ *for* $rt > S$*, then* $x(t) \to X_+$ *as* $t \to \infty$.

Using the preceding lemmas we can conclude the following:

Lemma 5 *Let* $K \subset \mathbb{B}(X_+, \lambda_+)$ *be compact. Then there exists an* $S > 0$ *such that if* $x(T) \in K$ *for* $rT > S$*, then* $x(t) \to X_+$ *as* $t \to \infty$.

Now we are ready to prove the generalization of statements 2 and 3 of Theorem 1:

Theorem 2 *Suppose we have a system of the form* (1.3) *for any* $n \in \mathbb{N}$*. Let* $(X(s), \Lambda(s))$ *be a stable path. Set* $X_\pm = \lim_{s\to\pm\infty} X(s)$.

1. *If there is another stable path* $(Y(s), \Lambda(s))$ *with* $Y_\pm = \lim_{s\to\pm\infty} Y(s)$ *such that* $Y_+ \neq X_+$ *and there are* $u < v$ *such that*
$$X(u) \in \mathbb{B}(Y(v), \Lambda(v)),$$
then $(X(s), \Lambda(s))$ *is not FBS and there is a parameter shift* $\tilde{\Lambda} \in \mathcal{P}(\lambda_-, \lambda_+)$ *such that there is R-tipping away from* X_- *for this* $\tilde{\Lambda}$ *for some* $r > 0$.
2. *If there is a* $Y_+ \neq X_+$ *such that* Y_+ *is an attracting equilibrium of* (1.1) *for* $\lambda = \lambda_+$ *and*
$$X_- \in \mathbb{B}(Y_+, \lambda_+),$$
then $(X(s), \Lambda(s))$ *is not FBS and there is R-tipping away from* X_- *for this* Λ *for all sufficiently large* $r > 0$.

Proof We will prove statement 1 first. Based on the assumptions, it is clear that $(X(s), \Lambda(s))$ is not forward basin stable. Pick $\epsilon > 0$ such that $K = \overline{B_\epsilon(X(u))} \subset \mathbb{B}(Y(v), \Lambda(v))$. By Lemma 2.3 of Ashwin et al. [3], there is an $r_0 > 0$ such that for all $r \in (0, r_0)$, $|x^r(s/r) - X(s)| < \epsilon/2$ for all $s \in \mathbb{R}$. Likewise, there exists an $r_1 > 0$ such that for all $r \in (0, r_1)$, if $x^r(v/r) \in K$, then $x^r(t) \to Y_+$ as $t \to \infty$. Now fix $r \in (0, \min\{r_0, r_1\})$.

Following the proof of Theorem 3.2 in Ashwin et al. [3], we will construct a reparametrization
$$\tilde{\Lambda}(s) := \Lambda(\sigma(s))$$
using a monotonic increasing $\sigma \in C^2(\mathbb{R}, \mathbb{R})$ that increases rapidly from $\sigma(s) = u$ to $\sigma(s) = v$ but increases slowly otherwise. In particular, for any $M > 1$ and $\eta > 0$ we choose a smooth monotonic function $\sigma(s)$ such that
$$\begin{aligned} \sigma(s) = s \quad &\text{for} \quad s < u \\ 1 \le \tfrac{d}{ds}\sigma(s) \le M \quad &\text{for} \quad u \le \sigma(s) \le u + \eta \\ \tfrac{d}{ds}\sigma(s) = M \quad &\text{for} \quad u + \eta < \sigma(s) < v - \eta \\ 1 \le \tfrac{d}{ds}\sigma(s) \le M \quad &\text{for} \quad v - \eta \le \sigma(s) \le v, \text{ and} \\ \tfrac{d}{ds}\sigma(s) = 1 \quad &\text{for} \quad \sigma(s) > v \end{aligned} \tag{3.1}$$

Let $x^{[r,\tilde{\Lambda}]}(t)$ denote the pullback attractor to (X_-, λ_-) with parameter change $\tilde{\Lambda}$. By construction, we know that $x^{[r,\tilde{\Lambda}]}(u/r) \in B_{\epsilon/2}(X(u))$. By choosing $M > 1$ sufficiently large and $\eta > 0$ sufficiently small, we can guarantee that $x^{[r,\tilde{\Lambda}]}(v/r) \in B_\epsilon(X(u)) \subset K$. This guarantees that $x^{[r,\tilde{\Lambda}]}(t) \to Y_+$ as $t \to \infty$.

Now we will prove statement 2. Pick $\epsilon > 0$ such that $B_{3\epsilon}(X_-) \subset \mathbb{B}(Y_+, \lambda_+)$. By Lemma 1, there is an $S_1 > 0$ such that the pullback attractor $x^r(t)$ to X_- satisfies $x^r(t) \in B_\epsilon(X_-)$ if $rt < -S_1$. By Lemma 5, there is some $S_2 > 0$ such that if $x^r(t) \in \overline{B_{2\epsilon}(X_-)}$ for $rt > S_2$ then $x^r(t) \to Y_+$ as $t \to \infty$. Take $S = \max\{S_1, S_2\}$.

By continuity, there is some $M > 0$ such that $|f(x, \lambda)| < M$ for all $x \in \overline{B_{2\epsilon}(X_-)}$ and $\lambda \in [\lambda_-, \lambda_+]$. Pick any

$$r > 2\frac{MS}{\epsilon}.$$

Suppose for the sake of contradiction that $x^r(S/r) \notin B_{2\epsilon}(X_-)$. We know $x^r(-S/r) \in B_\epsilon(X_-)$, so let $s' = \inf\{s \in (-S, S] : x^r(s/r) \notin B_{2\epsilon}(X_-)\}$. Then in fact s' is a minimum and $s' > -S$. By the Mean Value Theorem, there is some $s^* \in (-S, s')$ and $\lambda \in [\lambda_-, \lambda_+]$ such that

$$\begin{aligned} |x^r(-S/r) - x^r(s'/r)| &= |f(x^r(s^*/r), \lambda)| \left| -\frac{S}{r} - \frac{s'}{r} \right| \\ &< M\frac{2S}{r} \\ &< \epsilon \end{aligned}$$

Because $x^r(-S/r) \in B_\epsilon(X_-)$, this implies that $x^r(s'/r) \in B_{2\epsilon}(X_-)$, which is a contradiction. Therefore, $x^r(S/r) \in B_{2\epsilon}(X_-)$. As shown above, this implies that $x^r(t) \to Y_+$ as $t \to \infty$. Hence, there is R-tipping away from X_- for all sufficiently large $r > 0$. □

Example 1 We can apply Theorem 2 to the Lorenz equations:

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z \end{aligned} \tag{3.2}$$

As in Sparrow [11], we will fix $\sigma = 10$ and $\beta = 8/3$, but we will allow ρ to vary with time. The corresponding augmented system for (3.2) is

$$\begin{aligned} \dot{x} &= 10(y - x) \\ \dot{y} &= x(\Lambda(s) - z) - y \\ \dot{z} &= xy - \frac{8}{3}z \\ \dot{s} &= r \end{aligned}$$

for $r > 0$ and $\Lambda \in \mathcal{P}(15, 23)$. We will allow ρ to monotonically increase from 15 to 23, so $\lambda_- = 15$ and $\lambda_+ = 23$. As explained in Doedel et al. [4] and Sparrow [11], in this parameter regime there are three equilibria, one at the origin and the other two

$$\begin{aligned} C_1 &= \left(\sqrt{8/3(\rho - 1)}, \sqrt{8/3(\rho - 1)}, \rho - 1\right) \\ C_2 &= \left(-\sqrt{8/3(\rho - 1)}, -\sqrt{8/3(\rho - 1)}, \rho - 1\right). \end{aligned}$$

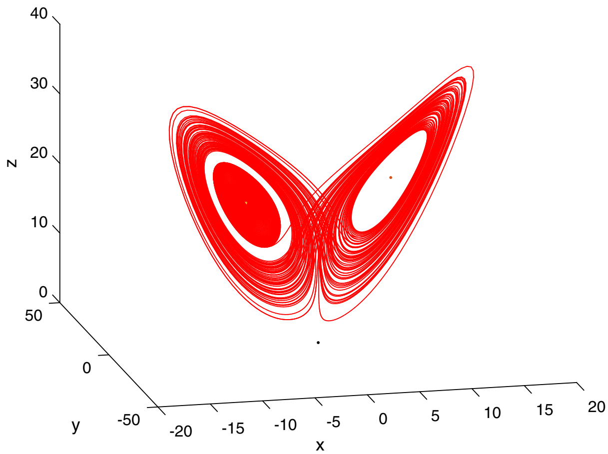


Fig. 1 Approximate solution curve to (3.2) with $\rho = 22.9$ and initial condition $\left(\sqrt{8/3(14.1)}, \sqrt{8/3(14.1)}, 14.1\right)$ (i.e. $C_1(s)$ when $\Lambda(s) = 15.1$). The trajectory converges to a point on the stable path $(C_2(s), \Lambda(s))$, indicating that $(C_1(s), \Lambda(s))$ is not forward basin stable

Both $C_{1,2}$ are attracting, and the origin is a saddle point. There are heteroclinic connections from the origin to $C_{1,2}$, and there are periodic orbits around $C_{1,2}$. There is no chaotic attractor for these values of ρ, although as $\rho \nearrow \rho_{het} \approx 24.0579$, the time it takes for the unstable manifold at the origin to approach $C_{1,2}$ increases without bound.

We will focus on the stable path

$$C_1(s) = \left(\sqrt{8/3(\Lambda(s) - 1)}, \sqrt{8/3(\Lambda(s) - 1)}, \Lambda(s) - 1\right)$$

with $C_{1\pm} = \lim_{s\to\pm\infty} C_1(s)$ and consider the possibility of R-tipping away from (C_{1-}, λ_-). From plotting solutions to (3.2) in MATLAB, we see that $(C_1(s), \Lambda(s))$ is not FBS (see Fig. 1). Therefore, according to Theorem 2, we can expect R-tipping for some choices of Λ and $r > 0$. Indeed if we choose

$$\Lambda(s) = 4\tanh(s) + 19$$

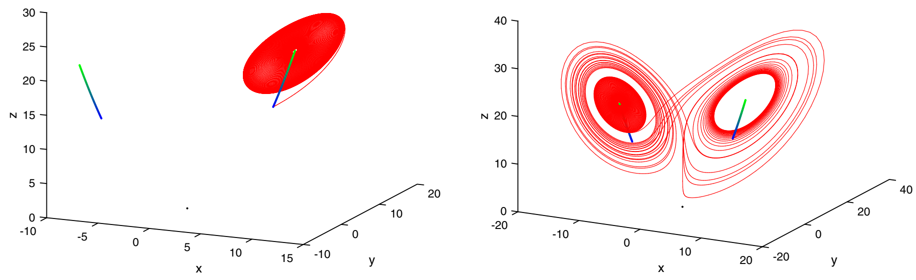
then for some values of $r > 0$ the pullback attractor to (C_{1-}, λ_-) tracks $(C_1(s), \Lambda(s))$ and for some values of $r > 0$ it tips to $(C_2(s), \Lambda(s))$ (see Fig. 2).

4 Forward Basin Stability and Forward Inflowing Stability

Now that we have successfully generalized statements 2 and 3 of Theorem 1, we will turn our attention to statement 1, which says that if a path is FBS in a 1-dimensional system, then there will be no R-tipping away from that path. However, as the next example shows, FBS is not enough to prevent R-tipping in systems where $n > 1$.

Example 2 Consider the following 2-dimensional system (which we have adapted from Example 5.11 of [10]):

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= -(x-\lambda) + 2(x-\lambda)^3 - y((x-\lambda)^2 - (x-\lambda)^4 - y^2) \end{aligned} \tag{4.1}$$



(a) $r = 13$ **(b)** $r = 15$

Fig. 2 In both figures, the blue/green dots mark the positions of $(C_{1,2}(s), \Lambda(s))$. The blue dots correspond to small values of s, and the green dots correspond to large values of s. The red curve is the pullback attractor to (C_{1-}, λ_-). When $r = 13$, the trajectory endpoint tracks $(C_1(s), \Lambda(s))$, but when $r = 15$ it does not (Color figure online)

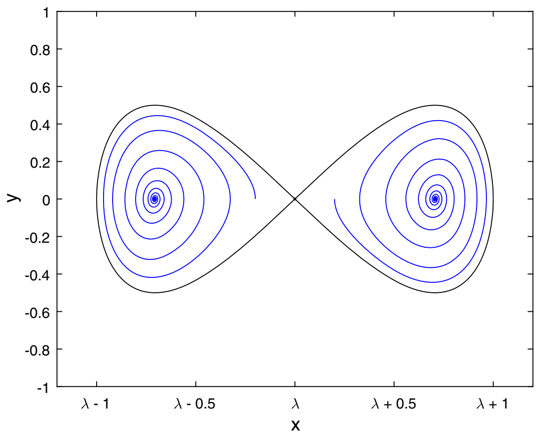


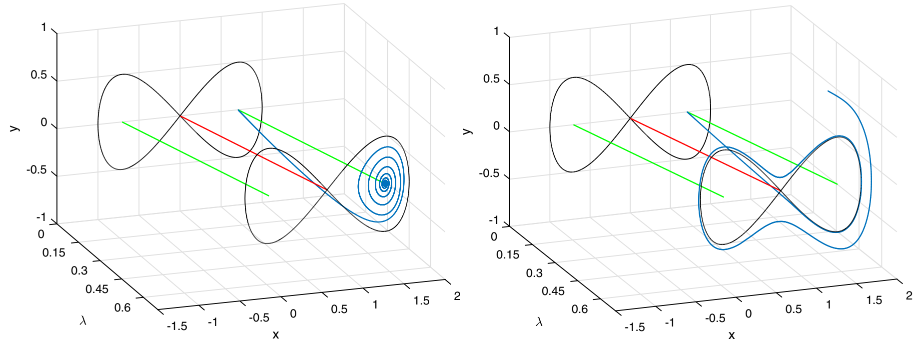
Fig. 3 Phase portrait for system (4.1). $(\lambda, 0)$ is a saddle point with two homoclinic orbits (shown in black). Both $\left(\lambda \pm \frac{1}{\sqrt{2}}, 0\right)$ are attracting equilibria; their basins of attraction are the regions inside the homoclinic loops

Then (4.1) has fixed points at $(\lambda, 0)$ and $\left(\lambda \pm \frac{1}{\sqrt{2}}, 0\right)$. There are two homoclinic orbits at $(\lambda, 0)$ defined by the curves $y = \pm\sqrt{(x-\lambda)^2 - (x-\lambda)^4}$. Both $\left(\lambda \pm \frac{1}{\sqrt{2}}, 0\right)$ are attracting, and their basins of attraction are the regions inside the corresponding homoclinic orbits. See Fig. 3.

Then we will let λ change with time at a rate $r > 0$ by setting $\lambda = \Lambda(s)$ and $s = rt$:

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= -(x - \Lambda(s)) + 2(x - \Lambda(s))^3 - y((x - \Lambda(s))^2 - (x - \Lambda(s))^4 - y^2) \\ \dot{s} &= r \end{aligned} \tag{4.2}$$

For Λ we will take $\Lambda(s) = \frac{13}{40}(1 + \tanh(s))$ so that $\lambda_- = 0$ and $\lambda_+ = 0.65 < \frac{1}{\sqrt{2}}$. Let $X(s) = \left(\Lambda(s) + \frac{1}{\sqrt{2}}, 0\right)$. Then $X_- = \left(\frac{1}{\sqrt{2}}, 0\right)$ and $X_+ = \left(\frac{1}{\sqrt{2}} + \frac{13}{20}, 0\right)$. Because $0 < \Lambda(s) < \frac{1}{\sqrt{2}}$ for all s, the stable path $(X(s), \Lambda(s))$ is FBS. Nevertheless, R-tipping can occur away from X_-. See Fig. 4.



(a) $r = 1$ **(b)** $r = 5$

Fig. 4 In both figures, the green lines represent the stable paths $\left(-1/\sqrt{2} + \Lambda(s), 0\right)$ and $X(s) = \left(1/\sqrt{2} + \Lambda(s), 0\right)$, and the red line represents the unstable path $(\Lambda(s), 0)$. The black loops show the positions of the homoclinic orbits in (4.1) when $\lambda = 0$ and 0.65. The blue curve is the pullback attractor to X_-. When $r = 1$, the pullback attractor endpoint tracks $X(s)$. When $r = 5$, and the pullback attractor diverges to infinity (does not endpoint track $X(s)$). Therefore, R-tipping has occurred. This shows that R-tipping can occur even when a path is forward basin stable in multi-dimensional systems (Color figure online)

Example 2 shows that FBS is not enough to guarantee against R-tipping in 2-dimensional systems. The reason that FBS is not sufficient in a 2-dimensional (or higher) system is that a point x might be in the basin of attraction of a fixed point p, but the velocity vector at x may not point toward p. The more dimensions there are in a system, the more directions there are to move, so in a sense this makes R-tipping more likely to happen. Although Example 2 is an example of a 2-dimensional system, it would not be difficult to construct a system of higher dimension in which there can be R-tipping away from a path that is FBS.

Therefore, since FBS is not enough to prevent R-tipping in systems of dimension greater than 1, we want to find a different condition that is sufficient to prevent R-tipping. We propose a condition called *forward inflowing stability* (FIS) which guarantees that R-tipping cannot happen away from a stable path. In what follows, we will assume that we have a system of the form (1.3) with a stable path $(X(s), \Lambda(s))$.

Definition 2 We say the stable path $(X(s), \Lambda(s))$ is *forward inflowing stable* if for each $s \in \mathbb{R}$ there is a compact set $K(s)$ such that

1. $X(s) \in \text{Int}\, K(s)$ for all $s \in \mathbb{R}$;
2. if $s_1 < s_2$, then $K(s_1) \subset K(s_2)$;
3. if $x \in \partial K(s)$, then $\exists t_0 > 0$ such that $x \cdot_{\Lambda(s)} t \in \text{Int}\, \overline{K(s)}$ for all $t \in (0, t_0)$;
4. $X_\pm \in \text{Int}\, K_\pm$ where $K_- = \bigcap_{s\in\mathbb{R}} K(s)$ and $K_+ = \overline{\bigcup_{s\in\mathbb{R}} K(s)}$; and
5. $K_+ \subset \mathbb{B}(X_+, \lambda_+)$ is compact.

Just as the notion of FBS compares the positions of equilibria along a path to basins of attraction later on in the path, FIS compares the positions of equilibria along the path to forward invariant sets (sets for which solutions "flow in") later on down the path.

Proposition 1 *If the stable path $(X(s), \Lambda(s))$ is FIS, then there is no R-tipping away from X_- for this Λ.*

Proof Fix $r > 0$. By FIS, there exist sets $K(s)$ satisfying the requirements of Definition 2. Set $K = \cup_{s\in\mathbb{R}} K(s) \times \{s\}$. If we pick a point x on the boundary of K when $s = s_0$, then there

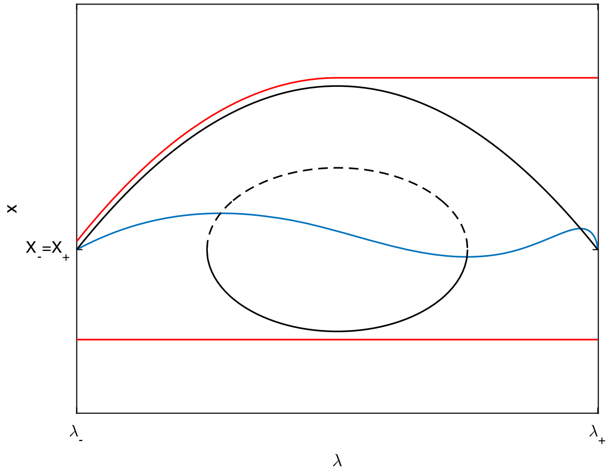


Fig. 5 Here there three paths: two stable (solid black lines) and one unstable (dashed black line). Let $(X(s), \Lambda(s))$ be the stable path that is defined for all λ-values. The red curves define $K(s)$ for each s if $\Lambda(s)$ is one-to-one. The pullback attractor for (X_-, λ_-) is shown in blue. Notice that the pullback attractor is fully contained in $K = \cup_{s\in\mathbb{R}} K(s) \times \{s\}$ and hence endpoint tracks $(X(s), \Lambda(s))$ (Color figure online)

exists a $t_0 > 0$ such that $x \cdot_{\Lambda(s_0)} t \in \text{Int } K(s_0)$ for all $t \in (0, t_0)$. Since $K(s_0) \subset K(s)$ if $s_0 < s$ and $\frac{ds}{dt} = r > 0$, there is some $t_1 > 0$ such that $x(t) \in \text{Int } K$ for all $t \in (0, t_1)$, where $x(t)$ is the solution through x in (1.3). Therefore, K is forward invariant under the flow of (1.3).

Let $x^r(t)$ be the pullback attractor to X_-. Because $X_- \in \text{Int } K_-$ and $K_- = \bigcap_{s\in\mathbb{R}} K(s)$, there is a $T \in \mathbb{R}$ such that $x^r(t) \in K(rt)$ for all $t < T$. Since K is forward invariant, this implies that $x^r(t) \in K(rt)$ for all $t \in \mathbb{R}$. In particular, $x^r(t) \in K_+$ for all $t \in \mathbb{R}$.

We know $K_+ \subset \mathbb{B}(X_+, \lambda_+)$ is compact. By Lemma 5 this implies $x^r(t) \to (X_+, \lambda_+)$ as $t \to \infty$. Therefore, $x^r(t)$ endpoint tracks the stable branch $(X(s), \Lambda(s))$ regardless of $r > 0$, so there is no R-tipping. □

Example 3 Consider Fig. 5. We will assume that $\Lambda(s)$ is injective, so that s and λ are in one-to-one correspondence. Let $(X(s), \Lambda(s))$ be the stable path that is defined for all λ-values. The red curves specify a choice of $K(s)$ in the following way: $K(s)$ is the closed interval between the two red curves when $\lambda = \Lambda(s)$. Based on what is shown in the figure, $\{K(s)\}$ satisfies the requirements in Definition 2, which shows that $(X(s), \Lambda(s))$ is FIS. The set $K = \cup_{s\in\mathbb{R}} K(s) \times \{s\}$ forms a forward invariant "tube" around the stable path $(X(s), \Lambda(s))$. As shown in Proposition 1, the pullback attractor for X_- is always contained in K. There can be no R-tipping away from X_- for this reason.

In general, FBS and FIS are conditions that are independent of each other. The path in Example 3 is not FBS but is FIS. Hence, FIS does not imply FBS. Likewise, FBS does not imply FIS, as shown in Fig. 6. Also note that in multi-dimensional systems FBS cannot imply FIS, as FIS prevents R-tipping, but FBS does not.

5 Monotone Systems

We will now focus our attention on rate-induced tipping in a special class of systems called *monotone systems*. The benefit of monotone systems is that their extra structure enables us

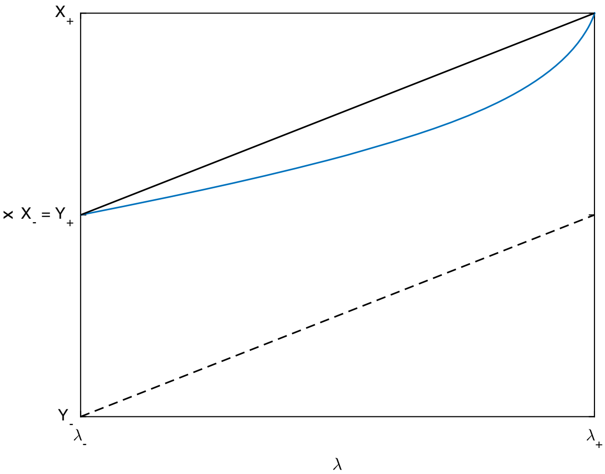


Fig. 6 The solid black curve is the stable path $(X(s), \Lambda(s))$, and the dashed black curve is an unstable path $(Y(s), \Lambda(s))$ satisfying $X_- = Y_+$. The blue curve is the pullback attractor to X_-. Assuming that $\Lambda(s)$ is one-to-one, $(X(s), \Lambda(s))$ is forward basin stable. However, it is not forward inflowing stable, since X_- is on the boundary of $\mathbb{B}(X_+, \lambda_+)$. Any possible choice of K_- must contain a neighborhood of X_-. Since $K_- \subset K_+$, K_+ cannot be fully contained in $\mathbb{B}(X_+, \lambda_+)$ (Color figure online)

in Proposition 2 to prove when rate-induced tipping can happen without having to calculate the basins of attraction of the equilibria (which can be chaotic in systems of dimension 3 or more, such as in Lorenz '63–see [4]). Likewise, in Proposition 3 we will be able to prove when rate-induced tipping cannot happen, using a simpler condition than inflowing stability.

In the context of functions from $\mathbb{R}$ to $\mathbb{R}$, monotonicity refers to the preserving (or the reversing) of the ordering on the real numbers. To extend this notion to systems of dimension higher than 1, we need to define an ordering on multi-dimensional spaces. We will use the following partial ordering on $\mathbb{R}^n$:

Suppose $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Then

$$\begin{aligned} x \leq y &\iff x_i \leq y_i \text{ for all } i \\ x < y &\iff x \leq y \text{ and } x \neq y \\ x \ll y &\iff x_i < y_i \text{ for all } i. \end{aligned}$$

If $K, L \subset \mathbb{R}^n$ are sets, we will write $K \ll L$ to mean $x \ll y$ for all $x \in K$ and $y \in L$. Furthermore, a set K is *p-convex* if for every $x, y \in K$ satisfying $x \leq y$, the line segment joining them also belongs to K.

Intuitively, a monotone system is a dynamical system that preserves this partial ordering. More specifically, a system of the form

$$\dot{x} = f(x) \tag{5.1}$$

for $f : U \to \mathbb{R}^n$, $U \subset R^n$ open, is *monotone* if $x \cdot t \leq y \cdot t$ whenever $x \leq y \in \mathbb{R}^n$ and $t \geq 0$. Section 3 of Hirsch and Smith [7] demonstrates that if $f \in C^1$ and U is p-convex, system (5.1) is monotone if and only if

$$\frac{\partial f_i}{\partial x_j} \geq 0 \; \forall i \neq j. \tag{5.2}$$

Then we have the following result about forward invariant sets in monotone systems which we will use later to prove a result about R-tipping in monotone systems:

Lemma 6 *Suppose* (5.1) *is a monotone system where* $f \in C^1$ *and* U *is p-convex. For any* $p \in U$*, define* $K_1(p) = \{x \in U : x \leq p\}$ *and* $K_2(p) = \{x \in U : x \geq p\}$*.*

1. *If* $f_i(p) < 0$ *for all* i*, then the vector field* f *points into* K_1 *on the boundary of* K_1*, so* K_1 *is forward invariant.*
2. *If* $f_i(p) > 0$ *for all* i*, then the vector field* f *points into* K_2 *on the boundary of* K_2*, so* K_2 *is forward invariant.*

Proof By the assumptions on system (5.1), we know that $\frac{\partial f_i}{\partial x_j} \geq 0$ for all $i \neq j$.

The result is trivial if $n = 1$, so we will assume $n \geq 2$. We will prove the statement for K_2; the proof for K_1 is similar.

Assume $f_i(p) > 0$ for all i. Pick any point $y = (y_1, \ldots, y_n)$ not equal to p on the boundary of K_2. The line ℓ between y and p can be parametrized by t in the following way:

$$\ell(t) = ((y_1 - p_1)t + p_1, (y_2 - p_2)t + p_2, \ldots, (y_n - p_n)t + p_n).$$

We need to show that $\dot{x}_i > 0$ at y, or $f_i(\ell(1)) > 0$, for all i such that $y_i = p_i$. We know that $f_i(\ell(0)) > 0$, so it will suffice to show that $(f_i \circ \ell)'(t) \geq 0$. In general, we have

$$\begin{aligned}(f_i \circ \ell)'(t) &= Df_i(\ell(t)) \cdot \ell'(t) \\ &= \left(\frac{\partial f_i}{\partial x_1}, \frac{\partial f_i}{\partial x_2}, \ldots, \frac{\partial f_i}{\partial x_n}\right)(\ell(t)) \cdot (y_1 - p_1, y_2 - p_2, \ldots, y_n - p_n) \\ &\geq 0\end{aligned}$$

because each $\frac{\partial f_i}{\partial x_j}(\ell(t)) \cdot (y_j - p_j) \geq 0$. □

Using Lemma 6 we can establish the following result about R-tipping in monotone systems:

Proposition 2 *Suppose we have a system of the form* (1.3) *where* $\dot{x} = f(x, \lambda)$ *is a monotone system on* U *(*U *is p-convex) for each* $\lambda \in [\lambda_-, \lambda_+]$*. Let* $(p(s), \Lambda(s))$ *be a path and* $(q(s), \Lambda(s))$ *a stable path; denote* $p_\pm = \lim_{s\to\pm\infty} p(s)$ *and* $q_\pm = \lim_{s\to\pm\infty} q(s)$*. Suppose for all* s *(including in the limits)* $Df(p(s))$ *has a positive eigenvalue whose associated eigenvector has all positive components.*

1. *If* $q(s) \ll p(s)$ *(resp.* $q(s) \gg p(s)$*) for all* $s \in \mathbb{R}$*, including in the limits as* $s \to \pm\infty$*, and there is a* $u < v$ *such that* $q(u) \gg p(v)$ *(resp.* $q(u) \ll p(v)$*), then there is a parameter shift* $\tilde{\Lambda}$ *such that there will be R-tipping away from* q_- *for this* $\tilde{\Lambda}$ *for some* $r > 0$*.*
2. *If* $q(s) \ll p(s)$ *(resp.* $q(s) \gg p(s)$*) for all* $s \in \mathbb{R}$*, including in the limits as* $s \to \pm\infty$*, and* $q_- \gg p_+$ *(resp.* $q_- \ll p_+$*), then there will be R-tipping away from* q_- *for this* Λ *for all sufficiently large* $r > 0$*.*

Proof We will prove statement 1. The proof of statement 2 is similar but does not require any reparametrization. Suppose $q(s) \ll p(s)$ for all $s \in \mathbb{R}$ including in the limits and that $q(u) \gg p(v)$. Let $x^r(t)$ denote the pullback attractor to (q_-, λ_-).

Pick $\epsilon > 0$ such that $\overline{B_\epsilon(q(u))} \gg \{p(v)\}$. By Lemma 2.3 of Ashwin et al. [3], there is an $r_0 > 0$ such that for all $0 < r < r_0$, $|x^r(s/r) - q(s)| < \epsilon/2$ for all $s \in \mathbb{R}$.

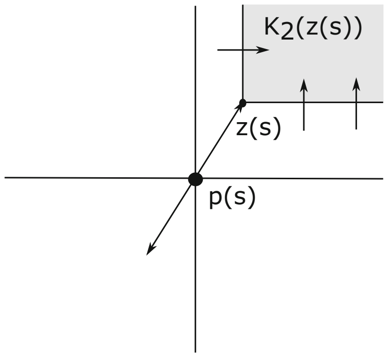


Fig. 7 If $Df(p(s))$ has a positive real eigenvalue whose associated eigenvector has all positive components, then there is a point $z(s)$ such that $\{z(s) \cdot_{\Lambda(s)} t\}_{t\leq 0} \gg \{p(s)\}$ and $f_i(z(s), \Lambda(s)) > 0$ for all i. Using $z(s)$, we can define the box $K_2(z(s))$ as in Proposition 2 satisfying $K_2(z(s)) \gg \{p(s)\}$ and such that the flow of (1.1) with $\lambda = \Lambda(s)$ is pointing in on all sides along the boundary of $K_2(z(s))$

For any $s \in \mathbb{R}$, since $Df(p(s))$ has a positive real eigenvalue with an eigenvector that has all positive components, by the Invariant Manifold Theorem (which is called "The Stable Manifold Theorem" in [8]) there is a point $z(s)$ such that $\{z(s) \cdot_{\Lambda(s)} t\}_{t\leq 0} \gg \{p(s)\}$ and $f_i(z(s), \Lambda(s)) > 0$ for all i. Then define $K_2(z(s)) = \{x \in U : x \geq z(s)\}$. By Lemma 6, the vector field $f(x, \Lambda(s))$ is pointing in on all sides along the boundary of $K_2(z(s))$. See Fig. 7 for an illustration.

Because $z(s)$ can be chosen to be arbitrarily close to $p(s)$, let us also say that $z(v)$ satisfies $\{z(v)\} \ll \overline{B_\epsilon(q(u))}$ and that $z(s)$ varies continuously in s.

Because the system converges as $s \to \infty$, there is an $S_0 > v$ such that the flow of the autonomous system (1.1) with $\lambda = \Lambda(s)$ points in along the boundary of $K_2(z(S_0))$ for every $s \geq S_0$ and $q_+ \notin K_2(z(S_0))$. Then $K_2(z(S_0)) \times [S_0, \infty)$ is forward invariant under the flow of (1.3) for any $r > 0$. Additionally, we can choose $r_1 > 0$ sufficiently small so that

$$\bigcup_{s\in[v,S_0]} K_2(z(s)) \times \{s\}$$

is forward invariant under the flow of (1.3). Now fix $r \in (0, \min\{r_0, r_1\})$.

As in the proof of Theorem 2, we can construct a reparametrization

$$\tilde{\Lambda}(s) := \Lambda(\sigma(s))$$

using a smooth monotonic increasing $\sigma \in C^2(\mathbb{R}, \mathbb{R})$ that increases rapidly from $\sigma(s) = u$ to $\sigma(s) = v$ but increases slowly otherwise. In particular, for any $M > 1$ and $\eta > 0$ we choose a smooth monotonic function $\sigma(s)$ that satisfies (3.1).

Let $x^{[r,\tilde{\Lambda}]}(t)$ denote the pullback attractor to (X_-, λ_-) with parameter change $\tilde{\Lambda}$. By construction, we know that $x^{[r,\tilde{\Lambda}]}(u/r) \in B_{\epsilon/2}(q(u))$. By choosing $M > 1$ sufficiently large and $\eta > 0$ sufficiently small, we can guarantee that $x^{[r,\tilde{\Lambda}]}(v/r) \in B_\epsilon(q(u)) \subset K_2(z(v))$. This implies that $x^{[r,\tilde{\Lambda}]}(t) \in K_2(z(S_0))$ for all sufficiently large t and therefore $x^{[r,\tilde{\Lambda}]}(t) \not\to q_+$ as $t \to \infty$. □

The power of Proposition 2 is in being able to determine when R-tipping will happen without having to know exactly where the basins of attraction are. Since the systems in question are monotone, it suffices to check the relative positions of different equilibria. In this sense, checking for the possibility of R-tipping in monotone systems is much like looking for R-tipping in 1-dimensional systems because in 1 dimension, the basins of attraction are completely determined by the positions of the equilibria. (And in fact all 1-dimensional systems are monotone.)

Now we will see how the idea of forward inflowing stability can be applied to monotone systems to show that there will not be rate-induced tipping. As we have already seen, in multi-dimensional systems there are many different "directions" in which a trajectory can tip. It will be useful for the next result if we narrow our focus from tipping in general to tipping in a particular direction. We make the following definition:

Definition 3 Let $(q(s), \Lambda(s))$ be a stable path in system (1.3), and let $x^r(t)$ denote the pullback attractor to q_-. Let $L \subset U$ be closed and $q_+ \notin L$. We say that $x^r(t)$ *tips to* L if there is some $r > 0$ and $T > 0$ such that $x^r(t) \in L$ for all $t \geq T$.

Proposition 3 *Suppose we have a system of the form* (1.3) *where* $\dot{x} = f(x, \lambda)$ *is a monotone system on* U *(*U *is p-convex) for each* $\lambda \in [\lambda_-, \lambda_+]$*. Let* $(p(s), \Lambda(s))$ *be a path and* $(q(s), \Lambda(s))$ *be a stable path; denote* $p_\pm = \lim_{s\to\pm\infty} p(s)$ *and* $q_\pm = \lim_{s\to\pm\infty} q(s)$*. Suppose for all* $s \in \mathbb{R}$ *(including in the limits)* $Df(p(s))$ *has a positive eigenvalue whose associated eigenvector has all positive components. If*

$$q(s_1) \ll p(s_2) \text{ (resp. } q(s_1) \gg p(s_2))$$

for all $s_1 \leq s_2$ *(including in the limits as* $s_1 \to -\infty$ *and* $s_2 \to \infty$*) then there cannot be R-tipping away from* q_- *to* $\{x : x \geq p_+\}$ *(resp. to* $\{x : x \leq p_+\}$*) for this* Λ*.*

Proof We will assume that $q(s_1) \ll p(s_2)$ for all $s_1 \leq s_2$ and prove the corresponding result. The proof of the other result is similar.

Because each $Df(p(s))$ has a positive real eigenvalue whose associated eigenvector has all positive components, by the Invariant Manifold Theorem there is a $z(s) \ll p(s)$ such that $z(s) \cdot_{\Lambda(s)} t \to p(s)$ as $t \to -\infty$ and $f_i(z(s), \Lambda(s)) < 0$ for all i. By changing $z(s)$ if necessary, we also can guarantee that $q(s) \ll z(s) \ll p(s)$ for all $s \in \mathbb{R}$, including in the limits and that $z(s_1) \leq z(s_2)$ for all $s_1 \leq s_2$.

Now define $K(s) = \{x \in U : x \leq z(s)\}$ for all s, including in the limits. Then the $\{K(s)\}$ satisfy all the conditions in Definition 2 except they are not compact, and we do not know that $K_+ \subset \mathbb{B}(q_+, \lambda_+)$. Nevertheless, arguments like those in Proposition 1 show that the pullback attractor $x^r(t)$ to q_- must satisfy $x^r(t) \in K_+$ for all $t \in \mathbb{R}$. Now $K_+ \ll \{x : x \geq p_+\}$, so $x^r(t)$ does not tip to $\{x : x \geq p_+\}$ for any $r > 0$. □

Proposition 3 significantly simplifies the conditions for showing that a system will not have R-tipping. In general, our method is to establish that a path is FIS, which can be quite difficult, but when the systems in question are monotone, it suffices to check the relative positions of the equilibria. Once again, this makes checking for R-tipping in monotone systems similar to checking for R-tipping in 1-dimensional systems because it reduces to comparing the positions of equilibria.

Notice that in Proposition 3 we cannot conclude that rate-induced tipping does not happen at all; it is possible that the parameter change in system (1.3) may cause rate-induced tipping to happen away from q_- in another direction. But given a particular monotone system, one could perhaps apply Proposition 3 along with some other arguments to conclude that no rate-induced tipping is possible for a given parameter change.

6 An Example

Here we give an example of a two-dimensional monotone system that will allow us to apply the things proven in this paper, particularly in Sect. 5. For $x = (x_1, x_2) \in \mathbb{R}^2$, consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + ax_1^2 + bx_1 - a(b+1) + x_2 \\ \dot{x}_2 &= x_1 - x_2\end{aligned} \tag{6.1}$$

for any $b > -1$ and a satisfying $|a| < \sqrt{b+1}$. Let

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \begin{pmatrix} -x_1^3 + ax_1^2 + bx_1 - a(b+1) + x_2 \\ x_1 - x_2 \end{pmatrix}$$

denote the vector field generated by (6.1). The fixed points of (6.1) are $p_1 = \left(-\sqrt{b+1}, -\sqrt{b+1}\right)$, $p_2 = (a, a)$, and $p_3 = \left(\sqrt{b+1}, \sqrt{b+1}\right)$. The derivative matrix at a point $x = (x_1, x_2)$ is

$$Df(x) = \begin{pmatrix} \alpha(x) & 1 \\ 1 & -1 \end{pmatrix} \tag{6.2}$$

where $\alpha(x) = -3x_1^2 + 2ax_1 + b$. This shows that (6.1) is monotone. If $\alpha(x) < -1$, then (6.2) has two negative eigenvalues, and if $\alpha(x) > -1$, then (6.2) has one positive and one negative eigenvalue. In our parameter regime,

$$\alpha\left(\pm\sqrt{b+1}, \pm\sqrt{b+1}\right) < -1,$$

so p_1 and p_3 are attracting equilibria, whereas $\alpha(a, a) > -1$, so p_2 is a saddle node. An eigenvector associated with the positive eigenvalue λ_+ of $Df(p_2)$ is

$$\begin{pmatrix} \lambda_+ + 1 \\ 1 \end{pmatrix},$$

which points in the all-positive direction.

Now, let us consider the possibility of rate-induced tipping in (6.1). To do this, we let a and b depend on a parameter that can vary with time:

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + a(\Lambda(s))x_1^2 + b(\Lambda(s))x_1 - a(\Lambda(s))(b(\Lambda(s)) + 1) + x_2 \\ \dot{x}_2 &= x_1 - x_2 \\ \dot{s} &= r,\end{aligned} \tag{6.3}$$

where $\Lambda \in \mathcal{P}(\lambda_-, \lambda_+)$ for some $\lambda_- < \lambda_+$ and $a, b : [\lambda_-, \lambda_+] \to \mathbb{R}$ are smooth functions satisfying $b(\lambda) > -1$ and $|a(\lambda)| < \sqrt{b(\lambda)+1}$ for all $\lambda \in [\lambda_-, \lambda_+]$. We will use the notation

$$f(x, \lambda) = \begin{pmatrix} f_1(x, \lambda) \\ f_2(x, \lambda) \end{pmatrix} = \begin{pmatrix} -x_1^3 + a(\lambda)x_1^2 + b(\lambda)x_1 - a(\lambda)(b(\lambda) + 1) + x_2 \\ x_1 - x_2 \end{pmatrix}$$

to denote the first two components of the vector field of (6.3). There are two stable paths in this augmented system: $(p_1(s), \Lambda(s))$ and $(p_3(s), \Lambda(s))$. The path $(p_2(s), \Lambda(s))$ is unstable. For ease of notation, we will define

$$\lim_{s \to \pm\infty} p_i(s) = p_{i\pm}.$$

We have the following result about the possibility of R-tipping in system (6.3):

Proposition 4 1. *If there exist* $s_1 < s_2$ *such that*

$$-\sqrt{b(\Lambda(s_1)) + 1} > a(\Lambda(s_2)) \ (\textit{resp. } \sqrt{b(\Lambda(s_1)) + 1} < a(\Lambda(s_2)))$$

then there is a parameter shift $\tilde{\Lambda} \in \mathcal{P}(\lambda_-, \lambda_+)$ *such that there is R-tipping away from* p_{1-} *(resp.* p_{3-}*) for this* $\tilde{\Lambda}$ *for some* $r > 0$.

2. *If, for all* $s_1 < s_2$ *(including in the limit as* $s_1 \to -\infty$ *or* $s_2 \to \infty$*),*

$$-\sqrt{b(\Lambda(s_1)) + 1} < a(\Lambda(s_2)) \text{ (resp. } \sqrt{b(\Lambda(s_1)) + 1} > a(\Lambda(s_2)))$$

then $(p_1(s), \Lambda(s))$ *(resp.* $(p_3(s), \Lambda(s))$*) is forward inflowing stable and there cannot be R-tipping away from* (p_{1-}, λ_-) *(resp.* (p_{3-}, λ_-)*) for this* Λ*.*

Remark 1 Notice that Proposition 4 gives a nearly exhaustive description of whether a given parameter change will lead to R-tipping or not for (6.3). The only cases left out are boundary cases when, for instance, $\sqrt{b(\Lambda(s_1)) + 1}$ is equal to, but never greater than, $a(\Lambda(s_2))$ for some $s_1 < s_2$.

Proof Statement 1 is a consequence of Proposition 2. The proof of statement 2 requires more work. We will show that there can be no R-tipping away from (p_{1-}, λ_-) if

$$-\sqrt{b(\Lambda(s_1)) + 1} < a(\Lambda(s_2))$$

for all $s_1 < s_2$. The proof of the corresponding statement is similar.

Let $r > 0$ and let $x^r(t)$ be the pullback attractor to p_{1-}. By Proposition 3, $x^r(t)$ does not tip to $\{x : x \geq p_{2-}\}$, but in fact the proof of Proposition 3 shows something stronger: there is a z satisfying $p_{1+} \ll z \ll p_{2+}$ such that $x^r(t) \in \{x : x \leq z\}$ for all t.

Choose $c < \inf_{s\in\mathbb{R}} \left\{-\sqrt{b(\Lambda(s)) + 1}\right\}$. Then

$$0 < -c^3 + a(\Lambda(s))c^2 + (b(\Lambda(s)) + 1)c - a(\Lambda(s))(b(\Lambda(s)) + 1)$$

for all values of s (including in the limits as $s \to \pm\infty$), and therefore

$$c^3 - a(\Lambda(s))c^2 - b(\Lambda(s))c + a(\Lambda(s))(b(\Lambda(s)) + 1) < c.$$

Choose any d satisfying

$$c^3 - a(\Lambda(s))c^2 - b(\Lambda(s))c + a(\Lambda(s))(b(\Lambda(s)) + 1) < d < c$$

for all values of s (including in the limits as $s \to \pm\infty$) and set $p = (c, d)$. Then $p \ll p_1(s)$ for all s, and $f_i(p, \lambda) > 0$ for all i and $\lambda \in [\lambda_-, \lambda_+]$. If we set $K(s) \equiv \{x : p \leq x \leq z\}$, then $\{K(s)\}$ clearly satisfies the first 4 conditions of Definition 2 to show that $(p_1(s), \Lambda(s))$ is forward inflowing stable. The only thing that remains to be shown is that $K_+ \subset \mathbb{B}(p_{1+}, \lambda_+)$.

What we are going to show is that $K_+ = \{x : p \leq x \leq z\}$ is in the basin of attraction of p_{1+} under the flow of (6.1) when $\lambda = \lambda_+$. Fix some $x \in K_+$. As shown above, K_+ is a forward invariant box, so $\{x \cdot t\}_{t\geq 0}$ stays in K_+ for all time. Hence, $\omega(x)$ is nonempty and compact. By Theorem 3.22 of Hirsch and Smith [7], $\omega(x)$ is a fixed point, and therefore must be p_1. Because x was an arbitrary point in K_+, K_+ is in the basin of attraction of p_{1+}.

Therefore, $(p_1(s), \Lambda(s))$ is FIS, and there cannot be R-tipping away from (p_{1-}, λ_-). □

Let us look at a couple of specific examples to illustrate Proposition 4.

Example 4 Let $\Lambda(s) = \frac{1}{2}(1 + \tanh(s))$. Then $\lambda_- = 0$ and $\lambda_+ = 1$. We define the dependence of a and b on the parameter λ to be $a(\lambda) = 2\lambda$ and $b(\lambda) = 8\lambda$. This means that $b(\Lambda(s)) > -1$ and $a(\Lambda(s)) < \sqrt{b(\Lambda(s)) + 1}$ for all s. However,

$$\sqrt{b(\Lambda(-5)) + 1} < a(\Lambda(5)),$$

which implies by Proposition 4 there will be R-tipping away from p_{3-} for some parameter shift $\tilde{\Lambda}$ and some $r > 0$. In this case, $\tilde{\Lambda}(s) = \Lambda(s)$ works. See Fig. 8a. Note that because

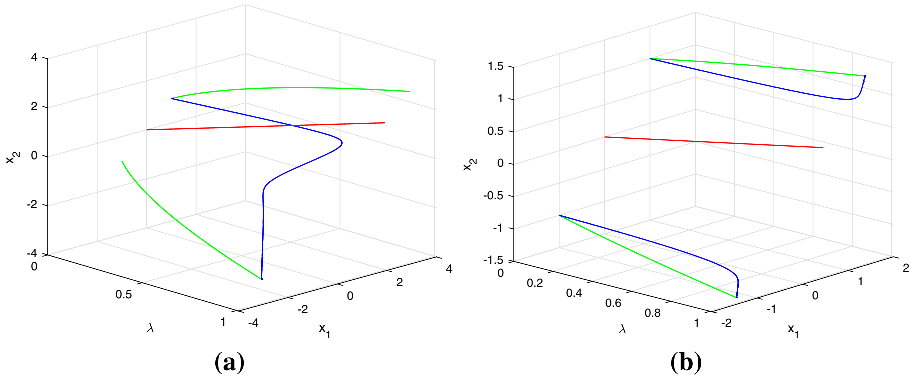


Fig. 8 In both pictures, the green curves represent the stable paths $p_1(s) = -\sqrt{b(\Lambda(s)) + 1}$ and $p_3(s) = \sqrt{b(\Lambda(s)) + 1}$ and the red curve represents the unstable path $p_2(s) = a(\Lambda(s))$ for different values of $\lambda = \Lambda(s)$. The blue curves are pullback attractors to the stable paths. In (**a**), the position of $p_3(s)$ for small s-values is smaller than the position of $p_2(s)$ for larger s-values. Therefore, R-tipping is possible away from p_{3-}. In (**b**), $p_1(s_1) < p_2(s_2) < p_3(s_1)$ for all $s_1 < s_2$ (including in the infinite limits), so R-tipping is not possible away from either p_{1-} or p_{3-}. Pullback attractors are shown in blue for both paths (Color figure online)

Λ is a one-to-one function of s, we can plot the positions of the trajectory and the quasi-stable/unstable equilibria against $\lambda = \Lambda(s)$ rather than s. This is convenient because the range of s is infinite, but the range of Λ is bounded.

Example 5 Once again, let $\Lambda(s) = \frac{1}{2}(1 + \tanh(s))$, but this time define $a(\lambda) = \frac{1}{2}\lambda$ and $b(\lambda) = \lambda$. Then $b(\Lambda(s)) > -1$ and $a(\Lambda(s)) < \sqrt{b(\Lambda(s)) + 1}$ for all s. Furthermore,

$$-\sqrt{b(\Lambda(s)) + 1} < -1 < 0 < a(\Lambda(s)) < \frac{1}{2} < 1 < \sqrt{b(\Lambda(s)) + 1}$$

for all s, so by Proposition 4 there can be no R-tipping away from either p_{1-} or p_{3-}. See Fig. 8b.

7 Conclusion

In summary, we have shown that R-tipping results are more complicated in multi-dimensional systems than in one-dimensional systems. R-tipping can happen anytime a path is not forward basin stable of a certain type, and sometimes there can be R-tipping even if a path is FBS. We proposed forward inflowing stability as a condition that prevents R-tipping in systems of all dimensions. One drawback is that it is difficult to know when a path is FIS because it requires knowledge about the autonomous systems with fixed parameter values and what sort of forward invariant sets exist around the equilibria. One future direction we could take is to give more concrete results about how to determine whether a path is FIS.

In this paper, we focused on stable paths of equilibria. However, in multi-dimensional systems, there could be stable paths of other attracting invariant sets, such as periodic orbits. In such a situation, different kinds of R-tipping are possible, known as partial tipping and full tipping (see [1]). Not much work has been done to determine when these kinds of tipping can or cannot happen, but the FBS and FIS methods could be used as a starting point, as we believe the results given in this paper are generalizable to invariant sets other than fixed points.

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Appendix: Proofs of Lemmas from Sect. 3

The proof of Lemma 1 closely follows the proof of Theorem 2.2 in Ashwin et al. [3]:

Proof Let us assume for the sake of simplicity that $X_- = 0$. Let

$$\omega(\epsilon) := \sup\{|d_x f(x, \Lambda(s)) - d_x f(0, \Lambda(s))| : s \in \mathbb{R}, |x| < \epsilon\}$$
$$\delta(S) := \max\left\{\sup_{s<-S} |f(0, \Lambda(s))|, \sup_{s<-S} |d_x f(0, \Lambda(s)) - d_x f(0, \lambda_-)|\right\}.$$

Note that $\omega(\epsilon) \to 0$ as $\epsilon \to 0$ and $\delta(S) \to 0$ as $S \to \infty$.

By the linear stability of X_-, the eigenvalues of

$$A := d_x f(0, \lambda_-)$$

have negative real parts, so there are $K > 0$ and $\alpha > 0$ such that $\left|e^{tA}\right| \le K e^{-\alpha t}$ for $t \ge 0$ (see Lemma 3.3.19 of [5]).

Now set $h(x, s) = f(x, \Lambda(s)) - Ax$ so that

$$\dot{x} = Ax + h(x, s) \tag{7.1}$$

Then

$$\begin{aligned} d_x h(x, s) &= d_x f(x, \Lambda(s)) - A \\ &= [d_x f(x, \Lambda(s)) - d_x f(0, \Lambda(s))] + \left[d_x f(0, \Lambda(s)) - d_x f(0, \lambda_-)\right] \end{aligned}$$

Therefore, if $s < -S$, we have

$$|h(0, s)| \le \delta(S), \quad |d_x h(x, t)| \le \omega(|x|) + \delta(S).$$

Consider the inequalities

$$\begin{aligned} 4K\alpha^{-1}\omega(\epsilon) &\le 1 \\ 2K\alpha^{-1}\delta(S) &\le \epsilon \\ 4K\alpha^{-1}\delta(S) &\le 1, \end{aligned} \tag{7.2}$$

for $\epsilon, S > 0$. If we choose ϵ sufficiently small, we can find some $S_0 > 0$ to satisfy (7.2). Now, we know that $\delta(S) \to 0$ as $S \to \infty$, so there is an $S_1 > 0$ such that $S \ge S_1$ implies that $\delta(S) \le \delta(S_0)$. Then if $S \ge S_1$, (7.2) is satisfied.

Now, fix any $r > 0$. We will show that the pullback attractor $x^r(t)$ to $X_- = 0$ satisfies $|x^r(t)| < \epsilon$ as long as $rt < -S_1$.

Let $\mathcal{P}$ be space of continuous functions $x(t)$ defined for $t < -\frac{S_1}{r}$ such that $|x(t)| \le \epsilon$. We define $\hat{x}$ for $x \in \mathcal{P}$ by

$$\hat{x}(t) = \int_{-\infty}^{t} e^{(t-u)A} h(x(u), ru)\, du.$$

Then if $x \in \mathcal{P}$, $\hat{x} = x$ if and only if $x(t)$ is a solution of (7.1). Also,

$$|\hat{x}(t)| \le \epsilon,$$

so $x \mapsto \hat{x}$ is a map from $\mathcal{P}$ to itself. Furthermore, if $x_1, x_2 \in \mathcal{P}$, then

$$||\hat{x}_1 - \hat{x}_2|| \leq \frac{1}{2}||x_1 - x_2||,$$

where $||x|| := \sup_{t<-S_1/r} |x(t)|$. Thus, $x \mapsto \hat{x}$ is a contraction mapping on $\mathcal{P}$, so there is a unique $x(t) \in \mathcal{P}$ such that $x(t) = \hat{x}(t)$. This $x(t)$ is a solution to (7.1) and satisfies $|x(t)| \leq \epsilon$ for all $t < -S_1/r$. This $x(t)$ must be the pullback attractor to $X_- = 0$, since the pullback attractor is the only trajectory of (1.3) that stays within a small neighborhood of X_- for all backward time. (See Theorem 2.2 of [3].) □

Proof of Lemma 2 Fix $r > 0$. For each $n \in \mathbb{N}$, there exists some $N_n \in \mathbb{N}$ such that $m \geq N_n$ implies $|y \cdot_{\lambda_+} s_m - z| < \frac{1}{n}$. (If $n \geq 2$, choose $N_n > N_{n-1}$.) Since $\Lambda(rt) \to \lambda_+$ as $t \to \infty$, there exists some $T_n > 0$ and $\delta_n > 0$ such that if $|x(t) - y| < \delta_n$ and $t > T_n$, then $|x(t + s_{N_n}) - y \cdot_{\lambda_+} s_{N_n}| < \frac{1}{n}$. There exists some $M_n \in \mathbb{N}$ such that $t_{M_n} > T_n$ and if $m \geq M_n$, then $|x(t_m) - y| < \delta_n$. (Again, if $n \geq 2$, choose $M_n > M_{n-1}$.)

Now, set $u_n = s_{N_n} + t_{M_n}$. Then,

$$\begin{aligned}
|x(u_n) - z| &= |x(t_{M_n} + s_{N_n}) - z| \\
&\leq |x(t_{M_n} + s_{N_n}) - y \cdot_{\lambda_+} s_{N_n}| + |y \cdot_{\lambda_+} s_{N_n} - z| \\
&< \frac{1}{n} + \frac{1}{n} \\
&= \frac{2}{n}
\end{aligned}$$

Therefore, $x(u_n) \to z$ as $n \to \infty$. □

Proof of Lemma 3 Let us assume for the sake of simplicity that $X_+ = 0$ and $\lambda_+ = 0$. Since $(X_+, \lambda_+) = (0, 0)$ is attracting, all eigenvalues of $A = d_x f(0, 0)$ have negative real part, so there is some $k > 0$ such that $\mathrm{Re}(\mu) < -k$ for every eigenvalue μ of A. We can choose an inner product $\langle , \rangle$ on U such that $\langle Ax, x\rangle \leq -k\langle x, x\rangle$ for all $x \in U$ (see the lemma in Chapter 7, Section 1 of [6]). This defines a norm $||x|| = \langle x, x\rangle^{1/2}$. By Taylor's formula in several variables we can write

$$f(x, \lambda) = Ax + \alpha(x, \lambda) + \beta(x),$$

where $||\alpha(x, \lambda)|| \leq \gamma(x, \lambda)|\lambda|$ for a positive continuous γ, and $||\beta(x)|| \leq \delta(x)||x||$, where δ is positive, continuous and $\delta(x) \to 0$ as $x \to 0$. Then we can write (1.3) as

$$\begin{aligned}
\frac{dx}{dt} &= Ax + \alpha(x, \Lambda(s)) + \beta(x) \\
\frac{ds}{dt} &= r
\end{aligned}$$

For a given $\epsilon > 0$ and $S > 0$, define $N_{\epsilon,S} = B_\epsilon(0) \times [S, \infty)$, where $B_\epsilon(0) = \{x \in U : ||x|| < \epsilon\}$. Note that

$$\begin{aligned}
\frac{d}{dt}\left(||x||^2\right) &= 2\langle \dot{x}, x\rangle \\
&= 2\langle Ax, x\rangle + 2\langle \alpha(x, \Lambda(s)), x\rangle + 2\langle \beta(x), x\rangle \\
&\leq -2k\langle x, x\rangle + 2||\alpha(x, \Lambda(s))|| \cdot ||x|| + 2||\beta(x)|| \cdot ||x|| \\
&\leq -2k||x||^2 + 2\gamma(x, \Lambda(s))|\Lambda(s)| \cdot ||x|| + 2\delta(x) \cdot ||x||^2
\end{aligned}$$

$$= 2||x||^2 \left(-k + 2\gamma(x, \Lambda(s)) \frac{|\Lambda(s)|}{||x||} + \delta(x) \right)$$

Choose $\epsilon > 0$ such that if $||x|| \leq \epsilon$, $\delta(x) < \frac{k}{2}$. Then choose $S > 0$ such that if $s \geq S$, $|\Lambda(s)| < \frac{k\epsilon}{4M}$ where $M > \sup_{||x|| \leq \epsilon, \lambda \in [\lambda_-, \lambda_+]} \gamma(x, \lambda)$. Thus, if $||x|| = \epsilon$ and $s \geq S$,

$$\begin{aligned} \frac{d}{dt}(||x||)^2 &< 2\epsilon^2 \left(-k + 2\gamma(x, \Lambda(s)) \left(\frac{k\epsilon}{4M\epsilon} \right) + \frac{k}{2} \right) \\ &< 2\epsilon^2 \left(-k + \frac{k}{2} + \frac{k}{2} \right) \\ &= 0 \end{aligned}$$

Therefore, for sufficiently small $\epsilon > 0$ there exists an $S > 0$ such that the vector field of (1.3) points into $N_{\epsilon,S}$ on its boundary, so $N_{\epsilon,S}$ is forward invariant. □

Proof of Lemma 4 Pick an $\epsilon > 0$ sufficiently small for Lemma 3. Make ϵ smaller if necessary so that $\overline{B_\epsilon(X_+)} \subset \mathbb{B}(X_+, \lambda_+)$. Then by Lemma 3, there exists an $S > 0$ such that if $x(t) \in \overline{B_\epsilon(X_+)}$ for $rT > S$, then $x(t) \in \overline{B_\epsilon(X_+)}$ for all $t \geq T$. Now fix $r > 0$. Since $\overline{B_\epsilon(X_+)}$ is compact, there is some $y \in \overline{B_\epsilon(X_+)}$ such that $x(t_n) \to y$ as $t_n \to \infty$. But $y \in \mathbb{B}(X_+, \lambda_+)$ by assumption, so $y \cdot t \to X_+$ in the autonomous system (1.1). Therefore, by Lemma 2, there exists a $\{u_n\} \to \infty$ such that $x(u_n) \to X_+$.

Now pick any $\delta \in (0, \epsilon)$. Then by Lemma 3, there exists some $S_\delta > 0$ such that if $x(T) \in B_\delta(X_+)$ for $rT > S_\delta$, then $x(T) \in B_\delta(X_+)$ for all $t \geq T$. By the previous paragraph, there is a $u_{n_\delta} > S_\delta / r$ such that $|x(u_{n_\delta}) - X_+| < \delta$. Therefore, $|x(t) - X_+| < \delta$ for all $t \geq u_{n_\delta}$. Hence $x(t) \to X_+$ as $t \to \infty$. □

Proof of Lemma 5 By Lemma 4, there is an $\epsilon > 0$ and an $S_1 > 0$ such that if $|x(t) - X_+| < \epsilon$ for $rt > S_1$, then $x(t) \to X_+$ as $t \to \infty$. Since $K \subset \mathbb{B}(X_+, \lambda_+)$ is compact, there is some $T_0 > 0$ such that $y \cdot_{\lambda_+} t \in B_{\epsilon/2}(X_+)$ for any $y \in K$ and $t \geq T_0$. Also, there is some $S_2 > 0$ such that if $x(T) = y_0 \in K$ for $rT > S_2$, then $|x(T + T_0) - y_0 \cdot_{\lambda_+} T_0| < \epsilon/2$ for any $y_0 \in K$.

Take $S = \max\{S_1, S_2\}$. Then, suppose $x(T) \in K$ for $rT > S$. If $x(T) = y_0$, then

$$\begin{aligned} |x(T + T_0) - X_+| &\leq |x(T + T_0) - y_0 \cdot_{\lambda_+} T_0| + |y_0 \cdot_{\lambda_+} T_0 - X_+| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

Therefore, $x(t) \to X_+$ as $t \to \infty$. □

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