M. Chu and A. Kolpakov (2020) “A Hyperbolic Counterpart to Rokhlin’s Cobordism Theorem,”
International Mathematics Research Notices, Vol. 00, No. 0, pp. 1–24
doi:10.1093/imrn/rnaa158

A Hyperbolic Counterpart to Rokhlin’s Cobordism Theorem

Michelle Chu[1] **and Alexander Kolpakov**[2,*]
[1]Department of Mathematics, University of Illinois at Chicago, 851 S. Morgan Street, Chicago, IL 60607-7045, USA and [2]Institut de mathématiques, Université de Neuchâtel, Rue Emile-Argand 11, CH-2000 Neuchâtel, Suisse/Switzerland

**Correspondence to be sent to: e-mail: kolpakov.alexander@gmail.com*

The purpose of the present paper is to prove existence of *super-exponentially* many compact orientable hyperbolic arithmetic n-manifolds that are geometric boundaries of compact orientable hyperbolic $(n+1)$-manifolds, for any $n \geq 2$, thereby establishing that these classes of manifolds have the same growth rate with respect to volume as all compact orientable hyperbolic arithmetic n-manifolds. An analogous result holds for non-compact orientable hyperbolic arithmetic n-manifolds of finite volume that are geometric boundaries for $n \geq 2$.

In homage to V. Rokhlin on his 100th anniversary.

1 Introduction

A classical result by V. Rokhlin states that every compact orientable 3-manifold bounds a compact orientable 4-manifold, and thus the 3-dimensional cobordism group is trivial. Rokhlin also proved that a compact orientable 4-manifold bounds a compact orientable 5-manifold if and only if its signature is zero, which is true for all closed orientable hyperbolic 4-manifolds. One can recast the question of bounding in the setting of hyperbolic geometry, which generated plenty of research directions over the past decades.

Received June 18, 2019; Revised March 14, 2020; Accepted May 22, 2020
Communicated by Prof. Igor Rivin

A hyperbolic manifold is a manifold endowed with a Riemannian metric of constant sectional curvature -1. Throughout the paper, hyperbolic manifolds are assumed to be connected, orientable, complete, and of finite volume, unless otherwise stated. We refer to [34, 45] for the definition of an arithmetic hyperbolic manifold.

A connected hyperbolic n-manifold $\mathcal{M}$ is said to *bound geometrically* if it is isometric to $\partial\mathcal{W}$ for a hyperbolic $(n+1)$-manifold $\mathcal{W}$ with totally geodesic boundary.

Indeed, some interest in hyperbolic manifolds that bound geometrically was kindled by the works of Long, Reid [28, 29], and Niemershiem [36], motivated by a preceding work of Gromov [15, 16] and a question by Farrell and Zdravkovska [13]. This question is also related to hyperbolic instantons, as described in [40, 41].

As [28] shows many closed hyperbolic 3-manifolds do not bound geometrically: a necessary condition is that the η-invariant of the 3-manifold must be an integer. The 1st example of a closed hyperbolic 3-manifold known to bound geometrically was constructed by Ratcliffe and Tschantz in [40] and has volume of order 200.

The 1st examples of knot and link complements that bound geometrically were produced by Slavich in [38, 39]. However, [25] implies that there are plenty of cusped hyperbolic 3-manifolds that cannot bound geometrically, with the obstruction being the geometry of their cusps.

In [30], by using arithmetic techniques, Long and Reid built infinitely many orientable hyperbolic n-manifolds $\mathcal{N}$ that bound geometrically an $(n+1)$-manifold $\mathcal{M}$, in every dimension $n \geqslant 2$. Every such manifold $\mathcal{N}$ is obtained as a cover of some n-orbifold $O_{\mathcal{N}}$ geodesically immersed in a suitable $(n+1)$-orbifold $O_{\mathcal{M}}$. However, this construction gives no control on the volume of the manifolds.

In [4], Belolipetsky, Gelander, Lubotzky, and Shalev showed that the growth rate of all orientable arithmetic hyperbolic manifolds, up to isometry, with respect to volume is super-exponential, in all dimensions $n \geq 2$. Their lower bound used a subgroup counting technique due to Lubotzky [32]. In the present paper, we shall use the ideas of [30] together with the subgroup counting argument due to Lubotzky [32] (also used in [4]), together with the more combinatorial colouring techniques from [27] in order to prove the following facts:

Proposition 1.1. Let $\kappa_n(x)$ = the number of non-isometric non-orientable compact arithmetic hyperbolic n-manifolds of volume $\leq x$. Then we have that $\kappa_n(x) \asymp x^x$ for any $n \geq 3$.

Proposition 1.2. Let $\nu_n(x)$ = the number of non-isometric non-orientable cusped arithmetic hyperbolic n-manifolds of volume $\leq x$. Then we have that $\nu_n(x) \asymp x^x$ for any $n \geq 3$.

Above, the notation "$f(x) \asymp x^x$" for a function $f(x)$ is a shorthand for "there exist positive constants A_1, B_1, A_2, B_2, and x_0, such that $A_1 x^{B_1 x} \leq f(x) \leq A_2 x^{B_2 x}$, for all $x \geq x_0$."

The techniques of [4, 32] provide us with super-exponentially many manifolds of volume $\leq x$ (for x sufficiently large) by employing a retraction of the manifold's fundamental group into a free group. In our case, we need however to take extra care in order to arrange for the kernel of such retraction comprise an orientation-reversing element. Here, Coxeter polytopes and reflection groups come into play as natural sources of orientation-reversing isometries, as well as building blocks for manifolds.

Then by using the embedding technique from [24] and the techniques for constructing torsion-free subgroups from [30] (see Lemma 3.1, also Lemma 3.2 below), we obtain the following theorems establishing that the growth rate with respect to volume of arithmetic hyperbolic manifolds bounding geometrically is the same as that over all arithmetic hyperbolic manifolds.

Theorem 1.3. Let $\beta_n(x)$ = the number of non-isometric orientable compact arithmetic hyperbolic n-manifolds of volume $\leq x$ that bound geometrically. Then we have that $\beta_n(x) \asymp x^x$ for $n \geq 3$.

Theorem 1.4. Let $\gamma_n(x)$ = the number of non-isometric orientable cusped arithmetic hyperbolic n-manifolds of volume $\leq x$ that bound geometrically. Then we have that $\gamma_n(x) \asymp x^x$ for $n \geq 3$.

As a by-product, we provide a different proof to a part of the results in [26] and construct a few new Coxeter polytopes not otherwise available on the literature. For dimensions $n = 2, \ldots, 6$ in the compact case and dimensions $n = 2, \ldots, 13$ in the cusped case, we construct explicit examples of retractions onto free groups. More involved computations may be performed in dimensions $n = 14, 15$ (using the polytopes from [1]) and $n = 18, 19$ (using the polytopes from [21]). However, the general case follows from the main result of Bergeron, Haglund, Wise [5] on virtually retractions of arithmetic groups of simplest type onto geometrically finite subgroups.

It is also worth mentioning that a linear lower bound with respect to volume for the number of isometry classes of compact orientable bounding hyperbolic 3-manifolds was obtained previously in [33] by extending the techniques from [23] and comparing the Betti numbers of the resulting manifolds.

Given the present question's background, one may think of Theorem 1.3 as a "hyperbolic counterpart" to Rokhlin's theorem. Indeed, not every compact orientable arithmetic hyperbolic 3-manifold bounds geometrically, but the number of those that do has the same growth rate as the number of all compact orientable arithmetic hyperbolic 3-manifolds. In the light of Wang's theorem [46] and the results of [9], an analogous statement can be formulated for geometrically bounding hyperbolic 4-manifolds without arithmeticity assumption.

As for the closed hyperbolic surfaces that bound geometrically, it follows from the work of Brooks [6] that for each genus $g \geq 2$ the ones that bound form a dense subset of the Teichmüller space. Thus, there are infinitely many of them in each genus $g \geq 2$. However, there are only finitely many arithmetic ones by [4]. The argument of Theorem 1.3 applies in this case, and we obtain:

Theorem 1.5. Let $\alpha(g)$ = the number of non-isometric orientable closed arithmetic surfaces of genus $\leq g$ that bound geometrically. Then $(cg)^{\frac{g}{8}} \leq \alpha(g) \leq (dg)^{2g}$, for some constants $0 < c \leq d$.

Remark 1.6. An analogous statement holds for finite-area non-compact surfaces if we substitute the genus g with the area x. Namely, then $(cx)^{\frac{x}{32\pi}} \leq \alpha(x) \leq (dx)^{\frac{x}{2\pi}}$, for some $0 < c \leq d$.

This adds many more (albeit not very explicit) examples to the ones obtained by Zimmermann in [47, 48].

The manifolds that we construct in abundance in order to prove Theorem 1.3–Theorem 1.5 all happen to be orientation double covers. An easy observation implies that any closed orientable manifold M that is an orientation cover bounds topologically: consider $W' = M \times [0, 1]$ and quotient one of its boundary components by an orientation-reversing fixed point free involution that M necessarily has in this case. The resulting manifold W is orientable with boundary $\partial W \cong M$. Indeed, these are the manifolds that are *not* orientation covers that may make the cobordism group non-trivial.

Concerning geometrically bounding manifolds, we are not aware at the moment of any that does bound geometrically and that is not an orientation cover, in both compact and finite-volume cases.

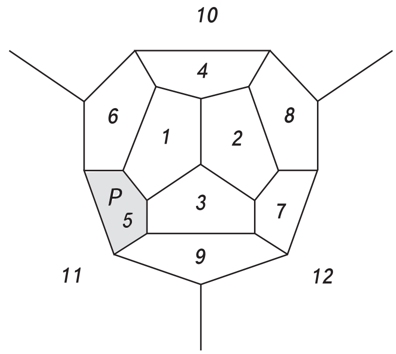


Fig. 1. A face labelling for the dodecahedron $\mathcal{D}$.

2 Constructing Geodesic Boundaries by Colourings

2.1 The right-angled dodecahedron

Let $\mathcal{D} \subset \mathbb{H}^3$ be a right-angled dodecahedron. By Andreev's theorem [2], it is realizable as a regular compact hyperbolic polyhedron. Suppose that the faces of $\mathcal{D}$ are labelled with the numbers 1, ..., 12 as shown in Figure 1. Let s_i be the reflection in the supporting hyperplane of the i-th facet of $\mathcal{D}$, for $i = 1, \ldots, 12$, and let $\Gamma_{12} = \mathrm{Ref}(\mathcal{D}) = \langle s_1, s_2, \ldots, s_{12} \rangle$ be the corresponding reflection group.

Let P be the pentagonal 2-dimensional face of $\mathcal{D}$ labelled 5, and let $\Gamma_4 = \langle s_1, s_3, s_9, s_{11} \rangle$ be an infinite-index subgroup of Γ_{12}, which we may consider as a reflection group acting on the supporting hyperplane of P, which is isometric to $\mathbb{H}^2$. There is a retraction R of Γ_{12} onto Γ_4 given by

$$R : s_i \mapsto \begin{cases} s_i, & \text{if } i \in \{1, 3, 9, 11\}, \\ \mathrm{id}, & \text{otherwise.} \end{cases}$$

The group Γ_4 is virtually free: it contains $F_3 \cong \langle x, y, z \rangle$, a free group of rank 3, as an index 8 normal subgroup. Indeed, with $x = s_1 s_{11}$, $y = (s_1 s_9)^2$, $z = s_1 s_3 s_{11} s_3$, we have F_3 realized as a subgroup of Γ_4, which is the fundamental group of a 2-sphere with four disjoint closed discs removed, as depicted in Figure 2.

Let P be a simple n-dimensional polytope (not necessarily hyperbolic) with m facets labelled by distinct elements of $\Omega = \{1, 2, \ldots, m\}$. A colouring of P, according to [12, 14, 20, 42, 43], is a map $\lambda : \Omega \to \mathbb{Z}_2^n$. A colouring is called proper if the colours of facets around each vertex of P are linearly independent vectors of $V = \mathbb{Z}_2^n$.

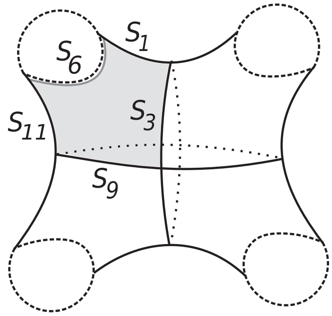


Fig. 2. The sphere $\mathbb{S}^2$ with four disjoint closed discs removed and one of eight tiles associated to Γ_4 shaded. The reflection side of s_6 completes this tile to the pentagon P and removing it is equivalent to cutting out a closed disc.

Proper colourings of compact right-angled polytopes $P \subset \mathbb{H}^n$ give rise to interesting families of hyperbolic manifolds [14, 23, 42, 43]. Such polytopes P are necessarily simple.

In [23], the notion of a colouring is extended to let $V = \mathbb{Z}_2^s$, $s \geq 2$, be a finite-dimensional vector space over $\mathbb{Z}_2$, and in [27] the notion of colouring is extended to polytopes that are not necessarily simple but rather satisfy a milder constraint of being simple at edges.

A polytope $P \subset \mathbb{H}^n$ is called simple at edges if each edge belongs to exactly $(n-1)$ facets. In the case of a finite-volume right-angled polytope $P \subset \mathbb{H}^n$, P is simple if P is compact, and P is simple at edges if it has any ideal vertices.

A colouring of a polytope $P \subset \mathbb{H}^n$ that is simple at edges is a map $\lambda : \Omega \to V$, where $V = \mathbb{Z}_2^s$, $s \geq n$, is a finite-dimensional vector space over $\mathbb{Z}_2$. A colouring λ is proper if the following two conditions are satisfied:

1. *Properness at vertices:* if v is a simple vertex of P, then the n colours of facets around it are linearly independent vectors of V.
2. *Properness at edges:* if e is an edge of P, then the $(n-1)$ colours of facets around e are linearly independent.

Given a fixed labelling Ω of the facets of a finite-volume right-angled polytope $P \subset \mathbb{H}^n$, we shall write its colouring as a vector $\lambda = (\lambda_1, \ldots, \lambda_m)$, where $\lambda_i = \sum_{k=0}^{\dim V-1} \lambda(i)_k \cdot 2^k$ is a binary representation of the vector $\lambda(i) \in V$ for all $i \in \Omega$.

Let s_i be a reflection in the supporting hyperplane of the i-th facets of P. Then a proper colouring $\lambda : \Omega \to V$ defines a homomorphism from the reflection group $\Gamma = \mathrm{Ref}(P) = \langle s_1, s_2, \ldots, s_m \rangle$ of P to V, such that $\ker \lambda$ is a torsion-free subgroup of Γ [27].

Let us consider one of the colourings of $\mathcal{D}$ defined in [14, Table 1] that gives rise to a non-orientable manifold cover of the orbifold $\mathbb{H}^3 / \Gamma_{12}$. Namely, choose $\lambda = (1,2,4,4,2,6,3,5,5,3,1,7)$, so that the i-th component of λ corresponds to the colour λ_i of the i-th face of $\mathcal{D}$. As follows from [23, Corollary 2.5], this colouring is indeed non-orientable, since $\lambda_1 + \lambda_2 + \lambda_7 = \mathbf{0}$ in $\mathbb{Z}_2^3$. Thus, $M = \mathbb{H}^3 / \Gamma$, with $\Gamma = \ker\lambda$ a torsion-free subgroup of Γ_{12}, is a non-orientable compact hyperbolic 3-manifold.

The reflection group Γ_{12} is an index 120 subgroup in the reflection group $\mathrm{Ref}(T)$ of the orthoscheme $T = [4,3,5]$, which is arithmetic. Thus, Γ_{12} is also arithmetic. Moreover, $\mathrm{Ref}(T) = O^+(q, \mathbb{Z}[\omega])$, with $\omega = \frac{1+\sqrt{5}}{2}$, for the quadratic form $q = -\omega x_0^2 + x_1^2 + x_2^2 + x_3^2$, as described in [3, §7] and, initially, in [7].

Next, let $\rho : \Gamma \to R(\Gamma)$ be the restriction of R. Observe that $R(\Gamma) = \Gamma_4$, and thus $\rho : \Gamma \to \Gamma_4$ is an epimorphism. Here, we use the fact that $s_1 = \rho(s_1 s_2 s_7)$, $s_3 = \rho(s_3 s_4)$, $s_9 = \rho(s_8 s_9)$, and $s_{11} = \rho(s_2 s_{10} s_{11})$, where all the respective products of s_i's belong to $\Gamma = \ker\lambda$.

For any subgroup $K \leq F_3$ of index n, let us consider $\rho^{-1}(K) = R^{-1}(K) \cap \Gamma$. Then K has index $8n$ in Γ_4, and $H = \rho^{-1}(K)$ has index $8n$ in Γ.

Moreover, we produce an orientation-reversing element $\delta \in \Gamma$ such that $\delta \in H$ for every such H.

Having established these facts, we know that there are $\asymp n^n$ non-conjugate in $\mathrm{Isom}(\mathbb{H}^3)$ subgroups of Γ by using the argument of [4, §5.2], and thus there are $\asymp x^x$ non-isometric non-orientable compact arithmetic 3-manifolds $M = \mathbb{H}^3 / H$ of volume $\leq x$ (for $x > 0$ big enough). This proves the 3-dimensional case of Proposition 1.1.

Now, observe that $x = s_1 s_{11}$, and $\lambda(x) = (1,0,0)^t + (1,0,0)^t = \mathbf{0}$ in $\mathbb{Z}_2^3$. Similarly, $\lambda(y) = \lambda(z) = \mathbf{0}$. Also, R maps x, y, and z respectively to themselves. Thus, $F_3 = \langle x, y, z \rangle \subset \rho(\Gamma)$. Finally, the element $\delta = s_2 s_4 s_6$ is such that $\lambda(\delta) = (0,1,0)^t + (1,0,0)^t + (1,1,0)^t = \mathbf{0}$, and $\rho(\delta) = \mathrm{id}$, so that $\delta \in \Gamma$ and $\delta \in H = \rho^{-1}(K)$, for every $K \leq F_3$.

Given that $H \leq O^+(q, \mathbb{Z}[\omega])$ for an admissible quadratic form q, we have that the argument in the proof of [24, Corollary 1.5] applies in this case, and thus, the non-orientable compact manifold $M = \mathbb{H}^3 / H$ embeds into a compact orientable manifold $N = \mathbb{H}^4 / G$, for some arithmetic torsion-free $G \leq O^+(Q, \mathbb{Z}[\tau])$, with $Q = q + x_4^2$. Then cutting N along M produces a manifold $N//M$, which is connected since N is orientable while M is not. Also, since M is a one-sided submanifold of N, the boundary $\partial N'$ is isometric to $\widetilde{M}$, the orientation cover of M. Thus, we obtain a collection of $\asymp n^n$ orientable arithmetic 3-manifolds $\widetilde{M}$ that bound geometrically. However, some of them can be isometric, since the same manifold $\widetilde{M}$ can be the orientation cover of several distinct non-orientable manifolds $N_1, \ldots, N_m$.

In order to estimate m, observe that each N_i is a quotient of $\widetilde{M}$ by a fixed point free orientation-reversing involution. Let the number of such involutions for $\widetilde{M}$ be $I(\widetilde{M})$. Then $m \leq I(\widetilde{M}) \leq |\mathrm{Isom}(\widetilde{M})| \leq c_1 \cdot \mathrm{Vol}(\widetilde{M}) \leq c_2 \cdot n = c_3 x$. Indeed, the isometry group of $\widetilde{M}$ is finite, and by the Kazhdan–Margulis theorem [22], there exists a lower bound for the volume of the orbifold $\widetilde{M} / \mathrm{Isom}(\widetilde{M}) \geq c_0 > 0$, from which the final estimate follows. Thus, we have at least $\asymp n^n/(c_2 n) \asymp n^n \asymp x^x$ non-isometric compact orientable arithmetic hyperbolic 3-manifolds $\widetilde{M}$ of volume $\leq x$ that bound geometrically. The upper-bound of the same order of growth follows from [4]. This proves the 3-dimensional case of Theorem 1.3.

2.2 The right-angled 120-cell

Let $\mathcal{C} \subset \mathbb{H}^4$ be the regular right-angled 120-cell. This polytope can be obtained by the Wythoff construction with the orthoscheme $[4,3,3,5]$ that uses the vertex stabilizer subgroup $[3,3,5]$ of order $(120)^2 = 14400$. The polytope $\mathcal{C}$ is compact, and each of its 3-dimensional facets is a regular right-angled dodecahedron isometric to $\mathcal{D}$ defined above.

Let us choose a facet F of $\mathcal{C}$ and label it 120. Since F is isometric to $\mathcal{D}$, we can label the neighbouring facets of F as follows:

- choose an isometry φ between F and $\mathcal{D}$ and transfer the labelling of 2-dimensional faces of $\mathcal{D}$ depicted in Figure 1 from $\mathcal{D}$ to F via φ,
- if F', a facet of $\mathcal{C}$, shares a 2-face labelled $i \in \{1, 2, \ldots, 12\}$ with F, label F' with i.

The remaining facets of $\mathcal{C}$ can be labelled with the numbers in $\{13, \ldots, 119\}$ in an arbitrary way. Let s_i denote the reflection on the supporting hyperplane of the i-th facet of $\mathcal{C}$, and let $\Gamma_{120} = \mathrm{Ref}(\mathcal{C}) = \langle s_1, s_2, \ldots, s_{120} \rangle$.

Now define a colouring Λ of $\mathcal{C}$ by using the colouring λ of $\mathcal{D}$ defined above. Namely, we set

$$\Lambda(s_i) = \begin{cases} \lambda_i, & \text{for } 1 \leq i \leq 12, \\ 2^{i-10}, & \text{for } 13 \leq i \leq 120. \end{cases}$$

Observe that Λ is a proper colouring of $\mathcal{C}$, as defined in [23], and thus $\Gamma = \ker \Lambda$ is torsion free. Also, Λ is a non-orientable colouring. As in the case of $\mathcal{D}$, we use the retraction R in order to map Γ_{120} onto Γ_4 that contains F_3 as a finite-index subgroup. By taking preimages $H = \rho^{-1}(K)$ in Γ of index n subgroups $K \leq F_3$ and applying our argument from the previous section, we complete the proof of Proposition 1.1 in the 4-dimensional case and obtain $\asymp n^n$ non-isometric non-orientable compact arithmetic

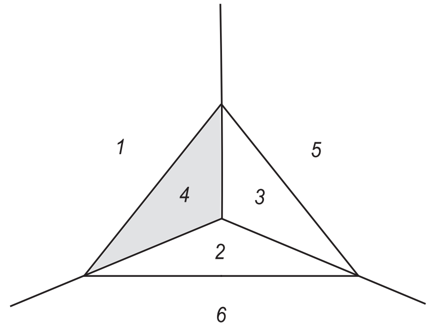


Fig. 3. A face labelling for the bi-pyramid $\mathcal{R}_3$. The compact vertices in $\mathbb{H}^3$ are the central one and the one at ∞. All other vertices are ideal and belong to $\partial\mathbb{H}^3$.

hyperbolic 4-manifolds $M = \mathbb{H}^4 \diagup H$. The rest of the argument follows from [24, Theorem 1.4]. Thus, the 4-dimensional case of Theorem 1.3 is also proven.

2.3 Non-compact right-angled polytopes

Let $\mathcal{R}_3$ be a right-angled bi-pyramid depicted in Figure 3, which is the 1st polytope in the series described by L. Potyagaĭlo and È. Vinberg in [37]. The construction in [37] produces a series of polytopes $\mathcal{R}_n \subset \mathbb{H}^n$, for $n = 3, \dots, 8$, of finite volume, with both finite and ideal vertices, such that each facet of $\mathcal{R}_n$ is isometric to $\mathcal{R}_{n-1}$. Each $\mathcal{R}_n$ is produced by Whythoff's construction from the quotient of $\mathbb{H}^n$ by the reflective part of $O^+(f_n, \mathbb{Z})$, with $f_n = -x_0^2 + \sum_{k=1}^{n} x_k^2$, for $n = 3, \dots, 8$.

If we provide a non-orientable proper colouring of $\mathcal{R}_3$, as defined in [27], we can apply our previous reasoning in order to prove Proposition 1.2 and Theorem 1.4 as consequence. Let us label the faces of $\mathcal{R}_3$ as shown in Figure 3, and let the colouring be $\lambda = (1, 1, 4, 7, 5, 2)$. It is easy to check that λ is indeed proper, since we need to check only the colours around the finite vertices and edges of $\mathcal{R}_3$. Also, λ is non-orientable, since $\lambda_4 + \lambda_5 + \lambda_6 = \mathbf{0}$ in $\mathbb{Z}_2^3$. Let $\Gamma = \ker \lambda$.

Let s_i be the reflection in the i-th facet of $\mathcal{R}_3$, and $\Gamma_6 = \langle s_1, \dots, s_6 \rangle$, and $\Delta = \langle s_1, s_2, s_3 \rangle$. Observe that Δ contains a free group of rank 2 as a normal subgroup of index 4. Indeed, $F_2 = \langle x, y \rangle$, with $x = s_1 s_2$, $y = s_3 s_1 s_2 s_3$ is such a subgroup.

Let R be a retraction $\Gamma_6 \to \Delta$ given by

$$R : s_i \mapsto \begin{cases} s_i, & \text{if } i \in \{1, 2, 3\}, \\ \text{id}, & \text{otherwise.} \end{cases}$$

Since R maps x and y respectively to themselves, and $\lambda(s_1s_2) = (1,0,0)^t + (1,0,0)^t = \mathbf{0}$ in $\mathbb{Z}_2^3$, we have that $F_3 \subset R(\Gamma)$. Moreover, for $\delta = s_4s_5s_6$ it holds that $\lambda(\delta) = \mathbf{0}$, as already verified above, and $R(\delta) = \mathrm{id}$. Then the argument from the previous case of the right-angled dodecahedron applies verbatim.

For the induction step from $\mathcal{R}_{n-1}$ to $\mathcal{R}_n$, we just need to enhance the colouring in the way completely analogous to the extension of a non-orientable colouring of the dodecahedron $\mathcal{D}$ to a non-orientable colouring of the 120-cell $\mathcal{C}$. Again, the rest of the argument proceeds verbatim in complete analogy to the previous cases.

2.4 Surfaces that bound geometrically

Let $\mathcal{P} \subset \mathbb{H}^2$ be a compact regular right-angled octagon, with sides labelled anti-clockwise 1, 6, 2, 7, 3, 8, 4, 5. Let s_i be the reflection in the i-th side of $\mathcal{P}$, and $\Gamma_8 = \mathrm{Ref}(\mathcal{P}) = \langle s_1, s_2, \ldots, s_8 \rangle$ be its reflection group. Also, let $\Gamma_4 = \langle s_1, s_2, s_3, s_4 \rangle$ and $F_3 = \langle x, y, z \rangle$, with $x = s_1s_2$, $y = s_1s_3$, $z = s_1s_4$, be a free subgroup of Γ_4 of index 2. The retraction of Γ_8 onto Γ_4 is given by

$$R : s_i \mapsto \begin{cases} s_i, & \text{if } i \in \{1,2,3,4\}, \\ \mathrm{id}, & \text{otherwise.} \end{cases}$$

Let us choose a colouring $\lambda = (1,1,1,1,2,3,5,6)$ for $\mathcal{P}$, which is a proper and non-orientable one, since $\lambda_1 + \lambda_5 + \lambda_6 = \mathbf{0} \in \mathbb{Z}_2^3$. Let $\Gamma = \ker \lambda$. An easy check ensures that $F_3 \subset R(\Gamma)$, as well as that $R(\delta) = \mathrm{id}$ for an orientation-reversing element $\delta = s_6s_7s_8 \in \Gamma$. Then the lower bound $\alpha(x) \geq (cx)^{\frac{x}{32\pi}}$, for some constant $c > 0$, for the number of geometrically bounding surfaces of area $\leq x$ (for x large enough) follows immediately: the area of $\mathcal{P}$ equals 2π, Γ has index 8 in Γ_8, and the orientation cover of a non-orientable surface has twice its area. We also use the fact that the rank $d \geq 2$ free group F_d has $\geq (n!)^{d-1}$ subgroups of index $\leq n$, for n large enough. The upper bound $\alpha(x) \leq (dx)^{\frac{x}{2\pi}}$, for some constant $d \geq c > 0$, follows from [4]. Since area $= 4\pi(g-1)$, for an orientable genus $g \geq 2$ surface, this proves Theorem 1.5. The case of non-compact finite-area surfaces mentioned in Remark 1.6 proceeds by analogy.

3 Constructing Geodesic Boundaries by Arithmetic Reductions

We start by recalling the following lemma of Long and Reid [30, Lemma 2.2] (c.f. also the remark after its proof).

Lemma 3.1 (Subgroup lemma). Let $\Gamma < O^+(n,1)$ be a subgroup of hyperbolic isometries defined over a number field K and δ an element of Γ. Let $\theta_1, \theta_2 : \Gamma \to F_i$ be two homomorphisms of Γ onto a group F_i, with torsion-free kernels. Let $\Theta(g) = \big(\theta_1(g), \theta_2(g)\big) : \Gamma \to F_1 \times F_2$. Suppose that $\theta_i(\delta)$ has order $k_i < \infty$, $i = 1, 2$, and any prime dividing $gcd(k_1, k_2)$ appears with distinct exponents in k_1 and k_2. Then $\Theta^{-1}\langle(\theta_1(\delta), \theta_2(\delta))\rangle$ is a torsion-free subgroup in Γ of finite index that contains δ.

The following lemma is used in order to show that the maps that we choose in the sequel as θ_i, $i = 1, 2$, in the subgroup lemma above have torsion-free kernels. Its proof is very similar to that of [30, Lemma 2.4].

Lemma 3.2 (No torsion lemma). Let $\Gamma < O^+(n, 1)$ be a finite subgroup defined over the ring of integers $\mathcal{O}_K$ of a number field K, and let $p \in \mathcal{O}_K$ be an odd rational prime that does not divide the order of Γ. Then the reduction of Γ modulo the ideal $\mathcal{J} = (p)$ is isomorphic to Γ.

Proof. A non-trivial element g of the kernel of the reduction map $\Gamma(\mathcal{O}_K) \to \Gamma(\mathcal{O}_K/\mathcal{J})$ can be written in the form $g = \mathrm{id} + p^r h$, where h is a matrix not all of whose entries are divisible by p^r, with r some positive integer. Let $q < \infty$ be the order of an element $g \in \Gamma$. Then we get

$$\mathrm{id} = g^q = \mathrm{id} + qp^r h + \sum_{t=2}^{q} \binom{q}{t} p^{rt} h^t,$$

and thus

$$qh = \mathbf{0} \mod p^r.$$

The latter implies p^r divides q, since $h \neq \mathbf{0} \mod p^r$. Thus, p divides q, and q divides the order of Γ, since q is the order of an element of Γ. The latter is a contradiction, and thus the reduction map has trivial kernel. ∎

As shown by Vinberg in [44], in some cases for an admissible quadratic form q of signature $(n, 1)$ defined over a totally real number field K with ring of integers O_K it holds that $O^+(q, O_K) = \mathrm{Ref}(P) \rtimes \mathrm{Sym}(P)$, where $P \subset \mathbb{H}^n$ is a finite-volume polytope. Here, $\mathrm{Ref}(P)$ denotes the associated reflection group, and $\mathrm{Sym}(P)$ is the group of symmetries of P. Also, we assume that O_K is a principal ideal domain in order to keep our account simpler. We refer the reader to [17] for more details.

If the above presentation of $O^+(q, O_K)$ takes place for some finite-volume polytope $P \subset \mathbb{H}^n$, the form q is called *reflective*, and the polytope P is called its *associated polytope*.

An algorithm introduced by Vinberg in [44] and implemented in [18] by Guglielmetti allows us to find the associated polytope $P \subset \mathbb{H}^n$ in finite time for any reflective admissible quadratic form of signature $(n, 1)$.

3.1 Compact polytopes in dimensions 5 and 6

Let $\omega = \frac{1+\sqrt{5}}{2}$ and let $P_n \subset \mathbb{H}^n$ be the polytopes associated to the quadratic forms

$$q_5 = -(-1 + 2\omega)x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2,$$

$$q_6 = -2\omega x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2.$$

The polytopes P_5 and P_6 are apparently new and were found by using AlVin and CoxIter software [18, 19]. They differ substantially from the polytopes that appear in [7, 8] and have fewer facets.

Let $\Gamma_n = \mathrm{Ref}(P_n)$ be the reflection group of P_n with generators s_i, $i \in I_n$, where I_n is the set of outer normals to the facets of P_n or, equivalently, the set of nodes of the Coxeter diagram of P_n. (The notation used is as follows: a dashed edge means two reflection hyperplanes have a common perpendicular, a solid edge means parallel (at the ideal boundary) hyperplanes, a double edge means label 4, a single edge means label 3, any other edge has a label on it describing the corresponding dihedral angle. The colours are used for convenience only.) With standard basis $\{v_0, v_1, \ldots, v_n\}$, the Vinberg algorithm determines the outer normals for $n = 5, 6$ that are given in the Appendix.

The associated reflection group Γ_n is arithmetic and contains a virtually free parabolic subgroup

$$\Delta = \begin{cases} \langle s_5, s_6, s_9 \rangle, & \text{for } n = 5, \\ \langle s_6, s_9, s_{17} \rangle, & \text{for } n = 6. \end{cases}$$

Indeed, Δ is isomorphic to the $(2, \infty, \infty)$-triangle group (here Δ is not actually generated by reflections in the sides of a hyperbolic finite-area triangle but is rather only abstractly isomorphic to such a group; however, we are interested in its algebraic rather than geometric properties, regarding its subgroup growth), which contains F_2 as a subgroup of index 4.

The retraction $R : \Gamma_n \to \Delta$ is defined by sending all but three generators of Γ_n to id, with the only generators mapped identically being those of $\Delta < \Gamma_n$.

In order for R being well defined, we essentially need that the generators of Δ be connected to the rest of the diagram by edges with even labels only, since any two generators connected by a path of odd-labelled edges are conjugate. This folds, for instance, if the facets corresponding to the reflections generating Δ are *redoubleable* in terms of [1].

The element

$$\delta = \begin{cases} s_1 s_2 s_3 s_4 s_7, & \text{for } n = 5, \\ s_7 s_{13} s_{18}, & \text{for } n = 6, \end{cases}$$

is orientation-reversing, as it is a product of an odd number of reflections in $\mathbb{H}^n$. Moreover, $\delta \in \ker R$.

We shall set the map Θ from Lemma 3.1 to be a pair of reductions modulo various rational primes, and then use Lemma 3.2 in order to ensure that their kernels are torsion free. Indeed, we need to choose such an odd prime $p \in \mathbb{Z}$ that it does not divide the order of any finite parabolic subgroup in the diagram of P_n, $n = 5, 6$. The least common multiples of orders of finite parabolic subgroups for P_n, $n = 5, 6$ are given in Table 1 (the orders of all finite parabolic subgroups associated with P_n can be obtained by using `CoxIter` [18] with the `-debug` option).

For $\Gamma \in \mathrm{GL}(n+1, \mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of a number field K, let ϕ_p denote the homomorphism $\Gamma \to \mathrm{GL}(n+1, \mathcal{O}_K/\mathcal{J})$ induced by reduction modulo $\mathcal{J} = (p)$, the principal ideal generated by a rational integer p.

Let us consider the reductions ϕ_7 and ϕ_{11} as defined above, and let $\Theta = (\phi_7, \phi_{11})$. For $n = 5$, the order of $\phi_7(\delta)$ equals $800 = 2^5 \cdot 5^2$, while the order of $\phi_{11}(\delta)$ equals $8052 = 2^2 \cdot 3^1 \cdot 11^1 \cdot 61^1$, as follows by straightforward computations, c.f. [10, 11]. For $n = 6$, the order of $\phi_7(\delta)$ equals $8 = 2^3$, while the order of $\phi_{11}(\delta)$ equals $44 = 2^2 \cdot 11^1$.

Then Lemma 3.1 and Lemma 3.2 apply, and $\Gamma = \Theta^{-1}\langle(\phi_7(\delta), \phi_{11}(\delta))\rangle$ is a torsion-free subgroup of finite index in Γ_n that contains the orientation-reversing element δ and retracts onto the free group $\Gamma \cap \Delta$. Then the argument analogous to that of Section 2 applies.

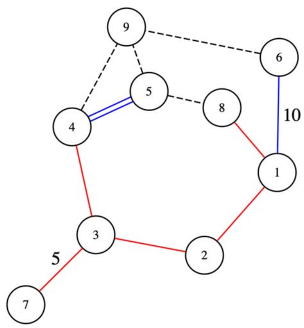
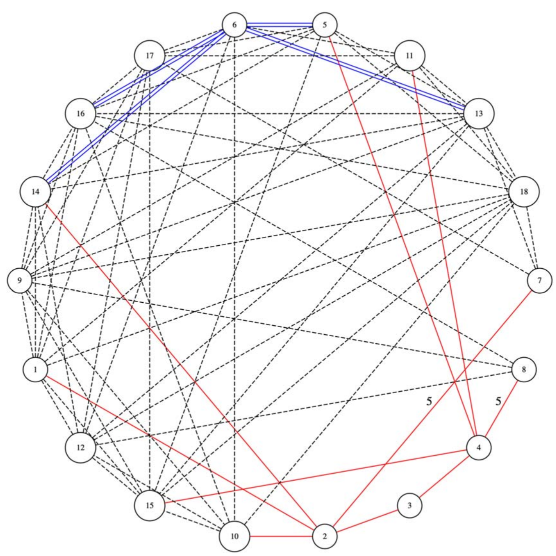
3.2 Right-angled cusped polytopes in dimensions 4 to 8

Let $P_n \subset \mathbb{H}^n$ be the right-angled polytopes associated to the principal congruence subgroups of level 2 for the quadratic forms

$$f_n = -x_0^2 + x_1^2 + x_2^2 + \cdots + x_n^2, \text{for } n = 4, \ldots, 8.$$

Let $\Gamma_n = \mathrm{Ref}(P_n)$ be the associated reflection group, with generators s_i, $i \in I_n$, where I_n is the set of outer normals to the facets of P_n. With standard basis

Table 1 Polytopes P_n, $n = 5, 6$, their Coxeter diagrams, and the least common multiple (LCM) of the orders of their parabolic finite subgroups

Polytope	Diagram	LCM
P_5		$57600 = 2^8{\cdot}3^2{\cdot}5^2$
P_6		$230400 = 2^{10} \cdot 3^2 \cdot 5^2$

$\{v_0, v_1, \ldots, v_n\}$, the Vinberg algorithm starts with the 1st n outer normals being

$$e_i = -v_i, \text{ for } 1 \le i \le n,$$

and continues with the next $\binom{n}{2}$ outer normals

$$e_{i,j} = v_0 + v_i + v_j, \text{ for } 1 \le i < j \le n,$$

all of them being 1-roots, as is necessary for determining the reflective part of the principle congruence level 2 subgroup rather than that of the whole group of units for

f_n. Let us set $e_{n+1} = e_{1,2} = v_0 + v_1 + v_2$ and $e_{n+2} = e_{3,4} = v_0 + v_3 + v_4$ as a more convenient notation.

Such Γ_n is arithmetic, and it contains a virtually free parabolic subgroup $\Delta = \langle s_3, s_4, s_{n+2} \rangle$. Indeed, Δ is isomorphic to the $(2, \infty, \infty)$-triangle group that contains F_2 as a subgroup of index 4. Consider the retraction

$$R : s_i \mapsto \begin{cases} s_i, & \text{if } i \in \{3, 4, n+2\}, \\ \mathrm{id}, & \text{otherwise.} \end{cases}$$

The element $\delta = s_1 s_2 s_{n+1}$ is orientation-reversing, as it is a product of three reflections in $\mathbb{H}^n$. Moreover, $\delta \in \ker R$.

For $\Gamma < GL(n+1, \mathbb{Z})$, let ϕ_m denote the homomorphism $\Gamma \to \mathrm{GL}(n+1, \mathbb{Z}/m\mathbb{Z})$ induced by reduction modulo a positive integer m. By [35, Theorem IX.7], we know that the kernel of ϕ_m is torsion-free for $m > 2$. The reduction of δ modulo 3 has order $4 = 2^2$, while its reductions modulo 4 has order 2, c.f. [10, 11]. Letting $\Theta = (\phi_3, \phi_4)$, Lemma 3.1 applies, and $\Gamma = \Theta^{-1}\langle(\phi_3(\delta), \phi_4(\delta))\rangle$ is a torsion-free subgroup of finite index in Γ_n that contains the orientation-reversing element δ and retracts onto the free group $\Gamma \cap \Delta$.

3.3 Cusped polytopes in dimensions 9 to 13

Let $P_n \subset \mathbb{H}^n$ be the polytopes from Table 7 in [44] associated to the quadratic forms

$$f_n = -2x_0^2 + x_1^2 + x_2^2 + \cdots + x_n^2 \text{ for } n = 9, 10, \text{ and } 13,$$

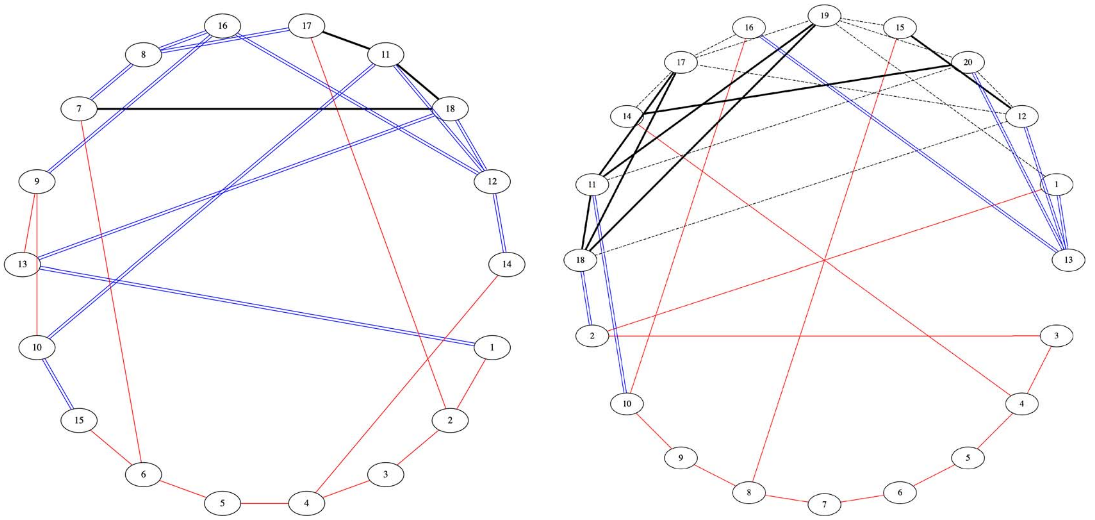
while P_n, for $n = 11, 12$, be the polytopes with Coxeter diagrams given in Figure 4, respectively. The latter ones appear to be new and were found by using `AlVin` [19].

Let $\Gamma_n = \mathrm{Ref}(P_n)$ be the reflection group of P_n with generators s_i, $i \in I_n$, where I_n is the set of nodes in the Coxeter diagram of P_n. Such Γ_n is arithmetic, and it contains a virtually free parabolic subgroup Δ indicated in Table 2.

Since Δ is generated by reflections in redoubleable facets, we can define a retraction $R : \Gamma_n \to \Delta$, as before, that send all the generators of Γ_n to id, except of those of Δ.

The orientation-reversing element $\delta_n \in \ker R$ is defined by

$$\delta_n = \begin{cases} s_1 s_2 s_3 \ldots s_8 s_{n+2}, & \text{for } n = 9, 10, 13, \\ s_7 s_8 s_{16}, & \text{for } n = 11, \\ s_2 s_{11} s_{18}, & \text{for } n = 12. \end{cases}$$



(a) $f_{11} = -2x_0^2 + x_1^2 + \ldots + x_8^2 + 2x_9^2 + 2x_{10}^2 + 2x_{11}^2$ (b) $f_{12} = -2x_0^2 + x_1^2 + \ldots + x_{11}^2 + 2x_{12}^2$

Fig. 4. The Coxeter diagrams for (a) P_{11} and (b) P_{12}, together with their associated quadratic forms.

Table 2 A virtually free parabolic subgroup $\Delta < \Gamma_n$, for $n = 9, \ldots, 13$

n	Generators of Δ	Triangle group $\cong \Delta$
9	s_9, s_{10}, s_{12}	$(2, \infty, \infty)$
10	s_{10}, s_{11}, s_{13}	$(2, 4, \infty)$
11	s_{11}, s_{12}, s_{18}	$(4, 4, \infty)$
12	s_{12}, s_{13}, s_{20}	$(4, 4, \infty)$
13	s_{13}, s_{14}, s_{19}	$(2, \infty, \infty)$

Table 3 Orders of the reductions of δ and their prime factorizations, c.f. [10, 11]

n	m_1	$k_1 =$ order of $\phi_{m_1}(\delta_n)$	m_2	$k_2 =$ order of $\phi_{m_2}(\delta_n)$
9, 10, 13	3	$84 = 2^2 \cdot 3^1 \cdot 7^1$	4	$34 = 2^1 \cdot 17^1$
11, 12	3	$6 = 2^1 \cdot 3^1$	4	$4 = 2^2$

Letting $\Theta = (\phi_{m_1}, \phi_{m_2})$, Lemma 3.1 applies with m_1 and m_2 as in Table 3. Here, we also notice that δ_n for $n = 10, 13$ is an extension of δ_9 by the identity map, which simplifies the computations.

Then $\Gamma = \Theta^{-1}\langle(\phi_{m_1}(\delta), \phi_{m_2}(\delta))\rangle$ is a torsion-free subgroup of finite index in Γ_n that contains the orientation-reversing element δ and retracts onto the free group $\Gamma \cap \Delta$.

4 Constructing Geometric Boundaries from Virtual Retracts

Let q_n be an admissible quadratic form of signature $(n, 1)$ defined over a totally real number field K with ring of integers O_K, and let $q_{n+1} = q_n + x_{n+1}^2$. Suppose also that $\Gamma_n < O^+(q_n, O_K)$ is a torsion-free subgroup either of finite co-volume or co-compact.

Now assume that there exists a retraction $R_n : \Gamma_n \to \Delta$ of Γ_n onto a virtually free subgroup Δ such that $\ker R_n$ contains an orientation-reversing element δ.

By [24, Proposition 7.1], there exists a torsion-free finite-index subgroup $\Gamma'_{n+1} \leq O^+(q_{n+1}, O_K)$, such that $\Gamma_n < \Gamma'_{n+1}$. Moreover, we may assume that $\mathbb{H}^n/\Gamma_n$ is a properly embedded totally geodesic submanifold of $\mathbb{H}^{n+1}/\Gamma_{n+1}$. Thus, the group Γ_n is a geometrically finite subgroup of Γ'_{n+1}. By [5, Theorem 1.4], there is a finite index subgroup $G < \Gamma'_{n+1}$, such that G virtually retracts to its geometrically finite subgroups. In particular, G virtually retracts to $G \cap \Gamma_n$. However, since Γ'_{n+1} is linear, the arguments of [31, Theorem 2.10] apply to give a virtual retraction from Γ'_{n+1} onto Γ_n. Let Γ_{n+1} be the finite index subgroup of Γ'_{n+1} that retracts onto Γ_n. Then the composition $R_{n+1} : \Gamma_{n+1} \to \Gamma_n \to \Delta$ is a retraction of Γ_{n+1} onto Δ, such that $\delta \in \ker R_{n+1}$.

All the previous arguments from Section 3 apply, and we obtain Theorems 1.4–1.5 for all $n \geq 2$, since we can use any of our examples worked out in Sections 2–3 as a basis for the above inductive procedure.

A.1 Outer normals for compact P_5

(subsection 3.1)

$e_i = -v_i + v_{i+1}$ for $1 \leq i \leq 4$,

$e_5 = -v_5$,

$e_6 = \omega v_0 + (2+\omega)v_1$,

$e_7 = \omega(v_0 + v_1 + v_2 + v_3)$,

$e_8 = (1+\omega)(v_0 + v_1) + \omega(v_2 + v_3 + v_4 + v_5)$.

A.2 Outer normals for compact P_6

(subsection 3.1)

$e_i = -v_i + v_{i+1}$ for $1 \leq i \leq 5$,

$e_6 = -v_6$,

$e_7 = v_0 + w(v_1 + v_2)$,

$e_8 = \omega(v_0 + v_1 + v_2 + v_3 + v_4)$,

$e_9 = \omega v_0 + 2\omega v_1$,

$e_{10} = (1+\omega)(v_0 + v_1 + v_2) + \omega(v_3 + v_4 + v_5 + v_6)$,

$e_{11} = (1+2\omega)v_0 + (1+3\omega)v_1 + (1+\omega)(v_2 + v_3 + v_4) + \omega(v_5 + v_6)$,

$e_{12} = (1+2\omega)v_0 + (2+3\omega)v_1 + \omega(v_2 + v_3 + v_4 + v_5 + v_6)$,

$e_{13} = (2+2\omega)v_0 + (1+2\omega)(v_1 + v_2 + v_3 + v_4 + v_5) + v_6$,

$e_{14} = (2+3\omega)v_0 + (2+4\omega)v_1 + (2+2\omega)v_2 + (1+2\omega)(v_3 + v_4 + v_5) + v_6$,

$e_{15} = (2+3\omega)v_0 + (3+4\omega)v_1 + (1+2\omega)(v_2 + v_3 + v_4) + 2\omega v_5$,

$e_{16} = (2+4\omega)v_0 + (3+6\omega)v_1 + (1+2\omega)(v_2 + v_3 + v_4 + v_5) + v_6$,

$e_{17} = (3+4\omega)v_0 + (2+5\omega)v_1 + (2+3\omega)(v_2 + v_3 + v_4 + v_5) + \omega v_6$,

$e_{18} = (4+5\omega)v_0 + (4+6\omega)v_1 + (2+4\omega)(v_2 + v_3 + v_4 + v_5)$.

A.3 Outer normals for cusped P_4

(subsection 3.2)

$e_i = -v_i$ for $1 \le i \le 4$,

$e_i = v_0 + v_{j_1} + v_{j_2}$ for $5 \le i \le 10$ and $1 \le j_1 < j_2 \le 4$.

A.4 Outer normals for cusped P_5

(subsection 3.2)

$e_i = -v_i$ for $1 \le i \le 5$,

$e_i = v_0 + v_{j_1} + v_{j_2}$ for $6 \le i \le 15$ and $1 \le j_1 < j_2 \le 5$,

$e_{16} = 2v_0 + v_1 + v_2 + v_3 + v_4 + v_5$.

A.5 Outer normals for cusped P_6

(subsection 3.2)

$e_i = -v_i$ for $1 \le i \le 6$,

$e_i = v_0 + v_{j_1} + v_{j_2}$ for $7 \le i \le 21$ and $1 \le j_1 < j_2 \le 6$,

$e_i = 2v_0 + v_{j_1} + \cdots + v_{j_5}$ for $22 \le i \le 27$ and $1 \le j_1 < j_2 < j_3 < j_4 < j_5 \le 6$.

A.6 Outer normals for cusped P_7

(subsection 3.2)

$e_i = -v_i$ for $1 \le i \le 7$,

$e_i = v_0 + v_{j_1} + v_{j_2}$ for $8 \le i \le 28$ and $1 \le j_1 < j_2 \le 7$,

$e_i = 2v_0 + v_{j_1} + \cdots + v_{j_5}$ for $29 \le i \le 49$ and $1 \le j_1 < j_2 < j_3 < j_4 < j_5 \le 7$,

$e_i = 3v_0 + 2v_{j_1} + v_{j_2} + \cdots + v_{j_7}$ for $50 \le i \le 56$ and $1 \le j_1 < j_2 < j_3 < j_4 < j_5 < j_6 < j_7 \le 7$.

A.7 Outer normals for cusped P_8

(subsection 3.2)

$e_i = -v_i$ for $1 \le i \le 8$,

$e_i = v_0 + v_{j_1} + v_{j_2}$ for $9 \le i \le 36$ and $1 \le j_1 < j_2 \le 8$,

$e_i = 2v_0 + v_{j_1} + \cdots + v_{j_5}$ for $37 \le i \le 92$ and $1 \le j_1 < j_2 < j_3 < j_4 < j_5 \le 8$,

$e_i = 3v_0 + 2v_{j_1} + v_{j_2} + \cdots + v_{j_7}$ for $93 \le i \le 148$,

$e_i = 4v_0 + 2(v_{j_1} + \cdots + v_{j_3}) + v_{j_4} + \cdots + v_{j_7}$ for $149 \le i \le 204$
and $1 \le j_1 < j_2 < j_3 < j_4 < j_5 < j_6 < j_7 \le 7$,

$e_i = 5v_0 + 2(v_{j_1} + \cdots + v_{j_6}) + v_{j_7} + v_{j_8}$ for $205 \le i \le 232$,

$e_i = 6v_0 + 3v_{j_1} + v_{j_2} + \cdots + v_{j_8}$ for $233 \le i \le 240$
and $1 \le j_1 < j_2 < j_3 < j_4 < j_5 < j_6 < j_7 < j_8 \le 8$.

A.8 Outer normals for cusped P_9

(subsection 3.3)

$e_i = -v_i + v_{i+1}$, for $1 \le i \le 8$,

$e_9 = -v_9$,

$e_{10} = v_0 + 2v_1$,

$e_{11} = v_0 + v_1 + v_2 + v_3 + v_4$,

$e_{12} = 2v_0 + v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8 + v_9$.

A.9 Outer normals for cusped P_{10}

(subsection 3.3)

$e_i = -v_i + v_{i+1}$, for$1 \le i \le 9$,

$e_{10} = -v_{10}$,

$e_{11} = v_0 + v_1 + v_2 + v_3 + v_4$,

$e_{12} = v_0 + 2v_1$,

$e_{13} = 2v_0 + v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8 + v_9 + v_{10}$.

A.10 Outer normals for cusped P_{11}

(subsection 3.3)

$e_i = -v_i + v_{i+1}$, for $1 \leq i \leq 7$, and $i = 9, 10$,

$e_i = -v_i$, and $i = 8, 11$,

$e_{12} = v_0 + v_9 + v_{10} + v_{11}$,

$e_{13} = v_0 + 2v_1 + v_9$,

$e_{14} = v_0 + v_1 + v_2 + v_3 + v_4$,

$e_{15} = 2v_0 + v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_9 + v_{10}$,

$e_{16} = 2v_0 + v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8 + v_9$,

$e_{17} = 3v_0 + 2(v_1 + v_2) + v_3 + v_4 + v_5 + v_6 + v_7 + v_8 + v_9 + v_{10} + v_{11}$,

$e_{18} = 4v_0 + 2(v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7) + v_9 + v_{10} + v_{11}$.

A.11 Outer normals for cusped P_{12}

(subsection 3.3)

$e_i = -v_i + v_{i+1}$, for $1 \leq i \leq 10$,

$e_i = -v_i$, for $i = 11, 12$,

$e_{13} = v_0 + 2v_1 + v_{12}$,

$e_{14} = v_0 + v_1 + v_2 + v_3 + v_4$,

$e_{15} = 2v_0 + v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8 + v_{12}$,

$e_{16} = 2v_0 + v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8 + v_9 + v_{10}$,

A.11 Outer normals for cusped P_{12}

(subsection 3.3)

$e_i = -v_i + v_{i+1}$, for $1 \leq i \leq 10$,

$e_i = -v_i$, for $i = 11, 12$,

$e_{13} = v_0 + 2v_1 + v_{12}$,

$e_{14} = v_0 + v_1 + v_2 + v_3 + v_4$,

$e_{15} = 2v_0 + v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8 + v_{12}$,

$e_{16} = 2v_0 + v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8 + v_9 + v_{10}$,

$e_{17} = 3v_0 + v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8 + v_9 + v_{10} + v_{11} + 2v_{12}$,

$e_{18} = 3v_0 + 2(v_1 + v_2) + v_3 + v_4 + v_5 + v_6 + v_7 + v_8 + v_9 + v_{10} + v_{11} + v_{12}$,

$e_{19} = 3(v_0 + v_1) + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8 + v_9 + v_{10} + v_{11}$,

$e_{20} = 5v_0 + 2(v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8 + v_9 + v_{10} + v_{11} + v_{12})$.

A.12 Outer normals for cusped P_{13}

(subsection 3.3)

$e_i = -v_i + v_{i+1}$, for $1 \le i \le 12$,

$e_{13} = -v_{13}$,

$e_{14} = v_0 + v_1 + v_2 + v_3 + v_4$,

$e_{15} = v_0 + 2v_1$,

$e_{16} = 2v_0 + v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8 + v_9 + v_{10}$,

$e_{17} = 3v_0 + 3v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8 + v_9 + v_{10} + v_{11} + v_{12}$,

$e_{18} = 3v_0 + 2v_1 + 2v_2 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8 + v_9 + v_{10} + v_{11} + v_{12} + v_{13}$,

$e_{19} = 5v_0 + 2(v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8 + v_9 + v_{10} + v_{11} + v_{12} + v_{13})$.

Acknowledgments

The 1st author thanks Darren Long (University of California, Santa Barbara) for many helpful discussions. Both authors would like to thank Mikhail Belolipetsky (IMPA), Vincent Emery (Universität Bern), and Ruth Kellerhals (Université de Fribourg), as well as the Mathematisches Forschungsinstitut Oberwolfach administration, for organizing the mini-workshop "Reflection Groups in Negative Curvature" (1915b) in April 2019, during which the results of this paper were presented and discussed. They also thank the organizers of the Borel Seminar (Les Diablerets, Switzerland) in December 2018, where this work was started, for stimulating research atmosphere, and the Swiss Doctoral Program—CUSO, Swiss Mathematical Society, and ETH Zürich for the financial support of the event. A word of special gratitude is addressed to the creators and maintainers of SAGE Math, and to Rafael Guglielmetti, the author of `CoxIter` and `AlVin`. The aforementioned software made most of the calculations in the present work possible and provided means for additional verification of those previously done. We also thank the anonymous referees for their suggestions and criticism that helped us improving this paper.

References

[1] Allcock, D. "Infinitely many hyperbolic Coxeter groups through dimension 19." *Geom. Topol.* 10 (2006): 737–58.

[2] Andreev, E. M. "On convex polyhedra in Lobacevskiĭ space." *Math. USSR Sb.* 10 (1970): 413–40.

[3] Belolipetsky, M. "Arithmetic hyperbolic reflection groups." *Bull. Am. Math. Soc.* 53 (2016): 437–75.

[4] Belolipetsky, M., T. Gelander, A. Lubotzky, and A. Shalev. "Counting arithmetic lattices and surfaces." *Ann. of Math. (2)* 172 (2010): 2197–221.

[5] Bergeron, N., F. Haglund, and D. T. Wise. "Hyperplane sections in arithmetic hyperbolic manifolds." *J. Lond. Math. Soc.* 83 (2011): 431–48.

[6] Brooks, R. "Circle packings and co-compact extensions of Kleinian groups." *Invent. Math.* 86 (1986): 461–9.

[7] Bugaenko, V. O. "Groups of automorphisms of unimodular hyperbolic quadratic forms over the ring $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$." *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* 5 (1984): 6–12.

[8] Bugaenko, V. O. "Arithmetic Crystallographic Groups Generated by Reflections, and Reflective Hyperbolic Lattices." In *Lie Groups, Their Discrete Subgroups, and Invariant Theory*, 33–55. Providence, RI: Adv. Soviet Math., 8, Amer. Math. Soc, 1992.

[9] Burger, M., T. Gelander, A. Lubotzky, and S. Mozes. "Counting hyperbolic manifolds." *Geom. Funct. Anal.* 12 (2002): 1161–73.

[10] Chu, M. and A. Kolpakov. "*Auxiliary Wolfram Mathematica Notebooks*." https://doi.org/10.7910/DVN/N0DP7R.

[11] Chu, M. and A. Kolpakov. "*Auxiliary SageMath Worksheets*." https://doi.org/10.7910/DVN/N0DP7R.

[12] Davis, M. and T. Januszkiewicz. "Convex polytopes, Coxeter orbifolds and torus actions." *Duke Math. J.* 62 (1991): 417–51.

[13] Farrell, F. T. and S. Zdravkovska. "Do almost flat manifolds bound?" *Michigan Math. J.* 30 (1983): 199–208.

[14] Garrison, A. and R. Scott. "Small covers of the dodecahedron and the 120-cell." *Proc. Amer. Math. Soc.* 131 (2003): 963–71.

[15] Gromov, M. "Manifolds of negative curvature." *J. Differ. Geom.* 13 (1987): 223–30.

[16] Gromov, M. "Almost flat manifolds." *J. Differ. Geom.* 13 (1978): 231–41.

[17] Guglielmetti, R. "Hyperbolic isometries in (in-)finite dimensions and discrete reflection groups: theory and computations." *PhD thesis no. 2008*, Université de Fribourg, 2017.

[18] Guglielmetti, R. "CoxIter – Computing invariants of hyperbolic Coxeter groups." *LMS J. Comput. Math.* 18 (2015): 754–73.

[19] Guglielmetti, R. "*AlVin: a C++ implementation of the Vinberg algorithm for diagonal quadratic forms*." https://rgugliel.github.io/AlVin/.

[20] Izmestiev, I. V. "Three-dimensional manifolds defined by coloring a simple polytope." *Math. Notes* 69 (2001): 340–6.

[21] Kaplinskaya, I. M. and È. B. Vinberg. "On the groups O_18,1 (Z) and O_19,1 (Z)." *Dokl. Akad. Nauk SSSR* 238 (1978): 1273–5.

[22] Kazhdan, D. A. and G. A. Margulis. "A proof of Selberg's hypothesis." *Math. Sbornik* 75 (1968): 162–8.

[23] Kolpakov, A., B. Martelli, and S. T. Tschantz. "Some hyperbolic three-manifolds that bound geometrically." *Proc. Am. Math. Soc.* 143 (2015): 4103–11.

[24] Kolpakov, A., A. Reid, and L. Slavich. "Embedding arithmetic hyperbolic manifolds." *Math. Res. Lett.* 25 (2018): 1305–28.

[25] Kolpakov, A., A. Reid, and S. Riolo. "Many cusped hyperbolic 3-manifolds do not bound geometrically." *Proc. Amer. Math. Soc.* 148 (2020): 2233–2243.

[26] Kolpakov, A. and S. Riolo. “Counting cusped hyperbolic three-manifolds that bound geometrically.” *Trans. Amer. Math. Soc.* 373 (2020): 229–247.

[27] Kolpakov, A. and L. Slavich. “Hyperbolic 4-manifolds, colourings and mutations.” *Proc. Lond. Math. Soc.* 113 (2016): 163–84.

[28] Long, D. D. and A. W. Reid. “On the geometric boundaries of hyperbolic 4-manifolds.” *Geom. Topol.* 4 (2000): 171–8.

[29] Long, D. D. and A. W. Reid. “All flat manifolds are cusps of hyperbolic orbifolds.” *Alg. Geom. Topol.* 2 (2002): 285–96.

[30] Long, D. D. and A. W. Reid. “Constructing hyperbolic manifolds which bound geometrically.” *Math. Res. Lett.* 8 (2001): 443–56.

[31] Long, D. D. and A. W. Reid. “Subgroup separability and virtual retractions of groups.” *Topology* 47 (2008): 137–59.

[32] Lubotzky, A. “Subgroup growth and congruence subgroups.” *Invent. Math.* 119 (1995): 267–95.

[33] Ma, J. and F. Zheng. “*Geometrically bounding 3-manifolds, volume and Betti numbers*.” arXiv:1704.02889.

[34] Maclachlan, C. and A. W. Reid. *The Arithmetic of Hyperbolic 3-Manifolds*. Graduate Texts in Mathematics, vol. 219. New York: Springer, 2003.

[35] Newman, M. “Integral Matrices.” In *Pure and Applied Mathematics*, vol. 45. New York: Academic Press, 1972.

[36] Nimershiem, B. E. “All flat three-manifolds appear as cusps of hyperbolic four-manifolds.” *Topol. Its Appl.* 90 (1998): 109–33.

[37] Potyagaĭlo, L. and È. Vinberg. “On right-angled reflection groups in hyperbolic spaces.” *Comment. Math. Helv.* 80 (2005): 63–73.

[38] Slavich, L. “A geometrically bounding hyperbolic link complement.” *Alg. Geom. Topol.* 15 (2015): 1175–97.

[39] Slavich, L. “The complement of the figure-eight knot geometrically bounds.” *Proc. Am. Math. Soc.* 145 (2017): 1275–85.

[40] Ratcliffe, J. G. and S. T. Tschantz. “Gravitational instantons of constant curvature.” *Class. Quantum Grav.* 15 (1998): 2613–27.

[41] Ratcliffe, J. G. “On the growth of the number of hyperbolic gravitational instantons with respect to volume.” *Class. Quantum Grav.* 17 (2000): 2999–3007.

[42] Vesnin, A. Y. “Three-dimensional hyperbolic manifolds of Löbell type.” *Siberian Math. J.* 28 (1987): 731–3.

[43] Vesnin, A. Y. “Three-dimensional hyperbolic manifolds with a common fundamental polyhedron.” *Math. Notes* 49 (1991): 575–7.

[44] Vinberg, È. B. “On groups of unit elements of certain quadratic forms.” *Math. USSR – Sbornik* 16 (1972): 17–35.

[45] Vinberg, È. B. and O. V. Shvartsman. “Discrete Groups of Motions of Spaces of Constant Curvature.” *Geometry II, Enc. Math. Sci*, vol. 29, 139–248. Berlin: Springer, 1993.

[46] Wang, H. C. "Topics on Totally Discontinuous Groups." In *Symmetric Spaces: Short Courses at Washington University*, 8, 459–87. New York: Marcel Dekker, 1993.

[47] Zimmermann, B. "Hurwitz groups and finite group actions on hyperbolic 3-manifolds." *J. Lond. Math. Soc.* 52 (1995): 199–208.

[48] Zimmermann, B. "A note on surfaces bounding hyperbolic 3-manifolds." *Monats. Math.* 142 (2004): 267–73.