Introducing An RME-Based Task Sequence to Support the Guided Reinvention of Vector Spaces

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In this paper, we introduce an RME-based (Freudenthal, 1991) task sequence intended to support the guided reinvention of the linear algebra topic of vector spaces. We also share the results of a paired teaching experiment (Steffe & Thompson, 2000) with two students. The results show how students can leverage their work in the problem context to develop more general notions of Null Space. This work informs further revisions to the task statements for using these materials in a whole-class setting.

Keywords: Linear Algebra, Inquiry-Oriented, Vector Spaces, Realistic Mathematics Education

Linear Algebra is a critical course for many majors in Science, Technology, Engineering, and Mathematics: students learn important computational methods, as well as begin to work with mathematical definitions, proofs, and theory. A survey of research on the teaching and learning of linear algebra identified significant bodies of recent research in the areas of span, linear independence, basis, and eigentheory (Stewart et al., 2019). Notably less work has been done in the importantly related areas of vector spaces and subspaces such as null and column spaces.

Literature Review

Researchers who have written about student learning related to the topic of vector spaces and subspaces have highlighted the challenges inherent and somewhat unique to teaching the topic, which was "not introduced to solve a specific open problem but rather to solve different problems with the same tools in an economic formal way" (Grenier-Boley, 2014, p. 439). A central theme in this literature focuses on the importance of how students reason about sets, binary operations, closure, and linear combinations of vectors (Britton & Henderson, 2009; Maracci, 2008; Mutambara & Bansilal, 2018; Parraguez & Oktac, 2010; Wawro et al., 2011). This is perhaps unsurprising when one considers vector spaces to basically be sets of vectors that are closed under linear combinations. In our work, we follow the recommendations of the Linear Algebra Curriculum Study Groups in not taking on an abstract treatment of vector spaces as a focus for a topic in a first course, but rather focus on subspaces of $\mathbf{R}^n$ (Carlson et al., 1993; Stewart et al, 2021).

Our curricula are based on Realistic Mathematics Education (RME), a curriculum design theory that relies on design heuristics, specifically, didactical phenomenology, guided reinvention, and emergent models (Freudenthal, 1991; Gravemeijer, 1999; Van den Heuvel-Panhuizen, 2020). In this paper, we describe our development of a sequence of tasks designed to support students' reinvention of vector spaces and present results from a paired teaching experiment implementing those tasks. Consistent with the RME design heuristic of didactical phenomenology, during our development of the tasks, we identified key conceptual goals for supporting students' conceptualization of vector spaces: understanding vector spaces and subspaces as spans of sets of vectors and understanding vector spaces and subspaces as sets closed under linear combinations.

We also sought to identify an approach to these views of vector space and subspace that connected to students' prior experiences in the Inquiry-Oriented Linear Algebra [IOLA] materials. Accordingly, we identified null spaces and column spaces of linear transformations as appropriate entry points into reasoning about vector spaces and subspaces as sets closed under linear combinations (or, equivalently, as the span of a set of vectors).

With this in hand, our general strategy for these materials is to approach the content objectives of Subspaces of $\mathbf{R}^n$ via an exploration of null spaces and column spaces. In this unit, students' solutions to problems will benefit from the development of special sets of vectors (which correspond to null spaces), which we anticipate generalizing to the broader notion of subspaces as: (a) sets that are closed under scalar multiples and vector addition and (b) sets that can be written as the span of some subset of $\mathbf{R}^n$. Our development team identified a problem context and iteratively refined a task sequence to support core conceptualizations for null spaces that we saw as promising because: (a) it would extend closed loops reasoning about linear dependence from prior tasks for making sense of null spaces in a new context and (b) this reasoning might then be extended to identify generalized solutions for non-homogeneous systems within the problem context (affine spaces). Our team conjectured that the notion of closure under linear combinations and closed loop reasoning would be useful for students, but we were unsure if and how they would engage in that reasoning. Given this approach to the guided reinvention of vector spaces and our current iteration of the task sequence, we developed the following Research Question:

What meanings do students develop for null spaces and subspaces from their engagement in the task sequence that we designed?

Methods

This study was conducted as part of a broader NSF-funded grant focused on expanding research-based curricula for inquiry-oriented linear algebra. Our data is in the context of subspaces of $\mathbf{R}^n$, with particular emphasis on the idea of null spaces. Our paired teaching experiment (PTE; Steffe & Thompson, 2000), or experiment involving interviews with one teacher-researcher and two participants, was organized around a sequence of four central tasks. PTEs can be helpful in seeing how students learn and reason about a concept. The teacher-researcher's role is to elicit and test participants' ideas. This approach is also helpful in designing and refining tasks; in this case, the task sequence leverages hallway closures in a school to better understand subspaces (null spaces), which we will detail in a following section. PTEs can be useful regarding guided reinvention (e.g., Lockwood & Purdy, 2019; Swinyard, 2011), where the goal of our PTE was for students to reinvent ways of organizing or thinking about subspaces.

Participants, Data Sources, and Analysis Methods

The participants in this study were two white male undergraduate students (which we have given the pseudonyms Carson and Drew) at a predominantly minority public institution in the Southeastern United States. All students who completed this class were invited to participate. However, Carson and Drew were the only students to volunteer at the end of the semester who also met the age constraints of the IRB protocol. Both Carson and Drew had just successfully completed a semester of inquiry-oriented linear algebra that did not include explicit instruction about subspaces. The course content did include the topics of span, linear independence, matrices as linear transformations, composition and inverses, eigenvectors, and eigenvalues. The two students participated as a pair in four, 90-minute problem-solving sessions that took place across four different days within a one-week timeframe. The first author was the instructor for the course as well as the teacher-researcher for the interviews.

The sessions were conducted and video recorded via Microsoft Teams, with the interviewer screen sharing a presentation of problem statements, annotating student ideas on that shared screen, and students typing additional work, responses, and ideas into the chat or holding their written work up to their cameras. During each interview session, a second member of the research team was present to ask additional questions about the participants' thinking, to witness student work, and to provide additional insight into how to plan for subsequent interviews. After each interview, the teacher-researcher, the observing team member, and at least one other research team member met virtually to discuss the day's progress and revise the planned materials for the next interview session.

In addition to the audio-video recordings of the meetings and collection of participant work and responses, the team members kept concurrent notes of the interview sessions and recorded thoughts shared during the debriefing sessions after each interview. Through this process, we developed areas of focus for better understanding the participants' reasoning as they worked through each task in the interviews. We specifically identified one key construct that continually emerged throughout the participants' discussions. Based on these conversations, we identified instances of the participants using the construct throughout the recordings and documented the evolution of how the students leveraged the construct from each task to the next. The field notes, post-interview discussions, and recordings provide triangulation to support the construct's importance throughout the series of interviews.

Task Sequence

Drawing on the design heuristic of didactical phenomenology (Freudenthal, 1991; Van den Heuvel-Panhuizen, 2020), the research team worked to identify a context to draw out aspects of subspaces, especially considering students' anticipated mathematics at the point in the semester at which these materials are planned to be implemented. As an entire research team, we have organized the instructional units so that these tasks would occur after students have learned about linear in/dependence, span, solving systems of equations, and matrices as linear transformations, including composition and inverses of linear transformations. The authors of this paper developed the following task sequence based on the idea of One-Way Hallways, which we think provides an experientially real starting point that is consistent with directed graphs, or graphs made of edges and vertices where the edges have an associated direction to specific vertices (Figure 1). Students are first presented with a diagram of the west wing of Ida B. Wells High School, with an arrow drawn along each corridor and an explanation of the diagram.

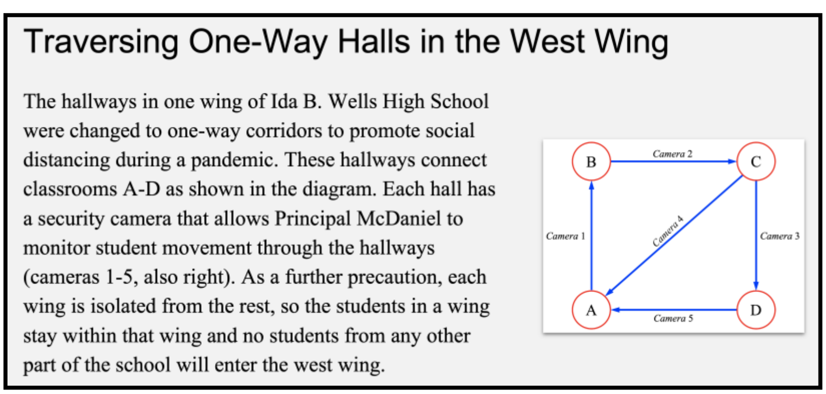

Figure 1. The setup for the first task in the sequence.

Task 1. The task sequence begins by focusing on one individual student's possible paths between two rooms and from one room, back to that same room. Students are asked to represent paths with column vectors that show how many times a student passed by a camera in each of the five hallways. So, for instance, the vector <1, 1, 0, 0, 0> represents a student passing by camera 1 and camera 2, but no other cameras. Students are first asked to identify all routes that a student could possibly take to start a journey in Room A and end the journey in Room C and write these possibilities as efficiently as possible. There are an infinite number of such paths if the student in the task repeats their trip down some of the hallways. For instance, the journeys described by the vectors <3,3,1,1,1> and <17,17,12,5,12> would also result in the student traveling from Room A to Room C. After this, students are asked to find routes that describe all journeys one student could take from Room C back to Room C while also considering the change in populations for each room. Students then are asked to consider both journeys (from A to C and from C to C) for 5 students traveling the hallways, once again also considering the population changes for each room. At the end of the task, students are prompted to consider the set of vectors they have developed for each of the four trips and any comparisons they can make between the trips.

Tasks 2. Task 2 is intended to extend the students' reasoning toward an understanding in which the hallway diagram encodes a mapping from "camera vectors" (5-tuples in which the k^{th} entry is the number of students who pass the k^{th} camera) to "classroom change vectors" (4-tuples, in which the first, second, third, and fourth entry is the change in student population for room A, B, C, and D, respectively). To promote this shift, we ask students to consider the effect that given "camera vectors" would have on classroom populations as well as identify possible "camera vectors" that would result in given changes in the four classrooms. This activity builds from the first activity by abstracting the vectors from being associated with any specific journey within the problem context and instead focuses on the input/output relationship between the camera vectors and classroom vectors. Specifically, Task 2 presents students with a room capacity constraint that requires none of the room populations change throughout the day. This anticipates a homogeneous system within this problem context.

Task 3. In Task 3, students are asked to develop a matrix that corresponds to the mapping defined by the hallway diagram and extend their reasoning about the input/output relationship by connecting it to their existing understanding of linear transformations. This task then asks students to reason about the set of vectors that result in no change in room populations as well as the set of all possible vectors that could describe changes in the room populations. In other words, this task leverages the problem context to prompt students to reason about the null space and column space of the linear transformation. Specifically, students are asked whether these sets are closed under scalar multiplication and whether they are closed under linear combinations. At this point, an instructor using these materials would define vector spaces and subspaces as sets of vectors that are closed under both operations and, equivalently, as sets of vectors that can be described as the span of some subset of the set.

Task 4. To generalize students' activity up to this point, Task 4 presents them with the matrix in Figure 2. This matrix represents students passing through hallways to a different set of classrooms in another wing of the school (the East Wing). Students are first asked to identify how many hallways and classrooms must be in the East Wing. Students are then prompted to figure out all possible hallway flows that would leave the populations unchanged, all possible hallway flows resulting in a particular vector, and all possible changes in population for each classroom in this wing. At the end of the task, students must decide if this latter set of vectors is a subspace of $\mathbf{R}^5$. The goal of the last part of this task is to formalize ideas in the context by having students think about them in terms of a matrix equation and set of vectors, while also linking these to the concept of subspaces.

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & -1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \end{pmatrix}$$

Figure 2. A matrix representing students passing through hallways of the East Wing of Ida B. Wells High School.

Results and Discussion

As conjectured, Carson and Drew developed robust reasoning during the initial task that extended into the later tasks. Specifically, the two participants identified closed loops within the hallway diagram as a helpful construct to make sense of multiple solutions within the problem context. The participants relied on this loop reasoning when discussing solutions to the homogeneous system as well as to non-homogeneous systems. Stated formally, this conceptualization of the problem allowed the participants to reason about the null space as well as an affine space that mapped to a non-zero vector.

When solving the Task 1, Carson and Drew took time to make sense of the given diagram and the vector <1,1,0,0,0>. The participants discussed how to interpret the vector as a journey through the hallways, eventually agreeing that the vector did represent a path from classroom A, down the hallway containing camera 1, past classroom B, and down the hallway containing camera 2 to arrive at classroom C. Following this discussion, the teacher-researcher asked the participants, “Can you think of another vector that might show a path from A to C?” Drew responded by saying,

> Well, the only other way you could get a path from A to C is if you go back to A from C and then you could just have like a loop going. So you could have it where it's like 221 - or no, 2201 [sic]. So it goes through that 4th camera goes through that 4th hallway diagonally and then back up and to the right and back to C.

The teacher-researcher asked Drew to clarify what he meant by the description, which prompted Drew to trace a triangle in the air while describing the journey, adding, “And then you can repeat that as many times as you want.” When asked to explain his last point, Drew said, “Yeah, so you could put a scalar before the vector and you can make it whatever you want and you would still get you to A or - from A to C every time.”

The teacher-researcher then asked the two participants to each write their understanding of what Drew had just described. Rather than writing anything, Carson stated that Drew was just describing going in a bunch of loops around A, B, and C. Then Carson said, “What's like, it's kind of like the homogeneous vector you go back to where you are. When we talk about linear transformations - that's basically what you're doing. You're going, you're go in an entire loop where you go somewhere, and then you go back.” This imagery references existing IOLA materials, specifically the “Getting Back Home” task (Unit 1 Task 3 of IOLA materials), which is reasonable considering that Drew and Carson had recently completed those materials.

At this point, the second researcher in the interview asked the participants to provide two different vectors that they though would be in that set. The teacher-researcher then elaborated on this by suggesting the participants provide generic vectors or other solutions. In response to this, Carson explained that he could see another path, but did not know how to describe that as a vector, saying, “You can go A, B, C, D and then back to A.” In this description, Carson is identifying a different loop than the one Drew suggested before. Meanwhile, Drew had typed into the meeting chat the vector expressions “x<1,1,1,0,1> + <1,1,0,0,0>” and “x<1,1,0,1,0> + <1,1,0,0,0>,” adding that his first expression represented the journey that Carson described. The two participants eventually determined that both loops could be used in combination to describe all possible journeys of one student traveling from room A to room C.

Throughout the remainder of the task and, indeed, the subsequent sessions of the teaching experiment, the participants continually referred to the loops as ways to generate additional solutions to problems in which they were asked to identify paths that would result in given room changes. This included later in the first interview when they were asked to find all possible journeys that would result in one student traveling from room C back to room C, five students traveling from room A to room C, and five students traveling from room C back to room C. This culminated in the development of four sets of solutions, which the participants compared. Drew and Carson correctly identified that the set of solutions for one student passing from Room C back to Room C was equivalent to the set of solutions for five students passing from Room C back to Room C. Similarly, the participants identified parallels for the affine sets they had found for the 1-person and 5-person journeys from A to C, noticing that the constant vector <1,1,0,0,0> would be scaled by 5 for the 5-person journey, but any combination of loops could be used.

During Tasks 2 and 3, the participants extended this loop reasoning to apply to any relationship between camera vectors and room change vectors. They also developed the appropriate matrix that is consistent with the vector mapping from $\mathbf{R}^5$ to $\mathbf{R}^4$ and explored the row-reduced echelon form of that matrix to further reason about solutions both for given camera vectors and for given room change vectors. We further identified an interesting result in response to Task 4 when the participants described and subsequently completed two different solution approaches to making sense of the East Wing matrix. Carson suggested row-reducing the matrix to identify solutions to the system. Drew, on the other hand, suggested constructing a map of the East Wing based on the given matrix and identifying loops in that diagram to find solutions to specific systems. Throughout the task sequence, the participants’ loop reasoning continually proved useful, including when considering new systems. In future implementations of the task, we anticipate further revising the materials to better support these conceptualizations. We are also transitioning to generating instructor support materials that will incorporate results from this PTEs as well as planned whole-class implementations.

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