Split cuts in the plane

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Abstract

We provide a polynomial time cutting plane algorithm based on split cuts to solve integer programs in the plane. We also prove that the split closure of a polyhedron in the plane has polynomial size.

1 Introduction

In this paper, we work in $\mathbb{R}^2$ and we always implicitly assume that all polyhedra, cones, half-planes, lines are rational. Given a polyhedron $P \subseteq \mathbb{R}^2$, we let $P_I := \mathrm{conv}(P \cap \mathbb{Z}^2)$, where $\mathrm{conv}(\cdot)$ denotes the convex hull operator.

Given $\pi \in \mathbb{Z}^2 \setminus \{0\}$ and $\pi_0 \in \mathbb{Z}$, let H_0 and H_1 be the half-planes defined by $\pi x \leq \pi_0$ and $\pi x \geq \pi_0 + 1$, respectively. Given a polyhedron $P \subseteq \mathbb{R}^2$, we let $P_0 := P \cap H_0$, $P_1 := P \cap H_1$ and $P^{\pi,\pi_0} := \mathrm{conv}(P_0 \cup P_1)$. P^{π,π_0} is a polyhedron that contains $P \cap \mathbb{Z}^2$. An inequality $cx \leq d$ is a *split inequality* (or *split cut*) for P if there exist $\pi \in \mathbb{Z}^2 \setminus \{0\}$ and $\pi_0 \in \mathbb{Z}$ such that the inequality $cx \leq d$ is valid for P^{π,π_0}. The vector (π, π_0), or the set $H_0 \cup H_1$, is a *split disjunction*, and we say that $cx \leq d$ is a split inequality for P with respect to (π, π_0). The closed complement of a split disjunction, i.e., the set defined by $\pi_0 \leq \pi x \leq \pi_0 + 1$, is called a *split set*. If one of P_0, P_1 is empty, say $P_1 = \emptyset$, the split inequality $\pi x \leq \pi_0$ is called a *Chvátal inequality*.

A Chvátal inequality $\pi x \leq \pi_0$ where π is a primitive vector (i.e., its coefficients are relatively prime) has the following geometric interpretation: Let $z := \max_{x \in P} \pi x$ and let H be the half-plane defined by $\pi x \leq z$. Then $P \subseteq H$ and $\pi_0 = \lfloor z \rfloor$, because $\pi_0 \in \mathbb{Z}$ and $P_1 = \emptyset$. Since π is a primitive vector, H_I is defined by the inequality $\pi x \leq \pi_0$. We will say that the inequality $\pi x \leq \pi_0$ defines the *Chvátal strengthening* of the half-plane H. If $z = \pi_0$, then we say H is *Chvátal strengthened*.

An *inequality description* of a polyhedron $P \subseteq \mathbb{R}^2$ is a system $Ax \leq b$ such that $P = \{x \in \mathbb{R}^2 : Ax \leq b\}$, where $A \in \mathbb{Z}^{m \times 2}$ and $b \in \mathbb{Z}^m$ for some positive integer m. The *size* of the description of P, i.e., the number of bits needed to encode the linear system, is $O(m \log \|A\|_\infty + m \log \|b\|_\infty)$. (Notation $\|\cdot\|_\infty$ indicates the infinity-norm of a vector or a matrix, i.e., the maximum absolute

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value of its entries.) It follows from the above argument that when the coefficients in each row of A are relatively prime integers, the inequalities defining P are Chvátal strengthened.

Given a polyhedron P, a *cutting plane* or *cut* is an inequality that defines a half-plane H such that $P \not\subseteq H$ but $P_I \subseteq H$. A *cutting plane algorithm* is a procedure that, given a polyhedron $P \subseteq \mathbb{R}^2$ and a vector $c \in \mathbb{Z}^2$, solves the integer program $\max\{cx : x \in P \cap \mathbb{Z}^2\}$ by adding at each iteration a cut that eliminates an optimal vertex of the current continuous relaxation until integrality is achieved or infeasibility is proven.

Integer programming in the plane is the problem $\max\{cx : Ax \leq b,\ x \in \mathbb{Z}^2\}$ where $c \in \mathbb{Z}^2, A \in \mathbb{Z}^{m\times 2}$ and $b \in \mathbb{Z}^m$. In Section 2 we provide a cutting plane algorithm for this problem that uses split inequalities as cutting planes and such that the number of iterations (i.e., cutting planes computed) is $O(m(\log \|A\|_\infty)^2)$. (The derivation of every cutting plane can be carried out in polynomial time but involves a constant number of gcd computations.)

We note that integer programming in the plane is well-studied and understood. In particular, given a polyhedron $P \subseteq \mathbb{R}^2$, Harvey [10] gave an efficient procedure to produce an inequality description of P_I. Eisenbrand and Laue [9] gave an algorithm to solve the problem that makes $O(m + \max\{\log \|A\|_\infty, \log \|b\|_\infty, \log \|c\|_\infty\})$ arithmetic operations.

As split cuts are widely used in integer programming solvers, the scope of the present research is to prove that this class of integer programs can also be solved in polynomial time with a cutting plane algorithm based on split cuts (albeit not as efficiently as in [9]).

The second part of this paper deals with the complexity of the split closure of a polyhedron in the plane. Given a polyhedron $P \subseteq \mathbb{R}^2$, the *split closure* P^{split} of P is defined as follows:

$$P^{\text{split}} := \bigcap_{\pi\in\mathbb{Z}^2\setminus\{0\},\pi_0\in\mathbb{Z}} P^{\pi,\pi_0}.$$

Cook, Kannan and Schrijver [6] proved that P^{split} is a polyhedron. Polyhedrality results for cutting plane closures, such as the above split closure result, have a long history in discrete optimization starting from the classical result that the *Chvátal closure* of a rational polytope (the intersection of all Chvátal inequalities) is polyhedral (see, e.g., Theorem 23.1 in [12]), with several more recent results [2, 7, 8, 3, 13, 11], to sample a few. The *complexity* of cutting plane closures, i.e., the number of facets and the bit complexity of the facets, is relatively less understood. One of the most well-known results in this direction is due to Bockmayr and Eisenbrand [4], who showed that the complexity of the Chvátal closure of a rational polytope is polynomial in the description size of the polytope, if the dimension is a fixed constant (see Theorem 21 in Section 3 below). It has long remained an open question whether the split closure is of polynomial complexity as well, even in the case of two dimensions. We settle this question in the affirmative in this paper; see Theorem 20 in Section 3.

Finally, as again shown in [6], if one defines $P^0 := P$ and recursively $P^i := (P^{i-1})^{\text{split}}$, then $P^t = P_I$ for some t. The *split rank* of P is the smallest t for which this occurs. It is a folklore result that if $P \subseteq \mathbb{R}^2$ is a polyhedron, its split rank is at most 2; we will observe in Remark 34 that this also follows from the arguments used in this paper.

Some notation Let $\dim(Q)$ denote the dimension of any polyhedron Q. A polyhedron in $\mathbb{R}^2$ which is the intersection of two non parallel half-planes is a *full-dimensional translated pointed cone*.

However, to simplify terminology we will often refer to such a polyhedron as a *translated cone*. Its unique vertex is the *apex* of the cone. We will use the notation $(H_1, H_2) := H_1 \cap H_2$ to denote the translated cone formed by the intersection of two half-planes H_1, H_2. Given a half-plane H, we let $H^=$ denote its boundary.

2 Tilt cuts and the clockwise algorithm

To simplify the presentation, throughout the paper the notions of facet and facet defining inequality of a polyhedron will be interchangeably used.

Definition 1. *(Tilt of a facet about a pivot with respect to a translated cone) Let $C = (H_1, H_2)$ be a translated cone with apex not in $\mathbb{Z}^2$, and assume that H_1 is Chvátal strengthened.*

Let $\widehat{H}$ be the line in H_1 parallel to $H_1^=$ and closest to $H_1^=$ such that $\widehat{H} \cap \mathbb{Z}^2 \neq \emptyset$. Let $p \in H_1^= \cap C \cap \mathbb{Z}^2$ and $q \in (H_1^= \setminus C) \cap \mathbb{Z}^2$ be the unique points such that the open line segment (p, q) contains no integer point. Let $x \in \widehat{H} \cap C \cap \mathbb{Z}^2$ and $y \in (\widehat{H} \setminus C) \cap \mathbb{Z}^2$ be the unique points such that the open line segment (x, y) contains no integer point (possibly $x \in H_2^=$).

Two parallel sides of the parallelogram $P := \mathrm{conv}(p, q, y, x)$ are contained in $H_1^= \cup \widehat{H}$. The other two sides of P define a split disjunction in the following way. Let W_0, W_1 be the half-planes such that $W_0^=$ is the line containing p and x, $W_1^=$ is the line containing q and y, and $W_0 \cap W_1 = \emptyset$. As P has integer vertices but contains no other integer point, P has area 1 and W_0, W_1 define a split disjunction (π, π_0).

Let F_1 be the facet of C induced by H_1. We now define the tilt *T of F_1 with* pivot *p* with respect to *C. If $C \cap W_1 = \emptyset$, then (π, π_0) defines a Chvátal cut for C (as in Fig. 1(i)), and we let T be this Chvátal cut. Otherwise let $x' \in W_1^= \cap C \cap \mathbb{Z}^2$ and $y' \in (W_1^= \setminus C) \cap \mathbb{Z}^2$ be the unique points such that the open line segment (x', y') contains no integer point and let q' be the point of intersection of $[x', y']$ and $H_2^=$ (possibly $q' = x'$), see Fig. 1(ii). We define T as the split cut for C with respect to (π, π_0) such that $T^=$ contains p and y'. (Note that in this case T is not the "best" split cut for C with respect to (π, π_0), as it does not define a facet of $\mathrm{conv}((C \cap W_0) \cup (C \cap W_1))$.)*

In the next two lemmas we refer to the notation introduced in Definition 1.

Lemma 2. *Let $ax \leq \beta$ and $dx \leq \delta$ be inequality descriptions of H_1 and H_2 respectively, where the coefficients of a are relatively prime. Then $dq - \delta \leq |a_1 d_2 - a_2 d_1|$.*

Proof. Let p be as in Definition 1. As a_1, a_2 are relatively prime, either $q = p + \begin{pmatrix} -a_2 \\ a_1 \end{pmatrix}$ or $q = p + \begin{pmatrix} a_2 \\ -a_1 \end{pmatrix}$. Taking an inner product with d, we obtain that $dq - dp \leq -a_2 d_1 + a_1 d_2$ in the first case and $dq - dp \leq a_2 d_1 - a_1 d_2$ in the second case. Since $p \in H_2$, $dp \leq \delta$. The result follows. □

Lemma 3. *Let $dx \leq \delta$ be an inequality description of H_2.*

(i) T is always Chvátal strengthened.

(ii) T defines a facet of C_I if and only if T is a Chvátal cut for C.

(iii) When $C \cap W_1 \neq \emptyset$, we have $0 < dy' - \delta \leq \frac{dq - \delta}{2}$.

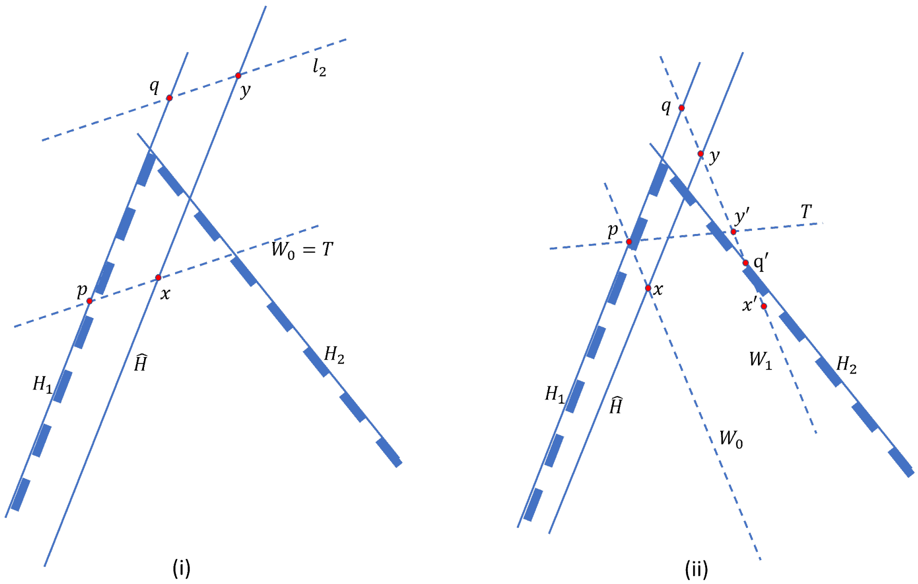

Figure 1: Illustration of Definition 1.

Proof. (i) As $p \in T^= \cap \mathbb{Z}^2$ and T is a rational half-plane, T is always Chvátal strengthened.

(ii) Recall that T is a Chvátal cut for C if and only if $C \cap W_1 = \emptyset$. When $C \cap W_1 = \emptyset$ we have $x \in T^= \cap C \cap \mathbb{Z}^2$, whereas when $C \cap W_1 \neq \emptyset$ we have $y' \in (T^= \setminus C) \cap \mathbb{Z}^2$ and there is no integer point in the open segment (p, y'). Since T is a facet of C_I if and only if $T^= \cap C$ contains an integer point different from p, this happens if and only if T is a Chvátal cut.

(iii) When $C \cap W_1 \neq \emptyset$, we have that $q, y, y' \in (W_1^= \cap \mathbb{Z}^2) \setminus C$, $x' \in (W_1^= \cap \mathbb{Z}^2) \cap C$, while $q' \in W_1^= \cap H_2^=$ and (x', y') has no integer points. Therefore the length of $[y', q']$ is at most half the length of $[q, q']$. This implies that $0 < dy' - \delta \leq \frac{dq - \delta}{2}$. $\square$

Remark 4. *The algorithm below uses the following fact: If $P \subseteq \mathbb{R}^2$ is not full-dimensional, the integer program $\max\{cx : x \in P \cap \mathbb{Z}^2\}$ can be solved by applying one Chvátal cut. Specifically, if $\dim(P) \leq 0$, the problem is trivial, and if $\dim(P) = 1$, with one cut we can certify infeasibility if $\text{aff}(P) \cap \mathbb{Z}^2 = \emptyset$ (where $\text{aff}(P)$ denotes the affine hull of P). The cut certifying infeasibility has boundary parallel to P in this case. Otherwise, if $\text{aff}(P) \cap \mathbb{Z}^2 \neq \emptyset$, the problem is unbounded if and only if $\max\{cx : x \in P\} = \infty$. Finally, if $\text{aff}(P) \cap \mathbb{Z}^2 \neq \emptyset$ and $\max\{cx : x \in P\}$ is finite, the integer program is either infeasible or admits a finite optimum: this can be determined by applying one Chvátal cut whose boundary is orthogonal to P. We note that, from standard results in integer programming, the Chvátal cuts considered above have polynomial encoding length.*

Definition 5. *(Late facet and early facet) Given an irredundant description $Ax \leq b$ of a full-dimensional pointed polyhedron P with m facets, we denote by F_i the facet of P defined by the the ith inequality $a^i x \leq b_i$ of the system $Ax \leq b$. Given a vector $c \in \mathbb{Z}^2 \setminus \{0\}$ such that the linear program $\max\{cx : x \in P\}$ has finite optimum, and a specified optimal vertex v, we assume that $a^1, \ldots, a^m$ are ordered clockwise so that $v \in F_m \cap F_1$ and c belongs to the cone generated by a^m and a^1. We call F_m the* late facet *and F_1 the* early facet *of P with respect to v. This ordered pair defines the translated cone (F_m, F_1) with apex v.*

Algorithm 1 The "clockwise" cutting plane algorithm

Input: A pointed polyhedron $P \subseteq \mathbb{R}^2$ and a vector $c \in \mathbb{Z}^2 \setminus \{0\}$ such that $\max\{cx : x \in P\}$ is finite.

Output: An optimal solution of the integer program $\max\{cx : x \in P \cap \mathbb{Z}^2\}$ or a certificate of infeasibility.

1: Initialize $Q := P$.

2: If $\dim(Q) \leq 1$, apply at most one Chvátal cut to output INFEASIBLE or an optimal solution.

3: Else solve the linear program $\max\{cv : v \in Q\}$ and let v^* be the optimal vertex. If $v^* \in \mathbb{Z}^2$, STOP and output v^*.

4: If $v^* \notin \mathbb{Z}^2$, number the facets of Q in clockwise order $F_1, \ldots, F_{m^Q}$ so that $v^* \in (F_{m^Q}, F_1)$. If F_{m^Q} is not Chvátal strengthened, let T be its Chvátal strengthening. Otherwise let T be the tilt of F_{m^Q} with respect to (F_{m^Q}, F_1). Update $Q := Q \cap T$ and go to Step 2.

In Algorithm 1, if Q is full-dimensional and has two vertices that maximize cx over Q, then $\arg\max_{x \in Q} cx$ is a bounded facet of Q ("optimal" facet). We assume that:

Assumption 6. *The optimal vertex v^* is the first vertex encountered when traversing the optimal facet in clockwise order.*

Therefore if two vertices maximize cx, the optimal facet is F_1 in our numbering. With this convention, inequality T computed in Step 4 at a given iteration will be tight for the vertex that is optimal at the successive iteration. Note that if $\arg\max_{x \in Q} cx$ is an unbounded facet of Q and the unique vertex is the first point encountered on this facet when traversing it clockwise, this facet is F_1 in our numbering, and if the unique vertex is the last point encountered on this facet when traversing it clockwise, this facet is F_m in our numbering. See Figure 2.

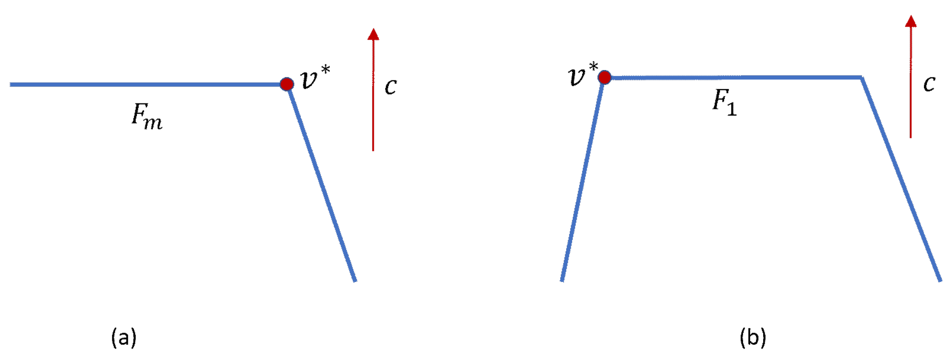

Figure 2: Illustration of Assumption 6

Remark 7. *A cutting plane algorithm typically works with a vertex solution, so it is natural to assume that P is pointed. (The integer program can be solved with at most one Chvátal cut when P is a polyhedron in the plane which is not pointed and it is known from standard results that this Chvátal cut has a polynomial size).*

If P is a pointed polyhedron but $\max\{cx : x \in P\}$ is unbounded, the integer program is either infeasible or unbounded. There are ways to overcome this, however it seems difficult to efficiently distinguish these two cases by only using cutting planes, even when $\dim(P) = 2$.

We will need the following theorem about the integer hulls of translated cones in the plane.

Theorem 8 ([10]). *Given a description $Ax \leq b$ of a translated cone $C \subseteq \mathbb{R}^2$, C_I has $O(\log \|A\|_\infty)$ facets. Furthermore, each facet of C_I has a description $ax \leq \beta$ where $\|a\|_\infty \leq \|A\|_\infty$.*

We also need the following lemma.

Lemma 9. *Let P be a pointed polyhedron in $\mathbb{R}^n$ such that $P_I \neq \emptyset$. Let u be the largest infinity norm of a vertex of P or P_I. Let $ax \leq \beta$ be an inequality which is valid for P_I but is not valid for P. Then $|\beta| \leq nu\|a\|_\infty$.*

Proof. As $P_I \neq \emptyset$, by Meyer's theorem (see e.g. Theorem 4.30 in [5]) P and P_I have the same recession cone. Therefore $\max_{x \in P} ax$ is finite and is larger than β, because $ax \leq \beta$ is not valid for P. Since finite maxima are attained at vertices, we have that $-nu\|a\|_\infty \leq \max_{x \in P_I} ax \leq \beta < \max_{x \in P} ax \leq nu\|a\|_\infty$, which proves the lemma. □

Next, we will prove that when the relaxation is a translated cone, the clockwise algorithm finds the solution in polynomial time. An important point of the proof is that when the tilt is not a Chvátal cut, the pivot remains unchanged. Otherwise, a new facet of the integer hull of the translated cone is obtained when the tilt is a Chvátal cut (see Lemma 3 (ii)). Since one has only a polynomial number of facets of the integer hull (see Theorem 8), it suffices to show that only a polynomial number of tilts are made about any one fixed pivot. This is achieved by appealing to Lemmas 2 and 3. We formalize this below.

Theorem 10. *Let $Ax \leq b$ be a description of a translated cone $C \subseteq \mathbb{R}^2$ and let $c \in \mathbb{Z}^2 \setminus \{0\}$ be such that $\max\{cx : x \in C\}$ is finite. Then the clockwise algorithm solves the integer program $\max\{cx : x \in C \cap \mathbb{Z}^2\}$ in $O((\log \|A\|_\infty)^2)$ iterations. Furthermore, there is a polynomial function $f(\cdot, \cdot)$ (independent of the data) such that every cut computed by the algorithm admits a description $ax \leq \beta$ where $\|a\|_\infty \leq \|A\|_\infty$ and $|\beta| \leq f(\|A\|_\infty, \|b\|_\infty)$.*

Proof. We use the same notation as in Definition 1 and the fact that since C is a translated cone, if $p \in \mathbb{Z}^2$ is a pivot element of a cut computed by the algorithm, then $p \in C$.

Let T_i be the cutting plane produced by the clockwise algorithm at iteration i, where we assume that T_0 is the Chvátal strengthening of H_1.

Claim 1. *If T_i defines the early facet of $C \cap T_0 \cap \cdots \cap T_i$, the clockwise algorithm computes an optimal solution in iteration $i + 1$.*

Proof of claim. In this case (T_{i-1}, T_i) is the new translated cone whose apex is the pivot element p_i of iteration i. As $p_i \in C \cap \mathbb{Z}^2$, at iteration $i + 1$ the algorithm determines that p_i is an optimal solution. ◇

T_i is the tilt (with pivot element p_i) with respect to the translated cone (T_{i-1}, H_2). Also recall that by Lemma 3 (i), T_i is Chvátal strengthened. This fact will be important because we will work with the translated cone (T_i, H_2) below and use notions from Definition 1 and results based on these notions, which assume that the facet $H_1 = T_i$ of the translated cone is Chvátal strengthened.

Claim 2. *There is a polynomial function $f(\cdot, \cdot)$ such that T_i admits a description $ax \leq \beta$ where $\|a\|_\infty \leq \|A\|_\infty$ and $|\beta| \leq f(\|A\|_\infty, \|b\|_\infty)$.*

Proof of claim. By induction on i. The base case $i = 0$ is trivial.

We first show that T_i admits a description $ax \le \beta$ where $\|a\|_\infty \le \|A\|_\infty$. If T_i defines a Chvátal cut with respect to the translated cone (T_{i-1}, H_2), as by induction T_{i-1} satisfies the claim, we are done by Lemma 3 (ii) and Theorem 8.

So we assume that $(T_{i-1}, H_2) \cap W_1 \neq \emptyset$ (where W_1 is as in Definition 1 with respect to (T_{i-1}, H_2)). Consider the translated cone (T_{i-1}, H_2'), where H_2' is the translation of H_2 through y' (see Figure 1(ii)). As T_i is the tilt with respect to (T_{i-1}, H_2), we have that $p_i, y' \in T_i^= \cap \mathbb{Z}^2$. Furthermore, T_{i-1} satisfies the claim by induction. It follows that T_i is a facet of the integer hull of (T_{i-1}, H_2') and, by Theorem 8, T_i admits a description $ax \le \beta$ where $\|a\|_\infty \le \|A\|_\infty$.

Let u be the largest infinity norm of a vertex of C or C_I. Then u is bounded by a polynomial function of $\|A\|_\infty$ and $\|b\|_\infty$ (see, e.g., [12, Theorems 10.2 and 17.1]). Therefore, by Lemma 9, there is a polynomial function $f(\cdot,\cdot)$ such that $|\beta| \le f(\|A\|_\infty, \|b\|_\infty)$. This completes the proof of the claim. ◇

We finally show that in $O((\log \|A\|_\infty)^2)$ iterations the clockwise algorithm finds an optimal solution to the program $\max\{cx : x \in C \cap \mathbb{Z}^2\}$. By Claim 1 if the cut T_i becomes the early facet in iteration i, then the algorithm finds an optimal solution in iteration $i+1$. By Claim 2 all cuts $T_1, \dots, T_i$ admit a description $ax \le \beta$ where $\|a\|_\infty \le \|A\|_\infty$ and $|\beta| \le f(\|A\|_\infty, \|b\|_\infty)$. Therefore, by Theorem 8, it suffices to show that within at most $O(\log \|A\|_\infty)$ iterations beyond any particular iteration i, the algorithm either finds an optimal solution or computes the facet adjacent to T_i of the integer hull of the translated cone (T_i, H_2). Note that, as all pivot elements are in C, this is also a facet of C_I.

By Lemma 3 (ii), the facet adjacent to T_i of the integer hull of the translated cone (T_i, H_2) is obtained when $W_1 \cap (T_i, H_2) = \emptyset$, i.e., when T_{i+1} is a Chvátal cut (this W_1 is as in Definition 1 with respect to the translated cone (T_i, H_2)). If $W_1 \cap (T_i, H_2) \neq \emptyset$, by Lemma 3 (iii), we have that $0 < dy' - \delta \le \frac{dq-\delta}{2}$, where $dx \le \delta$ is the inequality defining H_2 in the description of (T_i, H_2). We now observe that the new translated cone to be processed is (T_{i+1}, H_2) with apex y'. Moreover, the pivot remains p for the next iteration. Further, it is important that the facet H_2 has not changed. Thus, one may iterate this argument and since $dy' - \delta \in \mathbb{Z}$ at every iteration, by Lemma 2, after at most $O(\log \|A\|_\infty)$ iterations the algorithm will produce a Chvátal cut. □

Corollary 11. *Any cut derived during the execution of the clockwise cutting plane algorithm on any pointed, full-dimensional polyhedron P admits a description $ax \le \beta$ where $\|a\|_\infty \le \|A\|_\infty$ and $|\beta| \le f(\|A\|_\infty, \|b\|_\infty)$, where $f(\cdot,\cdot)$ is the function from Theorem 10.*

Proof. The argument in Claim 2 of the proof of Theorem 10 also applies here to show that $\|a\|_\infty \le \|A\|_\infty$, since we introduce new cutting planes by processing translated cones formed by original facet defining inequalities of P or the previous cuts.

To argue for the right hand side β, we consider two cases: when $P_I \neq \emptyset$ and when $P_I = \emptyset$. In the first case, the result follows from Lemma 9. In the second case, we again break the proof into two cases: when P is bounded and when P is unbounded. When P is bounded, note that all the cuts except for the last one must be valid for at least one vertex of P (otherwise, we would have proved infeasibility). For any such cut, let w be the vertex that is valid and let w' be a vertex that is cut off (which must exist because no cut in the algorithm is valid for P). Then, $-2\|w\|_\infty\|a\|_\infty \le aw \le \beta < \max_{x\in P} ax \le 2\|w'\|_\infty\|a\|_\infty$. This proves the claim for all cuts except for the last one. The last cut is obtained from a translated cone defined by original inequalities and/or previous cuts. Theorem 10 now applies.

We now tackle the case when P is unbounded and $P_I = \emptyset$. This implies that P has two unbounded facets. Let the half-planes corresponding to these unbounded facets be H_1 and H_2. We must have that $H_1 \cap H_2$ is a split set. Let the cuts introduced by the algorithms be $T_1, \ldots, T_k$. While at least one vertex of P survives, the argument from above can be used to bound $|\beta|$. Thus, let $i \in \{1, \ldots, k\}$ be the smallest index such that T_i cuts off every vertex of P. T_i must be derived from a translated cone using original inequalities and/or previous cuts and Theorem 10 applies. If $i = k$, then we are done. Otherwise, after iteration i we have a polyhedron with three facets H_1, T_i and H_2, in clockwise order. If the new optimal vertex is at the intersection of H_1 and T_i, then the Chvátal strengthening of H_1 gives an empty polyhedron and the algorithm terminates. Else, if the new optimal vertex is the intersection of T_i and H_2, then the new cutting plane T_{i+1} will be a tilt of T_i and will therefore also cut off every vertex of P. Repeating this argument, it follows that for all $i \leq j \leq k$, the half-plane T_j cuts off every vertex of P, and $T_j \cap P = T_j \cap H_1 \cap H_2$. Moreover, all the cuts $T_i, \ldots, T_{k-1}$ are valid for the integer hull of the translated cone formed by T_i and H_2, and Theorem 10 applies. Finally, T_k is either a Chvátal cut obtained by strengthening H_1, or is a cut valid for the translated cone formed by T_{k-1} and H_2. In either case, we are done by previous arguments. $\square$

We now extend Theorem 10 to handle the case of a general pointed polyhedron $P \subseteq \mathbb{R}^2$.

Definition 12. *(Facet ordering) We let Q_i be the polyhedron computed at the beginning of iteration i of the clockwise algorithm and T_i be the cutting plane computed at iteration i. We start our iterations at $i = 0, 1, \ldots$, so $Q_0 = P$ and $Q_i = Q_{i-1} \cap T_{i-1}$. When Q_i is full-dimensional, we let $F_{i,1}, \ldots, F_{i,m_i}$ be the facets of Q_i so that the optimal vertex of Q_i is the apex of $(F_{i,m_i}, F_{i,1})$ and T_i is either the Chvátal strengthening or the tilt of F_{i,m_i} with respect to $(F_{i,m_i}, F_{i,1})$.*

Note that, when Q_i is full-dimensional, T_i is either the early or the late facet of Q_{i+1}, as T_i defines the optimal vertex chosen by the algorithm.

Definition 13. *(Potentially late and potentially early facets) Given vectors a, b, we define $\angle(a, b)$ as the clockwise angle between a and b, starting from a. When Q_i is full-dimensional, let $a_{i,1}, \ldots, a_{i,m_i}$ be the normals of $F_{i,1}, \ldots, F_{i,m_i}$ (as defined in Definition 12). Then $\angle(a_{i,m_i}, c) < 180°$ and $\angle(c, a_{i,1}) < 180°$. We define a facet F of Q_i with normal a* potentially late *if either $F = F_{i,m_i}$ or $0 < \angle(a, c) < 180°$ and* potentially early *if either $F = F_{i,1}$ or $0 < \angle(c, a) \leq 180°$. Note that if a facet of Q_i satisfies $\angle(c, a) = 180°$, then it cannot define the optimal vertex of Q_j, $j > i$.*

Lemma 14. *Consider iterations i and $j > i$ and let Q_i and Q_j be the full-dimensional polyhedra defined in Definition 12 at these iterations. If F is a potentially early facet of Q_i, then it cannot become a potentially late facet of Q_j and if F is a potentially late facet of Q_i, then it cannot become a potentially early facet of Q_j.*

Proof. Let a be the normal of F. The result is obvious when $\angle(a, c) > 0$ and $\angle(c, a) > 0$. If $\angle(a, c) = 0$, i.e., $F = \arg\max_{x \in Q_i} cx$, then F remains potentially late or potentially early, by the choice of the optimal vertex; see Assumption 6. $\square$

Definition 15. *(Families of facets and tilts) Given a full-dimensional pointed polyhedron $P = Q_0 \subseteq \mathbb{R}^2$ and an objective vector $c \in \mathbb{Z}^2 \setminus \{0\}$ such that $\max\{cx : x \in P\}$ is finite, let $F_{0,1}, \ldots, F_{0,k}$ be the potentially early facets of Q_0 and $F_{0,k+1}, \ldots, F_{0,m_0}$ be the potentially late facets of Q_0. We say that facet $F_{0,\ell}$ of P belongs to* family ℓ *and we recursively assign a cut T_i produced at iteration*

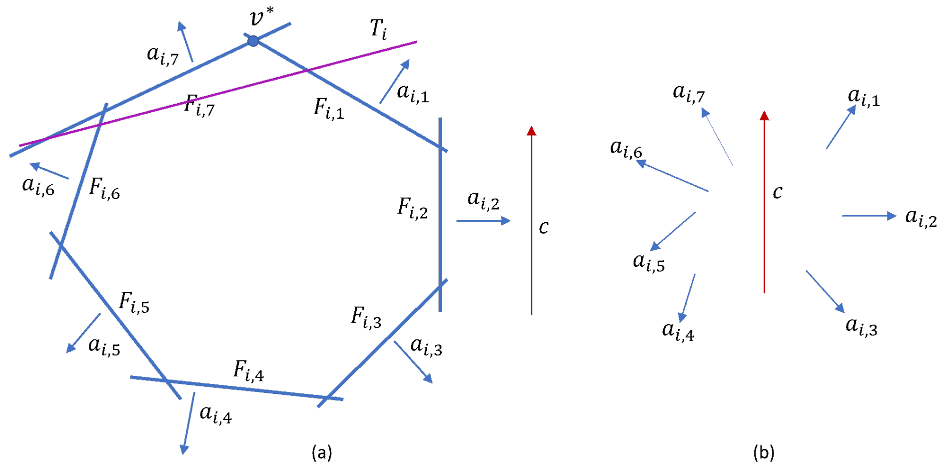

Figure 3: Illustration of the proof of Theorem 17. In (a), $m_i = 7$, $F_{i,1}$ is the early facet, and $F_{i,7}$ is the late facet as defined in Definition 5. In (b), the left-hand side normals correspond to potentially late families, while those at right-hand side belong to potentially early families. Thus, $E_i = 3$ (L_i will depend on the original set of facets).

i of the algorithm to the family of the late inequality that is used to produce T_i. We finally say that family ℓ is extinct *at iteration k if no facet of Q_k belongs to family ℓ.*

Remark 16. *By Lemma 14, no facet that is potentially early can become potentially late and vice versa; therefore, all cuts produced by the clockwise algorithm belong to the $m-k$ families associated with the potentially late facets of Q_0 (assuming the input to the algorithm is full-dimensional; otherwise, the algorithm terminates in at most two iterations —see Step 2 in Algorithm 1).*

Theorem 17. *Let $Ax \leq b$ be a description of a pointed polyhedron $P \subseteq \mathbb{R}^2$ with m facets, and $c \in \mathbb{Z}^2 \setminus \{0\}$ be such that $\max\{cx : x \in P\}$ is finite. Then the clockwise algorithm solves the integer program $\max\{cx : x \in P \cap \mathbb{Z}^2\}$ in $O(m(\log \|A\|_\infty)^2)$ iterations.*

Proof. We refer to the definitions of Q_i and T_i, and when Q_i is full-dimensional, to the definitions of $F_{i,1}, \ldots, F_{i,m_i}$ with corresponding normals $a_{i,1}, \ldots, a_{i,m_i}$ (see Definition 12 and Definition 13). In this case, we assume $F_{i,1}, \ldots, F_{i,k_i}$ are potentially early and $F_{i,k_i+1}, \ldots, F_{i,m_i}$ are potentially late. Moreover, let E_i be the number of facets of Q_i that are potentially early (i.e., $E_i = k_i$) and let L_i be the number of families that are not extinct at iteration i and such that the last inequality added to the family is potentially late. See Figure 3 for an illustration of these notations. Figure 4 illustrates some iterations of a potential run of the algorithm.

By Theorem 10, there exists a function $z \mapsto g(z)$, where $g \in O((\log z)^2)$, such that the clockwise algorithm applied to any translated cone with description $A'x \leq b'$ terminates in $g(\|A'\|_\infty)$ iterations. Define $t := g(\|A\|_\infty)$.

Claim. *Assume $\dim(Q_i) = 2$. Let j be the largest natural number such that at iteration $i + j$, $\dim(Q_{i+j}) = 2$, $F_{i+j,m_{i+j}}$ is in the same family as F_{i,m_i}, and $F_{i+j,1}$ and $F_{i,1}$ are both defined by the same normal. Then*

1. *$j \leq t$, and*

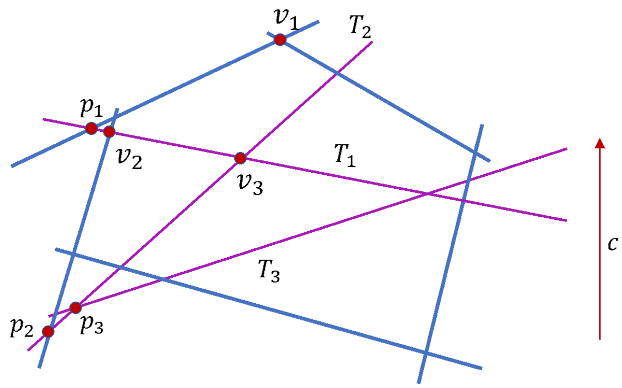

Figure 4: Illustration of an instance P with 3 iterations of the algorithm producing cuts T_1, T_2 and T_3 with pivots p_1, p_2 and p_3 respectively. The consecutive LP optimal solutions are labeled as v_1, v_2 and v_3.

2. *either* $\dim(Q_{i+j+1}) \leq 1$*, or the algorithm terminates at iteration* $i+j+1$*, or* $E_{i+j+1} + 2L_{i+j+1} \leq E_i + 2L_i - 1$.

Proof of claim. 1. follows from Theorem 10 and Corollary 11, after observing that during iterations $i, \dots, i+j$ the algorithm computes the same cuts as those that it would compute if the polyhedron at iteration i was the translated cone $(F_{i,m_i}, F_{i,1})$.

We now prove 2. Suppose $\dim(Q_{i+j+1}) = 2$. We will establish that either the algorithm terminates at iteration $i+j+1$ or $E_{i+j+1} + 2L_{i+j+1} \leq E_i + 2L_i - 1$. Let $(F_{i+j+1,m_{i+j+1}}, F_{i+j+1,1})$ be the translated cone at iteration $i+j+1$. Recall that, by the choice of the optimal vertex, T_{i+j} must be either $F_{i+j+1,m_{i+j+1}}$ or $F_{i+j+1,1}$. We distinguish two cases.

Case 1 Assume $T_{i+j} = F_{i+j+1,m_{i+j+1}}$. Since T_{i+j} is in the same family as $F_{i+j,m_{i+j}}$, which is in the same family as F_{i,m_i}, by definition of q we must have that $F_{i,1}$ is distinct from $F_{i+j+1,1}$. Then $F_{i,1}$ is a redundant inequality for Q_{i+j+1}. Therefore, by Lemma 14, $E_{i+j+1} \leq E_i - 1$ and $L_{i+j+1} = L_i$. Thus, $E_{i+j+1} + 2L_{i+j+1} \leq E_i + 2L_i - 1$.

Case 2 Assume $T_{i+j} = F_{i+j+1,1}$. Let p be the pivot element of the translated cone $(F_{i+j,m_{i+j}}, F_{i+j,1})$. If p is feasible, i.e., $p \in P \cap \mathbb{Z}^2$, then p will be the optimal integral vertex in the iteration $i+j+1$ and the algorithm terminates. Else p is infeasible. This means that p must violate the inequality defining $F_{i+j+1,m_{i+j+1}}$. Consider the facet F of P that is the original facet of P from the same family as T_{i+j}. We will now show that F and all the inequalities in this family except for T_{i+j} are redundant for $(F_{i+j+1,m_{i+j+1}}, T_{i+j})$ and therefore for $Q_{i+j+1} \subseteq (F_{i+j+1,m_{i+j+1}}, T_{i+j})$. Since we have processed this family during the algorithm, there must have been an optimal vertex defined by (F, F') for some inequality F' that is facet defining at some point during the algorithm. Let Q' be the polyhedron computed as in Definition 12 at the iteration of the algorithm when the vertex v defined by (F, F') was optimal.

Since $F_{i+j+1,m_{i+j+1}}$ is not redundant for Q' (as it is not redundant for Q_{i+j+1} which is a subset of Q'), and the inequality defining $F_{i+j+1,m_{i+j+1}}$ is valid for the optimal vertex v of Q', its normal vector cannot be contained in the cone generated by the normals of F and F'. Since the vertex defined by (F, F') was optimal for the relaxation Q', c is contained in the cone generated by the normals of F and F'. Thus, the normal of $F_{i+j+1,m_{i+j+1}}$ cannot be contained in the cone generated by the normal of F and c. This means that the normal of F is contained in the cone between the normal of $F_{i+j+1,m_{i+j+1}}$ and c, since both F and $F_{i+j+1,m_{i+j+1}}$ are late facets at some time during the algorithm. Moreover, the normals of the inequalities in the family of F are contained in the cone generated by the normal of F and T_{i+j}, and therefore, in the cone generated by the normals of $F_{i+j+1,m_{i+j+1}}$ and T_{i+j}. Since our current optimal vertex is defined by $F_{i+j+1,m_{i+j+1}}$ and T_{i+j}, all these inequalities from the family must be redundant for $(F_{i+j+1,m_{i+j+1}}, T_{i+j})$.

Thus, we have established that F and all the inequalities in its family except for T_{i+j} are redundant for Q_{i+j+1}. Since T_{i+j} is from the same family and is early at iteration $i+j+1$, we must have $L_{i+j+1} \leq L_i - 1$ by Lemma 14. Moreover, T_{i+j} is the only new early facet, and therefore, $E_{i+j} \leq E_i + 1$ by Lemma 14. Thus, $E_{i+j+1} + 2L_{i+j+1} \leq E_i + 2L_i - 1$. This completes the proof of the claim. ◇

By the above claim, in at most $O(\log \|A\|_\infty^2)$ iterations after iteration i, either the algorithm terminates or the number $E_i + 2L_i$ must decrease by at least 1. Since the maximum value of $E_i + 2L_i$ is at most $2m$, we have the result. □

Remark 18. *The upper bound on the number of iterations given in the above theorem does not depend on c.*

Remark 19. *By Lemma 3, when $C \cap W_1 \neq \emptyset$, the tilt T produced by the algorithm may not be a facet of the split closure of the translated cone C. However, this property is crucial for the convergence of the algorithm. Indeed, if the "best cut" is used, the algorithm may not converge as shown by the following example.*

Define $p_0 := 3$ and $p_{i+1} := 2p_i - 2$ for all integers $i \geq 1$. Given $i \geq 0$, consider the following integer program, which we denote by P_i:

$$\max\ x_2 \tag{1}$$
$$s.t.\ x_1 \leq 4 \tag{2}$$
$$(2p_i - 1)x_1 - (4p_i - 4)x_2 \geq 0 \tag{3}$$
$$5x_1 - 8x_2 \geq 0 \tag{4}$$
$$x_1,\ x_2 \in \mathbb{Z} \tag{5}$$

(Note that for $i = 0$ the inequalities Eq. (3) and Eq. (4) coincide.) The optimal solution of the continuous relaxation is $\left(4, \frac{2p_i-1}{p_i-1}\right) \notin \mathbb{Z}^2$, which is the unique point satisfying both constraints Eq. (2) and Eq. (3) at equality.

We will use the same notation as in Definition 1, where H_1 and H_2 are the two half-planes defined by Eq. (3) and Eq. (2), respectively. Since $(2p_i - 1)$ and $(4p_i - 4)$ are coprime numbers, $\hat{H}$ is defined by the equation $(2p_i - 1)x_1 - (4p_i - 4)x_2 = 1$. Combined with the fact that the pivot is $p = (0,0)$, this implies that $q = (4p_i - 4, 2p_i - 1)$.

We claim that $x = (-2p_i + 3, -p_i + 1)$ and $y = (2p_i - 1,\ p_i)$. This follows from the following three observations: (i) these two points are integer and belong to $\hat{H}$; (ii) $-2p_i + 3 < 4 < 2p_i - 1$ (because this is true for $i = 0$ and p_i increases as i increases); (iii) $y - x = q - p$.

It follows that $W_1^=$ is defined by the equation $(p_i - 1)x_1 - (2p_i - 3)x_2 = 1$. This line intersects the edge $\left\{(x_1, x_2) : x_1 = 4,\ x_2 \leq \frac{2p_i-1}{p_i-1}\right\}$ of the continuous relaxation at $q' = \left(4, \frac{4p_i-5}{2p_i-3}\right)$ (note that $\frac{4p_i-5}{2p_i-3} < \frac{2p_i-1}{p_i-1}$). Thus the strongest cut would be $(4p_i - 5)x_1 - (8p_i - 12)x_2 \geq 0$. Since $p_{i+1} = 2p_i - 2$, *the cut can be written as* $(2p_{i+1}-1)x_1-(4p_{i+1}-4)x_2 \geq 0$. *When we add this cut to the continuous relaxation, Eq.* (3) *becomes redundant and we obtain problem* P_{i+1}. *Then this procedure never terminates.*

3 Polynomiality of the split closure

In this section we prove the following result:

Theorem 20. *Let* $Ax \leq b$ *be a description of a polyhedron* $P \subseteq \mathbb{R}^2$ *consisting of* m *inequalities. Then the split closure of* P *admits a description whose size is polynomial in* m, $\log \|A\|_\infty$ *and* $\log \|b\|_\infty$.

We will make use of the following result, which holds in any fixed dimension.

Theorem 21. *[4] Let* $d \geq 1$ *be a fixed integer and let* $Ax \leq b$ *be a description of a polyhedron* $P \subseteq \mathbb{R}^d$ *consisting of* m *inequalities. Then the Chvátal closure of* P *admits a description whose size is polynomial in* m, $\log \|A\|_\infty$ *and* $\log \|b\|_\infty$.

Because of Theorem 21, in order to prove Theorem 20 it is sufficient to show that the intersection of all the split cuts for P that are not Chvátal cuts is a polyhedron that admits a description of polynomial size.

We now start the proof of Theorem 20. We can assume that $P \subseteq \mathbb{R}^2$ is pointed, as otherwise it is immediate to see that the split closure of P is P_I and is defined by at most two inequalities. The following result holds in any dimension.

Lemma 22 ([1]; see also [5, Corollary 5.7])**.** *The split closure of* P *is the intersection of the split closures of all the corner relaxations of* P *(i.e., relaxations obtained by selecting a feasible or infeasible basis of the system* $Ax \leq b$*).*

Since there are at most $\binom{m}{2}$ corner relaxations of P (i.e., bases of $Ax \leq b$), because of Lemma 22 in the following we will work with a corner relaxation of P, which we denote by C. Thus C is a full-dimensional translated pointed cone. We denote its apex by v.

Definition 23. *(Effective split sets and effective split cuts) We say that a split set* S *is* effective *for* C *if* v *lies in its interior; note that this happens if and only if there is a split cut for* C *derived from* S *that cuts off* v. *Such a split cut will also be called effective.*

Since C is a translated cone, for every effective split disjunction (π, π_0) we have $C^{\pi,\pi_0} = C \cap H$ for a unique split cut H derived from this disjunction. In the following, whenever we say "*the* split cut derived from a given disjunction" we refer to this specific split cut. Note that when the boundary of an effective split set S intersects the facets of C in precisely two points, the split cut derived from S is delimited by the line containing these two points, while when the boundary of S intersects the facets of C in a single point, the line delimiting the split cut derived from S contains this point and is parallel to the lineality space of S. (In the latter case, the split cut is necessarily a Chvátal cut.)

In the following, we let $\mathrm{intr}(X)$ denote the interior of a set $X \subseteq \mathbb{R}^2$.

Lemma 24. *Every effective split cut for* C *is of one of the following types:*

1. *a Chvátal cut;*

2. *a cut derived from a split set S such that $S \cap \operatorname{intr}(C_I) \neq \emptyset$; in this case, both lines delimiting S intersect the same facet of C_I.*

Proof. Consider any effective split cut given by a split set S. We look at two cases: the recession cone of C contains a recession direction of S, or not. In the second case, one of the boundary of S intersects with both facets of C, and the other one does not intersect with C since S is assumed to be effective. Then the split cut is a Chvátal cut. In the first case, if the recession direction of S is on the boundary of the recession cone of C, the two boundaries of S are parallel to a facet of C. In this case as well, the split cut is a Chvátal cut. Finally, suppose that the interior of the recession cone of C contains a recession direction of S. Since C and C_I have the same recession cone, S intersects the interior of C_I. As no vertex of C_I can be in the interior of S, the bounding lines of S must intersect the same facet of C_I. □

Definition 25. *Let $F_I^1, \dots, F_I^n$ be the facets of C_I. For every $i \in \{1, \dots, n\}$ we denote by ℓ_I^i the line containing F_I^i. Furthermore, we define $\widehat{\ell_I^i}$ as the unique line with the following properties:*

1. *$\widehat{\ell_I^i}$ is parallel to ℓ_I^i;*

2. *$\widehat{\ell_I^i}$ contains integer points;*

3. *there is no integer point strictly between ℓ_I^i and $\widehat{\ell_I^i}$;*

4. *$\widehat{\ell_I^i} \cap C_I = \emptyset$.*

Given two split cuts H, H' for C, we say that H dominates H' if $C \cap H \subseteq C \cap H'$.

Lemma 26. *Fix $i \in \{1, \dots, n\}$ and define the split set $S := \operatorname{conv}(\ell_I^i, \widehat{\ell_I^i})$. If $v \in \operatorname{intr}(S)$, then the split cut for C derived from S is a Chvátal cut that dominates every split cut derived from a split set intersecting F_I^i.*

Proof. Since $v \in \operatorname{intr}(S)$, the facets of C do not intersect $\widehat{\ell_I^i}$. This implies that the split cut for C derived from S is a Chvátal cut.

Let S' be any split set intersecting F_I^i, and denote by h_1, h_2 the two lines delimiting S'. Since there is no integer point in $\operatorname{intr}(S')$, both h_1 and h_2 intersect F_I^i. We denote by x^1 (resp., x^2) the intersection point of h_1 (resp., h_2) and F_I^i. Also we define y^1 (resp., y^2) as the intersection point of h_1 (resp., h_2) and $\widehat{\ell_I^i}$. Since $x^1, x^2 \in C$ and $y^1, y^2 \notin C$ (as $\widehat{\ell_I^i}$ does not intersect C), h_1 and h_2 intersect the facets of C in two points contained in $\operatorname{intr}(S)$. This implies that any split cut derived from S' is dominated by the Chvátal cut derived from S. □

Definition 27. *(Unit interval) Given a line ℓ containing integer points, we call each closed segment whose endpoints are two consecutive integer points of ℓ a* unit interval *of ℓ.*

Observation 28. *Any split set can intersect at most one unit interval of a given line not parallel to the split.*

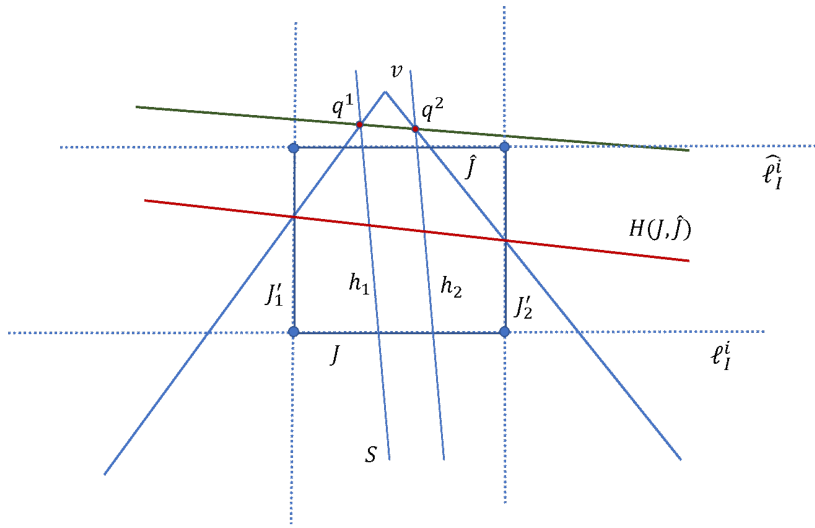

Figure 5: Illustration of the notation used in the proof of Lemma 31. The red cutting plane $H(J, \hat{J})$ dominates the dark green cutting plane derived from the split S.

Definition 29. *Fix $i \in \{1, \ldots, n\}$. Given a unit interval J of ℓ_I^i and a unit interval $\hat{J}$ of $\widehat{\ell_I^i}$, there exists a unique parallelogram of area 1 having J and $\hat{J}$ as two of its sides. We denote by $S(J, \hat{J})$ the split set delimited by the lines containing the other two sides of this parallelogram. If $S(J, \hat{J})$ is effective, we denote by $H(J, \hat{J})$ the split cut for C derived from $S(J, \hat{J})$.*

Lemma 30. *Fix $i \in \{1, \ldots, n\}$ such that $v \notin \operatorname{intr}(\operatorname{conv}(\ell_I^i, \widehat{\ell_I^i}))$. Then there exists a unique unit interval $\hat{J}$ of $\widehat{\ell_I^i}$ such that $\hat{J} \cap C \neq \emptyset$. Furthermore, for each unit interval J of ℓ_I^i contained in F_I^i, $S(J, \hat{J})$ is an effective split set.*

Proof. The existence of $\hat{J}$ follows from the assumption $v \notin \operatorname{intr}(\operatorname{conv}(\ell_I^i, \widehat{\ell_I^i}))$. Furthermore, $\hat{J}$ is unique because, by definition of $\widehat{\ell_I^i}$, there are no integer points in $C \cap \widehat{\ell_I^i}$.

We now prove that for each unit interval J of ℓ_I^i contained in F_I^i, $S(J, \hat{J})$ is an effective split cut. Up to a unimodular transformation, we can assume that $J = \{x \in \mathbb{R}^2 : x_2 = 0,\ 0 \leq x_1 \leq 1\}$ and $\hat{J} = \{x \in \mathbb{R}^2 : x_2 = 1,\ 0 \leq x_1 \leq 1\}$. Then the split set $S(J, \hat{J})$ is defined by the inequalities $0 \leq x_1 \leq 1$.

Since the second coordinate of v is $v_2 \geq 1$ and C_I is contained in the half-plane defined by $x_2 \leq 0$ (as this inequality induces facet F_I^i of C_I), it follows that both facets of C intersect the lines defined by $x_2 = 0$ and $x_2 = 1$. Thus one facet of C contains points $(a_1, 0)$ and $(b_1, 1)$, and the other facet contains points $(a_2, 0)$ and $(b_2, 1)$, where $a_1 \leq 0$, $a_2 \geq 1$ and $0 < b_1 < b_2 < 1$. It is now straightforward to verify that v, which is the intersection point of the two facets, satisfies $0 < v_1 < 1$. This shows that $S(J, \hat{J})$ is an effective split set. $\square$

Lemma 31. *Fix $i \in \{1, \ldots, n\}$. Let S be a split set that gives an effective split cut of type 2 from Lemma 24, where S intersects F_I^i. Suppose that this split cut is not dominated by a Chvátal cut. Let J be the unit interval of ℓ_I^i that intersects both lines delimiting S (see Observation 28), and let $\hat{J}$ be the unit interval of $\widehat{\ell_I^i}$ such that $\hat{J} \cap C \neq \emptyset$ (see Lemma 30). Then:*

(i) both lines delimiting S intersect $\hat{J}$;

(ii) any cut produced by S is dominated by $H(J, \hat{J})$.

Proof. Up to a unimodular transformation, we can assume that $J = \{x \in \mathbb{R}^2 : x_2 = 0,\ 0 \le x_1 \le 1\}$ and $\hat{J} = \{x \in \mathbb{R}^2 : x_2 = 1,\ 0 \le x_1 \le 1\}$. Since the split cut derived from S is not dominated by a Chvátal cut, by Lemma 26 the apex v does not lie strictly between ℓ_I^i and $\widehat{\ell_I^i}$. In other words, $v_2 \ge 1$. Furthermore $0 < v_1 < 1$, as shown in the proof of Lemma 30.

Let h_1 and h_2 be the lines delimiting S, and define the segments $J_1' := \{x \in \mathbb{R}^2 : x_1 = 0,\ 0 \le x_2 \le 1\}$ and $J_2' := \{x \in \mathbb{R}^2 : x_1 = 1, 0 \le x_2 \le 1\}$. Since both h_1 and h_2 intersect J and there is no integer point strictly between h_1 and h_2, we have that $h_1 \cup h_2$ can contain points from the relative interior of at most one of J_1', J_2' and $\hat{J}$.

Assume that h_1 and h_2 intersect the relative interior of J_1'. Since h_1 and h_2 also intersect J, we have

$$h_1 = \{x \in \mathbb{R}^2 : x_2 = ux_1 + r_1\}, \qquad h_2 = \{x \in \mathbb{R}^2 : x_2 = ux_1 + r_2\},$$

for some $u < 0$, and $r_1, r_2 \in \mathbb{R}$.

Given any $\bar{x} \in h_1 \cap \{x \in \mathbb{R}^2 : x_2 \ge 1\}$, we have

$$\bar{x}_1 = \frac{\bar{x}_2 - r_1}{u} \le \frac{1 - r_1}{u} < 0.$$

Thus $h_1 \cap \{x \in \mathbb{R}^2 : x_2 \ge 1\} \subseteq \{x \in \mathbb{R}^2 : x_1 < 0\}$. Similarly, $h_2 \cap \{x \in \mathbb{R}^2 : x_2 \ge 1\} \subseteq \{x \in \mathbb{R}^2 : x_1 < 0\}$. As $0 < v_1 < 1$ and $v_2 \ge 1$, it follows that v does not lie strictly between h_1 and h_2, a contradiction.

A similar argument shows that h_1 and h_2 do not intersect the relative interior of J_2'. It follows that h_1 and h_2 intersect $\hat{J}$, and (i) is proven.

We now prove (ii). Since, by part (i), each of h_1 and h_2 intersects both J and $\hat{J}$, each of h_1 and h_2 intersects the boundary of C. Moreover, because S is an effective split set, $h_1 \cup h_2$ intersects the boundary of C in at most two points. It follows that each of h_1 and h_2 intersects the boundary of C in a single point, say q^1 and q^2, respectively. Note that $q_2^1 > 0$ and $q_2^2 > 0$, because h_1 and h_2 intersect J. Label q^1 and q^2 in such a way that q^1 (resp., q^2) belongs to the facet of C contained in the half-plane $x_1 \le v_1$ (resp., $x_1 \ge v_1$). See Figure 5.

If $0 < q_2^j < 1$ for some $j \in \{1, 2\}$, then $0 < q_1^j < 1$, because h_1 and h_2 intersect both J and $\hat{J}$. If $q_2^j \ge 1$, then again $0 < q_1^j < 1$, as $\{x \in C : x_2 \ge 1\} \subseteq \{x \in \mathbb{R}^2 : 0 < x_1 < 1\}$.

The split set $S(J, \hat{J})$ is effective by Lemma 30, and its boundary intersects the facets of C in two points r^1, r^2 that satisfy $r_1^1 = 0$ and $r_1^2 = 1$. Then r^1 (resp., r^2) is further from the apex than q^1 (resp., q^2) is, as q^1, q^2 and v all satisfy $0 < x_1 < 1$. It follows that the cut $H(J, \hat{J})$ dominates any split cut derived from S. □

Lemma 32. *Fix $i \in \{1, \dots, n\}$ such that $v \notin \operatorname{intr}(\operatorname{conv}(\ell_I^i, \widehat{\ell_I^i}))$, and let $\hat{J}$ be as in Lemma 30. Write $F_I^i = J_0 \cup \cdots \cup J_t$, where $J_0, \dots, J_t$ are the unit intervals contained in F_I^i ordered consecutively (i.e, $J_{k-1} \cap J_k$ only contains a single point for every $k \in \{1, \dots, t\}$). Then every point that violates $H(J_k, \hat{J})$ for some $k \in \{0, \dots, t\}$ also violates $H(J_0, \hat{J})$ or $H(J_t, \hat{J})$.*

See Figure 6 for an illustration of Lemma 32 and its proof.

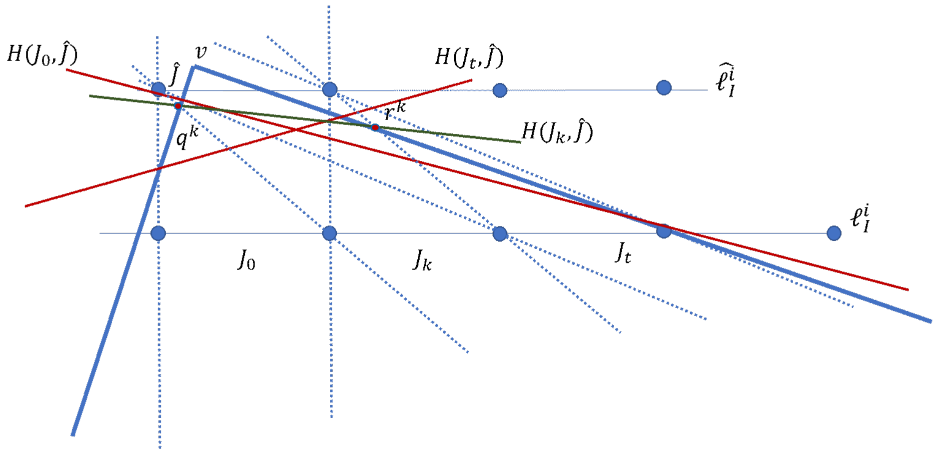

Figure 6: Illustration of the notation used in the proof of Lemma 32. The intersection of the two red cutting planes $H(J_0, \hat{J})$ and $H(J_t, \hat{J})$ from the leftmost and rightmost splits dominates all other split cuts $H(J_k, \hat{J}_0)$, $k = 0, \dots, t$, illustrated here in dark green.

Proof. Up to a unimodular transformation, we can assume that $\hat{J} = \{x \in \mathbb{R}^2 : x_2 = 1,\ 0 \le x_1 \le 1\}$ and $J_k = \{x \in \mathbb{R}^2 : x_2 = 0,\ k \le x_1 \le k+1\}$ for every $k \in \{0, \dots, t\}$. As argued in the proof of Lemma 30, one facet G_1 of C contains points $(a_1, 0)$ and $(b_1, 1)$, and the other facet G_2 contains points $(a_2, 0)$ and $(b_2, 1)$, where $a_1 \le 0$, $a_2 \ge 1$ (in fact, $a_2 - 1 \ge t \ge 0$) and $0 < b_1 < b_2 < 1$. Then the lines containing G_1 and G_2 are defined by the equations $x_1 + (a_1 - b_1)x_2 = a_1$ and $x_1 + (a_2 - b_2)x_2 = a_2$, respectively.

Given any $k \in \{0, \dots, t\}$, the lines delimiting the split set $S(J_k, \hat{J})$ are defined by the equations $x_1 + kx_2 = k$ and $x_1 + kx_2 = k + 1$. The intersection points of the former line with G_1 and of the latter line with G_2 are respectively the following:

$$q^k = \left(\frac{b_1 k}{b_1 - a_1 + k}, \frac{k - a_1}{b_1 - a_1 + k} \right), \qquad r^k = \left(\frac{b_2 k + b_2 - a_2}{b_2 - a_2 + k}, \frac{k + 1 - a_2}{b_2 - a_2 + k} \right).$$

Consider any $k \in \{0, \dots, t\}$. Since $H(J_k, \hat{J})$ is a half-plane that does not contain v, it is defined by an inequality of the form $c^k(x - v) \ge 1$, where $c^k \in \mathbb{R}^2$. Note that and $c^k(q^k - v) = c^k(r^k - v) = 1$, as q^k and r^k belong to the line delimiting $H(J_k, \hat{J})$.

Let $\bar{x}$ be any point in G_1 and $k \in \{0, \dots, t\}$. Then $\bar{x} = v + \mu(q^k - v)$ for some $\mu \ge 0$ and therefore

$$c^k(\bar{x} - v) = \mu c^k(q^k - v) = \mu = \frac{v_1 - \bar{x}_1}{v_1 - q_1^k}.$$

We claim that, for fixed $\bar{x} \in G_1$, the above right hand side is a concave function of k when k is considered as a continuous parameter in $[0, t]$. To simplify the argument, we will show that $\frac{v_1 - b_1}{v_1 - q_1^k}$ is a concave function of k: this is sufficient to establish the claim, as $v_1 - b_1 > 0$ and $v_1 - \bar{x}_1 > 0$.

We can calculate

$$v_1 - q_1^k = v_1 - \frac{b_1 k}{b_1 - a_1 + k} = \frac{(v_1 - b_1)k + v_1 b_1 - v_1 a_1}{b_1 - a_1 + k},$$

from which we obtain

$$\begin{aligned}\frac{v_1-b_1}{v_1-q_1^k}&=\frac{(v_1-b_1)k+(v_1-b_1)(b_1-a_1)}{(v_1-b_1)k+v_1b_1-v_1a_1}\\&=1+\frac{(v_1-b_1)(b_1-a_1)-(v_1b_1-v_1a_1)}{(v_1-b_1)k+v_1b_1-v_1a_1}\\&=1+\frac{b_1(a_1-b_1)}{(v_1-b_1)k+v_1b_1-v_1a_1}.\end{aligned}$$

Since $b_1>0$, $a_1-b_1<0$ and $v_1-b_1>0$, the last fraction above is of the form $\frac{\alpha}{\beta k+\gamma}$, where $\alpha<0$, $\beta>0$ and $\gamma\in\mathbb{R}$. It is immediate to verify that such a function of k is concave for $k>-\frac{\gamma}{\beta}$. In our context, this condition reads $k>-\frac{v_1(b_1-a_1)}{v_1-b_1}$, which is a negative number. Thus the function $k\mapsto c^k(\bar{x}-v)$ is concave over the domain $[0,t]$. A similar argument shows that, for any fixed $\bar{x}\in G_2$, the function $k\mapsto c^k(\bar{x}-v)$ is concave over $[0,t]$ (using the fact that $t\le a_2-1$).

If $\bar{x}$ is any point in C, then we can write $\bar{x}=\lambda x^1+(1-\lambda)x^2$, where $x^1\in G_1$, $x^2\in G_2$ and $0\le\lambda\le 1$. Then $c^k(\bar{x}-v)=\lambda c^k(x^1-v)+(1-\lambda)c^k(x^2-v)$ is a concave function of k. This implies that the minimum of the set $\{c^k(\bar{x}-v):k\in\{0,\dots,t\}\}$ is achieved for $k=0$ or $k=t$. In particular, if $\bar{x}$ violates $H(J_k,\hat{J})$ for some $k\in\{0,\dots,t\}$, then $\min\{c^0(\bar{x}-v),c^t(\bar{x}-v)\}\le c^k(\bar{x}-v)<1$, and thus $\bar{x}$ also violates $H(J_0,\hat{J})$ or $H(J_t,\hat{J})$. □

Theorem 33. *The number of facets of the split closure of a translated cone $C\subseteq\mathbb{R}^2$ is at most twice the number of facets of C_I plus the number of facets of the Chvátal closure of C.*

Proof. Let $E\subseteq\{1,\dots,n\}$ be the index set of the facets of C_I such that v does not lie strictly between ℓ_I^i and $\widehat{\ell_I^i}$. By Lemma 30, for every $i\in E$ there exists a unit interval $\hat{J}^i$ of $\widehat{\ell_I^i}$ such that $\hat{J}^i\cap C\neq\emptyset$. Moreover, let $J_0^i,\dots J_{t_i}^i$ be the unit intervals of ℓ_I^i contained in F_I^i, ordered consecutively. Let Q denote the Chvátal closure of C. We show that the split closure of C is given by

$$Q\cap\bigcap_{i\in E}\left(H(J_0^i,\hat{J}^i)\cap H(J_{t_i}^i,\hat{J}^i)\right),\tag{6}$$

which suffices to prove the theorem.

Consider any split cut H derived from a split set S. Since Eq. (6) is contained in Q, we may assume that H is not dominated by a Chvátal cut. Therefore it must be of type 2 in Lemma 24, and thus there is a facet F_I^i of C_I that intersects the two lines delimiting S. By Lemma 26, $i\in E$. By Lemma 31 part (ii), H is dominated by $H(J_k^i,\hat{J}^i)$ for some $k\in\{0,\dots,t_i\}$. By Lemma 32, $H(J_k^i,\hat{J}^i)$ is in turn dominated by $H(J_0^i,\hat{J}^i)\cap H(J_{t_i}^i,\hat{J}^i)$. □

To conclude the proof of Theorem 20, we note that by Lemma 22, Theorem 33 and Theorem 8, the number of inequalities needed to define the split closure of P is polynomial in m, $\log\|A\|_\infty$ and $\log\|b\|_\infty$. Furthermore, the above arguments show that the size of every inequality is polynomially bounded. (However, it is known that also in variable dimension every facet of the split closure of a polyhedron P is polynomially bounded; see, e.g., [5, Theorem 5.5].)

Remark 34. *Given a translated cone $C\subseteq\mathbb{R}^2$, the arguments used in this section show that, for every facet F_I^i of C_I, the split closure C' of C is strictly contained in the interior of the half-plane* *delimited by $\widehat{\ell}^i_I$ and containing C (where we adopt the notation introduced in Definition 25). This implies that the Chvátal closure of C' is C_I. In particular, the split rank of C is at most 2.*

Now let P be a polyhedron in $\mathbb{R}^2$. It is folklore that the integer hull of P is the intersection of the integer hulls of all the corner relaxations of P. (This is not true in higher dimensions.) Then, by the previous argument, the split rank of P is at most 2.

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