HIGHER ARITHMETIC DEGREES OF DOMINANT RATIONAL SELF-MAPS

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ABSTRACT. Suppose that $f\colon X \dashrightarrow X$ is a dominant rational self-map of a smooth projective variety defined over $\overline{\mathbf{Q}}$. Kawaguchi and Silverman conjectured that if $P \in X(\overline{\mathbf{Q}})$ is a point with well-defined forward orbit, then the growth rate of the height along the orbit exists, and coincides with the first dynamical degree $\lambda_1(f)$ of f if the orbit of P is Zariski dense in X.

In this note, we extend the Kawaguchi–Silverman conjecture to the setting of orbits of higher-dimensional subvarieties of X. We begin by defining a set of arithmetic degrees of f, independent of the choice of cycles, and we then develop the theory of arithmetic degrees in parallel to existing results for dynamical degrees. We formulate several conjectures governing these higher arithmetic degrees, relating them to dynamical degrees.

1. INTRODUCTION

Suppose that $f\colon X \dashrightarrow X$ is a dominant rational self-map of a d-dimensional smooth projective variety defined over $\overline{\mathbf{Q}}$. Let $X_f(\overline{\mathbf{Q}})$ denote the set of rational points for which the full forward orbit is well-defined. Fix a Weil height function $h_X\colon X(\overline{\mathbf{Q}}) \longrightarrow \mathbf{R}$ and set $h_X^+ \coloneqq \max\{h_X, 1\}$. Following Kawaguchi–Silverman [16], the *arithmetic degree* of a point $P \in X_f(\overline{\mathbf{Q}})$ is defined to be the limit

$$\alpha_f(P) = \lim_{n\to\infty} h_X^+(f^n(P))^{1/n},$$

supposing that it exists. This is a measure of the "arithmetic complexity" of the orbit of P.

Another basic invariant of f is its set of dynamical degrees. For $0 \leqslant k \leqslant d$, we define

$$\lambda_k(f) = \lim_{n\to\infty} ((f^n)^* H^k \cdot H^{d-k})^{1/n},$$

where H is an ample divisor on X. The limit is known to exist (see [8, 26, 5]). Also, note that it is independent of the choice of the ample divisor H. There is expected to be a close relationship between the arithmetic and dynamical degrees: a conjecture of Kawaguchi and Silverman [16] states that if P has Zariski dense f-orbit, then $\alpha_f(P) = \lambda_1(f)$.

Although the theory of dynamical degrees is comparably well-developed, many basic questions remain open about the arithmetic degree $\alpha_f(P)$ of a point P (see [15, 16, 19, 24, 20, 21, 18] for various results). The dynamical degrees of f satisfy certain inequalities, and there is a well-defined notion of relative dynamical degrees of a rational self-map preserving a fibration. We

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refer to [23, §3], [9, §4], and references therein for a comprehensive exposition on dynamical degrees, topological and algebraic entropies.

Our aim in this note is to develop a theory of higher arithmetic degrees in parallel to the existing theory for dynamical degrees, which was suggested in an unpublished note by Kawaguchi and Silverman (see, e.g., Silverman's talk [25] at a Simons Symposium in 2019). We introduce the so-called k-th arithmetic degree $\alpha_k(f)$ of a self-map f as a rough measure of the height growth of $(k-1)$-dimensional algebraic subvarieties of X, and show that a number of the basic properties of dynamical degrees extend to the arithmetic setting. Although the arguments are along familiar lines, considerable technical difficulties arise in dealing with the arithmetic intersection. Our higher arithmetic degrees of self-maps are defined independently of the choice of subvarieties (unlike the $\alpha_f(P)$ of Kawaguchi–Silverman, which is defined for each point P). On the other hand, we also formulate analogous invariants depending on cycles and a higher-dimensional analog of the Kawaguchi–Silverman conjecture.

1.1. **Examples.** Before giving the formal definition, we consider some illuminating examples in the first new case: the growth rate of heights of hypersurfaces under rational self-maps of projective spaces. In this case, the computations are quite concrete. Let $f\colon \mathbb{P}^d \dashrightarrow \mathbb{P}^d$ be a dominant rational self-map of $\mathbb{P}^d$ and let V be a hypersurface in $\mathbb{P}^d$, defined over $\overline{\mathbf{Q}}$. For the time being, we adopt the naive height $h(V)$ of V, i.e., the height of the defining homogeneous polynomial of V (see [12, §B.7]).

We then define the arithmetic degree of V by the limit $\alpha_d(f;V) = \lim_{n\to\infty} h((f^n)_*V)^{1/n}$; the more general and precise definition appears later as Definition 1.4.

The conjecture of Kawaguchi and Silverman predicts that if P is a point with Zariski dense orbit, then we have $\alpha_f(P) = \lambda_1(f)$. Considering some simple two-dimensional examples below shows that the most obvious analog of this conjecture is too optimistic: it may not be true that if a hypersurface V of $\mathbb{P}^d$ has Zariski dense orbit then $\alpha_d(f;V) = \lambda_d(f)$. In general, the $(d-1)$-th dynamical degree $\lambda_{d-1}(f)$ should also be taken into account.

Example 1.1. Let $f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a dominant rational self-map of $\mathbb{P}^2$ defined in an affine chart by $f(x,y) = (2y, xy)$. The iterates are given by $f^n(x,y) = \left(x^{F_{n-1}}(2y)^{F_n}, x^{F_n}(2y)^{F_{n+1}}/2\right)$, where $(F_n)_{n\in\mathbf{N}} = (0,1,1,2,3,5,\ldots)$ is the Fibonacci sequence. It is easy to verify that the map f is birational, its inverse is given by the formula $f^{-1}(x,y) = (2y/x, x/2)$, and the iterates of the inverse are given by

$$f^{-n}(x,y) = \left(\frac{x^{(-1)^n F_{n+1}}}{(2y)^{(-1)^n F_n}}, \frac{(2y)^{(-1)^n F_{n-1}}}{2x^{(-1)^n F_n}}\right).$$

Clearly, the dynamical degrees of f are given by

$$\lambda_0(f) = 1, \quad \lambda_1(f) = \frac{1+\sqrt{5}}{2}, \quad \lambda_2(f) = 1.$$

Consider now the height growth of some subvarieties of $\mathbb{P}^2$. If P is a point with Zariski dense orbit, then it is not hard to see that $\alpha_f(P) = \frac{1+\sqrt{5}}{2}$.

If V is a general curve on $\mathbb{P}^2$, its image under f^n can be computed by pulling back its defining equations via the formula for f^{-n}. From the definition of the height of a hypersurface in $\mathbb{P}^d$, we then have that $\alpha_2(f;V) = \frac{1+\sqrt{5}}{2} = \lambda_1(f) > \lambda_2(f)$.

Example 1.2. Let $f\colon \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ be the q-th power morphism of $\mathbb{P}^2$ for some integer $q \geq 2$, which is defined in coordinates via $f(x,y) = (x^q, y^q)$. Then $f^n(x,y) = (x^{q^n}, y^{q^n})$. Clearly, this f is a polarized endomorphism and has dynamical degrees

$$\lambda_0(f) = 1, \quad \lambda_1(f) = q, \quad \lambda_2(f) = q^2.$$

If P is a point with Zariski dense orbit, then we again have $\alpha_f(P) = q$. Consider now a line $V := \{x = 2\} \subset \mathbb{P}^2$. Then the image of V under f^n is just the line $\{x = 2^{q^n}\}$ and the induced morphism $f^n|_V$ has degree q^n. One can check that the arithmetic degree $\alpha_2(f;V)$ of V is equal to $\lim_{n\to\infty} h(q^n f^n(V))^{1/n} = q^2 = \lambda_2(f)$.

Observe that although in both examples we have $\alpha_f(P) = \lambda_1(f)$ predicted by the Kawaguchi–Silverman conjecture, it is not true in general that $\alpha_2(f;V) = \lambda_2(f)$. These examples, together with a heuristic from the function field case (see, e.g., [21]), naturally lend themselves to the speculation that $\alpha_k(f;V) = \max\{\lambda_k(f), \lambda_{k-1}(f)\}$. We note that in private communication, Joe Silverman indicated to us that he has additional examples using monomial maps on $\mathbb{P}^d$ suggesting the above relation between $\alpha_k(f;V)$, $\lambda_{k-1}(f)$, and $\lambda_k(f)$.

Regarding the function field case, suppose now that X is defined over the function field $\overline{\mathbf{Q}}(C)$, where C is a smooth projective curve over $\overline{\mathbf{Q}}$. Let $\pi\colon \mathcal{X} \longrightarrow C$ be a smooth model of X such that the generic fiber is isomorphic to X. Fix an ample divisor $H_{\mathcal{X}}$ on $\mathcal{X}$. Given an algebraic k-cycle $V \subset X$, take $\mathcal{V}$ to denote the closure of V in $\mathcal{X}$. The height of V is then defined to be the intersection number $\mathcal{V} \cdot H_{\mathcal{X}}^{k+1}$, computed on $\mathcal{X}$.

Now, suppose that $f\colon X \dashrightarrow X$ extends to a rational map $f_{\mathcal{X}}\colon \mathcal{X} \dashrightarrow \mathcal{X}$ and that the closure $\mathcal{V}$ of V is an ample cycle. Then we have

$$\begin{aligned}\alpha_{k+1}(f;V) &= \lim_{n\to\infty} h_X((f^n)_*V)^{1/n} = \lim_{n\to\infty}((f^n_{\mathcal{X}})_*\mathcal{V} \cdot H_{\mathcal{X}}^{k+1})^{1/n} \\ &= \lim_{n\to\infty}((f^n_{\mathcal{X}})^*H_{\mathcal{X}}^{k+1} \cdot \mathcal{V})^{1/n} = \lambda_{k+1}(f_{\mathcal{X}}).\end{aligned}$$

However, $\lambda_{k+1}(f_{\mathcal{X}})$ is related to the dynamical degrees of f itself by the product formula of Dinh–Nguyen [7]. Since $f_{\mathcal{X}}$ preserves a fibration π over the curve C, we have

$$\alpha_{k+1}(f;V) = \lambda_{k+1}(f_{\mathcal{X}}) = \max\{\lambda_{k+1}(f), \lambda_k(f)\}.$$

1.2. **Definitions of higher arithmetic degrees.** We turn at last to the general definitions. Let X be a smooth projective variety of dimension d defined over $\mathbf{Q}$. We choose an *integral model* $\pi\colon \mathfrak{X} \longrightarrow \operatorname{Spec}\mathbf{Z}$ of X, i.e., a projective scheme over $\operatorname{Spec}\mathbf{Z}$ such that π is flat and the generic fiber $\mathfrak{X}_{\mathbf{Q}}$ is isomorphic to X. In particular, $\mathfrak{X}$ is an arithmetic variety of relative dimension d over $\operatorname{Spec}\mathbf{Z}$. As we do not require π to be regular, such an integral model always exists.

We refer to Moriwaki's book [22] for an introduction to the Arakelov geometry; in particular, see [22, Chapter 5] for the arithmetic intersection theory and various arithmetic positivity properties. See also [3, Section 2].

Fix an arithmetically ample line bundle $\overline{\mathcal{H}} = (\mathcal{H}, \|\cdot\|)$ on $\mathfrak{X}$. The corresponding ample line bundle $\mathcal{H}_{\mathbf{Q}}$ on X will be simply denoted by H. For any k-cycle Z in $\mathfrak{X}$, the *Faltings height* $h_{\overline{\mathcal{H}}}(Z)$ of Z with respect to $\overline{\mathcal{H}}$ is defined by the Arakelov intersection

$$h_{\overline{\mathcal{H}}}(Z) := \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{H}})^k \cdot Z) := \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{H}}|_Z)^k),$$

where $\widehat{c}_1(\overline{\mathcal{H}}) \in \widehat{\mathrm{CH}}^1_D(\mathfrak{X})$ is the first arithmetic Chern class of $\overline{\mathcal{H}}$, and

$$\widehat{\deg}\colon \widehat{\mathrm{CH}}^{d+1}_D(\mathfrak{X}) \longrightarrow \mathbf{R}$$

is the arithmetic degree map. Note that we adopt the same convention for the height as in [10, Definition 2.5] without dividing by the algebraic degree of the cycle.

When $k \geq 1$, let $\deg_{\mathbf{Q}}(Z)$ denote the degree of $Z_{\mathbf{Q}}$ with respect to the ample line bundle H on X. Namely

$$\deg_{\mathbf{Q}}(Z) := (Z_{\mathbf{Q}} \cdot H^{k-1}).$$

Let $\mathfrak{f}\colon \mathfrak{X} \dashrightarrow \mathfrak{X}$ be a dominant rational self-map of $\mathfrak{X}$ extending a given dominant rational self-map f of X. Denote by $\Gamma_{\mathfrak{f}}$ the generic resolution of the graph of $\mathfrak{f}$ in $\mathfrak{X} \times_{\mathbf{Z}} \mathfrak{X}$; see [22, Theorem 5.1]. In particular, $\Gamma_{\mathfrak{f}}$ is an arithmetic variety. For $i = 1$ and 2, we let

$$\pi_i := \pi_{i,\mathfrak{f}}\colon \Gamma_{\mathfrak{f}} \longrightarrow \mathfrak{X}$$

denote the projection from $\Gamma_{\mathfrak{f}}$ onto the first and second component, respectively. We note that from now on, for the sake of simplifying the notation, we will drop the dependence on $\mathfrak{f}$ from the index of these two projection maps and simply refer to them as π_1 and π_2 instead of $\pi_{1,\mathfrak{f}}$ and $\pi_{2,\mathfrak{f}}$. We also denote the corresponding projection maps on the geometric fibers by π_1 and π_2 when there is no risk of confusion.

Recall that given an integer $0 \leqslant k \leqslant d$, the k-th (pure algebraic) degree of f with respect to H is defined by

$$\deg_k(f) := (\pi_2^* H^k \cdot \pi_1^* H^{d-k}).$$

Then the *k-th dynamical degree of f* is then interpreted as the growth rate of the k-th degree of f^n with respect to any ample divisor, i.e.,

$$\lambda_k(f) = \lim_{n\to\infty} \deg_k(f^n)^{1/n}. \tag{1.1}$$

The existence of the limit is non-trivial (see [8, 26, 5]). We can also consider analogous versions of these two quantities measuring the degree growth of an algebraic subvariety V of dimension k. Namely, we define the k-th degree of f along V as

$$\deg_k(f; V) := (\pi_2^* H^k \cdot \pi_1^* V)$$

and the *k-th dynamical degree of f along V* by

$$\lambda_k(f;V) := \limsup_{n\to\infty} \deg_k(f^n;V)^{1/n}. \tag{1.2}$$

Note that we are taking $\limsup$ since in general it is not known whether the limit actually exists. Similar phenomena also happen in the definition of arithmetic degrees below.

Definition 1.3. For any integer $0 \leqslant k \leqslant d+1$, let $\widehat{\deg}_k(f)$ denote the k-th degree of f with respect to an arithmetically ample line bundle $\overline{\mathcal{H}}$, i.e.,

$$\widehat{\deg}_k(f) := \widehat{\deg}\left(\widehat{c}_1(\pi_2^*\overline{\mathcal{H}})^k \cdot \widehat{c}_1(\pi_1^*\overline{\mathcal{H}})^{d+1-k}\right).$$

Then the *k-th arithmetic degree of f* is defined as

$$\alpha_k(f) := \limsup_{n\to\infty} \widehat{\deg}_k(f^n)^{1/n}. \tag{1.3}$$

Similarly, we define the arithmetic analog of the k-th degree of f along a subvariety.

Definition 1.4. Let V be a subvariety of dimension k in X which intersects properly the indeterminacy loci of f^n for all n. The strict transform $\pi_1^\circ V$ of V by π_1 has codimension $d-k$ on the graph $\Gamma_{\mathfrak{f}}(\mathbf{C})$ of f. Let

$$Z_V := \overline{\pi_{2*}\pi_1^\circ(V)}$$

be the Zariski closure in $\mathfrak{X}$ of the pushforward of the k-cycle $\pi_1^\circ(V)$ by π_2. We then similarly denote

$$\widehat{\deg}_{k+1}(f;V) := \widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{H}}|_{Z_V})^{k+1}\right),$$

and define the *$(k+1)$-th arithmetic degree of f along V* as

$$\alpha_{k+1}(f;V) := \limsup_{n\to\infty} \widehat{\deg}_{k+1}(f^n;V)^{1/n}. \tag{1.4}$$

When $\mathfrak{f}\colon \mathfrak{X} \longrightarrow \mathfrak{X}$ is a surjective morphism, it is not necessary to take a generic resolution of the graph when defining $\deg_k$ and $\widehat{\deg}_{k+1}$. In this case, we have

$$\deg_k(f;V) = (\pi_2^*H^k \cdot \pi_1^*V) = (H^k \cdot f_*V) = \deg_k(f_*V), \text{ and}$$
$$\widehat{\deg}_{k+1}(f;V) = \widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{H}}|_{\overline{f_*V}})^{k+1}\right),$$

where $\overline{f_*V}$ denotes the Zariski closure of f_*V in $\mathfrak{X}$. Thus $\lambda_k(f;V)$ (resp., $\alpha_{k+1}(f;V)$) measures the degree growth (resp., the height growth) of the iterates of an algebraic subvariety V.

1.3. **Conjectures and main results.** As we have mentioned before, it is known that the limit (1.1) defining $\lambda_k(f)$ indeed exists; in the other three cases (1.2)–(1.4), this is unknown and it is necessary to take the $\limsup$ in the definition. We will show later that the limit defining $\alpha_1(f)$ actually exists and is equal to $\lambda_1(f)$. We expect that the limits exist for the other $\alpha_k(f)$ as well. It is a more subtle question whether the limits exist for the invariants $\alpha_{k+1}(f;V)$, which depend on the particular subvarieties V, not just the map f. In the case of $\alpha_1(f;P)$, the existence of the limit is one part of Kawaguchi–Silverman conjecture. Even in the case of $\lambda_k(f;V)$ (a question with no arithmetic content), this seems to be an open question.

Conjecture 1.5. *Let $f\colon X \dashrightarrow X$ be a dominant rational map, and let V be a subvariety of dimension k in X which intersects properly the indeterminacy loci of f^n and is not contracted by f^n for all n. Then the limits defining $\lambda_k(f;V)$, $\alpha_k(f)$, and $\alpha_{k+1}(f;V)$ exist and are independent of the choices of H and $\overline{\mathcal{H}}$ as well as the integral model $\mathfrak{X}$.*

We also expect that $\alpha_{k+1}(f)$ provides an upper bound on the growth rate of the heights of algebraic subvarieties of dimension k.

Conjecture 1.6. *Let $f\colon X \dashrightarrow X$ be a dominant rational map, and let V be a subvariety of dimension k. Then $\alpha_{k+1}(f;V) \leqslant \alpha_{k+1}(f)$. Furthermore, if the orbit of V under f is Zariski dense in X, then $\alpha_{k+1}(f;V) = \alpha_{k+1}(f)$.*

Remark 1.7. We note that in our definition for $\alpha_{k+1}(f;V)$, we take into account the associated multiplicities in case f^n is ramified along V (see Remark 1.10 and also Example 1.11). Without counting these multiplicities, we could not expect that $\alpha_{k+1}(f;V) = \alpha_{k+1}(f)$ even when the orbit of V is Zariski dense in X. Indeed, Joe Silverman (whom we thank once again for his most valuable insight) showed us an example of a plane curve whose arithmetic degree computed using the strict transform $f^n(V)$ instead of the cycle-theoretic pushforward $f^n_*(V)$ is strictly less than $\alpha_2(f)$, even though the curve has Zariski dense orbit under f. See also his talk [25] at the Simons Symposium.

Although we are not able to prove that the $\limsup$ is actually a limit independent of any choice when $k \geqslant 2$, recent work due to Ikoma [13] based on the arithmetic Hodge index theorem of Yuan and Zhang [28] shows that the arithmetic intersection shares many similarities with the algebraic intersection. This allows us to extend a number of basic properties of dynamical degrees to the arithmetic setting. For example, it is shown that the Khovanski–Teissier inequalities hold in arithmetic intersection theory, and the log concavity of the arithmetic degrees follows. We list some basic properties of the arithmetic degrees in the next theorem.

Theorem A. *Let f be a dominant rational self-map of X. Let $\mathfrak{X}$ be any fixed integral model of X and $\overline{\mathcal{H}}$ any fixed arithmetically ample line bundle on $\mathfrak{X}$. Suppose that all arithmetic degrees $\alpha_k(f)$ of f are finite. Then the following assertions hold.*

(1) *The sequence $k \mapsto \alpha_k(f)$ is log-concave.*
(2) *If f is birational, then $\alpha_k(f) = \alpha_{d+1-k}(f^{-1})$.*

We also formulate an analog of the Kawaguchi–Silverman conjecture, which predicts that the height growth of cycles is controlled by dynamical degrees. In this case, the natural prediction derives from the case of function fields (see Section 1.1), in which it reduces to the familiar Dinh–Nguyen product formula for dynamical degrees (see [7]).

Conjecture 1.8. *Let f be a dominant rational self-map of X. Then for any integer $1 \leqslant k \leqslant d$, one has*

$$\alpha_k(f) = \max\{\lambda_k(f), \lambda_{k-1}(f)\}.$$

The conjunction of Conjectures 1.6 and 1.8 should be understood as the higher-dimensional formulation of the Kawaguchi–Silverman conjecture.

We prove this conjecture in the case that $\dim X = 2$ and f is birational (see Theorem 2.2) and in the case that $f\colon X \longrightarrow X$ is a polarized endomorphism (see Theorem 2.3).

Our next main result provides an inequality of Conjecture 1.8.

Theorem B. *Let f be a dominant rational self-map of X. Then for any integral model $\mathfrak{X}$ of X and any arithmetically ample line bundle $\overline{\mathcal{H}}$ on $\mathfrak{X}$, one has*

$$\alpha_k(f) \geqslant \max\{\lambda_k(f), \lambda_{k-1}(f)\},$$

for any integer $1 \leqslant k \leqslant d$.

The proof of Theorem B relies on an arithmetic Bertini argument (see Lemma 2.1) which allows us to relate the arithmetic degree with a height of a given particular cycle. Then one applies an inequality due to Faltings [10] to bound below the height of the cycles by its algebraic degree.

In the particular case when $k = 1$, we show that the first degree sequence $(\widehat{\deg}_1(f^n))_{n\geq 0}$ of f with respect to an arithmetically ample line bundle is submultiplicative.

Theorem C. *Let X, Y, Z be smooth projective varieties defined over $\mathbf{Q}$ of dimension d. Choose integral models $\mathfrak{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ of X, Y, and Z, respectively. Fix three arithmetically ample line bundles $\overline{\mathcal{H}}_{\mathfrak{X}}$, $\overline{\mathcal{H}}_{\mathcal{Y}}$, and $\overline{\mathcal{H}}_{\mathcal{Z}}$ on $\mathfrak{X}$, $\mathcal{Y}$, and $\mathcal{Z}$, respectively. Then there exists a constant $C > 0$ such that for any dominant rational maps $f\colon X \dashrightarrow Y$ and $g\colon Y \dashrightarrow Z$, one has*

$$\widehat{\deg}_1(g \circ f) \leqslant C\, \widehat{\deg}_1(f)\, \widehat{\deg}_1(g),$$

where the intersection numbers are taken with respect to the chosen arithmetically ample line bundles on each integral model.

The above result relies heavily on Yuan's work [27] on the properties of the arithmetic volume. From his arithmetic volume estimates, one deduces an effective way to obtain an arithmetically pseudo-effective line bundle. The above theorem yields the following consequence.

Corollary D. *The first arithmetic degree $\alpha_1(f)$ of f is equal to the limit, i.e.,*

$$\alpha_1(f) = \lim_{n\to\infty} \widehat{\deg}_1(f^n)^{1/n}.$$

Moreover, it is a finite number and independent of the choices of the integral model $\mathfrak{X}$ of X, the birational model of X, or the arithmetically ample line bundle $\overline{\mathcal{H}}$ on $\mathfrak{X}$.

More precisely, Theorem A shows that the growth rate of the sequence $(\widehat{\deg}_1(f^n))_{n\geq 0}$ is also a birational invariant. In this particular case, the relationship between the arithmetic degree and the dynamical degree is more constrained.

Theorem E. *Let f be a dominant rational self-map of X. Then for any integral model $\mathfrak{X}$ of X and any arithmetically ample line bundle $\overline{\mathcal{H}}$ on $\mathfrak{X}$, one has*

$$\alpha_1(f) = \lambda_1(f).$$

In particular, $\alpha_1(f)$ does not depend on the choice of the integral model $\mathfrak{X}$ nor on the choice of the arithmetically ample line bundle $\overline{\mathcal{H}}$.

Let us explain how one can understand the above theorem. Since Theorem B proves that $\lambda_1(f) \leqslant \alpha_1(f)$, the proof of this last result amounts in proving the converse inequality $\alpha_1(f) \leqslant \lambda_1(f)$. Using an arithmetic Bertini argument (see Section 2.2) and the integration by parts, we prove a more general statement (see Theorem 2.4), namely that for any positive integer n, there exists an algebraic cycle Z of dimension $k-1$ on X such that

$$\widehat{\deg}_k(f^n) \leqslant h_{\overline{\mathcal{H}}}(f^n_* Z) + C \deg_{k-1}(f^n)$$

for a positive constant $C > 0$ which is independent of n and Z. In the case where $k = 1$, the degree of f^n with respect to $\overline{\mathcal{H}}$ is controlled by the height of a point and the algebraic degree of f^n. We finally conclude using the fact that the height of the images of a rational point by f^n is controlled by the first dynamical degree of f (see [16, 19]).

1.4. **Remarks and further questions.** The above discussion leads us to the following natural question.

Question 1.9. Let V be a subvariety of dimension k in X which intersects properly the indeterminacy loci of f^n and is not contracted by f^n for all n. Then is it true that

$$\limsup_{n\to\infty} \left(h_{\overline{\mathcal{H}}}(f^n_* V)\right)^{1/n} = \alpha_{k+1}(f)$$

when V is generic?

Remark 1.10. In our notation, $f_* V$ denotes the cycle-theoretic pushforward of V, not just its strict transform; so, if f is ramified along V, then the class V gets multiplied by the degree of the ramification. Subtleties can arise when some iterates of f are ramified along V (for example, we refer the reader to [14, §7] for a construction of certain non-PCF endomorphisms f of projective spaces such that the ramification divisor R_f is irreducible and $f|_{R_f}$ is birational to the image $f(R_f)$), and when infinitely many different iterates are ramified along V the ramification is reflected in the arithmetic degree of the cycle: $f^n_* V$ is equal to the class of the strict transform $f^n(V)$ with a multiple of the ramification degree. Consider for contrast the following example.

Example 1.11. Define $f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ by setting $f(x, y) = (x^2, y)$ (on the affine part), and let V be the curve defined by $y = 0$. Then V is invariant under f and $f|_V$ is the squaring map on $\mathbb{P}^1$. As a result, $f^n_* V = 2^n V$, and we have $\alpha_2(f; V) = 2$. In particular, note that (with our definition) $\alpha_2(f; V)$ is not equal to 1 as one might think due to the fact that V is f-periodic (actually, it is even f-invariant). On the other hand, let W be the curve defined by $x = y$; this curve is linearly equivalent to V as an algebraic cycle and its iterates are also disjoint from the indeterminacy point $[0:1:0]$. Then $f^n(W)$ is defined by the equation $x = y^{2^n}$, so that $h(f^n_* W) = 2^n$. Thus we find that $\alpha_2(f;W) = 2$ as well.

Remark 1.12. It is also possible that f is ramified along a curve only a finite number of times, in which the extra factor corresponding to ramification disappears in the limit of $h(f^n_* V)^{1/n}$. It is also conceivable that f could ramify along V infinitely many times, but with such infrequency that the ramification is not reflected in the arithmetic degree. This is impossible, assuming the following special case of a generalization of the Dynamical Mordell–Lang conjecture (see [11, 1] for the original Dynamical Mordell–Lang Conjecture and also [2, 17] for a higher dimensional version of it).

Question 1.13. Suppose that $f\colon X \dashrightarrow X$ is a rational map and that $V \subset X$ is a subvariety with a well-defined forward orbit. Is

$$R(f,V) = \{n \in \mathbf{N} : f^n(V) \subseteq R_f\}$$

the union of a finite set and finitely many arithmetic progressions?

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2. On the k-th arithmetic degrees

2.1. **Proof of Theorem A.** Fix an integral model $\mathfrak{X}$ of X and an arithmetically ample line bundle $\overline{\mathcal{H}}$ on $\mathfrak{X}$. By abuse of notation, we also denote by f the dominant rational map on the integral model $\mathfrak{X}$. Let $\pi_{1,n}$ and $\pi_{2,n}$ be the projections from the generic resolution of the graph of f^n in $\mathfrak{X} \times_{\mathbf{Z}} \mathfrak{X}$ to the first and second components, respectively. By [13, Theorem 2.9(1)] applied to the arithmetic nef line bundles $\pi_{1,n}^*\overline{\mathcal{H}}$ and $\pi_{2,n}^*\overline{\mathcal{H}}$, we have that

$$\begin{aligned} \widehat{\deg}_k(f^n)^2 &= \widehat{\deg}\left(\widehat{c}_1(\pi_{2,n}^*\overline{\mathcal{H}})^k \cdot \widehat{c}_1(\pi_{1,n}^*\overline{\mathcal{H}})^{d+1-k}\right)^2 \\ &\geqslant \widehat{\deg}\left(\widehat{c}_1(\pi_{2,n}^*\overline{\mathcal{H}})^{k+1} \cdot \widehat{c}_1(\pi_{1,n}^*\overline{\mathcal{H}})^{d-k}\right) \widehat{\deg}\left(\widehat{c}_1(\pi_{2,n}^*\overline{\mathcal{H}})^{k-1} \cdot \widehat{c}_1(\pi_{1,n}^*\overline{\mathcal{H}})^{d+2-k}\right). \end{aligned}$$

We thus obtain that

$$\widehat{\deg}_k(f^n)^2 \geqslant \widehat{\deg}_{k+1}(f^n)\, \widehat{\deg}_{k-1}(f^n).$$

Taking the n-th root as n goes to infinity yields that

$$\alpha_k(f)^2 \geqslant \alpha_{k-1}(f)\alpha_{k+1}(f)$$

and the sequence $k \mapsto \alpha_k(f)$ is locally log-concave at each point. It is thus globally log-concave, as required.

The second assertion immediately follows from the definition of $\alpha_k(f)$, with the roles of π_1 and π_2 reversed. □

2.2. **A technical lemma.** In this section, we prove a technical lemma which relates the two different k-th degrees of a rational self-map f as well as the height of certain points on the f-orbit. To do so, let us introduce a few notations.

We fix a dominant rational map $f\colon X \dashrightarrow X$ defined over $\mathbf{Q}$. Denote by π_1 and π_2 the projections from a generically smooth birational model Γ_f of the graph of f in $\mathfrak{X} \times \mathfrak{X}$ onto the first and the second factors, respectively. We fix an arithmetically ample line bundle $\overline{\mathcal{H}}$ on $\mathfrak{X}$ and a positive integer $k \leqslant d$. Given $d+1-k$ sections $s_1, \ldots, s_{d+1-k}$ of $\overline{\mathcal{H}}$, we shall set cycles Z_i', Z_i as follows:

$$Z_i' \coloneqq \operatorname{div}(\pi_1^* s_i) \cdot \ldots \cdot \operatorname{div}(\pi_1^* s_{d+1-k}),$$
$$Z_i \coloneqq \operatorname{div}(s_i) \cdot \ldots \cdot \operatorname{div}(s_{d+1-k}),$$

for all $1 \leqslant i \leqslant d+1-k$.

By an arithmetic Bertini theorem [4, Theorem 1.1] applied to $\pi_1\colon \Gamma_f \longrightarrow \mathfrak{X}$, one can find $d+1-k$ sections $s_1, \ldots, s_{d+1-k}$ of $\overline{\mathcal{H}}$ satisfying the following conditions:

(i) For all i, the cycles Z_i are irreducible and generically smooth;
(ii) For all i, the horizontal parts of the cycles Z_i' are irreducible;
(iii) For all i, one has $\|s_i\| < 1$ uniformly.

We denote by ω_X the Kähler form on $X(\mathbf{C})$ associated to $\overline{\mathcal{H}}$. Observe that the supports of the cycles Z_i' form an increasing sequence of closed subvarieties of Γ_f. In the next lemma, we shall conventionally set $Z'_{d+2-k} \coloneqq \Gamma_f$.

Lemma 2.1. *With notation as above, one has*

$$\widehat{\deg}_k(f) = h_{\overline{\mathcal{H}}}(f_* Z_1) + \sum_{j=2}^{d+2-k} \int_{Z_j'(\mathbf{C})} (-\log \|\pi_1^* s_{j-1}\|) \pi_2^* \omega_X^k \wedge \pi_1^* \omega_X^{j-2}.$$

Proof. We first rewrite $\widehat{\deg}_k(f)$ as follows (see Definition 1.3):

$$\widehat{\deg}_k(f) = \widehat{\deg}\left(\widehat{c}_1(\pi_2^*\overline{\mathcal{H}})^k \cdot \widehat{c}_1(\pi_1^*\overline{\mathcal{H}})^{d-k} \cdot \widehat{\operatorname{div}}(\pi_1^* s_{d+1-k})\right).$$

By [3, Formula 2.3.8 of Proposition 2.3.1(vi)] applied to the arithmetic divisor $\widehat{\operatorname{div}}(s_{d+1-k})$, we have that

$$\begin{aligned}
\widehat{\deg}_k(f) &= \widehat{\deg}\left(\widehat{c}_1(\pi_2^*\overline{\mathcal{H}})^k \cdot \widehat{c}_1(\pi_1^*\overline{\mathcal{H}})^{d-k} \cdot \operatorname{div}(\pi_1^* s_{d+1-k})\right) \\
&\quad + \int_{\Gamma_f(\mathbf{C})} (-\log \|\pi_1^* s_{d+1-k}\|) \pi_2^* \omega_X^k \wedge \pi_1^* \omega_X^{d-k} \\
&= \widehat{\deg}\left(\widehat{c}_1(\pi_2^*\overline{\mathcal{H}})^k \cdot \widehat{c}_1(\pi_1^*\overline{\mathcal{H}})^{d-k} \cdot Z'_{d+1-k}\right) \\
&\quad + \int_{\Gamma_f(\mathbf{C})} (-\log \|\pi_1^* s_{d+1-k}\|) \pi_2^* \omega_X^k \wedge \pi_1^* \omega_X^{d-k}.
\end{aligned}$$

We then inductively apply [3, Formula 2.3.8 of Proposition 2.3.1(vi)] as follows:

$$\widehat{\deg}\left(\widehat{c}_1(\pi_2^*\overline{\mathcal{H}})^k\cdot\widehat{c}_1(\pi_1^*\overline{\mathcal{H}})^{d-k}\cdot Z'_{d+1-k}\right)=\widehat{\deg}\left(\widehat{c}_1(\pi_2^*\overline{\mathcal{H}})^k\cdot\widehat{c}_1(\pi_1^*\overline{\mathcal{H}})^{d-k-1}\cdot Z'_{d-k}\right)+\int_{Z'_{d+1-k}(\mathbf{C})}(-\log\|\pi_1^*s_{d-k}\|)\pi_2^*\omega_X^k\wedge\pi_1^*\omega_X^{d-k-1},$$

$$\widehat{\deg}\left(\widehat{c}_1(\pi_2^*\overline{\mathcal{H}})^k\cdot\widehat{c}_1(\pi_1^*\overline{\mathcal{H}})^{d-k-1}\cdot Z'_{d-k}\right)=\widehat{\deg}\left(\widehat{c}_1(\pi_2^*\overline{\mathcal{H}})^k\cdot\widehat{c}_1(\pi_1^*\overline{\mathcal{H}})^{d-k-2}\cdot Z'_{d-k-1}\right)+\int_{Z'_{d-k}(\mathbf{C})}(-\log\|\pi_1^*s_{d-k-1}\|)\pi_2^*\omega_X^k\wedge\pi_1^*\omega_X^{d-k-2},$$

$$\vdots$$

$$\widehat{\deg}\left(\widehat{c}_1(\pi_2^*\overline{\mathcal{H}})^k\cdot\widehat{c}_1(\pi_1^*\overline{\mathcal{H}})\cdot Z'_2\right)=\widehat{\deg}\left(\widehat{c}_1(\pi_2^*\overline{\mathcal{H}})^k\cdot Z'_1\right)+\int_{Z'_2(\mathbf{C})}(-\log\|\pi_1^*s_1\|)\pi_2^*\omega_X^k.$$

From the above sequence of equalities, we deduce that

$$\widehat{\deg}_k(f)=\widehat{\deg}\left(\widehat{c}_1(\pi_2^*\overline{\mathcal{H}})^k\cdot Z'_1\right)+\sum_{j=2}^{d+2-k}\int_{Z'_j(\mathbf{C})}(-\log\|\pi_1^*s_{j-1}\|)\pi_2^*\omega_X^k\wedge\pi_1^*\omega_X^{j-2}.$$

By the projection formula (cf. [3, Proposition 2.3.1(iv)]), the lemma follows. □

2.3. **Proof of Theorem B.** We shall use the following inequality due to Faltings [10, Proposition 2.16]:

There exists a constant $C>0$ depending only on the arithmetically ample line bundle $\overline{\mathcal{H}}$ on $\mathfrak{X}$ (and also depending on the supremum of the norm of a section of the corresponding line bundle at infinite places) such that

$$h_{\overline{\mathcal{H}}}(Z)\geqslant C(H^k\cdot Z),$$

for any effective k-cycle Z on X. By Lemma 2.1, for any positive integer n, we have

$$\widehat{\deg}_k(f^n)=h_{\overline{\mathcal{H}}}(f_*^nZ_1)+\sum_{j=2}^{d+2-k}\int_{Z'_j(\mathbf{C})}(-\log\|\pi_1^*s_{j-1}\|)\pi_2^*\omega_X^k\wedge\pi_1^*\omega_X^{j-2},$$

where Z_1, Z'_j, and s_j are defined in Section 2.2. Observe that Z_1 is a cycle on X, but the cycles Z'_j are on Γ_{f^n}. Since the supremum of the norm of the sections s_j is bounded above by 1, all the terms in the integral are non-positive. We thus have

$$\widehat{\deg}_k(f^n)\geqslant h_{\overline{\mathcal{H}}}(f_*^nZ_1).$$

Using the Faltings inequality, we thus deduce $\widehat{\deg}_k(f^n)\geqslant C(H^{k-1}\cdot f_*^nZ_1)$. Taking the n-th root as n goes to infinity finally yields that

$$\alpha_k(f)\geqslant\lambda_{k-1}(f).$$

For the other inequality, we observe that the roles played by the line bundles $\pi_1^*\overline{\mathcal{H}}$ and $\pi_2^*\overline{\mathcal{H}}$ are symmetric and the technical lemma also proves that $\widehat{\deg}_k(f^n)\geqslant Ch_{\overline{\mathcal{H}}}(f_*^{-n}Z)=Ch_{\overline{\mathcal{H}}}(\pi_{1*}\pi_2^*Z)$

for an appropriate complete intersection cycle of dimension k in $\mathfrak{X}$ and we conclude that $\alpha_k(f) \geqslant \lambda_k(f)$. We have thus proven that $\alpha_k(f) \geqslant \max\{\lambda_k(f), \lambda_{k-1}(f)\}$, as required. □

We have conjectured in fact that the inequality of Theorem B is always an equality (i.e., Conjecture 1.8). There are two cases in which we can prove this conjecture.

Theorem 2.2. *If $f\colon X \dashrightarrow X$ is a birational self-map of a surface X defined over* $\mathbf{Q}$, *then*

$$\alpha_k(f) = \max\{\lambda_k(f), \lambda_{k-1}(f)\}$$

for all $1 \leqslant k \leqslant 2$.

Proof. It is always true that $\lambda_0(f) = 1$, while $\alpha_1(f) = \lambda_1(f)$ for any map according to Theorem E. Besides, we have

$$\alpha_2(f) = \alpha_1(f^{-1}) = \lambda_1(f^{-1}) = \lambda_1(f).$$

Note that $\lambda_2(f) = 1$ as f is birational. We thus have $\alpha_2(f) = \max\{\lambda_1(f), \lambda_2(f)\}$. □

Recall that a surjective self-morphism $f\colon X \longrightarrow X$ is polarized, if there exists an ample line bundle L on X such that $f^*L \sim L^q$ for some integer $q > 1$.

Theorem 2.3. *Let $f\colon X \longrightarrow X$ be a polarized endomorphism with respect to the ample line bundle $H = \mathcal{H}_{\mathbf{Q}}$ on X, which extends to a polarizable endomorphism of an integral model $\mathfrak{X}$, then*

$$\alpha_k(f) = \max\{\lambda_k(f), \lambda_{k-1}(f)\}$$

for all $1 \leqslant k \leqslant d$.

Proof. We fix an isomorphism $\phi\colon \mathcal{H}^q \simeq f^*\mathcal{H}$ on the integral model $\mathfrak{X}$ of X, where $\overline{\mathcal{H}} = (\mathcal{H}, \|\cdot\|)$ is a fixed arithmetically ample line bundle on $\mathfrak{X}$. By [29, Theorem 2.2], there exists a unique metric $\|\cdot\|_0$ on $\mathcal{H}$ such that

$$\|\cdot\|_0 = (\phi^* f^* \|\cdot\|_0)^{1/q}.$$

This metric is obtained from the initial metric $\|\cdot\|$ on $\mathcal{H}$ using Tate's limiting argument. As the initial $\overline{\mathcal{H}}$ is arithmetically ample, so is the new arithmetic line bundle $\overline{\mathcal{H}}_0 := (\mathcal{H}, \|\cdot\|_0)$.

Let us compute the k-th degree of f^n with respect to the choice of the arithmetically ample line bundle $\overline{\mathcal{H}}_0$:

$$\widehat{\deg}_k(f^n) = \widehat{\deg}\left(\widehat{c}_1((f^n)^*\overline{\mathcal{H}}_0)^k \cdot \widehat{c}_1(\overline{\mathcal{H}}_0)^{d+1-k}\right) = q^{kn}\,\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{H}}_0)^{d+1}\right).$$

This proves that $\alpha_k(f) = q^k$. On the other hand, $\lambda_k(f) = q^k$ for any k. We hence conclude that

$$\alpha_k(f) = \max\{\lambda_k(f), \lambda_{k-1}(f)\},$$

as required. □

2.4. **Upper bound for the arithmetic degree.** We now state our next result, which gives an upper bound of the k-th degree of f^n.

Theorem 2.4. *Let $f\colon X \dashrightarrow X$ be a dominant rational map of a projective variety X defined over $\mathbf{Q}$. Then there exists a constant $C > 0$ such that for any positive integer $k \leqslant d+1$ and any positive integer n, there exists a $(k-1)$-cycle Z on X, whose support is not contained in the indeterminacy locus of f^n such that*

$$\widehat{\deg}_k(f^n) \leqslant h_{\overline{\mathcal{H}}}(f^n_* Z) + C \deg_{k-1}(f^n).$$

Moreover, Z is a cycle associated with an arithmetic cycle representing $\widehat{c}_1(\overline{\mathcal{H}})^{d+1-k}$ in the arithmetic Chow group of $\mathfrak{X}$.

Proof. We adopt the same notation of Section 2.2. Fix a positive integer n. We consider a generically smooth birational model Γ_{f^n} of the graph of f^n. By Lemma 2.1, we have

$$\widehat{\deg}_k(f^n) = h_{\overline{\mathcal{H}}}(f^n_* Z_1) + \sum_{j=2}^{d+2-k} \int_{Z'_j(\mathbf{C})} (-\log \|\pi_1^* s_{j-1}\|) \pi_2^* \omega_X^k \wedge \pi_1^* \omega_X^{j-2}.$$

We now find an upper bound for each integral in the above equality. To that end, we choose a generic section σ_k and use the Poincaré–Lelong formula

$$i\partial\bar{\partial} \log \|\sigma_k\| = \omega_X - [\sigma_k = 0].$$

Using the above and the Chern–Levine–Nirenberg inequalities [6, Proposition 4.6(a)], we have

$$\begin{aligned}
&\int_{Z'_j(\mathbf{C})} (-\log \|\pi_1^* s_{j-1}\|) \pi_2^* \omega_X^k \wedge \pi_1^* \omega_X^{j-2} \\
&= \int_{Z'_j(\mathbf{C})} (-\log \|\pi_1^* s_{j-1}\|) i\partial\bar{\partial} \pi_2^* \log \|\sigma_k\| \wedge \pi_2^* \omega_X^{k-1} \wedge \pi_1^* \omega_X^{j-2} \\
&\quad + \int_{Z'_j(\mathbf{C}) \cap \{\pi_2^* \sigma_k = 0\}} (-\log \|\pi_1^* s_{j-1}\|) \pi_2^* \omega_X^{k-1} \wedge \pi_1^* \omega_X^{j-2} \\
&\leqslant C \deg_{k-1}(f^n) + \int_{Z'_j(\mathbf{C}) \cap \{\pi_2^* \sigma_k = 0\}} (-\log \|\pi_1^* s_{j-1}\|) \pi_2^* \omega_X^{k-1} \wedge \pi_1^* \omega_X^{j-2}.
\end{aligned}$$

Note that the last inequality has on the right hand side an integral term of the same form. Applying the above argument inductively, we obtain that

$$\int_{Z'_j(\mathbf{C})} (-\log \|\pi_1^* s_{j-1}\|) \pi_2^* \omega_X^k \wedge \pi_1^* \omega_X^{j-2} \leqslant C' \deg_{k-1}(f^n),$$

where $C' > 0$ is a constant independent of n. This yields that

$$\widehat{\deg}_k(f^n) \leqslant h_{\overline{\mathcal{H}}}(f^n_* Z_1) + C'(d-k+1) \deg_{k-1}(f^n). \qquad \square$$

3. On the first arithmetic degree

3.1. **Proof of Theorem C.**

Our proof follows closely an algebraic approach for the existence of dynamical degrees for dominant rational self-maps or correspondences (cf. [26, 5]).

We shall use the following inequality on arithmetic line bundles due to Yuan [27].

Theorem 3.1 (cf. [27, Theorem B]). *Let $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ be two effective arithmetic line bundles over an arithmetic variety $\mathfrak{X}$ of relative dimension d. Then we have*

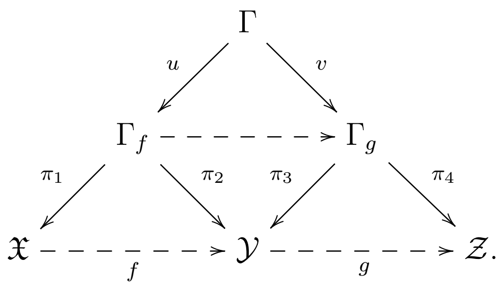
$$\widehat{\mathrm{vol}}(\overline{\mathcal{L}}+\overline{\mathcal{M}})^{1/(d+1)} \geq \widehat{\mathrm{vol}}(\overline{\mathcal{L}})^{1/(d+1)} + \widehat{\mathrm{vol}}(\overline{\mathcal{M}})^{1/(d+1)}.$$

An important consequence of the above theorem (see [27, Remark, p. 1459]) is that the difference $\overline{\mathcal{L}}-\overline{\mathcal{M}}$ is arithmetically big if

$$\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}})^{d+1}\right) > (d+1)\,\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}})^{d}\cdot \widehat{c}_1(\overline{\mathcal{M}})\right).$$

This inequality is the key ingredient to prove the submultiplicativity of the first degree sequence $(\widehat{\deg}_1(f^n))_{n\geq 0}$ of f with respect to an arithmetically ample line bundle.

Proof of Theorem C. We fix integral models $\mathfrak{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ of X, Y, and Z, respectively; fix three arithmetically ample line bundles $\overline{\mathcal{H}}_{\mathfrak{X}}$, $\overline{\mathcal{H}}_{\mathcal{Y}}$, and $\overline{\mathcal{H}}_{\mathcal{Z}}$, respectively. Let $f\colon X \dashrightarrow Y$ and $g\colon Y \dashrightarrow Z$ be two dominant rational maps. Take Γ_f, Γ_g the generic resolution of the graphs of $f\colon \mathfrak{X} \dashrightarrow \mathcal{Y}$ and $g\colon \mathcal{Y} \dashrightarrow \mathcal{Z}$, respectively. Consider also the generic resolution Γ of the graph of the rational map $\Gamma_f \dashrightarrow \Gamma_g$ induced by f. We also denote by π_1, π_2 the projection of Γ_f onto the $\mathfrak{X}$ and $\mathcal{Y}$, by u, v the projection of Γ onto Γ_f and Γ_g, and by π_3, π_4 the projections of Γ_g onto $\mathcal{Y}$ and $\mathcal{Z}$, respectively. We thus obtain the following diagram:



By applying Yuan's estimate to $\overline{\mathcal{L}} = u^*\pi_2^*\overline{\mathcal{H}}_{\mathcal{Y}}$ and $\overline{\mathcal{M}} = v^*\pi_4^*\overline{\mathcal{H}}_{\mathcal{Z}}$, the following class

$$(d+1)\frac{\widehat{\deg}\left(\widehat{c}_1(u^*\pi_2^*\overline{\mathcal{H}}_{\mathcal{Y}})^d\cdot \widehat{c}_1(v^*\pi_4^*\overline{\mathcal{H}}_{\mathcal{Z}})\right)}{\widehat{\deg}\left(\widehat{c}_1(u^*\pi_2^*\overline{\mathcal{H}}_{\mathcal{Y}})^{d+1}\right)} u^*\pi_2^*\overline{\mathcal{H}}_{\mathcal{Y}} - v^*\pi_4^*\overline{\mathcal{H}}_{\mathcal{Z}}$$

is pseudo-effective. Thus the intersection with a suitable power of the arithmetically nef class $u^*\pi_1^*\overline{\mathcal{H}}_{\mathfrak{X}}$ yields that

$$\widehat{\deg}_1(g\circ f) \leqslant (d+1)\frac{\widehat{\deg}\left(\widehat{c}_1(u^*\pi_2^*\overline{\mathcal{H}}_{\mathcal{Y}})^d\cdot \widehat{c}_1(v^*\pi_4^*\overline{\mathcal{H}}_{\mathcal{Z}})\right)}{\widehat{\deg}\left(\widehat{c}_1(u^*\pi_2^*\overline{\mathcal{H}}_{\mathcal{Y}})^{d+1}\right)}\,\widehat{\deg}_1(f). \tag{3.1}$$

Since $u^*\pi_2^* = v^*\pi_3^*$, we have

$$\frac{\widehat{\deg}\left(\widehat{c}_1(u^*\pi_2^*\overline{\mathcal{H}}_{\mathcal{Y}})^d \cdot \widehat{c}_1(v^*\pi_4^*\overline{\mathcal{H}}_{\mathcal{Z}})\right)}{\widehat{\deg}\left(\widehat{c}_1(u^*\pi_2^*\overline{\mathcal{H}}_{\mathcal{Y}})^{d+1}\right)} = \frac{\widehat{\deg}\left(\widehat{c}_1(v^*\pi_3^*\overline{\mathcal{H}}_{\mathcal{Y}})^d \cdot \widehat{c}_1(v^*\pi_4^*\overline{\mathcal{H}}_{\mathcal{Z}})\right)}{\widehat{\deg}\left(\widehat{c}_1(v^*\pi_3^*\overline{\mathcal{H}}_{\mathcal{Y}})^{d+1}\right)}.$$

Using the projection formula (cf. [3, Proposition 2.3.1(iv)]), we obtain that

$$\frac{\widehat{\deg}\left(\widehat{c}_1(v^*\pi_3^*\overline{\mathcal{H}}_{\mathcal{Y}})^d \cdot \widehat{c}_1(v^*\pi_4^*\overline{\mathcal{H}}_{\mathcal{Z}})\right)}{\widehat{\deg}\left(\widehat{c}_1(v^*\pi_3^*\overline{\mathcal{H}}_{\mathcal{Y}})^{d+1}\right)} = \frac{\widehat{\deg}\left(\widehat{c}_1(\pi_3^*\overline{\mathcal{H}}_{\mathcal{Y}})^d \cdot \widehat{c}_1(\pi_4^*\overline{\mathcal{H}}_{\mathcal{Z}})\right)}{\widehat{\deg}\left(\widehat{c}_1(\pi_3^*\overline{\mathcal{H}}_{\mathcal{Y}})^{d+1}\right)} = \frac{\widehat{\deg}_1(g)}{\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{H}}_{\mathcal{Y}})^{d+1}\right)}.$$

The above equalities together with the inequality (3.1) yield that

$$\widehat{\deg}_1(g\circ f) \leqslant C\,\widehat{\deg}_1(f)\,\widehat{\deg}_1(g),$$

where $C = (d+1)/\,\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{H}}_{\mathcal{Y}})^{d+1}\right)$, and hence Theorem C follows. □

3.2. **Proof of Theorem E.** By Theorem B applied to $k = 1$, we have

$$\alpha_1(f) \geqslant \max\{\lambda_1(f), \lambda_0(f)\} = \lambda_1(f).$$

For the converse inequality, we apply Theorem 2.4. There is a constant $C > 0$ such that for any positive integer n, there exists a zero-cycle Z_n on X such that $Z_n = \sum_i a_{n,i}[x_{n,i}] \in Z^d(X)$ associated to an arithmetic cycle representing the class $\widehat{c}_1(\overline{\mathcal{H}})^d$ (see §2.2), and

$$\widehat{\deg}_1(f^n) \leqslant \sum_i a_{n,i} h_{\overline{\mathcal{H}}}(f^n x_{n,i}) + C\deg_0(f^n).$$

By Matsuzawa's theorem (see [19, Theorem 1.4]), the growth rate of the height $h_{\overline{\mathcal{H}}}(f^n x_{n,i})$ of any point $x_{n,i}$ is bounded above by $C'(\lambda_1(f)+\epsilon)^n h_{\overline{\mathcal{H}}}(x_{n,i})$, so we obtain that

$$\widehat{\deg}_1(f^n) \leqslant C'(\lambda_1(f)+\epsilon)^n h_{\overline{\mathcal{H}}}(Z_n) + C.$$

Since $h_{\overline{\mathcal{H}}}(Z_n) \leqslant (\widehat{c}_1(\overline{\mathcal{H}})^{d+1})$, we get

$$\widehat{\deg}_1(f^n) \leqslant C'(\lambda_1(f)+\epsilon)^n (\widehat{c}_1(\overline{\mathcal{H}})^{d+1}) + C.$$

Taking the n-th root and letting $\epsilon \to 0$, we finally conclude that $\alpha_1(f) = \lambda_1(f)$. □

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