

# TRI-COBLE SURFACES AND THEIR AUTOMORPHISMS

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ABSTRACT. We construct some positive entropy automorphisms of rational surfaces with no periodic curves. The surfaces in question, which we term *tri-Coble surfaces*, are blow-ups of  $\mathbb{P}^2$  at 12 points which have contractions down to three different Coble surfaces. The automorphisms arise as compositions of lifts of Bertini involutions from certain degree 1 weak del Pezzo surfaces.

## 1. INTRODUCTION

Suppose that  $X$  is a projective surface over an algebraically closed field  $K$  and that  $\phi : X \rightarrow X$  is an automorphism of  $X$ . When  $K = \mathbb{C}$ , a theorem of Gromov and Yomdin asserts that  $\phi$  has positive topological entropy if and only if the spectral radius of  $\phi^* : N^1(X) \rightarrow N^1(X)$  is greater than 1, where  $N^1(X)$  denotes the (finite dimensional) real vector space of divisors on  $X$  modulo numerical equivalence. In a mild abuse of notation, for an arbitrary algebraically closed field  $K$  we will say that an automorphism  $\phi : X \rightarrow X$  has positive entropy if  $\phi^* : N^1(X) \rightarrow N^1(X)$  has spectral radius greater than 1.

Rational surfaces have proved to be an especially compelling source of examples of such automorphisms: although we do not attempt to provide an exhaustive bibliography, some representative constructions can be found in [2, 6, 10, 12]. McMullen asked whether any rational surface  $X$  admitting a positive entropy automorphism must have a pluri-anticanonical curve, i.e. a curve belonging to some linear system  $|-mK_X|$  [10, Question, pg. 87]. Since any automorphism  $f : X \rightarrow X$  preserves these linear systems, if one of them is nonempty then  $f$  must have invariant curves.

Bedford and Kim gave an elegant construction answering this question in the negative [3]. Considering the family of birational maps  $f_{a,b} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$  defined in affine coordinates by

$$(x, y) \mapsto \left( y, \frac{y+a}{x+b} \right),$$

they show that by carefully choosing values for the parameters  $a$  and  $b$  and passing to a suitable blow-up, one obtains an automorphism of a rational surface with no periodic curves at all.

In this note, we exhibit new positive entropy automorphisms of rational surfaces with no periodic curves. We hope that these examples may still be of interest, as they have some new features: the examples are easily understood geometrically, exist in a positive dimensional family, and can be defined over the rational numbers. (There are other examples of positive-dimensional families of rational surface automorphisms, for instance in [4], but these examples have invariant curves.)

The strategy underlying the construction is straightforward. Suppose that  $\mathbf{p} = \{p_1, \dots, p_r\}$  and  $\mathbf{q} = \{q_1, \dots, q_s\}$  are two configurations of points in  $\mathbb{P}^2$ , and let  $S_{\mathbf{p}}$  and  $S_{\mathbf{q}}$  denote the corresponding blow-ups. Suppose too that both these surfaces admit nontrivial automorphisms, say  $\phi_{\mathbf{p}} : S_{\mathbf{p}} \rightarrow S_{\mathbf{p}}$  and  $\phi_{\mathbf{q}} : S_{\mathbf{q}} \rightarrow S_{\mathbf{q}}$ . If  $\phi_{\mathbf{p}}$  fixes every point of  $\mathbf{q} \setminus \mathbf{p}$ , and  $\phi_{\mathbf{q}}$  fixes every point of  $\mathbf{p} \setminus \mathbf{q}$ , then both automorphisms  $\phi_{\mathbf{p}}$  and  $\phi_{\mathbf{q}}$  lift to automorphisms of the common resolution  $S_{\mathbf{pq}}$ . Even if  $\phi_{\mathbf{p}}$  and  $\phi_{\mathbf{q}}$  each has an invariant curve, there is no reason to expect that the composition  $\phi_{\mathbf{p}} \circ \phi_{\mathbf{q}}$  will fix either of these curves, let alone any other.

The difficulty lies in finding such configurations: for two automorphisms to each fix the base points of the other requires that these configurations be quite special (at least over  $\mathbb{C}$ ; we observe in §6 that finding such configurations of points over the fields  $\overline{\mathbb{F}}_p$  is essentially trivial).

We ultimately employ this approach using not two but three sets of points,  $\mathbf{p}$ ,  $\mathbf{q}$ , and  $\mathbf{r}$ . These will all be 8-tuples, but with six points common to all three. Each of the three blow-ups  $S_{\mathbf{p}}$ ,  $S_{\mathbf{q}}$ , and  $S_{\mathbf{r}}$  is a weak del Pezzo surface of degree 1, and the three configurations are chosen so that the corresponding Bertini involutions all lift to the common model  $S_{\mathbf{pqr}}$ . Although the composition of any two of the involutions has invariant curves, we show that the composition of all three has none. (A similar construction using certain de Jonquières involutions is given by Blanc [5]; however, that map fixes a curve pointwise.)

Before delving into the details, we give a quick description of the automorphism. First, we construct the rational surface  $X$ :

- (1) Choose three smooth quadrics  $Q_1$ ,  $Q_2$ , and  $Q_3$  in  $\mathbb{P}^3$  so that any pair  $Q_i$  and  $Q_j$  are tangent at two points. This determines three pairs of points,  $\mathbf{p}$ ,  $\mathbf{q}$ , and  $\mathbf{r}$ . (Such configurations can easily be visualized; see Figure 1.)
- (2) Choose a cubic surface  $S \subset \mathbb{P}^3$  which passes through all six points, and is tangent to both of the quadrics passing through each. Such cubics surely exist, as there is a 19-dimensional family of cubics and the tangency requirements impose only  $6 \times 3 = 18$  conditions.
- (3) Let  $X$  be the blow-up of  $S$  at the six points of tangency.

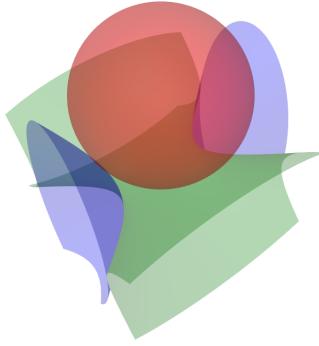


FIGURE 1. A configuration of three tangent quadrics. In this case, two of the quadrics are tangent along an entire curve.

Now, to the three pairs  $\mathbf{p}$ ,  $\mathbf{q}$ , and  $\mathbf{r}$ , we associate involutions  $\tau_{\mathbf{p}}$ ,  $\tau_{\mathbf{q}}$ , and  $\tau_{\mathbf{r}}$  as follows.

- (4) Given a general point  $z$  on  $S$ , let  $\Pi \subset \mathbb{P}^3$  denote the plane through  $z$  and the two points of  $\mathbf{p} = \{p_1, p_2\}$ .

- (5) The intersection  $C = S \cap \Pi$  is a smooth genus 1 curve in  $\Pi$ , passing through  $p_1, p_2$ , and  $z$ .
- (6) There exists a unique conic  $\gamma \subset \Pi$  which is tangent to  $C$  at  $p_1$  and  $p_2$  and passes through  $z$ .
- (7)  $\tau_p(z)$  is defined to be the residual sixth point of intersection of  $\gamma$  with  $C$ .
- (8) The maps  $\tau_q$  and  $\tau_r$  are defined analogously.

Our main result is then:

**Theorem 1.** *Let  $S$  be a cubic surface containing three pairs of points  $\mathbf{p}$ ,  $\mathbf{q}$ , and  $\mathbf{r}$  as above. Let  $X = S_{\mathbf{pqr}}$  be the blow-up of the cubic at 6 points. For general parameter choices, the maps  $\tau_p$ ,  $\tau_q$ , and  $\tau_r$  all lift to biregular involutions of  $X$ , and the composition  $\tau_p \circ \tau_q \circ \tau_r$  is an automorphism of positive entropy which has no periodic curves.*

## 2. COBLE SURFACES AND THE BERTINI INVOLUTION

We begin by recalling some classical geometry surrounding weak del Pezzo surfaces of degree 1, the Bertini involution, and Coble rational surfaces. A reference for most of the results in this section is [7, Chapter 8].

A del Pezzo surface of degree  $k$  is a surface for which  $-K_S$  is ample, i.e. for which  $(-K_S)^2 = k > 0$  and  $-K_S \cdot C > 0$  for any curve  $C$ ; such surfaces exist for  $1 \leq k \leq 9$ . Over an algebraically closed field, any del Pezzo surface can be realized as the blow-up of  $\mathbb{P}^2$  at a configuration of  $9 - k$  points, except in the case  $k = 8$  when  $S = \mathbb{P}^1 \times \mathbb{P}^1$  is also possible. Conversely, any blow-up of  $\mathbb{P}^2$  at a general configuration of points – for which no three points are collinear, etc. – is a del Pezzo surface.

A weak del Pezzo surface of degree  $k$  is a surface for which  $-K_S$  is big and nef and  $(-K_S)^2 = k > 0$  and  $-K_S \cdot C \geq 0$  for any curve  $C$ . The distinction is that there may now be a finite number of curves  $C$  for which  $-K_S \cdot C = 0$ . The weaker big and nef condition allows blow-ups of  $\mathbb{P}^2$  at points in mildly degenerate configurations, for example configurations with 3 collinear points but which are otherwise general.

If  $S$  is a del Pezzo surface of degree 3, the anticanonical linear system  $|-K_S|$  is ample and basepoint-free, and the corresponding map  $\phi_{|-K_S|} : S \rightarrow \mathbb{P}^3$  realizes  $S$  as a cubic surface in  $\mathbb{P}^3$ . Under this embedding, elements of  $|-K_S|$  are given by hyperplane sections of  $S$ . Conversely, any smooth cubic surface in  $\mathbb{P}^3$  is a del Pezzo surface of degree 3, and is isomorphic to the blow-up of  $\mathbb{P}^2$  at a suitable configuration of 6 points.

If  $S$  is a weak del Pezzo surface, the anticanonical map  $\phi_{|-K_S|}$  remains basepoint-free and determines a map  $\phi : S \rightarrow \mathbb{P}^3$ , but this map is not an isomorphism: any curves  $C$  with  $-K_S \cdot C = 0$  are contracted to singular points. In this setting, the image of  $\phi_{|-K_S|}$  is again a cubic surface, but it may have du Val singularities.

The geometry of degree 1 (weak) del Pezzos is also of central importance to us. A del Pezzo of degree 1 can be realized as the blow-up of  $\mathbb{P}^2$  at a suitable configuration of 8 points. Since any blow-up of  $\mathbb{P}^2$  at six general points is isomorphic to a cubic surface, any del Pezzo surface of degree 1 can be realized at the blow-up of a cubic surface at two points. Note that this representation is not unique: there are many choices for which two  $(-1)$ -curves should be contracted to obtain a del Pezzo of degree 3. The next lemma collects some basic facts about the geometry of degree 1 weak del Pezzo surfaces.

**Proposition 2** ([7, 8.3.2]). *Suppose that  $S$  is a weak del Pezzo surface of degree 1. Then*

- (1)  $|-K_S|$  is a pencil of genus 1 curves with one basepoint and smooth general member;

(2)  $|-2K_S|$  is 4-dimensional and basepoint-free, and the 2-anticanonical map  $\phi_{|-2K_S|} : S \rightarrow \mathbb{P}^3$  is generically 2-to-1, with image a quadric cone.

The Bertini involution  $\tau : S \rightarrow S$  is defined to be the covering involution associated to  $\phi_{|-2K_S|}$ , which extends to a biregular map.

The Bertini involution admits a simple description in terms of the pencil  $|-K_S|$ . Given a general point  $z \in S$ , there is a unique smooth genus 1 curve  $C \in |-K_S|$  passing through  $z$ . Since  $-2K_S \cdot C = 2$ , there is a unique point  $z'$  on  $C$  for which  $(-2K_S)|_C \otimes \mathcal{O}_C(-z - z')$  is trivial in  $\text{Pic}^0(C)$ , and so every element of  $|-2K_S|$  passing through  $z$  also passes through  $z'$ . This  $z'$  is the image of  $z$  under the Bertini involution. As a result, we get a convenient characterization of the fixed points of Bertini involution:

**Lemma 3.** *Suppose that  $S$  is a weak del Pezzo surface of degree 1 and that  $z$  is a point which lies on a smooth curve  $C \in |-K_S|$ . Then  $z$  is fixed by  $\tau$  if and only if  $(-2K_S)|_C \otimes \mathcal{O}_C(-2z) = 0$  in  $\text{Pic}^0(C)$ .*

A degree 1 del Pezzo can be obtained by blowing up 2 suitable points on a cubic surface, and it is easy to characterize when such two-point blow-ups are weak del Pezzo:

**Lemma 4.** *Suppose that  $S \subset \mathbb{P}^3$  is a smooth cubic surface and that  $S_{\mathbf{p}} = S$  is the blow-up of  $S$  at two points  $\mathbf{p} = \{p_1, p_2\}$ . Then  $S_{\mathbf{p}}$  is a weak del Pezzo surface if and only if the line between  $p_1$  and  $p_2$  is not contained in  $S$ .*

*Proof.* Let  $E_1$  and  $E_2$  be the exceptional divisors of  $\pi : \text{Bl}_{\mathbf{p}} \mathbb{P}^3 \rightarrow \mathbb{P}^3$ , and let  $H = \pi^* \mathcal{O}_{\mathbb{P}^3}(1)$ . Then  $-K_S = (H - E_1 - E_2)|_S$ . The divisor  $H - E_1 - E_2$  on  $\text{Bl}_{\mathbf{p}} \mathbb{P}^3$  is not nef, since it has intersection  $-1$  with the strict transform of the line through  $p_1$  and  $p_2$ . However, this is the only curve with which it has negative intersection, and so  $(H - E_1 - E_2)|_S$  is nef as long as this line is not contained in  $S$ . Since  $(-K_S)^2 = 1$ , we conclude that  $-K_S$  is big.  $\square$

Note that if  $S$  is the blow-up of a cubic in  $\mathbb{P}^3$  at two points  $p_1$  and  $p_2$ ,  $|-2K_S|$  consists of quadric surfaces which are tangent to the cubic at each blown up point. The family of such quadrics is 3-dimensional. This gives rise to a convenient description of the Bertini involution on such blow-ups [1, pg. 128]. Given a point  $z \in S$ , there is 2-dimensional family of quadric surfaces which are tangent to  $S$  at the two points of  $\mathbf{p}$  and pass through the point  $z$ . This net of quadrics has a unique basepoint on  $S$ , which coincides with  $\tau(z)$ .

The existence of the additional basepoint may be readily seen in this context. Consider the plane  $P$  passing through  $p_1$ ,  $p_2$ , and  $z$ . A quadric  $Q$  which is tangent to  $S$  at  $p_1$  and  $p_2$  and passes through  $z$  yields a plane conic  $Q \cap P$  in  $P$  which is tangent to the cubic curve  $S \cap P$  at both  $p_1$  and  $p_2$  and passes through  $z$ . Since there is only one such plane conic – we have imposed five linear conditions – all of these quadric surfaces must have the same intersection with  $P$ . The basepoint  $\tau(z)$  is the residual sixth point of intersection between  $Q \cap P$  and  $S \cap P$ .

According to Lemma 3, a point  $z$  is fixed by  $\tau$  if in the plane  $\Pi = \Pi_{p_1 p_2 z}$ , there exists a plane conic  $C \subset \Pi$  which is tangent to  $S \cap \Pi$  at the three points  $p_1$ ,  $p_2$ , and  $z$ .

Weak del Pezzo surfaces never admit automorphisms of positive entropy, and it is necessary to look at rational surfaces obtained by blowing up additional points. Central to our analysis are the *Coble surfaces*.

**Definition 1.** A *Coble surface*  $S$  is a smooth rational surface for which  $|-K_S|$  is empty but  $|-2K_S|$  is not. A *simple Coble surface* is a Coble surface for which  $|-2K_S|$  is represented by a smooth rational curve.

An application of the adjunction theorem shows that on a simple Coble surface, the rational curve  $C \in |-2K_S|$  has self-intersection  $-4$ . Such  $S$  can be obtained, for example, by blowing up the nodes of an irreducible, rational plane sextic with exactly ten nodes [6].

Suppose that  $S \subset \mathbb{P}^3$  is a smooth cubic surface and that  $\mathbf{p}$  is a quadruple of points on  $S$ . If there exists a quadric surface  $Q$  which is tangent to  $S$  at the points  $\mathbf{p}$  and such that  $Q \cap S$  has strict transform on  $S_{\mathbf{p}} = \text{Bl}_{\mathbf{p}} S$  which is smooth and irreducible, then  $S_{\mathbf{p}}$  is a simple Coble surface, with  $Q \cap S \in |-2K_{S_{\mathbf{p}}}|$ . As a result, simple Coble surfaces can sometimes be obtained by blowing up 4 special points on a smooth cubic.

To carry out the strategy outlined in the introduction, we would need a cubic surface  $S$  with two pairs of points  $\mathbf{p} = \{p_1, p_2\}$  and  $\mathbf{q} = \{q_1, q_2\}$  such that  $\tau_{\mathbf{p}}$  fixes  $\mathbf{q}$  and  $\tau_{\mathbf{q}}$  fixes  $\mathbf{p}$ . If this is the case, then both  $\tau_{\mathbf{p}}$  and  $\tau_{\mathbf{q}}$  lift to the blow-up  $S_{\mathbf{pq}}$ . One might expect that  $\tau_{\mathbf{p}} \circ \tau_{\mathbf{q}}$  has no invariant curves, but the next result shows that this is too much to hope for.

**Theorem 5.** *Suppose that  $S$  is a weak del Pezzo surface of degree 3 and that  $\mathbf{p}$  and  $\mathbf{q}$  are two disjoint pairs of points on  $S$  such that  $S_{\mathbf{p}}$  and  $S_{\mathbf{q}}$  are both weak del Pezzo surfaces of degree 1. Suppose further that two non-degeneracy conditions are satisfied:*

- (N1) *The element of  $|-K_S|$  through any three of the four points is smooth and irreducible, and these curves are distinct.*
- (N2) *When  $S$  is realized as a cubic in  $\mathbb{P}^3$  under the anticanonical embedding, the tangent plane to  $S$  at any of the four points does not pass through any other.*

*Then  $S_{\mathbf{pq}}$  is a Coble surface if and only if  $\tau_{\mathbf{p}}$  fixes each point of  $\mathbf{q}$  and  $\tau_{\mathbf{q}}$  fixes each point of  $\mathbf{p}$ .*

Since any automorphism of a Coble surface has an invariant curve, the unique element of  $|-2K_{S_{\mathbf{pq}}}|$ , the compositions  $\tau_{\mathbf{p}} \circ \tau_{\mathbf{q}}$  must have an invariant curve, and it will be necessary to refine our approach. Before giving the proof, we record a simple geometric lemma.

**Lemma 6.** *Suppose that  $p_0, p_1, p_2, p_3$  are four non-coplanar points in  $\mathbb{P}^3$ , and for  $0 \leq i \leq 3$ , let  $\Pi_i$  denote the plane passing through the three points other than  $p_i$ . Suppose that there exist four smooth conics  $C_i \subset \Pi_i$  such that:*

- (1) *Each  $C_i$  passes through the three points  $p_j$  which lie on  $\Pi_i$ ;*
- (2) *At the point  $p_i$ , we have  $\dim(T_{p_i}C_j + T_{p_i}C_k + T_{p_i}C_{\ell}) = 2$ .*

*Then there exists a quadric  $Q \subset \mathbb{P}^3$  so that  $C_i = Q \cap \Pi_i$ .*

*Proof.* We may choose coordinates  $[X_0, X_1, X_2, X_3]$  on  $\mathbb{P}^3$  so that the points are the four standard coordinate points. Then the plane  $\Pi_i$  is defined by  $X_i = 0$ .

That the conics each pass through three of these points and lie in a plane  $X_i = 0$  means that they are given by equations of the form

$$\begin{aligned} F_0 &= a_{12}X_1X_2 + a_{13}X_1X_3 + a_{23}X_2X_3 = 0, \\ F_1 &= b_{02}X_0X_2 + b_{03}X_0X_3 + b_{23}X_2X_3 = 0, \\ F_2 &= c_{01}X_0X_1 + c_{03}X_0X_3 + c_{13}X_1X_3 = 0, \\ F_3 &= d_{01}X_0X_1 + d_{02}X_0X_2 + d_{12}X_1X_2 = 0 \end{aligned}$$

To show that there exists a quadric  $Q$  as claimed, we must show that we may replace each  $F_i$  by a suitable multiple and arrange that any monomial  $X_iX_j$  has the same coefficient in both of the equations in which it appears. The quadric  $Q$  is then defined by the polynomial obtained by combining all of these monomials.

We next work out the conditions on the coefficients given by the assumed tangency. First consider the point  $p_0$ . In the affine chart on  $\mathbb{P}^3$  given by  $X_0 = 1$ , this point is the origin, and the three quadrics are given by

$$\begin{aligned} F_1 &= b_{02}X_2 + b_{03}X_3 + b_{23}X_2X_3 = 0 \\ F_2 &= c_{01}X_1 + c_{03}X_3 + c_{13}X_1X_3 = 0 \\ F_3 &= d_{01}X_1 + d_{02}X_2 + d_{12}X_1X_2 = 0 \end{aligned}$$

Tangent vectors to these curves at the origin in  $\mathbb{A}^3$  are then given by

$$(0, b_{03}, -b_{02}), \quad (c_{03}, 0, -c_{01}), \quad (d_{02}, -d_{01}, 0).$$

The vectors are coplanar if the determinant of the matrix with these as rows vanishes, and analogous computations at the other three coordinate points yield the four conditions

$$\begin{aligned} b_{02}c_{03}d_{01} - b_{03}c_{01}d_{02} &= 0, & a_{12}c_{13}d_{01} - a_{13}c_{01}d_{12} &= 0, \\ a_{12}b_{23}d_{02} - a_{23}b_{02}d_{12} &= 0, & a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} &= 0. \end{aligned}$$

That the conics are smooth implies that none of the coefficients vanish, and so after multiplying the equations by constants we may assume that the coefficients on the  $X_0X_1$  and  $X_2X_3$  terms are already equal, so that  $c_{01} = d_{01} = 1$  and  $a_{23} = b_{23} = 1$ . Our system of equations then becomes:

$$\begin{aligned} b_{02}c_{03} - b_{03}d_{02} &= 0 & a_{12}c_{13} - a_{13}d_{12} &= 0 \\ a_{12}d_{02} - b_{02}d_{12} &= 0 & a_{13}c_{03} - b_{03}c_{13} &= 0. \end{aligned}$$

This shows that

$$\frac{a_{12}}{d_{12}} = \frac{b_{02}}{d_{02}} = \frac{a_{13}}{c_{13}} = \frac{b_{03}}{c_{03}}.$$

Multiplying the equations  $F_2$  and  $F_3$  by this common value, we obtain multiples of the defining equations which have all corresponding coefficients equal.  $\square$

*Proof of Theorem 5.* The surface  $S$  can be mapped to  $\mathbb{P}^3$  by the anticanonical map. This map contracts the  $(-2)$ -curves on  $S$ , and the image is a cubic surface with du Val singularities. The assumption that  $S_{\mathbf{p}}$  and  $S_{\mathbf{q}}$  are weak del Pezzo implies that none of the blown up points lies on a  $(-2)$ -curve, so the four points of  $\mathbf{p}$  and  $\mathbf{q}$  are all mapped to smooth points. The distinctness assumption in (N1) implies that no four of these points have coplanar image.

One direction of the proof is simple. Suppose that  $S_{\mathbf{p}\mathbf{q}}$  is a Coble surface. Then there exists a quadric  $Q$  which is tangent to  $S$  at the four points  $\mathbf{p} \cup \mathbf{q}$ . According to our description of the Bertini involution on a 2-point blow-up of the cubic, the point  $q_1$  is fixed by  $\tau_{\mathbf{p}}$  if and only if there exists a conic  $C$  in the plane  $\Pi = \Pi_{\mathbf{p}q_1}$  such that  $C$  is double on the smooth cubic  $S \cap \Pi$  at these three points. But there certainly exists such a conic: we can simply take  $C = Q \cap \Pi$ , which is tangent at the points since  $Q$  and  $S$  are. The other fixed point conditions follow in the same way.

Suppose instead that  $\tau_{\mathbf{p}}$  fixes the points of  $\mathbf{q}$  and  $\tau_{\mathbf{q}}$  fixes the points of  $\mathbf{p}$ . Let  $\Pi$  denote the plane through  $\mathbf{p}$  and  $q_1$ . There exists a plane conic  $C \subset \Pi$  which is double on the plane cubic  $S \cap \Pi$  at  $p_1, p_2$ , and  $q_1$ .

We claim that  $C$  must be smooth. If not, then  $C$  is a union of two lines. There are two possibilities: either (i)  $C$  is the double of a line  $L$  passing through both  $p_1$  and  $p_2$  or (ii) one of the lines  $L$  is tangent to  $S \cap C$  at  $p_1$ . Case (i) is ruled out by (N1), since this would mean that  $q_1$  lies on the line between  $p_1$  and  $p_2$ , so that the four points are coplanar. Case (ii) is

ruled by (N2): if  $L$  passes through  $p_2$  or  $q$ , this would mean that the tangent plane to  $S$  at  $p_1$  passes through  $p_2$  or  $q$ . If  $L$  misses both these points, then the second line  $L'$  must be tangent to  $S \cap \Pi$  at both  $p_2$  and  $q$ , which again contradicts (N2).

Making the same argument for other triples of points, we conclude that in any plane  $\Pi$  through three of the four points in  $\mathbf{p} \cup \mathbf{q}$ , there exists a conic smooth  $C$  tangent to  $S \cap \Pi$  at these three points. Three such conics pass through each of the points in  $\mathbf{p} \cup \mathbf{q}$ , and at each point the three conics have coplanar tangent directions, since all the tangents are contained in the tangent plane to  $S$ .

It follows from Lemma 6 that there is in fact a quadric surface  $Q \subset \mathbb{P}^3$  which is tangent to  $S$  at each of the four points, which shows that  $|-2K_{S_{\mathbf{p}\mathbf{q}}}|$  is nonempty. Since (N1) implies that the four points are not coplanar,  $|-K_{S_{\mathbf{p}\mathbf{q}}}|$  is empty, and so the  $S_{\mathbf{p}\mathbf{q}}$  is a Coble surface.  $\square$

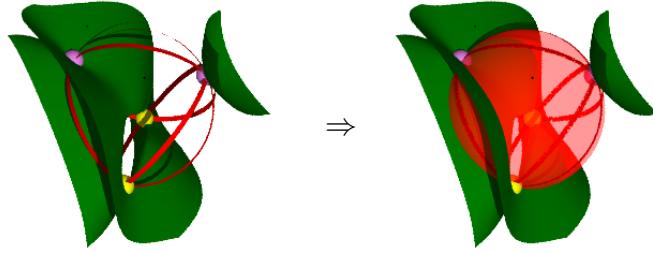


FIGURE 2.  $\tau_p$  fixes  $\mathbf{q}$  and  $\tau_q$  fixes  $\mathbf{p}$  if and only if  $S$  is tangent to four conic curves. According to Lemma 6, this is possible only if  $S$  is tangent to a conic surface, which means that  $S_{\mathbf{p}\mathbf{q}}$  is a Coble surface.

Blowing up two pairs of points on a cubic as in Theorem 5 yields a Coble surface, and every automorphism has an invariant curve. This is illustrated in Figure 2. We thus proceed to blow up a third pair of points, leading to the following definition.

**Definition 2.** Let  $S$  be a weak del Pezzo surface of degree 3 and suppose that there exist three disjoint pairs of points  $\mathbf{p}$ ,  $\mathbf{q}$ , and  $\mathbf{r}$  on  $S$  such that:

- (T1)  $S_{\mathbf{p}}$ ,  $S_{\mathbf{q}}$ , and  $S_{\mathbf{r}}$  are weak del Pezzo surfaces of degree 1, with corresponding Bertini involutions  $\tau_p$ ,  $\tau_q$  and  $\tau_r$ ;
- (T2)  $\tau_p$  fixes the points of  $\mathbf{q} \cup \mathbf{r}$ ,  $\tau_q$  fixes the points of  $\mathbf{p} \cup \mathbf{r}$ , and  $\tau_r$  fixes the points of  $\mathbf{p} \cup \mathbf{q}$ .
- (T3) Each of the 4-tuples  $\mathbf{p} \cup \mathbf{q}$ ,  $\mathbf{p} \cup \mathbf{r}$ , and  $\mathbf{q} \cup \mathbf{r}$  satisfies the nondegeneracy conditions (N1) and (N2).

Then we term the blow-up  $X = S_{\mathbf{pqr}}$  a *tri-Coble surface*. Notice that  $X$  may be contracted to each of  $S_{\mathbf{pq}}$ ,  $S_{\mathbf{pr}}$ , and  $S_{\mathbf{qr}}$ , which are all Coble surfaces according to Lemma 5. If any one of these three is a simple Coble surface, then we call  $X$  a *simple tri-Coble surface*.

Suppose that  $S$  is a smooth cubic surface in  $\mathbb{P}^3$ . For  $\tau_p$  to fix the points of  $\mathbf{q}$  and  $\tau_q$  to fix the points of  $\mathbf{p}$  means that there is a quadric  $Q_1$  which is tangent to  $S$  at the four points of  $\mathbf{p} \cup \mathbf{q}$ . Similarly, there must exist a quadric  $Q_2$  tangent to  $S$  at  $\mathbf{p} \cup \mathbf{r}$ , and a quadric  $Q_3$  tangent to  $S$  at  $\mathbf{q} \cup \mathbf{r}$ . If such configurations can be constructed on  $S$ , and the non-degeneracy conditions are satisfied, then the blow-up  $S_{\mathbf{pqr}}$  will be an example of a tri-Coble surfaces. If some intersection  $S \cap Q_i$  has strict transform on the blow-up which is smooth, then we obtain a simple tri-Coble surface.

## 3. NO INVARIANT CURVES

We will mostly be interested in the composition  $\phi = \tau_p \circ \tau_q \circ \tau_r$ . The bi-anticanonical curve on  $S_{\mathbf{pq}}$  is invariant under both  $\tau_p$  and  $\tau_q$ , but there are no obvious curves invariant under all three of these involutions, and so it seems reasonable to expect that  $\phi$  does not have any invariant curves at all.

**Theorem 7.** *Suppose that  $X = S_{\mathbf{pqr}}$  is a tri-Coble surface. Then each of the involutions  $\tau_p$ ,  $\tau_q$ , and  $\tau_r$  lifts to a biregular involution of  $X$ . Let  $\phi = \tau_p \circ \tau_q \circ \tau_r$  be the composition. Then  $\phi$  is an automorphism of positive entropy. If  $X$  is a simple tri-Coble surface, then  $\phi^*$  does not fix any pseudoeffective class. In particular,  $\phi$  has no periodic curves.*

*Proof.* First, observe that if  $C$  is a  $\phi$ -periodic curve, then  $\bigcup_n \phi^n(C)$  is a  $\phi$ -invariant (albeit reducible) curve, and so it suffices to show that there is no invariant curve.

The action of the Bertini involution on  $N^1(S)$  for a degree 1 weak del Pezzo surface was known classically [7]. If  $S$  is presented as a blow-up of  $\mathbb{P}^2$  at eight points with exceptional divisors  $E_1, \dots, E_8$ , then with respect to the basis  $H, E_1, \dots, E_8$ :

$$\tau^* = \begin{pmatrix} 17 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ -6 & -3 & -2 & -2 & -2 & -2 & -2 & -2 \\ -6 & -2 & -3 & -2 & -2 & -2 & -2 & -2 \\ -6 & -2 & -2 & -3 & -2 & -2 & -2 & -2 \\ -6 & -2 & -2 & -2 & -3 & -2 & -2 & -2 \\ -6 & -2 & -2 & -2 & -2 & -3 & -2 & -2 \\ -6 & -2 & -2 & -2 & -2 & -2 & -3 & -2 \\ -6 & -2 & -2 & -2 & -2 & -2 & -2 & -3 \end{pmatrix}.$$

The induced action of  $\tau_p$  on a tri-Coble surface is obtained by appending a  $4 \times 4$  identity matrix to the matrix  $\tau^*$ , corresponding to the 4 exceptional divisors above the points  $\mathbf{q} \cup \mathbf{r}$  which are invariant under  $\tau_p$ . The matrices for the other involutions are computed analogously, and multiplying all three together we obtain

$$\phi^* = \begin{pmatrix} 377 & 126 & 126 & 126 & 126 & 126 & 126 & 150 & 150 & 30 & 30 & 6 & 6 \\ -126 & -43 & -42 & -42 & -42 & -42 & -42 & -50 & -50 & -10 & -10 & -2 & -2 \\ -126 & -42 & -43 & -42 & -42 & -42 & -42 & -50 & -50 & -10 & -10 & -2 & -2 \\ -126 & -42 & -42 & -43 & -42 & -42 & -42 & -50 & -50 & -10 & -10 & -2 & -2 \\ -126 & -42 & -42 & -42 & -43 & -42 & -42 & -50 & -50 & -10 & -10 & -2 & -2 \\ -126 & -42 & -42 & -42 & -42 & -43 & -42 & -50 & -50 & -10 & -10 & -2 & -2 \\ -126 & -42 & -42 & -42 & -42 & -42 & -43 & -50 & -50 & -10 & -10 & -2 & -2 \\ -6 & -2 & -2 & -2 & -2 & -2 & -2 & -3 & -2 & 0 & 0 & 0 & 0 \\ -6 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -3 & 0 & 0 & 0 & 0 \\ -30 & -10 & -10 & -10 & -10 & -10 & -10 & -12 & -12 & -3 & -2 & 0 & 0 \\ -30 & -10 & -10 & -10 & -10 & -10 & -10 & -12 & -12 & -2 & -3 & 0 & 0 \\ -150 & -50 & -50 & -50 & -50 & -50 & -50 & -60 & -60 & -12 & -12 & -3 & -2 \\ -150 & -50 & -50 & -50 & -50 & -50 & -50 & -60 & -60 & -12 & -12 & -2 & -3 \end{pmatrix}.$$

A direct calculation shows that the characteristic polynomial is

$$\chi_{\phi^*}(t) = (t-1)(t+1)^{10}(t^2 - 110t + 1),$$

and so the first dynamical degree is  $\lambda_1(\phi) = 55 + 12\sqrt{21} \approx 109.99$  and  $\phi$  has positive entropy.

Moreover, the 1-eigenspace of  $\phi^*$  is 1-dimensional, spanned by the canonical class  $K_X$ . Consequently, the only possible  $\phi$ -invariant curve would be pluri-anticanonical, and to prove that  $\tau$  admits no periodic curve, it suffices to show that  $|-mK_X|$  is not effective for any  $m > 0$ . This is done in Corollary 10 below.  $\square$

In fact, we will show that the anticanonical class on a simple tri-Coble surface  $X$  is not even pseudoeffective, (i.e. numerically a limit of effective classes), which implies that  $|-mK_X|$  is empty for every  $m > 0$ . This is straightforward: the Coble surface  $S_{\mathbf{pq}}$  has anti-bicanonical class represented by an irreducible curve of negative self-intersection. A anti-bicanonical curve on  $X = S_{\mathbf{pqr}}$  would correspond to a member of  $|-2K_{S_{\mathbf{pq}}}|$  with nodes at the two points of  $\mathbf{r}$ . But this linear system contains only a single, smooth, rigid curve: it can not be deformed to have nodes at the points  $\mathbf{r}$ . This already shows that  $|-2K_X|$  is not effective, and the fact that it is not pseudoeffective is an easy extension of the argument.

**Lemma 8.** *Suppose that  $X$  is a smooth projective surface containing an irreducible curve  $C_1$  with  $C_1^2 < 0$ . Let  $C_2$  be another curve satisfying  $C_1 \cdot C_2 \geq 0$ . Then for any  $\epsilon > 0$ , the class  $C_1 - \epsilon C_2$  is not pseudoeffective.*

*Proof.* Let  $A$  be an ample class on  $X$ . We show first that any effective representative of a class  $C_1 - \epsilon C_2 + \delta A$  for small  $\delta$  must have  $C_1$  in its support with multiplicity close to 1. Indeed, suppose that  $C_1 + \epsilon A + \delta A \equiv rC_1 + F$ , where  $F$  is an effective divisor whose support does not contain  $C_1$ . Then  $F = (1-r)C_1 - \epsilon C_2 + \delta A$ . Intersecting both sides with  $C_1$ , we obtain

$$\begin{aligned} F \cdot C_1 &= (1-r)C_1^2 - \epsilon C_1 \cdot C_2 + \delta A \cdot C_1 \\ 0 &\leq (1-r)C_1^2 + \delta A \cdot C_1 \\ 1-r &\leq -\delta \frac{A \cdot C_1}{C_1^2}. \end{aligned}$$

The second line follows from the fact that  $F \cdot C_1 \geq 0$  while the third requires  $C_1^2 < 0$ . Now compute

$$\begin{aligned} F \cdot A &= (1-r)C_1 \cdot A - \epsilon C_2 \cdot A + \delta A^2 \\ &\leq \left( -\delta \frac{A \cdot C_1}{C_1^2} \right) C_1 \cdot A - \epsilon C_2 \cdot A + \delta A^2 \\ &= \delta \left( A^2 - \frac{A \cdot C_1}{C_1^2} \right) - \epsilon C_2 \cdot A. \end{aligned}$$

If  $\epsilon$  is fixed and  $\delta < \left( A^2 - \frac{A \cdot C_1}{C_1^2} \right)^{-1}$  ( $A \cdot C_2$ ) is taken sufficiently small, the right side is evidently negative, so that  $F \cdot A < 0$ , which is impossible if  $F$  is effective. It follows that  $C_1 - \epsilon C_2 + \delta A$  is not effective for sufficiently small  $\delta$ , so that  $C_1 - \epsilon C_2$  is not pseudoeffective.  $\square$

**Lemma 9.** *Suppose that  $S$  is a smooth surface and  $C \subset S$  is a smooth, irreducible curve with  $C^2 < 0$ . Let  $p$  be any point of  $S$ , and let  $\pi : S' \rightarrow S$  be the blow-up of  $S$  at  $p$ , with exceptional divisor  $E$ . Then the class  $\pi^*C - eE$  is not pseudoeffective for any  $e > 1$ .*

*Proof.* The strict transform  $C'$  of  $C$  on  $S'$  has class  $\pi^*C - bE$ , where  $b$  is either 0 or 1 (depending on whether  $p$  lies on  $C$ ). Then  $C'$  is a smooth, irreducible curve with  $(C')^2 < 0$ , and the claim follows from Lemma 8 taking  $C_1 = C'$  and  $C_2 = E$ .  $\square$

**Corollary 10.** *Let  $X$  be a simple tri-Coble surface. Then  $-K_X$  is not pseudoeffective, and every linear system  $|-mK_X|$  is empty.*

*Proof.* The simplicity hypothesis means that  $X$  is obtained by blowing up two points on  $S_{\mathbf{pq}}$ , where  $S_{\mathbf{pq}}$  has a unique irreducible rational curve in  $|-2K_{S_{\mathbf{pq}}}|$ . From the adjunction theorem, this curve is smooth of self-intersection  $-4$ , and since  $-2K_X = \pi^*(-2K_{S_{\mathbf{pq}}}) - 2(E_1 + E_2)$ , it follows from Lemma 9 that this class is not pseudoeffective.  $\square$

#### 4. THE EXISTENCE OF TRI-COBLE SURFACES

There is still one piece missing: we must prove that tri-Coble surfaces actually exist. The proof is by direct construction. By definition, a tri-Coble surface is a 6-point blow-up of a smooth degree-3 weak del Pezzo surface  $S$ . To construct a tri-Coble surface, we will take  $S$  to be a particular cubic surface in  $\mathbb{P}^3$  and then blow up three pairs of points on  $S$ , which arise as the tangency points  $S$  with three quadric surfaces, as described following Definition 2.

To actually find such configurations, it is helpful to invert our perspective. Rather than fixing a cubic surface  $S \subset \mathbb{P}^3$  and searching for six points  $\mathbf{p}, \mathbf{q}, \mathbf{r}$  such that there exist three quadrics each tangent at four of the six, we begin with three quadrics  $Q_1, Q_2, Q_3 \subset \mathbb{P}^3$  such that each pair of quadrics are tangent at two points. Only after fixing the quadrics and the six tangency points do we construct the cubic surface. For a cubic surface to pass through a point and have a given tangent plane there imposes 3 conditions. Since the space of cubic surfaces has dimension 19, we expect that given six points and prescribed tangent planes at those points, there should exist a  $19 - 6 \times 3 = 1$ -dimensional pencil of satisfactory cubic surfaces.

We must then check several non-degeneracy conditions to ensure that our surface is actually a simple tri-Coble surface:

- (1)  $S$  is smooth and irreducible;
- (2) the blow-ups  $S_{\mathbf{p}}, S_{\mathbf{q}}, S_{\mathbf{r}}$  are weak del Pezzo;
- (3) no four of the six points are coplanar;
- (4) the intersection of  $S$  with any plane through any three of the six points is a smooth cubic;
- (5) the tangent plane to  $S$  at any point does not pass through any other points;
- (6) the intersections  $Q_i \cap S_{\mathbf{pqr}}$  are smooth and irreducible.

(Here (1) and (2) show that (T1) is satisfied. (3) and (4) together imply that (N1) holds as required by (T2), while (5) checks (N2). At last, (6) implies that the surface is actually a *simple* tri-Coble surface, so that one of the three blow-down Coble surfaces has a smooth bi-anticanonical curve.)

With a bit of computer-aided experimentation, this strategy readily yields examples where the quadrics and the tangency points are all defined over  $\mathbb{Q}$ . Consider the following three quadric surfaces in  $\mathbb{P}^3$ , with coordinates  $(w : x : y : z)$ :

$$\begin{aligned} Q_1 &: 59w^2 + x^2 + y^2 - 20wz + z^2 = 0, \\ Q_2 &: -9w^2 - x^2 + 9y^2 + z^2 = 0, \\ Q_3 &: -9w^2 - x^2 + 9y^2 - 6z^2 = 0, \end{aligned}$$

as well as the six points

$$\begin{aligned} p_1 &= (1 : 4 : 0 : 5), & q_1 &= (1 : 0 : 5 : 6), & r_1 &= (12 : -15 : 13 : 0), \\ p_2 &= (1 : -4 : 0 : 5), & q_2 &= (1 : 0 : -5 : 6), & r_2 &= (-3 : 12 : 5 : 0). \end{aligned}$$

The following table gives the tangent spaces  $T_{p_i}Q_j$  at the six points, using the coordinates from the dual  $\mathbb{P}^3$ . An “X” in a column indicates that the surface does not pass through the point.

	$Q_1$	$Q_2$	$Q_3$
$p_1$	$(-9 : -4 : 0 : 5)$	$(-9 : -4 : 0 : 5)$	X
$p_2$	$(-9 : 4 : 0 : 5)$	$(-9 : 4 : 0 : 5)$	X
$q_1$	$(1 : 0 : -5 : 4)$	X	$(1 : 0 : -5 : 4)$
$q_2$	$(1 : 0 : 5 : 4)$	X	$(1 : 0 : 5 : 4)$
$r_1$	X	$(-36 : 5 : 39 : 0)$	$(-36 : 5 : 39 : 0)$
$r_2$	X	$(9 : -4 : 15 : 0)$	$(9 : -4 : 15 : 0)$

An examination of the table shows that these quadrics satisfy the required pointwise tangency conditions. It is then an exercise in linear algebra to write down a cubic surface with the prescribed tangent planes at all six of these points. There is in fact a 1-dimensional family of such cubics, with one such surface  $S$  defined by the vanishing of the equation

$$\begin{aligned} F = & 9963w^3 + 56187w^2x + 27707wx^2 + 3018x^3 + 12069w^2y + 366x^2y + 11457wy^2 \\ (*) \quad & + 3213xy^2 + 351y^3 - 5358w^2z - 11610wxz - 3002x^2z - 4140wyz + 18y^2z \\ & - 7643wz^2 - 1857xz^2 + 111yz^2 + 38z^3. \end{aligned}$$

*Remark.* An easy dimension count suggests that on a general cubic  $S$  it should be possible to find pairs  $\mathbf{p}$ ,  $\mathbf{q}$ , and  $\mathbf{r}$  so that the blow-up  $S_{\mathbf{pqr}}$  is a tri-Coble surface. However, actually finding such points, even on the Fermat cubic, leads to equations with no obvious solutions. The approach followed in this section seems to be more straightforward, even if it leads to a somewhat cumbersome cubic surface.

To show  $S_{\mathbf{pqr}}$  is tri-Coble surface, it is still necessary to verify non-degeneracy conditions (1)-(6) ensuring that the automorphisms  $\tau_{\mathbf{p}}$ ,  $\tau_{\mathbf{q}}$ , and  $\tau_{\mathbf{r}}$  are all well-defined and that their composition has no periodic curves. It seems unsurprising that these non-degeneracy conditions hold, but since the cubic  $S$  is constructed as the solution to an interpolation problem, it is difficult to directly control its geometry.

Although these verifications are tedious to carry out by hand, they are routine on a computer. Notice that checking (2) does not actually necessitate the relatively difficult task of finding the 27 lines on  $S$ : it is enough to check that the particular lines connecting pairs of points are not contained in  $S$ . The nondegeneracy properties are checked for the specific cubic  $(*)$  in the Sage script `nondegen.sage`, which can be obtained at [8]. Note that all of these are open conditions, and so the general cubic in the pencil constructed earlier also satisfies these conditions.

## 5. FURTHER PROPERTIES OF THE AUTOMORPHISM GROUP

In this section, we study the full group  $\text{Aut}(X)$  in somewhat more detail, proving the following:

**Theorem 11.** *Let  $X$  be a tri-Coble surface. Then:*

- (1) *The involutions  $\tau_p$ ,  $\tau_q$ , and  $\tau_r$  generate a free subgroup  $G \cong (\mathbb{Z}/2\mathbb{Z})^{*3} \subseteq \text{Aut}(X)$ .*
- (2) *Suppose that  $f \in G$  is an automorphism in this subgroup. Then  $f$  has an invariant curve if and only if it is conjugate to an iterate of one of the maps  $\tau_p$ ,  $\tau_q$ ,  $\tau_r$ ,  $\tau_p \circ \tau_q$ ,  $\tau_p \circ \tau_r$ , or  $\tau_q \circ \tau_r$ .*
- (3)  *$\text{Aut}(X)$  contains a subgroup isomorphic to  $\mathbb{Z} * \mathbb{Z}$ , every non-trivial element of which is an automorphism with no periodic curves.*

Let  $G \subset \text{Aut}(X)$  be the subgroup generated by  $\tau_p$ ,  $\tau_q$ , and  $\tau_r$ . The proof is based on constructing an action of  $G$  on a hyperbolic plane. We first describe a subspace of  $N^1(X)$  which is invariant under pulling back by elements of  $G$ . Write  $E_p$  for the sum of the two exceptional divisors obtained by blowing up the two points of  $p$ , and similarly for the other pairs. Consider first the involution  $\tau_p$ . It must be that  $\tau_p^*(K_X) = K_X$ . Similarly, we have  $\tau_p^*(E_q) = E_q$  and  $\tau_p^*(E_r) = E_r$ , since these are the exceptional divisors obtained by blowing up fixed points of  $\tau_p$ . From the matrix for the pullback  $\tau^*$  of the Bertini involution, choosing  $E_p$  to be the sum  $E_7 + E_8$ , we see that

$$\tau^*(E_p) = 12H - 4 \sum_{i=1}^8 E_i - E_p = -4K_X - E_p + 4E_q + 4E_r.$$

It follows that the four-dimensional subspace  $W$  spanned by the classes  $K_X$ ,  $E_p$ ,  $E_q$ , and  $E_r$  is invariant under  $\tau_p$ . By the same argument, this same subspace is invariant under  $\tau_q$  and  $\tau_r$  as well. Furthermore, since these automorphisms all preserve the intersection form on  $N^1(X)$ , the orthogonal complement  $V = K_X^\perp \subset W$  is also invariant. This is a 3-dimensional subspace, spanned by the three classes  $F_p = -4K_X + 3E_q + 3E_r$ ,  $F_q = -4K_X + 3E_p + 3E_r$ , and  $F_r = -4K_X + 3E_p + 3E_q$ . (It might seem simpler to choose the classes  $-4K_X + 6E_p$ , but we will see that the  $F_p$  are nef.)

With respect to the basis given by these classes, the three involutions act as:

$$(\tau_p|_V)^* = \begin{pmatrix} 1 & 5 & 5 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad (\tau_q|_V)^* = \begin{pmatrix} 0 & 0 & -1 \\ 5 & 1 & 5 \\ -1 & 0 & 0 \end{pmatrix}, \quad (\tau_r|_V)^* = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 5 & 5 & 1 \end{pmatrix}.$$

Based on these matrices, one sees that the pullbacks given by the involutions on  $V$  are given by reflection in the hyperplanes  $H_p^\perp$ ,  $H_q^\perp$ , and  $H_r^\perp$ , where

$$\begin{aligned} H_p &= 5F_p - F_q - F_r \\ H_q &= 5F_q - F_p - F_r \\ H_r &= 5F_r - F_p - F_q. \end{aligned}$$

Now, the intersection form on  $N^1(X)$  restricts to  $V$  as a form with signature  $(1, 2)$ . Since the automorphism group preserves the intersection form, considering the subset  $\Delta \subset V$  determined by classes  $x$  with  $\langle x, x \rangle = 1$ , we obtain an action of  $\text{Aut}(X)$  on a real hyperbolic plane. One can see in Figure 3 that the hyperplanes  $H_p^\perp$ ,  $H_q^\perp$ , and  $H_r^\perp$  do not intersect inside  $\Delta$ , so the action of  $G$  is given by reflections in disjoint hyperplanes. This implies that there are no relations among the involutions (e.g. by Poincaré's polygon theorem [9]), yielding a free group  $G \cong (\mathbb{Z}/2\mathbb{Z})^{*3} \subset \text{Aut}(X)$ . This proves the first claim of Theorem 11.

We next consider the three compositions  $\sigma_{pq} = \tau_p \circ \tau_q$ ,  $\sigma_{pr} = \tau_p \circ \tau_r$ , and  $\sigma_{qr} = \tau_q \circ \tau_r$ . Each of these maps is of positive entropy, but has an invariant curve: for  $\sigma_{pq}$ , this is the strict transform of the unique element of  $|-2K_{S_{pq}}|$ . Write  $C_{pq}$ ,  $C_{pr}$ , and  $C_{qr}$  for these three invariant curves.

The 2-dimensional subspace in  $V$  spanned by  $4F_p - F_r$  and  $4F_q - F_r$  is invariant under  $\sigma_{pq}$ , with similar formulas for the other pairs. As a result, pullback by  $\sigma_{pq}$  leaves invariant the geodesic line  $L_{pq} \subset \Delta$  obtained as the intersection of this subspace with  $\Delta$ , and  $\sigma_{pq}^*$  acts on  $L_{pq}$  by a translation. Lines  $L_{pr}$  and  $L_{qr}$  are constructed in the same way. At last, write  $\theta_{pq}^+$  and  $\theta_{pq}^-$  for the endpoints of these geodesics on the boundary of  $\Delta$ . These are nef classes which are eigenvectors for pullback by the corresponding automorphisms.

Since  $C_{pq}$  is invariant under  $\sigma_{pq}$ , we obtain  $\langle C_{pq}, \theta_{pq}^\pm \rangle = 0$ . Moreover, the form  $\langle C_{pq}, - \rangle$  is nontrivial on  $V$ , with  $\ker(\langle C_{pq}, - \rangle) \cap \Delta = L_{pq}$ . It follows that  $\text{Amp}(V) = V \cap \text{Amp}(X)$  has intersection with  $\Delta$  lying entirely on one side of the geodesic line  $L_{pq}$ ; classes in the shaded areas of Figure 3 are not nef, because they have negative intersection with one of  $C_{pq}$ ,  $C_{pr}$ , and  $C_{qr}$ .

Let  $\Sigma$  denote the right-angled hyperbolic hexagon with edges determined by  $H_p^\perp$ ,  $H_q^\perp$ ,  $H_r^\perp$ ,  $L_p$ ,  $L_q$ , and  $L_r$ . The six vertices of the hexagon are given by  $3F_p + F_q - F_r$  and its permutations. Figure 3 illustrates the important points and geodesics in  $\Delta$ .

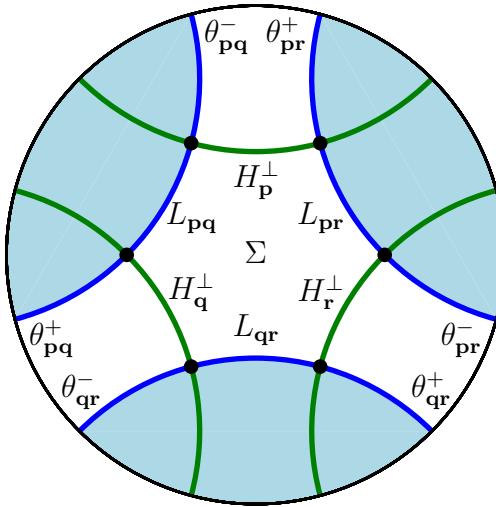


FIGURE 3. The action of  $G$  on  $\Delta$

We claim any class in the interior of  $\Sigma$  is ample. Indeed, such a class is a linear combination of the six eigenvector classes  $\theta_{pq}^\pm$ ,  $\theta_{pr}^\pm$ , and  $\theta_{qr}^\pm$ , hence is nef, and it has positive self-intersection. So it suffices to check that there is no curve  $B$  for which  $\langle \mu, B \rangle = 0$  for some  $\mu$  in the hexagon  $\Sigma$ . Suppose on the contrary that there is a curve  $B$  that has intersection 0 with a class  $\mu$  in  $\Sigma$ .

Suppose first that  $\langle B, - \rangle = 0$  identically on  $V$ , so that  $B$  lies in  $V^\perp$ . The action of  $G$  on  $V^\perp$  is of finite order, and so the union of the images of  $B$  under  $G$  is an invariant curve for the entire group. However, we have seen that there is no such invariant curve. Instead, suppose that  $\langle B, v \rangle = 0$  for some class in  $v \in V$ . Then the set of elements of  $\Delta$  orthogonal to  $B$  gives a geodesic  $\gamma$  which meets  $\Sigma$ . Some of the six classes  $\theta_{pq}^\pm$ ,  $\theta_{pr}^\pm$ ,  $\theta_{qr}^\pm$  lie on one side of  $\gamma$ , and some lie on the other. In particular, one of these classes has negative intersection with  $B$ , which is impossible since the classes are nef. We conclude that  $\Sigma \subset \text{Amp}(V)$ .

By taking the orbit of  $\Sigma$  under the maps  $\sigma_{pq}^*$ ,  $\sigma_{pr}^*$ , and  $\sigma_{qr}^*$ , we obtain many more ample regions in  $\Delta$ . Similarly, taking the orbit of the shaded regions yields many non-ample regions.

Indeed, the union of these two sets completely covers the interior of  $\Delta$ , and we obtain a complete picture of  $\text{Amp}(V)$ .

Let  $\Lambda$  be the limit set of  $G$  acting on  $\partial\Delta$ . This is a Cantor set, with complement the union of the shaded intervals in Figure 4. The hyperbolic convex hull  $\text{Conv}(\Lambda)$  is the closure of  $\text{Amp}(V)$ ; it is the unshaded region in the figure. Observe that the boundary of  $\text{Conv}(\Lambda)$  is given by curves of the form  $\gamma(L_{\mathbf{p}\mathbf{q}})$ ,  $\gamma(L_{\mathbf{p}\mathbf{r}})$ , and  $\gamma(L_{\mathbf{q}\mathbf{r}})$ , for automorphisms  $\gamma \in G$ .

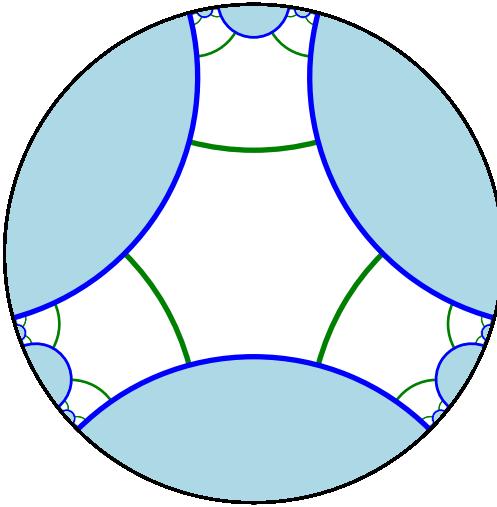


FIGURE 4. The set  $\text{Conv}(\Lambda)$

Since  $G \cong (\mathbb{Z}/2\mathbb{Z})^{*3}$ , the only elements of finite order are conjugate to the generating involutions. Suppose then that  $\psi \in G$  is of infinite order and that it has an invariant curve  $B$ . Consider  $B^\perp \subset \Delta$ . Since  $\Delta$  contains ample classes,  $B^\perp$  is a proper subset of  $\Delta$ , and so it is a geodesic. Since the orbit of an ample class under iterates of  $\psi$  must converge to  $B^\perp$ , this geodesic has nonempty intersection with the boundary of the nef cone. From the description of the boundary of the nef cone, the only possibility is that that geodesic  $B^\perp$  is of the form  $\gamma(L_{\mathbf{p}\mathbf{q}})$ ,  $\gamma(L_{\mathbf{p}\mathbf{r}})$ , or  $\gamma(L_{\mathbf{q}\mathbf{r}})$  for some  $\gamma \in G$ . If  $B^\perp = \gamma(L_{\mathbf{p}\mathbf{q}})$ , then  $\psi$  and  $\gamma \circ \sigma_{\mathbf{p}\mathbf{q}} \circ \gamma^{-1}$  both preserve  $\gamma(L_{\mathbf{p}\mathbf{q}})$ . Since  $G$  is discrete it follows that  $\psi$  is an iterate of  $\gamma \circ \sigma_{\mathbf{p}\mathbf{q}} \circ \gamma^{-1}$ . This proves the second part of Theorem 11.

At last we must exhibit a free subgroup of  $\text{Aut}(X)$  with no elements conjugate to any power of  $\tau_p, \tau_q, \tau_r, \sigma_{pq}, \sigma_{pr},$  or  $\sigma_{qr}.$  Let  $\xi = \tau_p \tau_q \tau_r.$  Define

$$\alpha = (\tau_p \tau_q) \xi (\tau_p \tau_q)$$

$$\beta = (\tau_p \tau_r) \xi (\tau_p \tau_r).$$

Let  $H = \langle \alpha, \beta \rangle \subset G \subseteq \text{Aut}(X)$  be the subgroup generated by  $\alpha$  and  $\beta$ . When forming any of the six products  $\alpha\alpha$ ,  $\alpha\beta$ ,  $\alpha\beta^{-1}$ ,  $\beta\beta$ ,  $\beta\alpha$ ,  $\beta\alpha^{-1}$ , or their inverses, there is at most one cancellation of two consecutive copies of the same involution, which guarantees that  $H$  is free.

Finally, we will show that  $H$  has trivial intersection with the conjugacy classes mentioned above. Any conjugacy class in the free product  $G \cong (\mathbb{Z}/2\mathbb{Z})^{*3}$  has a unique (up to cyclic permutation) representative in which the first and last letters are different (i.e. which is cyclically reduced). Suppose that

$$w = \alpha^{j_1} \beta^{j_1} \cdots \alpha^{j_n} \beta^{j_n}$$

is an element of  $H$ . After replacing  $w$  by a conjugate by some element of  $H$ , we may assume that it is cyclically reduced as an element of  $H$  with respect to the generators and  $\alpha$  and  $\beta$ . After expressing  $w$  as a word in  $\tau_p$ ,  $\tau_q$ , and  $\tau_r$ , from the definitions of  $\alpha$  and  $\beta$  we again see that cyclically permutation yields at most one additional cancellation. In particular, if  $w$  is nontrivial, then it is conjugate to a cyclically reduced word which contains within it either the product  $\xi = \tau_p \tau_q \tau_r$  or  $\xi^{-1}$ . However, no power of any of the six conjugacy classes of the statement has a cyclically reduced representative which involves all three generators. We conclude that  $w$  is not conjugate to any of these elements.

## 6. EXAMPLES OVER $\overline{\mathbb{F}}_p$

In general it seems difficult to find configurations  $\mathbf{p}$  and  $\mathbf{q}$  with positive-entropy automorphisms  $\phi_{\mathbf{p}}$  and  $\phi_{\mathbf{q}}$  for which  $\mathbf{q} \setminus \mathbf{p}$  is invariant under  $\phi_{\mathbf{p}}$  and  $\mathbf{p} \setminus \mathbf{q}$  is invariant under  $\phi_{\mathbf{q}}$ . However, we observe now that over the field  $k = \overline{\mathbb{F}}_p$ , the algebraic closure of a finite field, essentially any configuration will do. Most constructions of automorphisms of rational surfaces, such as those of Bedford–Kim [2] or McMullen [10], work perfectly well over these fields, although one must exercise some care that the characteristic is large enough to ensure that the configurations  $\mathbf{p}$  constructed there actually consist of sets of distinct points.

**Theorem 12.** *Let  $\mathbf{p}$  and  $\mathbf{q}$  be two configurations of points in  $\mathbb{P}^2$  for which there exist automorphisms  $\phi_{\mathbf{p}} : S_{\mathbf{p}} \rightarrow S_{\mathbf{p}}$  and  $\phi_{\mathbf{q}} : S_{\mathbf{q}} \rightarrow S_{\mathbf{q}}$ . Then there exist positive integers  $m$  and  $n$  so that  $\phi_{\mathbf{p}}^m \circ \phi_{\mathbf{q}}^n$  lifts to an automorphism of  $S_{\mathbf{pq}}$ .*

*Proof.*  $\phi_{\mathbf{p}} : S_{\mathbf{p}} \rightarrow S_{\mathbf{p}}$  is defined over an algebraically closed field  $\overline{\mathbb{F}}_p$ . There exists a finite field  $\mathbb{F}_q$  such that all the points of  $\mathbf{p}$  and  $\mathbf{q}$  and the maps  $\phi_{\mathbf{p}}$  and  $\phi_{\mathbf{q}}$  are all defined over  $\mathbb{F}_q$ .

Now, the number of  $\mathbb{F}_q$ -points on  $S_{\mathbf{p}}$  is finite, and these points are permuted by  $\phi_{\mathbf{p}}$ , so some iterate  $\phi_{\mathbf{p}}^m$  fixes all the points of  $\mathbf{q} \setminus \mathbf{p}$ . Similarly, an iterate  $\phi_{\mathbf{q}}^n$  fixes all the points in  $\mathbf{p} \setminus \mathbf{q}$ . Then the composition  $\phi_{\mathbf{p}}^m \circ \phi_{\mathbf{q}}^n$  of these two iterates lifts to an automorphism of the blow-up  $S_{\mathbf{pq}}$ .  $\square$

Let  $S_q^{\text{all}}$  denote the blow-up of  $\mathbb{P}^2_{\overline{\mathbb{F}}_p}$  at all  $\mathbb{F}_q$ -points for some prime power  $q = p^s$ . The proof of Theorem 12 shows that if  $\mathbf{p}$  is a configuration defined over  $\mathbb{F}_q$ , then any automorphism  $\phi : S_{\mathbf{p}} \rightarrow S_{\mathbf{p}}$  has an iterate that lifts to an automorphism of  $S_q^{\text{all}}$ , so it seems reasonable to expect that this group is quite large once  $q$  is sufficiently big.

**Question 1.** What can be said about the group  $\text{Aut}(S_q^{\text{all}})$ ? Is it finitely generated?

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