NOTIONS OF NUMERICAL IITAKA DIMENSION DO NOT COINCIDE

JOHN LESIEUTRE

Abstract. Let X be a smooth projective variety. The Iitaka dimension of a divisor D is an important invariant, but it does not only depend on the numerical class of D. However, there are several definitions of "numerical Iitaka dimension", depending only on the numerical class. In this note, we show that there exists a pseudoeffective $\mathbb{R}$-divisor for which these invariants take different values. The key is the construction of an example of a pseudoeffective $\mathbb{R}$-divisor D_+ for which $h^0(X, \lfloor mD_+ \rfloor + A)$ is bounded above and below by multiples of $m^{3/2}$ for any sufficiently ample A.

1. Introduction

Given a divisor D on a projective variety X, the Iitaka dimension of D is a fundamental invariant measuring the asymptotic growth of spaces of sections of mD.

Theorem-Definition (e.g. [10, Corollary 2.1.38])**.** Suppose that X is a smooth projective variety and D is a divisor on X. There exists an integer $\kappa(D)$, the *Iitaka dimension* of D, as well as constants $C_1, C_2 > 0$ such that for sufficiently large and divisible m,

$$C_1 m^{\kappa(D)} < h^0(X, mD) < C_2 m^{\kappa(D)}.$$

The most important case is when $D = K_X$ is the canonical class, in which case $\kappa(K_X)$ is simply the Kodaira dimension of X.

The Iitaka dimension has the inconvenient property that it is not a numerical invariant of D. It is possible, for example, that there exist two divisors D_1 and D_2 which have the same numerical class, but such that any multiple of D_1 is rigid, while D_2 moves in a pencil. In this case, $\kappa(D_1) = 0$ while $\kappa(D_2) \geq 1$ [11, Example 6.1].

One approach to constructing a numerical analog of the Iitaka dimension is to perturb each mD by a fixed ample divisor A, considering the dimensions $h^0(X, mD + A)$ as m increases. This growth of these sections does indeed yield an important numerical invariant, Nakayama's $\kappa_\sigma(D)$. There are a number of other possible definitions of numerical dimension, some of which we recall in the next section.

The main result of this paper is that, at least when D is an $\mathbb{R}$-divisor, the spaces of sections $h^0(X, \lfloor mD \rfloor + A)$ need not even grow polynomially in m.

Theorem 1. *There exists a smooth projective threefold X and a pseudoeffective $\mathbb{R}$-divisor D on X such that for any sufficiently ample class A, there exist constants $C_1, C_2 > 0$ so that*

$$C_1 m^{3/2} < h^0(X, \lfloor mD \rfloor + A) < C_2 m^{3/2}.$$

As a consequence of this calculation, we conclude that various notions of numerical dimension do not coincide in general, contrary to general expectation. The example is a pseudoeffective $\mathbb{R}$-divisor on a Calabi–Yau threefold X which has previously appeared in the work of Oguiso [15].

2. Preliminaries

We begin with some preliminary definitions. We work throughout over an algebraically closed field K of characteristic 0. Write $\equiv$ for the relation of numerical equivalence and $N^1(X)$ for the finite-dimensional $\mathbb{R}$-vector space of numerical classes of divisors on X. If D is a Cartier divisor, we will write $h^0(X, D)$ for $h^0(X, \mathcal{O}_X(D))$.

Definition 1 ([13, Ch. V])**.** The numerical dimension $\kappa_\sigma(D)$ is the largest integer k such that for some ample divisor A, one has

$$\limsup_{m\to\infty} \frac{h^0(X, \lfloor mD \rfloor + A)}{m^k} > 0.$$

If no such k exists, we take $\kappa_\sigma(D) = -\infty$. We will also consider a closely related invariant: $\kappa_\sigma^{\mathbb{R}}(D)$ is the supremum of the real numbers for which such an inequality holds. It will follow from our example that these two quantities may be distinct.

Remark 1. There are several variations on this definition. For example, one might replace the lim sup by a lim inf; this is the definition of κ_σ used in [4] and some older versions of [13]. Nakayama denotes this invariant by κ_σ^-. It remains unclear whether these values can be distinct.

It is also possible to ask for the smallest integer k for which

$$\limsup_{m\to\infty} \frac{h^0(X, \lfloor mD \rfloor + A)}{m^k} < \infty.$$

Nakayama denotes the resulting invariant by $\kappa_\sigma^+(D)$. This is the version of numerical dimension used in, for example, [9]. Our main example shows that this invariant is not equal to $\kappa_\sigma(D)$ in general.

An important result of Nakayama [13, Theorem V.1.12] states that if D is a pseudoeffective $\mathbb{R}$-divisor on X for which $h^0(X, \lfloor mD \rfloor + A)$ is not bounded in m (i.e. for which $D \neq N_\sigma(D)$), then for any sufficiently ample divisor A there is a constant C for which

$$h^0(X, \mathcal{O}_X(\lfloor mD \rfloor + A)) > Cm$$

for all m. The same result has been recovered in positive characteristic [4]. It follows that if $h^0(X, \mathcal{O}_X(\lfloor mD \rfloor + A))$ is not bounded, then $\kappa_\sigma^{\mathbb{R}}(D) \geq 1$.

A second definition of numerical dimension, Nakayama's $\kappa_\nu(D)$, is based on the notion of *numerical domination.*

Definition 2 ([13, Ch. V, §2], cf. [6])**.** Suppose that D is a pseudoeffective $\mathbb{R}$-divisor on X and $W \subset X$ is a subvariety. We say that D *numerically dominates* W (written $D \succeq W$) if there exists a birational morphism $\pi : \tilde{X} \to X$ such that $\pi^{-1}\mathcal{I}_W \cdot \mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(E)$ and for every positive b and every ample divisor A on $\tilde{X}$, there exist $x > b$ and $y > b$ such that the class $x \cdot \pi^* D - y \cdot E_W + A$ is pseudoeffective.

For discussion of this condition and some illuminating illustrations, we refer to the works of Nakayama [13] and Eckl [6].

Definition 3 ([13])**.** The numerical dimension $\kappa_\nu(D)$ is the minimum dimension of a subvariety $W \subset X$ for which D does not numerically dominate W.

A third definition is provided in terms of the positive intersection product. While we refer to [2] and [3] for the details of the construction, to a set of pseudoeffective divisors $D_1, \dots, D_k$ on X one associates a class in $N^k(X)$ which roughly measures the class of the intersection among the D_i which takes place away from their base loci. This positive intersection product is continuous on the big cone, but unlike the usual intersection form, is not linear.

Definition 4 ([2])**.** The numerical dimension $\nu_{\mathrm{BDPP}}(D)$ is the largest integer k for which the positive intersection product $\langle D^k \rangle$ is nonzero.

Remark 2. In the case that D is nef, the positive intersection product coincides with the usual intersection form, and Definition 4 coincides with the original definition of Kawamata [8]. In this case, it is proved by Nakayama that $\kappa_\sigma(D) = \kappa_\nu(D) = \nu_{\mathrm{BDPP}}(D)$.

3. Main example

Example 1 ([15, §6])**.** Let X be a smooth threefold in $\mathbb{P}^3 \times \mathbb{P}^3$ given as the intersection of general divisors of bidegrees $(1,1)$, $(1,1)$, and $(2,2)$. It follows from adjunction and the Lefschetz hyperplane theorem that X is a smooth, Calabi-Yau threefold of Picard rank 2. Let $\pi_i : X \to \mathbb{P}^3$ $(i = 1, 2)$ be the two projections. A basis for $N^1(X)$ is given by the numerical classes of the two divisors $H_i = \pi_i^* \mathcal{O}_{\mathbb{P}^3}(1)$.

The maps π_1 and π_2 are both generically 2 to 1, and so there are two associated birational covering involutions $\tau_i : X \dashrightarrow X$. The maps τ_i are not biregular, since the π_i have some positive-dimensional fibers. However, since K_X is trivial, these maps extend to pseudoautomorphisms of X, i.e. birational maps which are an isomorphism in codimension 1.

Oguiso checks that with respect to the basis above, we have:

$$\tau_1^* = \begin{pmatrix} 1 & 6 \\ 0 & -1 \end{pmatrix}, \quad \tau_2^* = \begin{pmatrix} -1 & 0 \\ 6 & 1 \end{pmatrix}.$$

The composite map $\phi = \tau_1 \circ \tau_2$ acts on $N^1(X)$ by

$$\phi^* = \tau_2^* \tau_1^* = \begin{pmatrix} -1 & -6 \\ 6 & 35 \end{pmatrix}.$$

Recall that for $0 \leq k \leq N = \dim X$, the k^{th} dynamical degree of ϕ is the number

$$\lambda_k(\phi) = \lim_{n \to \infty} \left((\phi^n)^*(H^k) \cdot H^{N-k} \right)^{1/n},$$

where H is a fixed ample divisor; in fact this limit exists and is independent of H [5]. In our case, the first dynamical degree is the spectral radius of ϕ^*, which is given by

$$\lambda = \lambda_1(\phi) = 17 + 12\sqrt{2} \approx 33.97\dots.$$

It is also useful to compute the nef and pseudoeffective cones, as well as certain subcones. The nef cone is spanned by the classes of the two divisors H_1 and H_2.

Lemma 2. *The pseudoeffective cone coincides with the movable cone and is spanned by the two eigenvectors of ϕ^*, which up to a choice of normalization are given by:*

$$\Delta_+ \equiv (1 - \sqrt{2})H_1 + (1 + \sqrt{2})H_2,$$
$$\Delta_- \equiv (1 + \sqrt{2})H_1 + (1 - \sqrt{2})H_2.$$

These satisfy $\phi^* \Delta_+ = \lambda \Delta_+$ and $\phi^* \Delta_- = \lambda^{-1} \Delta_-$.

Proof. Since $\phi^*(\overline{\mathrm{Mov}}(X)) = \overline{\mathrm{Mov}}(X)$ and $\phi^*(\overline{\mathrm{Eff}}(X)) = \overline{\mathrm{Eff}}(X)$, this follows from a form of the Perron–Frobenius theorem [1]. Indeed, suppose that Δ is a pseudoeffective divisor with nonzero components of each eigenvector. Then for each $n > 0$, the class $\frac{1}{\lambda^n}(\phi^*)^n(\Delta)$ also lies in the cone. Since the cone is closed, so too does the limit of these classes, which is a multiple of the eigenvector Δ_+. To see that it is on the boundary, notice that if Δ is non-pseudoeffective class, the analogous sequence of pullbacks yields a sequence of non-pseudoeffective divisors which also converge to Δ_+. The same argument applies to the movable cone, or to Δ_- after replacing ϕ by ϕ^{-1}. □

Let D_+ and D_- be the two $\mathbb{R}$-divisors in the span of H_1 and H_2 which represent the numerical classes Δ_+ and Δ_-. It is necessary to choose explicit $\mathbb{R}$-divisors rather than numerical classes in order to make sense of the round-downs $\lfloor mD_+ \rfloor$, but the resulting $\kappa_\sigma(\Delta_+)$ is ultimately independent of the choice.

It will also be convenient for us to work with the cone $\mathcal{C} \subset N^1(X)$ spanned by H_2 and $\tau_1^* H_2 = 6H_1 - H_2$. This cone has the property that if D is any divisor class lying in $\mathcal{C}$, then either D or $\tau_1^* D$ is big and nef.

Theorem 3. *The pseudoeffective* $\mathbb{R}$*-divisor* D_+ *satisfies:*

(1) $\nu_{BDPP}(D_+) = \kappa_\sigma(D_+) = 1$;
(2) $\kappa_\sigma^{\mathbb{R}}(D_+) = 3/2$;
(3) $\kappa_\sigma^+(D_+) = \kappa_\nu(D_+) = 2$.

The bulk of the work is dedicated to computing $h^0(X, \lfloor mD_+ \rfloor + A)$ and hence $\kappa_\sigma^{\mathbb{R}}(D_+)$; in fact, the computations of $\nu_{\mathrm{BDPP}}(D_+)$ and $\kappa_\nu(D_+)$ follow from this and the inequalities of [11] and [6]. Since these can also be computed directly, we include a derivation for the sake of completeness. The main complication is that the definition of κ_σ and $\kappa_\sigma^{\mathbb{R}}$ for $\mathbb{R}$-divisors requires working with round-downs, while the other notions do not; this makes it somewhat tedious to compute.

Heuristic. Before giving a proof, we briefly explain the calculation of $h^0(X, \lfloor mD_+ \rfloor + A)$. The variety X has the property that given any big divisor class D, there is a pseudoautomorphism (either ϕ^m or $\tau_1 \circ \phi^m$), such that the pullback of D under this map is big and nef. Since $h^0(X, D)$ is invariant under pulling back by a pseudoautomorphism, and $h^0(X, A)$ can be computed using the Riemann–Roch theorem if A is big and nef, it is possible to compute $h^0(X, D)$ for any big divisor D, even those such as $\lfloor mD_+ \rfloor + A$ which have complicated base loci and lie very close to the pseudoeffective boundary.

For simplicity, we work in the basis for $N^1(X)$ given by Δ_+ and Δ_-, the two extremal rays on $\overline{\mathrm{Eff}}(X)$. The pullback ϕ^* is given in this basis by $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$, and so it preserves a quadratic form, the product of the two coordinates of a class written with respect to this basis. Choosing a suitable scaling of Δ_+, we may assume that $A = \Delta_+ + \Delta_-$ is ample. With respect to this basis, the class $m\Delta_+ + A$ has coordinates $(m + 1, 1)$. The ample cone consists of divisors for which the two coordinates are approximately equal (more precisely, for which their ratio is contained in some bounded interval). Since pullback by ϕ^* preserves the product of the coordinates, the pullback $\phi^{k_m}(m\Delta_+ + A)$ which is ample must be roughly $\sqrt{m}\Delta_+ + \sqrt{m}\Delta_-$, which is the case when $k_m \approx -\frac{1}{2}\log_\lambda m$. We are then in position to compute

$$\begin{aligned} h^0(X, \lfloor mD_+ \rfloor + A) &= h^0(X, \phi^{*k_m}(\lfloor mD_+ \rfloor + A)) = \chi(\phi^{*k_m}(\lfloor mD_+ \rfloor + A)) \\ &\approx (\phi^{*k_m}(\lfloor mD_+ \rfloor + A))^3/6 \approx (\phi^{*k_m}(m\Delta_+ + A))^3/6 \\ &\approx \left(\sqrt{m}\Delta_+ + \sqrt{m}\Delta_-\right)^3/6 = Cm^{3/2}. \end{aligned}$$

The next few lemmas establish the bounds required to make this precise. For simplicity, we focus our computations on the particular variety X, but similar results can be obtained for more general contexts; see Lemma **??**. The proofs involve many constants whose precise values are not important; we will denote these constants by C_i, $C_{1,j}$ and $C_{2,k}$ as they appear.

It is convenient to introduce a new set of coordinates on $\mathrm{Big}(X)$. Given a big class $D = a_1\Delta_+ + a_2\Delta_-$ (which must have $a_1, a_2 > 0$), we set

$$L_1(D) = a_1a_2, \quad L_2(D) = \frac{a_1}{a_2}.$$

For an $\mathbb{R}$-divisor D, we write $L_i(D)$ for the corresponding value for the numerical class. These coordinates owe their convenience to the facts that

$$L_1(\phi^*D) = L_1(D), \quad L_2(\phi^*D) = \lambda^2 L_2(D).$$

Lemma 4. *Suppose that D is a big class on X. Then there exists an integer k so that $(\phi^*)^k(D)$ lies in the cone $\mathcal{C}$.*

Proof. The cone $\mathcal{C}$ is bounded by the two divisors

$$H_2 = \left(\frac{2+\sqrt{2}}{8}\right)\Delta_+ + \left(\frac{2-\sqrt{2}}{8}\right)\Delta_-,$$

$$\tau_1^*H_2 = \left(\frac{10-7\sqrt{2}}{8}\right)\Delta_+ + \left(\frac{10+7\sqrt{2}}{8}\right)\Delta_-,$$

and so

$$L_2(H_2) = \frac{2+\sqrt{2}}{2-\sqrt{2}} = 3+2\sqrt{2}$$

$$L_2(\tau_1^*H_2) = \frac{10-7\sqrt{2}}{10+7\sqrt{2}} = 99-70\sqrt{2}.$$

Then

$$\frac{L_2(H_2)}{L_2(\tau_1^*H_2)} = \frac{3+2\sqrt{2}}{99-70\sqrt{2}} = 577+408\sqrt{2} = \lambda^2.$$

We have seen that $L_2(\phi^*D) = \lambda^2 L_2(D)$, and the claim follows: explicitly, we may take

$$k = -\left\lfloor \frac{1}{2}\left(\log_\lambda L_2(D) - \log_\lambda L_2(\tau_1^*H_2)\right)\right\rfloor. \qquad \square$$

The next observation is that on this variety X, it is straightforward to compute $h^0(X, D)$ for any big and nef D.

Lemma 5. *There exist constants $C_{1,1}, C_{2,1} > 0$ such that if $D = a_1H_1 + a_2H_2$ is any big and nef Cartier divisor,*

$$C_{1,1}D^3 < h^0(X, D) < C_{2,1}D^3.$$

Proof. The intersection form on divisors on X is given by $H_1^3 = H_2^3 = 2$ and $H_1^2 H_2 = H_1 H_2^2 = 6$. Since X is a Calabi–Yau threefold, it follows from the Hirzebruch–Riemann–Roch theorem and Kawamata–Viehweg vanishing that for any big and nef class D,

$$h^0(X,D) = \chi(X,D) = \frac{D^3}{6} + \frac{c_2(X)\cdot D}{12}$$
$$\frac{h^0(X,D)}{D^3} = \frac{1}{6} + \frac{1}{12}\frac{c_2(X)\cdot D}{D^3}.$$

We have $c_2(X) = H_1^2 + 6H_1H_2 + H_2^2$, and so explicitly,

$$\frac{h^0(X,D)}{D^3} = \frac{1}{6} + \frac{1}{12}\frac{(H_1^2 + 6H_1H_2 + H_2^2)\cdot(a_1H_1 + a_2H_2)}{(a_1H_1 + a_2H_2)^3}$$
$$= \frac{1}{6} + \frac{1}{12}\frac{44(a_1+a_2)}{2a_1^3 + 18a_1^2a_2 + 18a_1a_2^2 + 2a_2^3} = \frac{1}{6} + \frac{11}{6}\frac{1}{(a_1+a_2)^2 + 6a_1a_2}.$$

Since a_1 and a_2 are non-negative integers, not both 0, the claim holds with $C_{1,1} = 1/6$ and $C_{2,1} = 2$. □

Lemma 6. *There exist constants $C_{1,2}, C_{2,2} > 0$ such that if D is any Cartier divisor contained in the cone $\mathcal{C}$,*

$$C_{1,2}L_1(D)^{3/2} < h^0(X,D) < C_{2,2}L_1(D)^{3/2}.$$

Proof. Suppose first that D is actually big and nef. We may write $D = a_1\Delta_+ + a_2\Delta_-$ where $a_1 = (L_1(D)L_2(D))^{1/2}$ and $a_2 = (L_1(D)/L_2(D))^{1/2}$. Using the intersection form given in the proof of Lemma 5, we have

$$\Delta_+^3 = \Delta_-^3 = -8,$$
$$\Delta_+^2\cdot\Delta_- = \Delta_+\cdot\Delta_-^2 = 56.$$

This yields

$$D^3 = L_1(D)^{3/2}\left(-8L_2(D)^{3/2} + 168L_2(D)^{1/2} + 168L_2(D)^{-1/2} - 8L_2(D)^{-3/2}\right).$$

The factor in parentheses is easily checked to be non-negative if

$$11 - 2\sqrt{30} \le L_2(D) \le 11 + 2\sqrt{30},$$

and is zero only at the endpoints. On the other hand, any big and nef divisor A satisfies

$$L_2(H_1) = 3 - 2\sqrt{2} \le L_2(A) \le 3 + 2\sqrt{2} = L_2(H_2).$$

This interval is strictly contained in the one for which $D^3 > 0$, which proves that there exist $C_{1,2}$ and $C_{2,2}$ which give the required bound for any big and nef classes by Lemma 5. If D lies in $\mathcal{C}$ but it is not big and nef, then τ_1^*D is big and nef. Since $h^0(X,\tau_1^*D) = h^0(X,D)$ and $L_1(\tau_1^*D)$ is bounded above and below by constant multiples of $L_1(D)$, the claim follows. □

The next lemma checks that rounding down does not have a large impact on $L_1(\lfloor D\rfloor + A)$.

Lemma 7. *There exist constants $C_1, C_{1,3}, C_{2,3} > 1$ such that for any pseudoeffective $\mathbb{R}$-divisor $D = a_1D_+ + a_2D_-$ and any ample divisor $A = b_1D_+ + b_2D_-$ with $b_i > C_1$, we have*

$$C_{1,3}L_1(D+A) \le L_1(\lfloor D\rfloor + A) \le C_{2,3}L_1(D+A).$$

Proof. Suppose that $D = a_1D_+ + a_2D_-$, and that $\lfloor D \rfloor = \tilde{a}_1D_+ + \tilde{a}_2D_-$. It is clear that there is a constant $c_1 > 0$ so that $|a_i - \tilde{a}_i| < c_1$: to compute the $\tilde{a}_i$, one expresses the divisor in terms of the basis H_1 and H_2, rounds down the coefficients, and then changes the basis back.

Since D_+ and D_- bound the pseudoeffective cone, the fact that D is pseudoeffective means that $a_i \geq 0$. The rounded coefficients $\tilde{a}_i$ may be negative, but the above shows that $\tilde{a}_i \geq -c_1$.

Then $D + A = (a_1 + b_1)D_+ + (a_2 + b_2)D_-$ and $\lfloor D \rfloor + A = (\tilde{a}_1 + b_1)D_+ + (\tilde{a}_2 + b_2)D_-$, and we find that

$$\frac{L_1(\lfloor D \rfloor + A)}{L_1(D + A)} = \frac{(\tilde{a}_1 + b_1)(\tilde{a}_2 + b_2)}{(a_1 + b_1)(a_2 + b_2)} = \frac{\tilde{a}_1 + b_1}{a_1 + b_1} \cdot \frac{\tilde{a}_2 + b_2}{a_2 + b_2}.$$

Take $C_1 = 1 + c_1 > 1$, and suppose that $b_i > C_1$. Then

$$\begin{aligned} a_i + b_i &\geq 0 + b_i > C_1 > 1, \\ \tilde{a}_i + b_i &> -c_1 + b_i > C_1 - c_1 = 1. \end{aligned}$$

Since both $a_i + b_i$ and $\tilde{a}_i + b_i$ are greater than 1,

$$\left| \log \left(\frac{\tilde{a}_i + b_i}{a_i + b_i} \right) \right| = |\log(\tilde{a}_i + b_i) - \log(a_i + b_i)| < |\tilde{a}_i - a_i| < C_1,$$

which implies that each of the factors on the right hand side of the preceding equation are bounded by multiplicative factors of e^{-C_1} and e^{C_1}. The result follows with $C_{1,3} = e^{-2C_1}$ and $C_{2,3} = e^{2C_1}$. □

Theorem 8 ($\Rightarrow$ Theorem 3, (2))**.** *Suppose that $A = b_1D_+ + b_2D_-$ is an ample Cartier divisor with $b_1, b_2 \geq C_1$. There exist constants $C_{1,4}$ and $C_{2,4}$ such that for all sufficiently large m,*

$$C_{1,4}m^{3/2} < h^0(X, \lfloor mD_+ \rfloor + A) < C_{2,4}m^{3/2}.$$

Proof. We have

$$L_1(mD_+ + A) = L_1((m + b_1)D_+ + b_2D_-) = (m + b_1)b_2.$$

It follows from Lemma 7 that

$$C_{1,3}(m + b_1)b_2 \leq L_1(\lfloor mD_+ \rfloor + A) \leq C_{2,3}(m + b_1)b_2.$$

According to Lemma 4, for every value of m, there exists a constant k_m for which $(\phi^{k_m})^*(\lfloor mD_+ \rfloor + A)$ lies in the cone $\mathcal{C}$, and since $L_1(-)$ is invariant under ϕ, this shows

$$C_{1,3}(m + b_1)b_2 \leq L_1(\phi^{k_m *}(\lfloor mD_+ \rfloor + A)) \leq C_{2,3}(m + b_1)b_2.$$

Since $h^0(X, \lfloor mD_+ \rfloor + A) = h^0(X, \phi^{k_m *}(\lfloor mD_+ \rfloor + A))$, Lemma 6 yields

$$C_{1,2} \left(C_{1,3}(m + b_1)b_2 \right)^{3/2} \leq h^0(X, \lfloor mD_+ \rfloor + A) \leq C_{2,2} \left(C_{2,3}(m + b_1)b_2 \right)^{3/2},$$

and the theorem follows. □

Remark 3. For any given value of m, it is straightforward to use a computer algebra system and the Riemann–Roch theorem for a Calabi-Yau threefold to determine the exact value of $h^0(X, \lfloor mD_+ \rfloor + A)$. For the ample divisor $A = H_1 + H_2$, taking $m = 2^k$ for $10 \leq k \leq 50$, we find that

$$24 \cdot m^{3/2} < h^0(X, \lfloor m\Delta_+ \rfloor + A) < 54 \cdot m^{3/2}.$$

The computations $\kappa_\sigma(D_+) = 1$, $\kappa_\sigma^-(D_+) = 1$, $\kappa_\sigma^{\mathbb{R}}(D_+) = 3/2$, and $\kappa_\sigma^+(D_+) = 2$ are immediate. It remains to compute $\nu_{\text{BDPP}}(D_+)$ and $\kappa_\nu(D_+)$.

Proof of Theorem 3, (1). If $\phi : X \dashrightarrow X$ is an isomorphism in codimension 1, then $\langle \phi^* D_1 \cdot \phi^* D_2 \rangle = \phi_1^*(\langle D_1 \cdot D_2 \rangle)$, where $\phi_1^* : N_1(X) \to N_1(X)$ is the pullback map on curve classes. Then for any value of n, we have

$$\begin{aligned} \left\langle (\phi^{n*}(D_+) + \phi^{n*}(D_-))^2 \right\rangle &= \phi_1^{n*}(\langle D_+ + D_- \rangle^2) \\ \left\langle (\lambda^n D_+ + \lambda^{-n} D_-)^2 \right\rangle &= \phi_1^{n*}(\langle D_+ + D_- \rangle^2) \\ \left\langle (D_+ + \lambda^{-2n} D_-)^2 \right\rangle &= \lambda^{-2n} \phi_1^{n*}(\langle D_+ + D_- \rangle^2). \end{aligned}$$

Since ϕ_1^* has spectral radius $\lambda < \lambda^2$, the quantity on the right approaches 0. On the other hand, the classes of the divisors $D_+ + \lambda^{-2n} D_-$ approach D_+ from an ample direction in $N^1(X)$. It follows from the definition of the positive intersection product for pseudoeffective classes [3, Definition 2.10] that the limit of the left side is $\langle D_+^2 \rangle$. Consequently $\langle D_+^2 \rangle = 0$, and so $\nu_{\mathrm{BDPP}}(D_+) = 1$. □

Proof of Theorem 3, (3). It is a result of Nakayama that $\kappa_\sigma(D) \le \kappa_\nu(D)$ for any pseudoeffective $\mathbb{R}$-divisor D [13, Proposition V.2.22(1)]. In fact, the proof *loc. cit.* applies equally well to $\kappa_\sigma^{\mathbb{R}}(D)$, and so $\kappa_\nu(D_+) \ge 3/2$. Since this invariant is integer-valued, and $\kappa_\nu(D) = \dim X = 3$ if and only if D is big, we conclude that $\kappa_\nu(D_+) = 2$.

□

Remark 4. The question of whether $\kappa_\sigma(D) = \kappa_\nu(D)$ in general originates with Nakayama. The general equality $\nu_{\mathrm{BDPP}}(D) = \kappa_\sigma(D) = \kappa_\nu(D)$ is asserted in the two papers [11] and [6]. These papers prove a number of remarkable inequalities between various notions of numerical dimension, but unfortunately each contains a gap: [11, Proposition 5.3] does not hold in general (see [6, §2.9] for some discussion), while the proof of [6, Proposition 3.4] fails because the middle row of the commutative diagram is not necessarily exact. This requires some additional corrections to the literature; see [7, Corrigendum].

Remark 5. Observe that Theorem 3 provides a counterexample to [11, Theorem 6.7, (7)]; it would be interesting to know whether for any pseudoeffective $\mathbb{R}$-divisor D, there exist constants C_1 and C_2 for which

$$C_1 m^{\kappa_\sigma^{\mathbb{R}}(D)} < h^0(X, \lfloor mD \rfloor + A) < C_2 m^{\kappa_\sigma^{\mathbb{R}}(D)}.$$

Remark 6. Although for simplicity we have preferred explicit computations on the variety X, the same strategy should suffice to compute the numerical dimension in many other contexts. According to the Kawamata–Morrison cone conjecture, if X is a Calabi–Yau threefold, then for any big divisor class D there exists a pseudoautomorphism $\phi : X \dashrightarrow X$ such that $\phi^* D$ lies in some fixed polyhedral subcone of $\mathrm{Big}(X)$, where the volume can likely be computed explicitly.

We now give a general computation in this vein, for another notion of numerical dimension, ν_{Vol}. This invariant is similar to κ_σ, but has two simplifying advantages: (i) one need not worry about the difference between $\chi(D)$ and $h^0(X, D)$ when X is not a Calabi–Yau, and (ii) it is not necessary to take the round-down of an $\mathbb{R}$-divisor, which in the case $\rho(X) > 2$ could push the divisor out of the 2-dimensional eigenspace for ϕ^* spanned by Δ_+ and Δ_-.

Definition 5 ([11])**.** Suppose that X is a projective variety and D is a pseudoeffective divisor class on X. Fix an ample divisor A. The numerical dimension $\nu_{\mathrm{vol}}(D)$ is the largest integer k for which there exists a constant C satisfying

$$C t^{\dim X - k} < \mathrm{vol}(L + tA)$$

for all $t > 0$. We also define $\nu_{\text{vol}}^{\mathbb{R}}(D)$ to be th largest real number k with this property.

Lemma 9. *Suppose that $\phi : X \dashrightarrow X$ is a pseudoautomorphism satisfying $\lambda_1(\phi) > 1$. Let $\lambda_1 = \lambda_1(\phi)$ and $\mu_1 = \lambda_1(\phi^{-1})$; it follows from the log-concavity of dynamical degrees that $\mu_1 > 1$ as well. Suppose that there exist a λ_1-eigenvector Δ_+ for ϕ^* and a λ_1-eigenvector Δ_- for ϕ^{-1*} with the property that $A = \Delta_+ + \Delta_-$ is ample. Then*

$$\nu_{vol}^{\mathbb{R}}(\Delta_+) = (\dim X)\left(1 + \frac{\log \mu_1}{\log \lambda_1}\right)^{-1}$$

Proof. Since ϕ^* preserves the volume of a divisor,

$$\begin{aligned}\operatorname{vol}(A) = \operatorname{vol}(\Delta_+ + \Delta_-) &= \operatorname{vol}((\phi^*)^n(\Delta_+ + \Delta_-)) = \operatorname{vol}\left(\lambda_1^n \Delta_+ + \mu_1^{-n}\Delta_-\right) \\ &= \operatorname{vol}\left((\lambda_1^n - \mu_1^{-n})\left(\Delta_+ + \frac{\mu_1^{-n}}{\lambda_1^n - \mu_1^{-n}}(\Delta_+ + \Delta_-)\right)\right) \\ &= (\lambda_1^n - \mu_1^{-n})^{\dim X} \operatorname{vol}\left(\Delta_+ + \frac{\mu_1^{-n}}{\lambda_1^n - \mu_1^{-n}}(\Delta_+ + \Delta_-)\right).\end{aligned}$$

Taking $A = \Delta_+ + \Delta_-$ and $t_n = \frac{\mu_1^{-n}}{\lambda_1^n - \mu_1^{-n}}$, we find that

$$\operatorname{vol}(\Delta_+ + t_n A) = (\lambda_1^n - \mu_1^{-n})^{-\dim X} \operatorname{vol}(A) = C_n t_n^{\nu},$$

where

$$\nu = (\dim X)\left(1 + \frac{\log \mu_1}{\log \lambda_1}\right)^{-1}$$

$$C_n = \frac{(\lambda_1^n - \mu_1^{-n})^{-\dim X} \operatorname{vol}(A)}{\left(\frac{\mu_1^{-n}}{\lambda_1^n - \mu_1^{-n}}\right)^{\left((\dim X)\left(1 + \frac{\log \mu_1}{\log \lambda_1}\right)^{-1}\right)}}$$

One may check that C_n is a decreasing function as n increases and that $\lim_{n\to\infty} C_n = \operatorname{vol}(A)$. In particular, for sufficiently large n we have

$$\operatorname{vol}(A)t_n^{\nu} < \operatorname{vol}(\Delta_+ + t_n A) < 2\operatorname{vol}(A)t_n^{\nu}.$$

Since $\operatorname{vol}(\Delta_+ + tA)$ is an increasing function in t, this implies that there exists a constant C such that $\operatorname{vol}(\Delta_+ + tA) < Ct^{\nu}$ for all $0 < t < 1$, and so

$$\nu_{\text{vol}}^{\mathbb{R}}(\Delta_+) = \nu = (\dim X)\left(1 + \frac{\log \mu_1}{\log \lambda_1}\right)^{-1} = (\dim X)\left(1 + \frac{\log \lambda_1(\phi^{-1})}{\log \lambda_1(\phi)}\right)^{-1}. \qquad \square$$

In the example of this section, $\lambda_1(\phi) = \lambda_1(\phi^{-1}) = \lambda$ and the formula yields $\nu_{\text{vol}}^{\mathbb{R}}(\Delta_+) = 3/2$, which coincides with $\kappa_{\sigma}^{\mathbb{R}}(\Delta_+)$.

Remark 7. It is not at all clear that the quantity $(\dim X)\left(1 + \frac{\log \lambda_1(\phi^{-1})}{\log \lambda_1(\phi)}\right)^{-1}$ should always be rational when $\Delta_+ + \Delta_-$ is ample, although I am not aware of any relevant counterexamples.

Remark 8. N. McCleerey has showed that $\nu_{\text{vol}}(D) = \nu_{\text{BDPP}}(D)$ in several cases, e.g. when $\nu_{\text{BDPP}}(D) = 0$ or $\nu_{\text{BDPP}}(D) = \dim X - 1$ [12] (when $\dim X = 3$, this covers all cases except that of $\nu_{\text{BDPP}}(D)$ which occurs for our main example). It would also be interesting to know whether $\kappa_{\sigma}(D) = \nu_{\text{vol}}(D)$ in general.

When ϕ is an automorphism with $\lambda_1(\phi) > 1$ (rather than just a pseudoautomorphism), it is possible to give a more precise computation of the numerical dimension of the eigenvector in terms of the dynamical degrees of ϕ. In this case, Δ_+ is nef, and the different definitions of numerical dimension coincide; in particular, the value is always an integer.

Let $J_k(\phi)$ denote the size of the largest Jordan block for $\phi^* : N^k(X) \to N^k(X)$, and take $\tilde{J}_k(\phi) = J_k(\phi) - 1$, so that for a general ample divisor H we have $(\phi^*)^m(H^k) \cdot H^{N-k} \sim \lambda_k(\phi)^m m^{\tilde{J}_k(\phi)}$. Here by $\sim$ we mean that the left quantity is bounded above and below by multiples of the right one.

Theorem 10. *Suppose that $\phi : X \to X$ is an automorphism with $\lambda_1(\phi) > 1$ and that Δ_+ is a leading eigenvector for ϕ, equal to $\Delta_+ = \lim_{n\to\infty} \frac{1}{\lambda_1(\phi)^n n^{\tilde{J}_1(\phi)}} (\phi^*)^n H$ for some ample H. Then $\nu_{BDPP}(\Delta_+) = \kappa_\sigma(\Delta_+) = \kappa_\nu(\Delta_+)$ with*

$$\kappa_\sigma(\Delta_+) = \max\left\{a : \lambda_a(\phi) = \lambda_1(\phi)^a \text{ and } \tilde{J}_a(\phi) = a\tilde{J}_1(\phi)\right\}.$$

Proof. Let H be an ample class and $N = \dim X$. Then $\Delta_+^a \neq 0$ if and only if $\Delta_+^a \cdot H^{N-a} > 0$, and we compute

$$\begin{aligned}
\Delta_+^a \cdot H^{N-a} &= \lim_{n\to\infty} \left(\frac{1}{\lambda_1(\phi)^{na} n^{a\tilde{J}_1(\phi)}} (\phi^*)^n (H)^a \right) \cdot H^{N-a} \\
&= \lim_{n\to\infty} \frac{1}{\lambda_1(\phi)^{na} n^{a\tilde{J}_1(\phi)}} \left((\phi^*)^n (H)^a \cdot H^{N-a} \right) \\
&= \lim_{n\to\infty} \frac{1}{\lambda_1(\phi)^{na} n^{a\tilde{J}_1(\phi)}} \left(\lambda_a(\phi)^n n^{\tilde{J}_a(\phi)} \right) \\
&= \lim_{n\to\infty} \left(\frac{\lambda_a(\phi)}{\lambda_1(\phi)^a} \right) \left(n^{\tilde{J}_a(\phi) - a\tilde{J}_1(\phi)} \right).
\end{aligned}$$

Consequently $\Delta_+^a \cdot H^{N-a} > 0$ if $\lambda_a(\phi) = \lambda_1(\phi)^a$ and $\tilde{J}_a(\phi) = a\tilde{J}_1(\phi)$. The claim follows. (Note that the first equality always holds if $=$ is changed to $\leq$, by log concavity of dynamical degrees; the same is true of the second in the case that the first is an equality.) □

Example 2. Suppose that X is a hyper-Kähler manifold of dimension $N = 2m$ and that $\phi : X \to X$ is an automorphism. It is shown by Oguiso [14] that $\lambda_a(\phi) = \lambda_1(\phi)^a$ for $a \leq m$, so that $\nu(\Delta_+) = m$ in this case.

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THE PENNSYLVANIA STATE UNIVERSITY, 204 MCALLISTER BUILDING, UNIVERSITY PARK, PA 16801
Email address: `jdl@psu.edu`