

THE UNBOUNDED INTEGRALITY GAP OF A SEMIDEFINITE RELAXATION OF THE TRAVELING SALESMAN PROBLEM*

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Abstract. We study a semidefinite programming relaxation of the traveling salesman problem introduced by de Klerk, Pasechnik, and Sotirov [*SIAM J. Optim.*, 19 (2008), pp. 1559–1573] and show that their relaxation has an unbounded integrality gap. In particular, we give a family of instances such that the gap increases linearly with n . To obtain this result, we search for feasible solutions within a highly structured class of matrices; the problem of finding such solutions reduces to finding feasible solutions for a related linear program, which we do analytically. The solutions we find imply the unbounded integrality gap. Further, these solutions imply several corollaries that help us better understand the semidefinite program and its relationship to other TSP relaxations. Using the same technique, we show that a more general semidefinite program introduced by de Klerk, de Oliveira Filho, and Pasechnik [*Handbook on Semidefinite, Conic and Polynomial Optimization*, Springer, New York, 2012, pp. 171–199.] for the k -cycle cover problem also has an unbounded integrality gap.

Key words. traveling salesman problem, approximation algorithms, semidefinite programming

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1. Introduction. The traveling salesman problem (TSP) is one of the most famous problems in combinatorial optimization. An input to the TSP consists of a set of n cities $[n] := \{1, 2, \dots, n\}$ and edge costs c_{ij} for each pair of distinct $i, j \in [n]$ representing the cost of traveling from city i to city j . Given this information, the TSP is to find a minimum-cost tour visiting every city exactly once. Throughout this paper, we implicitly assume that the edge costs are *symmetric* (so that $c_{ij} = c_{ji}$ for all distinct $i, j \in [n]$) and *metric* (so that $c_{ij} \leq c_{ik} + c_{kj}$ for all distinct $i, j, k \in [n]$). Hence, we interpret the n cities as vertices of the complete undirected graph K_n with edge costs $c_e = c_{ij}$ for edge $e = \{i, j\}$. In this setting, the TSP is to find a minimum-cost Hamiltonian cycle on K_n .

The TSP is well known to be NP-hard. It is even NP-hard to approximate TSP solutions in polynomial time to within any constant factor $\alpha < \frac{123}{122}$ (see Karpinski, Lampis, and Schmied [22]). For the general TSP (without any assumptions beyond metric and symmetric edge costs), the state-of-the-art approximation algorithm remains Christofides' 1976 algorithm [4]. The output of Christofides' algorithm is at most a factor of $\frac{3}{2}$ away from the optimal solution to any TSP instance.

A broad class of approximation algorithms begin by relaxing the set of Hamiltonian cycles. The prototypical example is the subtour elimination linear program (also referred to as the Dantzig–Fulkerson–Johnson relaxation [6] and the Held–Karp bound [19], and which we will refer to as the subtour LP). Let $V = [n]$ denote the

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set of vertices in K_n and let E denote the set of edges in K_n . For $S \subset V$, denote the set of edges with exactly one endpoint in S by $\delta(S) := \{e = \{i, j\} : |\{i, j\} \cap S| = 1\}$ and let $\delta(v) := \delta(\{v\})$. The subtour elimination linear programming relaxation of the TSP is

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{subject to} \quad & \sum_{e \in \delta(v)} x_e = 2, \quad v = 1, \dots, n, \\ & \sum_{e \in \delta(S)} x_e \geq 2, \quad S \subset V : S \neq \emptyset, S \neq V, \\ & 0 \leq x_e \leq 1, \quad e = 1, \dots, n. \end{aligned}$$

The constraints $\sum_{e \in \delta(v)} x_e = 2$ are known as the degree constraints, while the constraints $\sum_{e \in \delta(S)} x_e \geq 2$ are known as the subtour elimination constraints. Wolsey [33] and Shmoys and Williamson [30] show that solutions to this linear program are also within a factor of $\frac{3}{2}$ of the optimal, integer solution to the TSP.

Instead of linear programming relaxations, another approach is to consider relaxations that are semidefinite programs (SDPs). This avenue is considered by Cvetković, Čangalović, and Kovačević-Vujčić [5]. They introduce an SDP relaxation that searches for solutions that meet the degree constraints and that are at least as connected as a cycle with respect to algebraic connectivity (see section 4.4). Goemans and Rendl [13], however, show that the SDP relaxation of Cvetković, Čangalović, and Kovačević-Vujčić [5] is weaker than the subtour LP in the following sense: any solution to the subtour LP implies an equivalent feasible solution for the SDP of Cvetković et al. of the same cost. Since both optimization problems are minimization problems, the optimal value for the SDP of Cvetković, Čangalović, and Kovačević-Vujčić cannot be closer than the optimal solution of the subtour LP to the optimal solution to the TSP.

More recently, de Klerk, Pasechnik, and Sotirov [9] introduced another SDP relaxation of the TSP. This SDP can be motivated and derived through a general framework for SDP relaxations based on the theory of association schemes (see de Klerk, de Oliveira Filho, and Pasechnik [7]). Moreover, de Klerk, Pasechnik, and Sotirov [9] show computationally that this new SDP is incomparable to the subtour LP: there are cases for which their SDP provides a closer approximation to the TSP than the subtour LP and vice versa! Moreover, de Klerk, Pasechnik, and Sotirov [9] show that their SDP is stronger than the earlier SDP of Cvetković, Čangalović, and Kovačević-Vujčić [5]: any feasible solution for the SDP of de Klerk, Pasechnik, and Sotirov [9] implies a feasible solution for the SDP of Cvetković, Čangalović, and Kovačević-Vujčić [5] of the same cost.

We analyze the SDP relaxation of de Klerk, Pasechnik, and Sotirov [9]; our main result is that the integrality gap of this SDP is unbounded. To show this result, we introduce a family of instances corresponding to a *cut semimetric*: a subset $S \subset V$ such that $c_{ij} = 1$ if $\{i, j\} \in \delta(S)$ and $c_{ij} = 0$ otherwise. We will take $|S| = \frac{n}{2}$. Equivalently, $n/2$ of the cities are located at the point $(0) \in \mathbb{R}^1$, the remaining $n/2$ cities are located at $(1) \in \mathbb{R}^1$, and the cost c_{ij} is the Euclidean distance between the locations of city i and city j . We show that for these instances the integrality gap grows linearly in n . The feasible solutions we introduce to bound the integrality gap, moreover, have the same algebraic connectivity as a Hamiltonian cycle on n vertices, even though their cost becomes arbitrarily far from that of a Hamiltonian cycle (see section 4.4) as n grows.

We introduce the SDP of de Klerk, Pasechnik, and Sotirov [9] in section 2. In section 3 we discuss our motivations and prove our result. The crux of our argument involves exploiting the symmetry of the instances we introduce. We consider a candidate class of solutions to the SDP respecting this symmetry and show that members

of this class are feasible solutions to the SDP if and only if they are feasible solutions for a simpler linear program, whose constraints enforce certain positive semidefinite inequalities. We then analytically find solutions to this linear program and show that these solutions imply the unbounded integrality gap. Next, in section 4, we discuss several corollaries of our main result. These corollaries shed new light on how the SDP relates to the subtour LP as well as to the earlier SDP of Cvetković, Čangalović, and Kovačević-Vujčić [5]. In section 5, we apply our technique for showing that the integrality gap is unbounded to a generalization of the SDP of de Klerk, Pasechnik, and Sotirov [9] for the minimum-cost k -cycle cover problem; when $k = 1$, this problem is exactly the same as the TSP. This more general SDP was introduced in de Klerk, de Oliveira Filho, and Pasechnik [7], and we show that it also has an unbounded integrality gap.

This work is related in spirit to Goemans and Rendl [14] and to de Klerk and Dobre [8]. Goemans and Rendl [14] study how to solve SDPs arising from association schemes using a linear program. Specifically, they show that an SDP of the form

$$\max \langle M_0, X \rangle \text{ subject to } \langle M_j, X \rangle = b_j \text{ for } j = 1, \dots, m, \quad X \succeq 0,$$

where the M_j are fixed input matrices forming an association scheme, can be solved using a linear program. Like Goemans and Rendl [14], the SDP we study is related to an association scheme and we obtain a result using a linear program. In contrast, however, to having input matrices that form an association scheme, the SDP we analyze seeks solutions that satisfy many properties of a certain, fixed association scheme (in particular, de Klerk, de Oliveira Filho, and Pasechnik [7] shows that the constraints of the SDP are satisfied by the association scheme corresponding to cycles; see section 2). Moreover, we only use a linear program to find feasible solutions to this SDP that are sufficient to imply an unbounded integrality gap: this SDP does not in general reduce to the LP we use.

De Klerk and Dobre [8] study the SDP of de Klerk, Pasechnik, and Sotirov [9] in a different context: that of the symmetric circulant TSP (SCTSP). In this setting, the matrix of edge costs $(c_{ij})_{i,j=1}^n$ (with $c_{ii} = 0$) is required to be a symmetric, circulant matrix (see section 2.2). The symmetry of such matrices allows de Klerk and Dobre [8] to write the SDP of de Klerk, Pasechnik, and Sotirov [9] as a linear program. In contrast, the specific instance we present does not have a circulant cost matrix and we use linear programming to find a single feasible solution.

2. A semidefinite programming relaxation of the TSP.

2.1. Notation and preliminaries. Throughout this paper we will use standard notation from linear algebra. We use J_m and I_m to denote the all-ones and identity matrices in $\mathbb{R}^{m \times m}$, respectively. When clear from context, we suppress the dependency on the dimension and just write J and I . We denote by e the column vector of all ones, so that $J = ee^T$. Also, we use S^m for the set of real, symmetric matrices in $\mathbb{R}^{m \times m}$ and $\otimes$ to denote the Kronecker product of matrices. $A \succeq B$ denotes that $A - B$ is a positive semidefinite (PSD) matrix (we will generally have A, B symmetric, in which case positive semidefiniteness is equivalent to all eigenvalues of $A - B$ being nonnegative). The trace of a matrix A , denoted $\text{trace}(A)$, is the sum of its diagonal entries so that for $A, B \in S^m$, $\text{trace}(AB) = \sum_{i=1}^m \sum_{j=1}^m A_{ij}B_{ij}$. $A \geq 0$ means that each entry of matrix A is nonnegative.

Our main result addresses the *integrality gap* of a relaxation, which represents the worst-case ratio of the original problem's optimal solution to the relaxation's

optimal solution. We are specifically interested in the gap of the SDP of de Klerk, Pasechnik, and Sotirov [9]; we will refer to this SDP as simply “the SDP” throughout. Let C denote a matrix of edge costs, so that $C = (c_{ij})$ is a symmetric matrix with diagonal entries equal to zero. Let $\text{OPT}_{\text{SDP}}(C)$ and $\text{OPT}_{\text{TSP}}(C)$ respectively denote the optimal solutions to the SDP and to the TSP for a given matrix of costs C . The integrality gap is then

$$\sup_C \frac{\text{OPT}_{\text{TSP}}(C)}{\text{OPT}_{\text{SDP}}(C)},$$

where we take the supremum over all valid cost matrices C (those whose constituent costs are metric and symmetric). This ratio is bounded below by 1, since the SDP is a relaxation of the TSP; we rederive this fact in section 2.2. We will show that the ratio cannot be bounded above by any constant. In contrast, the results we noted previously about the subtour LP imply that its integrality gap is bounded above by $\frac{3}{2}$.

Throughout the remainder of this paper we will take n to be even and let $d = \frac{n}{2}$. We use $A \setminus B$ for set minus notation, so that $A \setminus B = \{a \in A : a \notin B\}$. We take $x \in \mathbb{R}^{\binom{n}{2}}$ to mean that x is a vector whose entries are indexed by the edges of K_n .

2.2. Facts about the SDP. The SDP introduced by de Klerk, Pasechnik, and Sotirov [9] uses d matrix variables $X^{(1)}, \dots, X^{(d)} \in \mathbb{R}^{n \times n}$, with the cost of a solution depending only on $X^{(1)}$. It is

$$\begin{aligned} (1) \quad & \min \quad \frac{1}{2} \text{trace}(CX^{(1)}) \\ & \text{subject to} \quad X^{(k)} \geq 0, & k = 1, \dots, d, \\ & \sum_{j=1}^d X^{(j)} = J - I, \\ & I + \sum_{j=1}^d \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0, & k = 1, \dots, d, \\ & X^{(k)} \in S^n, & k = 1, \dots, d. \end{aligned}$$

Both de Klerk, Pasechnik, and Sotirov [9] and de Klerk, de Oliveira Filho, and Pasechnik [7] show that this is a relaxation of the TSP by showing that the following solution is feasible. For a simple, undirected graph G , let $A_k(G)$ be the k th distance matrix: the matrix with i, j th entry equal to 1 if and only if the shortest path between vertices i and j in G is of distance k , and equal to 0 otherwise. Let $\mathcal{C}_n$ be a cycle of length n (i.e., any Hamiltonian cycle on $[n]$). The solution where $X^{(k)} = A_k(\mathcal{C}_n)$ for $k = 1, \dots, d$ is feasible for the SDP (see Proposition 2.1). Hence, the optimal integer solution to the TSP has a corresponding feasible solution to the SDP. That SDP solution has the same value as the optimal integer solution to the TSP: each edge $e = \{i, j\}$ is represented twice in $X^{(1)}$ as both $X_{ij}^{(1)}$ and $X_{ji}^{(1)}$, but this is accounted for by the factor $\frac{1}{2}$ in the objective function.

These solutions are shown to be feasible in de Klerk, Pasechnik, and Sotirov [9] by noting that the $A_k(\mathcal{C}_n)$ form an association scheme and are therefore simultaneously diagonalizable. This allows for the positive semidefinite inequalities to be verified after computing the eigenvalues of each $A_k(\mathcal{C}_n)$. A more systematic approach is taken in de Klerk, de Oliveira Filho, and Pasechnik [7], where they introduce general results about association schemes. The constraints of the SDP then represent an application of these results to a specific association scheme: that of the distance matrices $A_k(\mathcal{C}_n)$. We begin by providing a new, direct proof that the SDP is a relaxation of the TSP.

PROPOSITION 2.1 (de Klerk, Pasechnik, and Sotirov [9]). *Setting $X^{(j)} = A_j(\mathcal{C}_n)$ for $j = 1, \dots, d$ yields a feasible solution to the SDP (1).*

We will use two lemmas in our proof. First, the main work in our proof involves showing that the positive semidefinite inequalities from (1) hold. We do so by noticing that $I + \sum_{j=1}^d \cos(\frac{2\pi jk}{n}) A_j(\mathcal{C}_n)$ has a very specific structure: that of a *circulant* matrix. A circulant matrix is a matrix of the form

$$M = \begin{pmatrix} m_0 & m_1 & m_2 & m_3 & \cdots & m_{n-1} \\ m_{n-1} & m_0 & m_1 & m_2 & \cdots & m_{n-2} \\ m_{n-2} & m_{n-1} & m_0 & m_1 & \ddots & m_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ m_1 & m_2 & m_3 & m_4 & \cdots & m_0 \end{pmatrix} = (m_{(s-t) \bmod n})_{s,t=1}^n.$$

The eigenvalues of circulant matrices are well understood, which will allow us to show that $I + \sum_{j=1}^d \cos(\frac{2\pi jk}{n}) A_j(\mathcal{C}_n)$ is a positive semidefinite matrix for each k by computing the eigenvalues of that linear combination. In particular, we have the following lemma.

LEMMA 2.2 (Gray [16]). *The circulant matrix $M = (m_{(s-t) \bmod n})_{s,t=1}^n$ has eigenvalues*

$$\lambda_t(M) = \begin{cases} \sum_{s=0}^{n-1} m_s e^{-\frac{2\pi s t \sqrt{-1}}{n}} & \text{if } t = 1, \dots, n-1, \\ \sum_{s=0}^{n-1} m_s & \text{if } t = n. \end{cases}$$

This is the only section where we will work with imaginary numbers, and to avoid ambiguity with index variables, we explicitly write $\sqrt{-1}$ and reserve i and j as index variables.

Our second lemma is a trigonometric identity that we will use repeatedly in later proofs.

LEMMA 2.3. *Let n be even and $0 < k < n$ be an integer. Then*

$$\sum_{j=1}^d \cos\left(\frac{2\pi jk}{n}\right) = \frac{-1 + (-1)^k}{2}.$$

Proof. Our identity is a consequence of Lagrange's trigonometric identity (see, e.g., Identity 14 in section 2.4.1.6 of Jeffrey and Dai [21]), which states, for $0 < \theta < 2\pi$, that

$$\sum_{j=1}^m \cos(j\theta) = -\frac{1}{2} + \frac{\sin((m + \frac{1}{2})\theta)}{2 \sin(\frac{\theta}{2})}.$$

Taking $\theta = \frac{2\pi k}{n}$ and using $n = 2d$, we obtain

$$\begin{aligned} \sum_{j=1}^d \cos\left(\frac{2\pi k}{n} j\right) &= -\frac{1}{2} + \frac{\sin(\pi k + \frac{\pi k}{n})}{2 \sin \frac{\pi k}{n}} \\ &= -\frac{1}{2} + (-1)^k \frac{1}{2}, \end{aligned}$$

where we recall that $\sin(\pi + \theta) = -\sin(\theta)$. □

Notice that when $k = 0$ or $k = n$, the sum is d .

Proof of Proposition 2.1. We first remark that each $A_j(\mathcal{C}_n)$ is a nonnegative symmetric matrix. Moreover, $\sum_{j=1}^d A_j(\mathcal{C}_n) = J - I$. This follows because, in $\mathcal{C}_n$, the

shortest path between any pair of distinct vertices $u, v \in [n]$ is a unique element s of the set $[d]$. Hence, exactly one of the terms in the sum $\sum_{j=1}^d A_j(\mathcal{C}_n)$ has a one in its u, v entry, and all other terms have a zero. The diagonals of each $A_j(\mathcal{C}_n)$ consist of all zeros, since the shortest path from vertex i to itself has length $0 \notin [d]$.

Now for any fixed $k \in [d]$ we compute the eigenvalues of the matrix

$$M := I + \sum_{j=1}^d \cos\left(\frac{2\pi jk}{n}\right) A_j(\mathcal{C}_n).$$

First, suppose the vertices are labeled so that the cycle $\mathcal{C}_n$ is $1, 2, 3, \dots, n-1, n, 1$. We will later note why this is without loss of generality.

Then M is circulant with, for $j = 1, \dots, d$, entries m_j and m_{n-j} given exactly by the coefficient of the j th term in the sum. Namely,

$$m_0 = 1, \quad m_d = \cos\left(\frac{2\pi kd}{n}\right), \quad m_j = m_{n-j} = \cos\left(\frac{2\pi jk}{n}\right), \quad j = 1, \dots, d-1.$$

We can directly compute the t th eigenvalue of M using Lemma 2.2. Our later proofs will include similar computations, so we pay particular emphasis to the details of our algebraic manipulation. For $t = 1, \dots, n-1$, the t th eigenvalue of M is

$$\begin{aligned} \lambda_t(M) &= \sum_{s=0}^{n-1} m_s e^{-\frac{2\pi s t \sqrt{-1}}{n}} \\ &= 1 + \cos\left(\frac{2\pi kd}{n}\right) e^{-\frac{2\pi d t \sqrt{-1}}{n}} + \sum_{s=1}^{d-1} \cos\left(\frac{2\pi s k}{n}\right) \left(e^{-\frac{2\pi s t \sqrt{-1}}{n}} + e^{-\frac{2\pi (n-s) t \sqrt{-1}}{n}}\right), \end{aligned}$$

where we have first written the terms when $s = 0$ and $s = d$. We rewrite terms so that our sum is to d and simplify exponentials:

$$\begin{aligned} \lambda_t(M) &= 1 - \cos\left(\frac{2\pi kd}{n}\right) e^{\frac{2\pi d t \sqrt{-1}}{n}} + \sum_{s=1}^d \cos\left(\frac{2\pi s k}{n}\right) \left(e^{-\frac{2\pi s t \sqrt{-1}}{n}} + e^{\frac{2\pi s t \sqrt{-1}}{n}}\right) \\ &= 1 - (-1)^k (-1)^t + 2 \sum_{s=1}^d \cos\left(\frac{2\pi s k}{n}\right) \cos\left(\frac{2\pi s t}{n}\right). \end{aligned}$$

Recalling the product-to-sum identity for cosines (that $2 \cos(\theta) \cos(\phi) = \cos(\theta + \phi) + \cos(\theta - \phi)$), we get

$$\lambda_t(M) = 1 - (-1)^{k+t} + \sum_{s=1}^d \cos\left(\frac{2\pi s}{n}(k+t)\right) + \sum_{s=1}^d \cos\left(\frac{2\pi s}{n}(k-t)\right).$$

Using Lemma 2.3 and $(-1)^{k+t} = (-1)^{k-t}$, we have

$$\lambda_t(M) = \begin{cases} 1 - (-1)^{2d} + 2d & \text{if } k = t = d, \\ -\frac{1}{2} + (-1)^{k+t} \frac{1}{2} + d & \text{if } k \neq d, t \in \{k, n-k\}, \\ 1 - (-1)^{k+t} - \frac{1}{2} + (-1)^{k+t} \frac{1}{2} - \frac{1}{2} + (-1)^{k-t} \frac{1}{2} & \text{otherwise} \end{cases}$$

$$= \begin{cases} 2d & \text{if } k = t = d, \\ d & \text{if } k \neq d, t \in \{k, n-k\}, \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalue λ_n is

$$\begin{aligned}\lambda_n(M) &= \sum_{s=0}^{n-1} m_s \\ &= 1 - \cos\left(\frac{2\pi kd}{n}\right) + 2 \sum_{s=1}^d \cos\left(\frac{2\pi sk}{n}\right) \\ &= 1 - (-1)^k - 1 + (-1)^k \\ &= 0.\end{aligned}$$

The matrix M thus has all nonnegative eigenvalues, so the positive semidefinite constraints hold for each $k \in \{1, \dots, d\}$.

Finally, we note that our assumption that the cycle $\mathcal{C}_n$ is $1, 2, 3, \dots, n-1, n, 1$ was without loss of generality: we can replace the $A_j(\mathcal{C}_n)$ with $P^T A_j(\mathcal{C}_n) P = P^{-1} A_j(\mathcal{C}_n) P$ for a permutation matrix P that permutes the labels of the vertices so that the cycle is $1, 2, 3, \dots, n-1, n, 1$. Then M and $P^{-1} M P$ are similar matrices and share the same spectrum. Thus M is positive semidefinite if and only if $P^{-1} M P$ is positive semidefinite; $P^{-1} M P$ is the circulant matrix above, with

$$m_0 = 1, \quad m_d = \cos\left(\frac{2\pi kd}{n}\right), \quad m_j = m_{n-j} = \cos\left(\frac{2\pi jk}{n}\right), \quad j = 1, \dots, d-1,$$

and thus both $P^{-1} M P$ and M are positive semidefinite. $\square$

We briefly remark that de Klerk, Pasechnik, and Sotirov [9] also use the eigenvalue properties of circulant matrices in proving that the SDP is a relaxation of the TSP. They use the fact that each individual $A_k(\mathcal{C}_n)$ is circulant to compute the eigenvalues of each $A_k(\mathcal{C}_n)$, while we use the fact that the linear combination of those matrices denoted above by M is circulant.

3. The unbounded integrality gap. To show that the SDP has an arbitrarily bad integrality gap, we demonstrate a family of instances of edge costs for which we can upper bound the SDP's objective value. We consider an instance with two groups of $n/2$ vertices. The costs associated with intergroup edges will be expensive (1), while the costs of intragroup edges will be negligible (0). As noted in the introduction, this instance is equivalent to both a cut semimetric and an instance where the costs are given by Euclidean distances in $\mathbb{R}^1$. Explicitly, we will use the cost matrix

$$\hat{C} := \begin{pmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d.$$

Notice that the edge costs embedded in this matrix are metric.

Throughout this paper, we reserve U and W to refer to the two groups of vertices, so that $|U| = |W| = d$ and $V = U \cup W$. In a Hamiltonian cycle $\delta(U) \geq 2$, so that any feasible solution to the TSP must use the expensive intergroup edges at least twice. We can achieve a tour costing 2 with a tour that starts in U , goes through all the

vertices in U , crosses to W , goes through the vertices in W , and then returns to U . Hence $\text{OPT}_{\text{TSP}}(\hat{C}) = 2$.

We state our main result, the proof of which culminates in section 3.3.

THEOREM 3.1.

$$\text{OPT}_{\text{SDP}}(\hat{C}) \leq \frac{\pi^2}{2n} \text{OPT}_{\text{TSP}}(\hat{C}).$$

As a consequence, we have the following corollary.

COROLLARY 3.2. *The SDP (1) has an unbounded integrality gap. That is, there exists no constant $\alpha > 0$ such that*

$$\frac{\text{OPT}_{\text{TSP}}(C)}{\text{OPT}_{\text{SDP}}(C)} \leq \alpha$$

for all cost matrices C .

To prove Theorem 3.1, we construct a family of feasible SDP solutions whose cost becomes arbitrarily small as n grows. We will specifically search for solutions respecting the symmetry of $\hat{C}$: matrices $X^{(j)}$ that place a weight of a_j on each intragroup edge and a weight of b_j on each intergroup edge. Moreover, we choose¹ the b_i so as to enforce that the row sums of the $X^{(j)}$ match those of the distance matrices $A_j(\mathcal{C}_n)$ introduced earlier: $X^{(j)}e = A_j(\mathcal{C}_n)e = 2e$ for $j = 1, \dots, d-1$ and $X^{(d)}e = A_d(\mathcal{C}_n)e = e$. Since every vertex is incident to $d-1$ edges in its group (with weight a_i) and d edges in the other group (with weight b_i), we have

$$(d-1)a_i + db_i = \begin{cases} 2 & \text{if } i = 1, \dots, d-1, \\ 1 & \text{if } i = d. \end{cases}$$

Rearranging for the b_i lets us express the j th solution matrix of this form as

$$(2) \quad X^{(j)} = \left(\begin{pmatrix} a_j & b_j \\ b_j & a_j \end{pmatrix} \otimes J_d \right) - a_j I_n, \quad b_j = \begin{cases} \frac{4}{n} - \left(1 - \frac{2}{n}\right) a_j & \text{if } j = 1, \dots, d-1, \\ \frac{2}{n} - \left(1 - \frac{2}{n}\right) a_j & \text{if } j = d, \end{cases}$$

where we subtract $a_j I_n$ so that the diagonal is zero. The cost of such a solution is entirely determined by the $(n/2)^2$ intergroup edges, each of cost b_1 . Each edge is accounted for twice in $\text{trace}(\hat{C}X^{(1)})$, but the objective scales by $1/2$, so the cost of this solution is

$$\left(\frac{n}{2}\right)^2 b_1.$$

Theorem 3.1 will then follow from the claim below.

CLAIM 3.3. *Choosing the parameters*

$$a_i = \frac{2}{n-2} \left(\cos\left(\frac{\pi i}{d}\right) + 1 \right), \quad i = 1, \dots, d,$$

so that

$$b_i = \begin{cases} \frac{2}{n} (1 - \cos(\frac{\pi i}{d})) & \text{if } i = 1, \dots, d-1, \\ \frac{2}{n} & \text{if } i = d, \quad i = 1, \dots, d, \end{cases}$$

leads to a feasible solution for the SDP (1) with matrices $X^{(j)}$ as given in (2).

¹Note that de Klerk, Pasechnik, and Sotirov [9] actually show that every feasible solution must satisfy $X^{(i)}e = 2e$ for $i = 1, \dots, d-1$ and $X^{(i)}e = e$ for $i = d$ (when n is even). The fact that every feasible solution matches these row sums is not something we will need, though we implicitly use it to inform the solutions we search for. We provide an alternative, direct proof that all feasible solutions must satisfy these row sums in the extended version of this paper Gutekunst and Williamson [18].

In particular $b_1 = \frac{2}{n}(1 - \cos(\frac{\pi}{d}))$. Basic facts from calculus will show that this is roughly $\frac{1}{n^3}$, so that the cost of our solution is $(n/2)^2 b_1$, which is roughly $\frac{1}{n}$, which gets arbitrarily small with n .

The main work in proving this claim involves showing that the $X^{(j)}$ satisfy the PSD constraints. We first characterize the choices of the a_i that lead to feasible SDP solutions of the form in (2); this is done in section 3.1. There we exploit the structure of matrices in the form of (2) to write the PSD constraints on the $X^{(j)}$ as linear constraints on the a_i ; these linear constraints will imply that all eigenvalues of the term $I + \sum_{i=1}^d \cos(\frac{2\pi ik}{n}) X^{(i)}$ are nonnegative. To finish proving the claim, in section 3.2 we show that the claimed a_i are indeed feasible.

3.1. Finding structured solutions to the SDP via linear programming.

In this section we prove the following proposition.

PROPOSITION 3.4. *For the SDP (1), finding a minimum-cost feasible solution of the form*

$$X^{(j)} = \left(\begin{pmatrix} a_j & b_j \\ b_j & a_j \end{pmatrix} \otimes J_d \right) - a_j I_n, \quad \text{where } b_j = \begin{cases} \frac{4}{n} - (1 - \frac{2}{n}) a_j & \text{if } j = 1, \dots, d-1, \\ \frac{2}{n} - (1 - \frac{2}{n}) a_j & \text{if } j = d, \end{cases}$$

for $j = 1, \dots, d$ is equivalent to solving the following linear program:

$$\begin{array}{ll} \max & a_1 \\ \text{subject to} & \sum_{i=1}^d \cos\left(\frac{2\pi ik}{n}\right) a_i \geq -\frac{2}{n-2}, \quad k = 1, \dots, d, \\ & \sum_{i=1}^d \cos\left(\frac{2\pi ik}{n}\right) a_i \leq 1, \quad k = 1, \dots, d, \\ & \sum_{i=1}^d a_i = 1, \\ & a_i \leq \frac{4}{n-2}, \quad i = 1, \dots, d-1, \\ & a_d \leq \frac{2}{n-2}, \\ & a_i \geq 0, \quad i = 1, \dots, d. \end{array}$$

Proof. First we notice that maximizing a_1 is equivalent to minimizing b_1 , which is in turn equivalent to minimizing the cost $(\frac{n}{2})^2 b_1$ of the SDP solution. The $X^{(i)}$ are nonnegative if and only if $a_i \geq 0$, $b_i \geq 0$ for $i = 1, \dots, d$. The constraints $a_i \geq 0$ are explicit in the linear program, and $b_i \geq 0$ is equivalent to $a_i \leq \frac{4}{n-2}$, $i = 1, \dots, d-1$ and $a_d \leq \frac{2}{n-2}$. Finally, the constraint that the $X^{(j)}$ sum to $J - I$ is equivalent to $\sum_{i=1}^d a_i = 1$ and $\sum_{i=1}^d b_i = 1$. However, $\sum_{i=1}^d b_i = 1$ follows from requiring that $\sum_{i=1}^d a_i = 1$:

$$\begin{aligned} \sum_{i=1}^d b_i &= \sum_{i=1}^{d-1} \left(\frac{4}{n} - \left(1 - \frac{2}{n}\right) a_i \right) + \left(\frac{2}{n} - \left(1 - \frac{2}{n}\right) a_d \right) \\ &= (d-1) \frac{4}{n} + \frac{2}{n} - \left(1 - \frac{2}{n}\right) \sum_{i=1}^d a_i \\ &= 2 - \frac{2}{n} - \left(1 - \frac{2}{n}\right) \\ &= 1. \end{aligned}$$

It remains to show that the k th SDP constraint is equivalent to

$$-\frac{2}{n-2} \leq \sum_{i=1}^d \cos\left(\frac{2\pi ik}{n}\right) a_i \leq 1, \quad k = 1, \dots, d.$$

The k th SDP constraint is

$$I + \sum_{i=1}^d \cos\left(\frac{2\pi ik}{n}\right) X^{(i)} \succeq 0.$$

Using properties of the Kronecker product (see Chapter 4 of Horn and Johnson [20]) and the structure of our $X^{(j)}$, we simplify this:

$$\begin{aligned} I_n + \sum_{i=1}^d \cos\left(\frac{2\pi ik}{n}\right) X^{(i)} &= I_n + \sum_{i=1}^d \cos\left(\frac{2\pi ik}{n}\right) \left(\begin{pmatrix} a_i & b_i \\ b_i & a_i \end{pmatrix} \otimes J_d - a_i I_n \right) \\ &= \left(1 - \sum_{i=1}^d \cos\left(\frac{2\pi ik}{n}\right) a_i \right) I_n + \left(\sum_{i=1}^d \cos\left(\frac{2\pi ik}{n}\right) \begin{pmatrix} a_i & b_i \\ b_i & a_i \end{pmatrix} \right) \otimes J_d \\ &= (1 - a^{(k)}) I_n + \begin{pmatrix} a^{(k)} & b^{(k)} \\ b^{(k)} & a^{(k)} \end{pmatrix} \otimes J_d, \end{aligned}$$

where

$$a^{(k)} = \sum_{i=1}^d \cos\left(\frac{2\pi ik}{n}\right) a_i, \quad b^{(k)} = \sum_{i=1}^d \cos\left(\frac{2\pi ik}{n}\right) b_i$$

depend on the full sequences $a_1, \dots, a_d, b_1, \dots, b_d$ and on k .

To explicitly write the eigenvalues of the k th SDP constraint, we use several helpful facts from linear algebra.

- The pq eigenvalues of $A \otimes B$ with $A \in \mathbb{R}^{p \times p}$ and $B \in \mathbb{R}^{q \times q}$ are $\lambda_i(A)\lambda_j(B)$ for $i = 1, \dots, p$ and $j = 1, \dots, q$. See Theorem 4.2.12 in Chapter 4 of Horn and Johnson [20].
- The rank one matrix $J_d = ee^T$, with e of dimension d , has one eigenvalue d corresponding to eigenvector e , and all other eigenvalues are zero. (Choose, e.g., any $d-1$ linearly independent vectors that are orthogonal to e .)
- $\lambda(A)$ is an eigenvalue of A with eigenvector v if and only if $\lambda(A) + c$ is an eigenvalue of $A + cI$ with eigenvector v . This follows by direct computation.
- The eigenvalues of $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ are $a+b$ and $a-b$ with respective eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

From these facts, we obtain that the eigenvalues of $I + \sum_{i=1}^d \cos(\frac{2\pi ik}{n}) X^{(i)}$ are

$$1 - a^{(k)}, \quad 1 - a^{(k)} + \frac{n}{2} (a^{(k)} + b^{(k)}), \quad \text{and} \quad 1 - a^{(k)} + \frac{n}{2} (a^{(k)} - b^{(k)}).$$

For example, $1 - a^{(k)}$ has multiplicity $n-2$. It corresponds to the $d-1$ zero eigenvalues of J_d , each of which gives rise to 2 zero eigenvalues of

$$\begin{pmatrix} a^{(k)} & b^{(k)} \\ b^{(k)} & a^{(k)} \end{pmatrix} \otimes J_d.$$

Therefore, for the k th PSD constraint of the SDP (1) to hold, it suffices that the following three linear inequalities hold:

$$(3) \quad 1 - a^{(k)} \geq 0, \quad 1 - a^{(k)} + \frac{n}{2} (a^{(k)} + b^{(k)}) \geq 0, \quad 1 - a^{(k)} + \frac{n}{2} (a^{(k)} - b^{(k)}) \geq 0.$$

We have thus derived a system of inequalities on the a_i , b_i that, if satisfied, imply a set of feasible solutions to the SDP. We can further simplify these by writing the b_i in terms of the a_i . As in Proposition 2.1, we begin by writing the sum so that we can use Lemma 2.3. We compute

$$\begin{aligned} b^{(k)} &= \sum_{i=1}^d \cos\left(\frac{2\pi i k}{n}\right) b_i \\ &= \left(\sum_{i=1}^{d-1} \cos\left(\frac{2\pi i k}{n}\right) \left(\frac{4}{n} - \left(1 - \frac{2}{n}\right) a_i\right)\right) + \cos\left(\frac{2\pi d k}{n}\right) \left(\frac{2}{n} - \left(1 - \frac{2}{n}\right) a_d\right) \\ &= \frac{4}{n} \left(\sum_{i=1}^d \cos\left(\frac{2\pi i k}{n}\right)\right) - \left(1 - \frac{2}{n}\right) \left(\sum_{i=1}^d \cos\left(\frac{2\pi i k}{n}\right) a_i\right) - \cos(\pi k) \left(\frac{2}{n}\right). \end{aligned}$$

Using Lemma 2.3, we have

$$\begin{aligned} b^{(k)} &= \frac{4}{n} \left(\frac{-1 + (-1)^k}{2}\right) - \left(1 - \frac{2}{n}\right) a^{(k)} - (-1)^k \left(\frac{2}{n}\right) \\ &= -\left(1 - \frac{2}{n}\right) a^{(k)} - \frac{2}{n}. \end{aligned}$$

We use this relationship to simplify the second and third inequalities in (3) by writing them only in terms of $a^{(k)}$. We obtain

$$1 - a^{(k)} + \frac{n}{2} (a^{(k)} + b^{(k)}) = 1 - a^{(k)} + \frac{n}{2} \left(a^{(k)} - \left(1 - \frac{2}{n}\right) a^{(k)} - \frac{2}{n}\right) = 0$$

and

$$1 - a^{(k)} + \frac{n}{2} (a^{(k)} - b^{(k)}) = 1 - a^{(k)} + \frac{n}{2} \left(a^{(k)} + \left(1 - \frac{2}{n}\right) a^{(k)} + \frac{2}{n}\right) = 2 + (n-2)a^{(k)}.$$

Hence, the three inequalities in (3) become

$$-\frac{2}{n-2} \leq a^{(k)} \leq 1,$$

and these inequalities are equivalent to ensuring that the k th PSD constraint of the SDP in (1) holds. $\square$

COROLLARY 3.5. *Consider a possible solution to the SDP (1) of the form*

$$X^{(j)} = \left(\begin{pmatrix} a_j & b_j \\ b_j & a_j \end{pmatrix} \otimes J_d \right) - a_j I_n, \quad \text{where } b_j = \begin{cases} \frac{4}{n} - \left(1 - \frac{2}{n}\right) a_j & \text{if } j = 1, \dots, d-1, \\ \frac{2}{n} - \left(1 - \frac{2}{n}\right) a_j & \text{if } j = d. \end{cases}$$

The k th PSD constraint $I + \sum_{j=1}^d \cos\left(\frac{2\pi j k}{n}\right) X^{(j)} \succeq 0$ is equivalent to $-\frac{2}{n-2} \leq a^{(k)} \leq 1$.

3.2. Analytically finding solutions to the linear program. We now show that the following choice of the a_i leads to $X^{(j)}$ that are feasible for the SDP (1):

$$a_i = \frac{2}{n-2} \left(\cos \left(\frac{\pi i}{d} \right) + 1 \right), \quad i = 1, \dots, d.$$

As argued above, to show feasibility we need only verify that the constraints of the linear program in Proposition 3.4 hold. Notice that $-1 \leq \cos(\pi i/d) \leq 1$ so that, for $i = 1, \dots, d-1$, we have $0 \leq a_i \leq \frac{4}{n-2}$. Moreover, $a_d = 0$. Hence we need only show that $\sum_{i=1}^d a_i = 1$ and that the $a^{(k)}$ live in the appropriate range.

CLAIM 3.6. For $a_i = \frac{2}{n-2} (\cos(\frac{\pi i}{d}) + 1)$,

$$\sum_{i=1}^d a_i = 1.$$

Proof. We directly compute $\sum_{i=1}^d a_i$ using Lemma 2.3 with $k = 1$. Then we have

$$\sum_{i=1}^d a_i = \frac{2}{n-2} \sum_{i=1}^d \left(\cos \left(\frac{\pi i}{d} \right) + 1 \right) = \frac{2}{n-2} (-1 + d) = 1. \quad \square$$

CLAIM 3.7. For $a_i = \frac{2}{n-2} (\cos(\frac{\pi i}{d}) + 1)$,

$$a^{(k)} = \begin{cases} \frac{d-2}{n-2} & \text{if } k = 1, \\ -\frac{2}{n-2} & \text{otherwise.} \end{cases}$$

Proof. As in Proposition 2.1, we use the product-to-sum identity for cosines and then do casework using Lemma 2.3. We have

$$\begin{aligned} a^{(k)} &= \sum_{i=1}^d \cos \left(\frac{2\pi i k}{n} \right) a_i \\ &= \frac{2}{n-2} \sum_{i=1}^d \left(\cos \left(\frac{2\pi i k}{n} \right) + \cos \left(\frac{2\pi i k}{n} \right) \cos \left(\frac{\pi i}{d} \right) \right) \\ &= \frac{2}{n-2} \sum_{i=1}^d \left(\cos \left(\frac{2\pi i k}{n} \right) + \frac{1}{2} \cos \left(\frac{2\pi i(k+1)}{n} \right) + \frac{1}{2} \cos \left(\frac{2\pi i(k-1)}{n} \right) \right). \end{aligned}$$

We can apply Lagrange's trigonometric identity except for when $k = 1$, so that

$$\begin{aligned} a^{(k)} &= \begin{cases} \frac{2}{n-2} \left(\frac{-1+(-1)^k}{2} + \frac{-1+(-1)^{k+1}}{4} + \frac{-1+(-1)^{k-1}}{4} \right) & \text{if } k > 1, \\ \frac{2}{n-2} (-1 + 0 + \frac{1}{2}d) & \text{if } k = 1 \end{cases} \\ &= \begin{cases} -\frac{2}{n-2} & \text{if } k > 1, \\ \frac{d-2}{n-2} & \text{if } k = 1. \end{cases} \quad \square \end{aligned}$$

Claim 3.7 and Corollary 3.5 now show that the claimed a_i imply feasible solutions satisfying the PSD constraints. Taken with Claim 3.6 and Proposition 3.4, we have that

$$a_i = \frac{2}{n-2} \left(\cos \left(\frac{\pi i}{d} \right) + 1 \right), \quad i = 1, \dots, d,$$

is feasible for the linear program in Proposition 3.4 and therefore implies feasible solutions for the SDP (1) of the form

$$X^{(j)} = \left(\begin{pmatrix} a_j & b_j \\ b_j & a_j \end{pmatrix} \otimes J_d \right) - a_j I_n, \quad \text{where } b_j = \begin{cases} \frac{4}{n} - \left(1 - \frac{2}{n}\right) a_j & \text{if } j = 1, \dots, d-1, \\ \frac{2}{n} - \left(1 - \frac{2}{n}\right) a_j & \text{if } j = d. \end{cases}$$

3.3. The unbounded integrality gap. We are now able to prove our main theorem.

THEOREM 3.1.

$$\text{OPT}_{\text{SDP}}(\hat{C}) \leq \frac{\pi^2}{2n} \text{OPT}_{\text{TSP}}(\hat{C}).$$

Proof. Earlier we saw that a feasible solution of the form in (2) had cost $\frac{n^2}{4} b_1$ and $\text{OPT}_{\text{TSP}}(\hat{C}) = 2$. Hence, assuming a feasible solution, we can bound

$$\frac{\text{OPT}_{\text{SDP}}(\hat{C})}{\text{OPT}_{\text{TSP}}(\hat{C})} \leq \frac{n^2 b_1}{8}.$$

We have found a feasible solution with parameter

$$a_1 = \frac{2}{n-2} \left(\cos\left(\frac{\pi}{d}\right) + 1 \right)$$

so that

$$b_1 = \frac{4}{n} - \left(1 - \frac{2}{n}\right) \frac{2}{n-2} \left(\cos\left(\frac{\pi}{d}\right) + 1 \right) = \frac{2}{n} \left(1 - \cos\left(\frac{\pi}{d}\right) \right).$$

Using a Taylor series with remainder,

$$\cos\left(\frac{\pi}{d}\right) = 1 - \frac{\pi^2}{2d^2} + \frac{1}{4!} \frac{\pi^4}{d^4} \cos(\xi_{1/d}) \geq 1 - \frac{\pi^2}{2d^2},$$

where $\xi_{1/d} \in [0, \frac{1}{d}]$

Hence, we bound as follows:

$$\begin{aligned} \frac{\text{OPT}_{\text{SDP}}(\hat{C})}{\text{OPT}_{\text{TSP}}(\hat{C})} &\leq \frac{n^2 b_1}{8} \\ &\leq \frac{n^2}{8} \frac{2}{n} \left(\frac{\pi^2}{2d^2} \right) \\ &= \frac{\pi^2}{2n}. \end{aligned} \quad \square$$

We note that, at best, the SDP (1) is an $\mathcal{O}(n)$ -approximation algorithm. We also notice the following.

REMARK 3.8. *Several hierarchies exist that strengthen convex relaxations of combinatorial optimization problems, including those of Sherali and Adams [29], Lovász and Schrijver [24], and Lasserre [23]. These hierarchies iteratively add constraints to the relaxation; after sufficiently many iterations, the surviving feasible solutions correspond exactly to convex combinations of integer solutions. See Chlamtac and Tulsiani [3] for a detailed survey.*

Cheung [2], for example, applies hierarchies to show that certain feasible solutions for the subtour LP survive applying the Lovász and Schrijver hierarchy any

constant number of times. In particular, those solutions violated certain constraints (2-matching inequalities) satisfied by Hamiltonian cycles. One might analogously wonder how long our solution survives iteratively adding constraints to an appropriate linear program. $X^{(1)}$ is not feasible for the subtour LP for sufficiently large n , so that it trivially doesn't survive any rounds of these hierarchies applied to the subtour LP. In contrast, it can be shown that the feasible $X^{(1)}$ we found is in the convex hull of cycle covers. Hence our solution would survive arbitrarily many rounds of any of these hierarchies applied to a linear program obtained by using only the degree constraints of the subtour LP.

4. Corollaries of Theorem 3.1. Theorem 3.1 and its proof imply several corollaries that help us better understand the SDP (1) and its relationship to other relaxations of the TSP. We list several corollaries in this section, first relating the SDP to the subtour LP (sections 4.1 through 4.3), and then relating the SDP to another SDP for the TSP in section 4.4.

4.1. Nonmonotonicity of solution costs. We begin with the counterintuitive result that adding vertices (in a way that retains costs being metric) can arbitrarily decrease the cost of some solutions to the SDP (1). We state this as a *nonmonotonicity* property that contrasts with both TSP and subtour LP solutions.

Consider an optimization problem whose variables correspond to edges of the complete graph K_n and whose input consists of edge costs and a size n . Let $S \subset [n]$ be a subset of the vertices. Let OPT denote the cost of the optimal solution to the optimization problem on the full set of vertices, and let $\text{OPT}[S]$ denote the cost of the optimal solution to the optimization problem *induced* on the set S . Formally, if C denotes the matrix of edge costs corresponding to the original input, then the induced problem on S uses the edge cost matrix $C[S]$ defined to be the principle submatrix of C obtained by deleting the rows and columns in $[n] \setminus S$. If $\text{OPT}[S] \leq \text{OPT}$ for all possible input costs, values of n , and subsets S , we say that the optimization property has a *monotonicity* property.

The TSP (as usual, assuming metric and symmetric edge costs) is well known to be monotonic (this can be seen as an application of *shortcutting*; see section 2.4 of Williamson and Shmoys [32] for details of shortcutting.) Moreover, Shmoys and Williamson [30] show that the subtour LP is also monotonic. Our example shows that the SDP of de Klerk, Pasechnik, and Sotirov [9], however, is not: the cost of our SDP solutions get arbitrarily small as n grows, and our instance on n' vertices can be viewed as induced from a larger instance on $n > n'$ vertices.²

COROLLARY 4.1. *The SDP (1) is not monotonic.*

4.2. The relationship of our SDP solutions to the minimum spanning tree polytope. The *minimum spanning tree* (MST) polytope is

$$\left\{ z \in \mathbb{R}^{\binom{n}{2}} : \sum_{e \in E} x_e = n - 1, \sum_{e \in E(S)} z_e \leq |S| - 1 \text{ for all } S \subset V, z \geq 0 \right\}.$$

²As a technical point, the lack of monotonicity requires showing that, given cost matrix C , the SDP (1) does not admit solutions of nonpositive cost (this is to rule out the case in which, for all n , the optimal solution to the SDP given cost matrix C is zero and hence does not arbitrarily decrease). One way to see that there cannot be solutions of cost zero is to use methods from spectral graph theory. As we will discuss in detail in section 4.4, for any feasible SDP solution, $X^{(1)}$ can be viewed as the weighted adjacency matrix of a graph that must have strictly positive algebraic connectivity. Hence, Cheeger's inequality can be used to get a strictly positive lower bound on $\delta(U)$, the weight of edges of cost 1 in any feasible SDP solution. See Spielman [31].

One nice, well-known property of the subtour LP is that any feasible solution to it, when appropriately scaled, is also feasible for the MST polytope (see, e.g., Gao [11] for a very similar argument). Conversely, solutions to the SDP cannot in general be scaled to be in the MST polytope. We show this directly using our feasible solutions.³

COROLLARY 4.2. *Let $x \in \mathbb{R}^{\binom{n}{2}}$ be defined by $x_e = X_{ij}^{(1)} = X_{ji}^{(1)}$ and denote by E the set of all edges in the complete graph on n vertices. There is no suitable scaling factor c such that cx is in the minimum spanning tree polytope (where c is allowed to be a function of n).*

Proof. Notice that $X^{(1)}e = 2e$ implies that $\sum_{e \in E} x_e = n$ so that we must set $c = \frac{n-1}{n}$. Again let U correspond to the set of vertices in one group. Then there are $\binom{d}{2}$ edges in $E(U)$, each of which has an assigned weight of a_1 in our solution. Hence,

$$\begin{aligned} \frac{n-1}{n} \sum_{e \in E(U)} x_e &= \frac{n-1}{n} \sum_{e \in E(U)} a_1 \\ &= \frac{n-1}{n} \binom{n/2}{2} \frac{2}{n-2} \left(\cos\left(\frac{\pi}{d}\right) + 1 \right) \\ &= \frac{n-1}{4} \left(\cos\left(\frac{\pi}{d}\right) + 1 \right) \\ &\geq \frac{n-1}{4} \left(2 - \frac{\pi^2}{2d^2} \right) \\ &= d - \frac{1}{2} + O\left(\frac{1}{n}\right) \\ &> |U| - 1 \end{aligned}$$

for all n sufficiently large. $\square$

4.3. The SDP and subtour elimination linear program when n is small.

When $n = 6$, our solution is

$$X^{(1)} = \begin{pmatrix} 3/4 & 1/6 \\ 1/6 & 3/4 \end{pmatrix} \otimes J_3 - \frac{3}{4} I_6.$$

Letting $U = \{1, 2, 3\}$ denote one of the two groups of vertices, we see that $\delta(U)$ has 9 edges in it, each of which is assigned a weight of $1/6$, so that the total weight crossing $\delta(U)$ in this solution is $9 * \frac{1}{6} = \frac{3}{2} < 2$. This violates the subtour elimination constraint for U . Hence, we see that the subtour LP and SDP have distinct feasible regions when $n = 6$. We can show, in contrast, that they are the same for $n \leq 5$. Doing so involves computations that are of a different spirit than what we have done so far; we defer this proof to the extended version of this paper (Gutekunst and Williamson [18]). We emphasize this result because, when $n \leq 5$, it is known that the feasible region of the subtour LP consists exactly of convex combinations of Hamiltonian cycles (see, for example, Grötschel and Padberg [17]). Hence this result lets us characterize the feasible region of the SDP when $n \leq 5$ as corresponding

³We briefly note that, if we could appropriately scale the SDP solutions to be in the MST polytope, we would be able to bound the integrality gap by a factor of 2 by using the standard tree-doubling approximation algorithm (see, e.g., section 2.4 of Williamson and Shmoys [32]); from this observation, and the fact that we have shown that the integrality gap is unbounded, it follows that our solutions cannot be scaled to lie in the MST polytope. Here we instead choose to provide a direct proof that reveals how far our solutions are outside of the MST polytope.

exactly to convex combinations of Hamiltonian cycles. We formalize and prove these results in Gutekunst and Williamson [18].

4.4. The relationship of our solution to an earlier TSP SDP. Previously we mentioned an earlier SDP relaxation for the TSP from Cvetković, Čangalović, and Kovačević-Vujčić [5] which was shown to be weaker than the subtour LP in Goemans and Rendl [13]. This relaxation has a single matrix variable X and takes the form

$$(4) \quad \begin{array}{ll} \min & \frac{1}{2} \text{trace}(CX) \\ \text{subject to} & Xe = 2e, \\ & X_{ii} = 0, & i = 1, \dots, n, \\ & X_{ij} \leq 1, & i, j = 1, \dots, n, \\ & 2I - X + \left(2 - 2 \cos\left(\frac{2\pi}{n}\right)\right)(J - I) \succeq 0, \\ & X \in S^n. \end{array}$$

The variable X can be interpreted as a weighted adjacency matrix, and the constraint that $Xe = 2e$ ensures that e is an eigenvector of X with corresponding eigenvalue 2. The term $2I - X$ in the PSD constraint can be interpreted as the Laplacian of X : let G be a weighted, undirected graph on n vertices with weighted adjacency matrix A . Let D be the degree matrix of G (i.e., D is diagonal with $D_{ii} = \sum_{j=1}^n A_{ij}$). The Laplacian of G is defined as

$$L := D - A.$$

With the interpretation of X as a weighted adjacency matrix, the constraint $Xe = 2e$ implies that the Laplacian corresponding to X is

$$L(X) := 2I - X,$$

where we make the dependence on X explicit. This observation, and machinery from spectral graph theory, motivates the positive semidefinite constraint in the SDP of Cvetković, Čangalović, and Kovačević-Vujčić [5]. (See Spielman [31] for a nice introduction to spectral graph theory.)

In more detail, let $h_n := 2 - 2 \cos(\frac{2\pi}{n})$ so that the positive semidefinite constraint is

$$L(X) + h_n(J - I) \succeq 0.$$

The value of h_n is known to be the second smallest eigenvalue of the Laplacian of a cycle on n vertices.⁴ The second smallest eigenvalue of the Laplacian is known as the *algebraic connectivity* of a graph.

The Laplacian of a weighted graph is known to be positive semidefinite (see Spielman [31], which represents the Laplacian as a quadratic form), so we can write the eigenvalues of $L(X)$ as $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. Since X is symmetric, we further assume that these eigenvalues correspond to an orthogonal basis of eigenvectors $v_1, \dots, v_n$, where v_i corresponds to eigenvalue λ_i . Moreover, we can choose to let $v_1 = e$ and $\lambda_1 = 0$, since $Xe = 2e$. The eigenvalues of $L(X) + h_n(J - I)$ are then

$$\lambda_1 + (n-1)h_n = (n-1)h_n, \lambda_2 - h_n, \dots, \lambda_n - h_n.$$

These follow by right-multiplying $L(X) + h_n(J - I)$ by v_i and noting that $Jv_i = ee^T v_i = 0$ if $i \neq 1$, and $Jv_1 = ee^T e = ne$. Since $h_n \geq 0$, the positive semidefinite

⁴The Laplacian of a cycle graph is also a circulant matrix, with $m_0 = 2$, $m_1 = m_{n-1} = -1$, and $m_i = 0$ otherwise. Its eigenvalues can be directly computed using Lemma 2.2.

constraint ensures that

$$\lambda_i - h_n \geq 0, \quad i = 1, \dots, n-1,$$

or equivalently that

$$(5) \quad \lambda_2 \geq h_n.$$

Hence, the PSD constraint introduced by Cvetković, Čangalović, and Kovačević-Vujčić [5] ensures that the algebraic connectivity of X is at least h_n , the algebraic connectivity of a cycle on n vertices.

One might wonder if our solution $X^{(1)}$ is also feasible for the SDP of Cvetković, Čangalović, and Kovačević-Vujčić [5]. The answer is yes because, as mentioned earlier, de Klerk, Pasechnik, and Sotirov [9] showed that any solution of (1) is feasible for the SDP of Cvetković, Čangalović, and Kovačević-Vujčić [5] in (4). Hence, Theorem 3.1 implies that the SDP (4) also has an unbounded integrality gap.

Here we show the result directly for our feasible solutions, as it turns out that our solution corresponds to an instance where (5) is tight. Thus our $X^{(1)}$ instance and cost matrix $\hat{C}$ provide an explicit example of a weighted graph that has exactly the same algebraic connectivity as a cycle and in which every vertex has degree 2, but has cost arbitrarily far from a minimum-cost Hamiltonian cycle.

PROPOSITION 4.3. *Taking*

$$X = X^{(1)} = \left(\begin{pmatrix} a_1 & b_1 \\ b_1 & a_1 \end{pmatrix} \otimes J_d \right) - a_1 I_n$$

with $a_1 = \frac{2}{n-2}(\cos(\frac{\pi}{d}) + 1)$ and $b_1 = \frac{2}{n}(1 - \cos(\frac{\pi}{d}))$ yields a feasible solution for the SDP (4). Moreover, the algebraic connectivity of X is exactly that of an n -cycle.

Proof. By construction, $X^{(1)}$ satisfies all conditions of (4) except possibly that

$$2I_n - X^{(1)} + h_n(J_n - I_n) \succeq 0.$$

By the argument above, it suffices to compute the second smallest eigenvalue of $2I_n - X^{(1)}$ and show that it is at least h_n . The eigenvalues of

$$2I_n - X^{(1)} = (2 + a_1)I_n - \left(\begin{pmatrix} a_1 & b_1 \\ b_1 & a_1 \end{pmatrix} \otimes J_d \right)$$

are $2 + a_1$, with multiplicity $n - 2$, and $2 + a_1 - d(a_1 \pm b_1)$, each with multiplicity 1. Simplifying these later eigenvalues, we have the two eigenvalues

$$2 + a_1 - d(a_1 + b_1) = 0, \quad 2 + a_1 - d(a_1 - b_1) = h_n.$$

Hence, the second smallest eigenvalue of $L(X^{(1)})$ is indeed h_n . $\square$

COROLLARY 4.4. *The SDP (4) has an unbounded integrality gap.*

COROLLARY 4.5. *The algebraic connectivity of $X^{(1)}$ is equal to the algebraic connectivity of cycle.*

5. The k -cycle cover problem. In the TSP, we try to find a minimum-cost cycle that covers all vertices. This problem is generalized in the k -cycle cover problem, which involves finding k equally sized cycles that cover all of the vertices (and assumes n is divisible by k). Just as in the TSP, the goal is to do so with minimum cost. As

for the TSP, there are algorithms for finding approximate solutions with a bounded integrality gap. Goemans and Williamson [15] give a 4-approximation algorithm for this problem.

De Klerk, de Oliveira Filho, and Pasechnik [7] notice that the SDP (1) can be modified to become a relaxation of the k -cycle problem by changing only the objective function. They prove the following result.

PROPOSITION 5.1. *The following SDP is a relaxation of the minimum-cost k -cycle cover problem:*

$$\begin{aligned}
 (6) \quad & \min \quad \frac{1}{2} \text{trace} (CX^{(k)}) \\
 & \text{subject to} \quad X^{(j)} \geq 0, \quad j = 1, \dots, d, \\
 & \quad \sum_{j=1}^d X^{(j)} = J - I, \\
 & \quad I + \sum_{j=1}^d \cos\left(\frac{2\pi ij}{n}\right) X^{(j)} \succeq 0, \quad i = 1, \dots, d, \\
 & \quad X^{(i)} \in S^n, \quad i = 1, \dots, d.
 \end{aligned}$$

Proof (from de Klerk, de Oliveira Filho, and Pasechnik [7]). The proof uses exactly the same feasible solutions as Proposition 2.1. The key observation is that the k th distance matrix $A_k(C_n)$ represents a partition of the vertices into k equally sized cycles. In particular, if C_n is the cycle $v_1, v_2, \dots, v_n, v_1$, then $A_k(C_n)$ consists of the cycles $v_i, v_{i+k}, v_{i+2k}, \dots, v_{i+(n-k)}, v_i$ for $i = 1, 2, \dots, k$ (see, for example, Figure 1). Any k -cycle cover of the vertices can similarly be represented as the k th distance matrix of some Hamiltonian cycle. $\square$

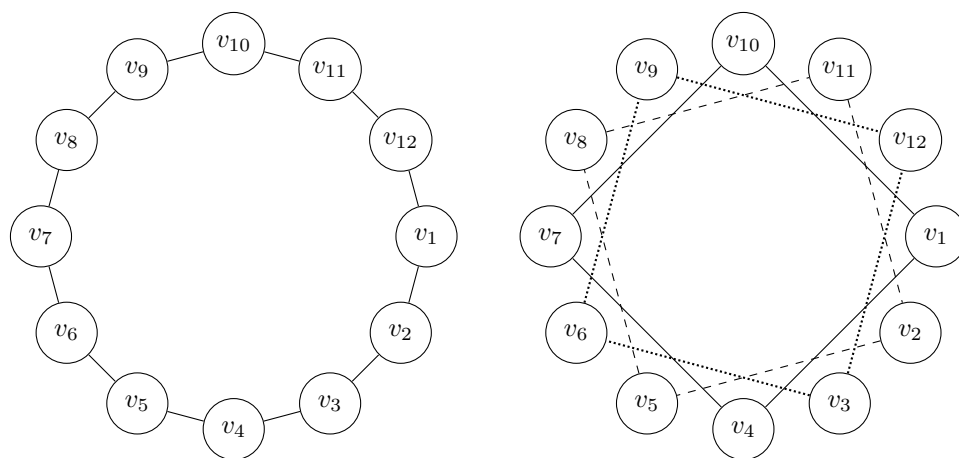


FIG. 1. The graphs corresponding to $A_1(C_n)$ and $A_3(C_n)$ when $n = 12$. Notice that the right-hand graph is a 3-cycle cover, and each cycle is drawn with a different edge style.

Since the SDP for the TSP is a special case of this SDP obtained by setting $k = 1$, it is natural to wonder if our technique also shows that this more general SDP has an unbounded integrality gap. Again the answer is in the affirmative. Let $\text{OPT}_{\text{SDP}}(C)$ and $\text{OPT}_{k\text{-cycle}}(C)$ respectively denote the optimal solutions to the SDP (6) and to the k -cycle cover problem for a given matrix of costs C and fixed $k \geq 2$. Our earlier result generalizes as follows.

THEOREM 5.2. *There exist cost matrices $\hat{C}$ with metric and symmetric edge costs such that*

$$\text{OPT}_{\text{SDP}}(\hat{C}) \leq \frac{\pi^2}{n} \frac{k}{k+1} \text{OPT}_{k\text{-cycle}}(\hat{C}).$$

COROLLARY 5.3. *The SDP (6) for the k -cycle cover problem has an unbounded integrality gap. That is, there exists no constant $\alpha > 0$ such that*

$$\frac{\text{OPT}_{k\text{-cycle}}(C)}{\text{OPT}_{\text{SDP}}(C)} \leq \alpha$$

for all cost matrices C .

The full proof of Theorem 5.2 uses almost exactly the same ideas as in the proof of Theorem 3.1: for sufficiently structured solutions, finding a feasible SDP solution is equivalent to finding a feasible solution to a related linear program. We find feasible solutions to such a linear program and use a Taylor series with remainder to obtain the stated bounds. Because the ideas are so similar, we only highlight the main steps of the proof. A more detailed sketch can be found in the appendix of the extended version of this paper, Gutekunst and Williamson [18].

We modify our example from the proof of Theorem 3.1 and consider cost matrices reflecting $k+1$ equally sized groups of vertices. Hence, we let $n = ck(k+1)$ and will scale n by scaling $c \in \mathbb{N}$ (to reduce casework, we also take c to be even when k is even). As before, the costs associated with intergroup edges will be 1, while the costs of intragroup edges will be 0. Our cost matrix is

$$\hat{C} := (J_{k+1} - I_{k+1}) \otimes J_{ck}.$$

Notice that any integer solution to the k -cycle problem will use cycles of length $c(k+1)$, while each group is of size ck . Hence, any cycle in any integer solution will need to use at least two expensive edges. This lower bounds the cost of $\text{OPT}_{k\text{-cycle}}(\hat{C})$ as $2k$. We can also achieve this cost by labeling the groups of ck vertices as $G_1, \dots, G_{k+1}$. For $i = 1, \dots, k$, we create a cycle $\mathcal{C}_i$ that visits all vertices in group G_i , then visits c vertices in G_{k+1} , and then returns to G_i for each $i = 1, \dots, k$. Each cycle $\mathcal{C}_i$ costs 2, so that the cost is indeed $2k$. Hence, regardless of n , $\text{OPT}_{k\text{-cycle}}(\hat{C}) = 2k$.

Our proof of Theorem 5.2 now proceeds as in the proof of Theorem 3.1. We find solutions whose structure respects the symmetry of $\hat{C}$: solutions that place a weight a_i on each intragroup edge, a weight b_i on each intergroup edge, and zeros on the diagonal. That is,

$$X^{(i)} = ((b_i J_{k+1} + (a_i - b_i) I_{k+1}) \otimes J_{ck}) - a_i I_n.$$

Also, note that in each of the n rows of $X^{(k)}$ there are $n - ck = ck^2$ entries that are b_k (occurring exactly where C has ones), and ck entries that are a_k (occurring exactly where C has zeros). Hence the cost of such a solution is now based entirely on b_k :

$$\frac{1}{2} \langle C, X^{(k)} \rangle = \frac{1}{2} n(n - ck) b_k = \frac{c^2}{2} (k+1) k^3 b_k.$$

The structure of these matrices (and their linear combinations) allows us to again explicitly write down the eigenvalues of each semidefinite constraint. We also enforce

the same constraints on the row sums of the $X^{(i)}$, which now become

$$b_i = \frac{1}{ck^2} \begin{cases} (2 - (ck - 1)a_i) & \text{if } i < d, \\ (1 - (ck - 1)a_i) & \text{if } i = d. \end{cases}$$

By only considering solutions with this structure, we obtain the following counterpart of Proposition 3.4.

PROPOSITION 5.4. *Finding a minimum-cost feasible solution to the SDP (6) of the form*

$$X^{(i)} = ((b_i J_{k+1} + (a_i - b_i) I_{k+1}) \otimes J_{ck}) - a_i I_n,$$

for $i = 1, \dots, d$ with

$$b_i = \frac{1}{ck^2} \begin{cases} (2 - (ck - 1)a_i) & \text{if } i < d, \\ (1 - (ck - 1)a_i) & \text{if } i = d, \end{cases}$$

is equivalent to solving the following optimization problem:

$$(7) \quad \begin{array}{ll} \max & a_k \\ \text{subject to} & \sum_{i=1}^d \cos\left(\frac{2\pi ij}{n}\right) a_i \geq -\frac{1}{ck-1}, \quad j = 1, \dots, d, \\ & \sum_{i=1}^d \cos\left(\frac{2\pi ij}{n}\right) a_i \leq 1, \quad j = 1, \dots, d, \\ & \sum_{i=1}^d a_i = 1, \\ & a_i \leq \frac{2}{ck-1}, \quad i = 1, \dots, d-1, \\ & a_d \leq \frac{1}{ck-1}, \\ & a_i \geq 0, \quad i = 1, \dots, d. \end{array}$$

To prove Theorem 5.2, it suffices to show that the following choice of a_i , for $i = 1, \dots, d$, leads to a feasible SDP solution:

$$a_i = \begin{cases} 0 & \text{if } i \not\equiv_k k, \\ \frac{2}{n-k-1} \left(\cos\left(\frac{\pi i}{d}\right) + k \right) & \text{if } i \equiv_k k, \ i \neq d, \\ \frac{1}{n-k-1} \left(\cos\left(\frac{\pi i}{d}\right) + k \right) & \text{if } i = d. \end{cases}$$

To verify that this choice of the a_i leads to a feasible solution for the linear program (7), we again use Lagrange's trigonometric identity and the product-to-sum identity for cosines. The computations requiring Lagrange's identity involve more bookkeeping (there are more cases involving summing cosines whose arguments are all integer multiples of 2π), but are otherwise in the same spirit as in Claims 3.6 and 3.7. These calculations are sketched in the appendix of the extended version of this paper, Gutekunst and Williamson [18].

Such feasible solutions bound the cost of the optimal SDP solution as

$$\text{OPT}_{\text{SDP}}(\hat{C}) \leq \frac{c^2}{2} (k+1) k^3 b_k.$$

Using $\cos(\frac{k\pi}{d}) \geq 1 - \frac{k^2\pi^2}{2d^2}$ we have

$$\begin{aligned} b_k &= \frac{2}{ck^2} \left(1 - (ck - 1) \frac{1}{n - k - 1} \left(\cos\left(\frac{\pi k}{d}\right) + k \right) \right) \\ &\leq \frac{2}{ck^2} \left(1 - \frac{ck - 1}{n - k - 1} \left(k + 1 - \frac{k^2\pi^2}{2d^2} \right) \right) \\ &= \frac{\pi^2}{cd^2(k + 1)}. \end{aligned}$$

For any fixed k , the denominator is again $\mathcal{O}(n^3)$. Recall also that

$$\text{OPT}_{k\text{-cycle}}(\hat{C}) = 2k.$$

Hence,

$$\begin{aligned} \frac{\text{OPT}_{\text{SDP}}(\hat{C})}{\text{OPT}_{k\text{-cycle}}(\hat{C})} &\leq \frac{\frac{c^2}{2}(k + 1)k^3b_k}{2k} \\ &\leq \frac{\pi^2}{4} \frac{ck^2}{d^2}. \end{aligned}$$

Using $4d^2 = n^2$ and $n = ck(k + 1)$,

$$\begin{aligned} \frac{\text{OPT}_{\text{SDP}}(\hat{C})}{\text{OPT}_{k\text{-cycle}}(\hat{C})} &= \pi^2 \frac{1}{c(k + 1)^2} \\ &= \pi^2 \frac{k}{k + 1} \frac{1}{n}. \end{aligned}$$

In the last line, we used that $c = n/(k(k + 1))$. Again, the SDP's integrality gap is unbounded; as n increases, solutions to the SDP become arbitrarily small. This is sufficient, as it will imply

$$b_k \leq \frac{\pi^2}{cd^2(k + 1)}.$$

For these solutions, it can then be shown that

$$\begin{aligned} \frac{\text{OPT}_{\text{SDP}}(\hat{C})}{\text{OPT}_{k\text{-cycle}}(\hat{C})} &\leq \frac{\frac{c^2}{2}(k + 1)k^3b_k}{2k} \\ &\leq \pi^2 \frac{k}{k + 1} \frac{1}{n}. \end{aligned}$$

6. Conclusion and open questions. In this paper, we have shown that an SDP for the TSP introduced in de Klerk, Pasechnik, and Sotirov [9] has an unbounded integrality gap. To do so, we considered instances when n was even and used highly structured feasible solutions that could be expressed as the Kronecker product of simple matrices. Such structure is not preserved for n odd, and we are unable to analyze this case without a significantly more complicated analysis. The solutions we found for n even implied several corollaries, and we used the same techniques to show that a related SDP, for the k -cycle cover problem and introduced in de Klerk, de Oliveira Filho, and Pasechnik [7], also has an unbounded integrality gap.

One open question relates to the relationship between the SDP of de Klerk, Pasechnik, and Sotirov [9] and the subtour LP. For the instance we constructed, the subtour LP outputs the exact cost of a solution tour. Is it the case that an approximation algorithm that runs both the SDP and subtour LP, then takes the best solution, has an integrality gap of $1.5 - \epsilon$ for $\epsilon > 0$?

A second open question relates to the performance of the SDP on special types of TSP instances. One example is the Euclidean TSP, where each city $i \in [n]$ corresponds to a point $x_i \in \mathbb{R}^2$, and the cost c_{ij} is given by the Euclidean distance between x_i and x_j . While no algorithm for the general TSP (with metric and symmetric edge costs) has been shown to have an integrality gap strictly less than 1.5, Arora [1] and Mitchell [25] give a polynomial time approximation scheme for the Euclidean TSP. Moreover, one can solve the Euclidean TSP in $\mathbb{R}^1$ exactly: if $x_m \in \min_{i \in [n]} x_i$ and $x_M \in \max_{i \in [n]} x_i$, then any optimal tour will cost $2(x_M - x_m)$; such a tour can be achieved by starting at x_m , iteratively visiting vertices in increasing order of x_i until reaching x_M , and returning to x_m . We noted that our instance corresponds to an instance of Euclidean TSP in $\mathbb{R}^1$ (and hence in $\mathbb{R}^2$ by lifting the points (0) and (1) in $\mathbb{R}^1$ to $(0, 0)^T$ and $(1, 0)^T$, respectively), so that the SDP has an unbounded integrality gap even when restricted to the Euclidean TSP (or the Euclidean TSP in $\mathbb{R}^1$).

Another class of instances that has received considerable attention is that of graphic TSP: here the input corresponds to a connected, undirected graph G on vertex set $[n]$, and for $i, j \in [n]$ the cost c_{ij} is the length of the shortest i - j path in G . Several recent papers have bounded the integrality gap on graphic TSP instances as strictly less than 1.5. (See, for example, Gharan, Saberi, and Singh [12] for a $1.5 - \epsilon$ bound, Mömke and Svensson [26] for a 1.461 bound, Mucha [27] for a $\frac{13}{9} \approx 1.444$ bound, and Sebő and Vygen [28] for a 1.4 bound.) It is not hard to show that the SDP has at most an integrality gap of 2 when restricted to graphic TSP instances. This follows because the minimum-cost Hamiltonian cycle is at most twice the cost of an MST (see section 2.4 of Williamson and Shmoys [32], for example), and the cost of an MST in a connected graph with unit edge weights is $n - 1$. Conversely, in graphic TSP the minimum-cost of an edge is 1. This means that $C \geq J - I$ (entrywise). Thus,

$$\frac{1}{2} \langle C, X^{(1)} \rangle \geq \frac{1}{2} \langle J - I, X^{(1)} \rangle = \frac{1}{2} \langle J, X^{(1)} \rangle = n.$$

Above, the first inequality follows from the fact that $X^{(1)}$ is nonnegative, the first equality follows from the fact that the diagonal of $X^{(1)}$ is zero, and the final equality follows from $X^{(1)}e = 2e$. An open question is to exactly compute the integrality gap of the SDP on graphic TSP; we conjecture that the integrality gap is at least 1.5 and is asymptotically achieved when G is a path.

Finally, de Klerk and Sotirov [10] present a stronger version of the SDP (1). It remains open to compute the integrality gap of this SDP relaxation of the TSP.

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