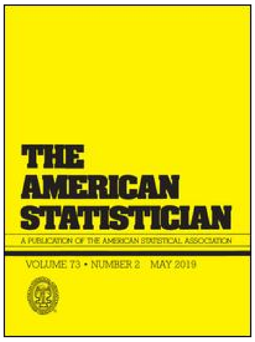

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The Impact of Application of the Jackknife to the Sample Median

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ABSTRACT
The jackknife is a reliable tool for reducing the bias of a wide range of estimators. This note demonstrates that even such versatile tools have regularity conditions that can be violated even in relatively simple cases, and that caution needs to be exercised in their use. In particular, we show that the jackknife does not provide the expected reliability for bias-reduction for the sample median, because of subtle changes in behavior of the sample median as one moves between even and odd sample sizes. These considerations arose out of class discussions in a MS-level nonparametrics course.

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1. Introduction

Suppose that T_n is the estimator of θ based on n independent and identically distributed (iid) observations. Quenouille (1956) suggested the jackknife technique for reducing the bias of T_n. Following Quenouille's spirit, suppose that the bias of T_n is of the form

$$E(T_n) - \theta = \frac{a}{n} + \frac{b}{n^2} + O(n^{-\frac{5}{2}}). \tag{1}$$

The Jackknife uses the relationship between biases for the statistic based on the entire sample, and subsamples in which one observation is removed, to estimate bias, and so to produce a less-biased estimator. Let $T^*_{n-1,i}$ be the estimator based on the sample of size $n-1$ with observation i omitted. Let $\bar{T}^*_n = \sum_{i=1}^n T^*_{n-1,i}/n$. Then $B = (n-1)(\bar{T}^*_n - T_n)$ is the Jackknife estimator of the bias of T_n. Since

$$E(B) = \frac{a}{n} + O(n^{-\frac{3}{2}}), \tag{2}$$

the bias of $T_n - B$ is $O(n^{-3/2})$.

Efron (1982) and Shao and Tu (1995) presented more recent surveys of jackknife techniques. Generally speaking, much of the utility of the jackknife lies in its applicability with only minimal mathematical analysis. When teaching about the jackknife, however, it is useful to examine analytically tractable situations. The simplest context for jackknife analysis is of the sample average as the estimate of the expectation of independent and identically distributed observations; in this case, the jackknife provides no correction. The next simple case is that of the sample median as an estimate of the population median. Students will likely be surprised at the complexity that arises in this seemingly simple case. Jeske and Sampath (2003) demonstrated a similar unexpected complexity arising from resampling techniques, and also demonstrable with elementary mathematical tools; their example arises out of an application of the bootstrap.

Let T_n be the sample median. A generally accepted definition of sample median T_n is the following: If n is odd, $T_n = X_{((n+1)/2)}$; if n is even, $T_n = (X_{(n/2)} + X_{(n/2+1)})/2$, where $X_{(i)}$ is the ith order statistic.

Consider the jackknife bias estimator for the sample median for data from a continuous distribution. For the sake of calculating $T^*_{n-1,i}$, suppose that the data $X_1, X_2, \ldots, X_n$ are in increasing order. When the sample size n is even, $T_n = (X_{(n/2)} + X_{(n/2+1)})/2$, and

$$T^*_{n-1,i} = \begin{cases} X_{(n/2+1)}, & \text{if } i \le n/2; \\ X_{(n/2)}, & \text{if } i \ge n/2+1. \end{cases}$$

Then $\bar{T}^*_n = (X_{(n/2)} + X_{(n/2+1)})/2 = T_n$, and the bias estimate is always 0.

Harrell and Davis (1982) presented a quantile estimator, applicable to the median, that is more efficient than the standard sample quantile; because our present investigation is primarily a pedagogical investigation of a simple application of the jackknife, we do not address this more efficient estimator. We investigate jackknife behavior for the sample median. Section 2 proves that under certain conditions on the density function $f(x)$, Equation (1) holds for the odd n case and the even n case separately, and the constant a in Equation (1) is the same, but the constant b in Equation (1) shows different forms in both cases. Section 2.3 summarizes the impact of application of the jackknife to the sample median and concludes that the jackknife is not conducive to reducing the bias of the sample median. Section 3 verifies this article in the case of the standard exponential distribution. Some technical details are provided in the appendices.

2. Main Result

Suppose that a sample $X_1, X_2, \ldots, X_n$ is independent and identically distributed, with a cumulative distribution function

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$F(x)$, and a density function $f(x)$ satisfying the following conditions:

- Condition 1: the fourth derivative of the density $f^{(4)}(x)$ exists for all x in $F^{-1}((0,1))$.
- Condition 2: the density $f(x) > 0$ on $F^{-1}((0,1))$. (Conditions 1 and 2 imply that $F(x)$ has an inverse function defined on $(0,1)$, denoted by $g(u) := F^{-1}(u)$, and that $g(u)$ has a fifth derivative.)
- Condition 3: there exists a nonnegative integer r such that $h(u) := u^r g(u)(1-u)^r$ has a bounded fifth derivative. (We will elaborate on the Condition 3 in the parts A, B, and C of the appendices.)

2.1. Case 1: n Is Odd

The sample median is the order statistic $X_{((n+1)/2)}$. Using the fact that the order statistic $X_{(k)}$ has the density function

$$\frac{n!}{(k-1)!(n-k)!} f(x)(F(x))^{k-1}(1-F(x))^{n-k}, \tag{3}$$

and the fact that the beta function $B(i, n+1-i) = (i-1)!(n-i)!/n!$, where n and i are positive integers and $i \le n$,

$$E\left(X_{(\frac{n+1}{2})}\right) = \int_{-\infty}^{+\infty} \frac{1}{B(\frac{n+1}{2},\frac{n+1}{2})} x f(x) F(x)^{\frac{n-1}{2}} (1-F(x))^{\frac{n-1}{2}} dx. \tag{4}$$

Change variables with $u = F(x)$. Recall that $g(u) = F^{-1}(u)$, the inverse function of $F(x)$. Using the fact that $g'(u) = 1/f(F^{-1}(u))$,

$$\begin{aligned} E\left(X_{(\frac{n+1}{2})}\right) &= \int_0^1 \frac{1}{B(\frac{n+1}{2},\frac{n+1}{2})} F^{-1}(u) u^{\frac{n-1}{2}}(1-u)^{\frac{n-1}{2}} du \\ &= \frac{1}{B(\frac{n+1}{2},\frac{n+1}{2})} \int_0^1 u^{\frac{n-1}{2}-r}(1-u)^{\frac{n-1}{2}-r} h(u) du. \end{aligned} \tag{5}$$

Let ω be the median of the population. Then $\omega = g(1/2)$. Under the assumed condition on $f(x)$, $h(u)$ has a fifth derivative. Expand $h(u)$ about $u = 1/2$ by a Taylor formula with Lagrange residual term to obtain

$$\begin{aligned} E\left(X_{(\frac{n+1}{2})}\right) &= \int_0^1 \frac{u^{\frac{n-1}{2}-r}(1-u)^{\frac{n-1}{2}-r}}{B(\frac{n+1}{2},\frac{n+1}{2})} \left(\left(\frac{1}{2}\right)^{2r}\omega \right.\\ &+ h'\left(\frac{1}{2}\right)\left(u-\frac{1}{2}\right) + \frac{1}{2}h^{(2)}\left(\frac{1}{2}\right)\left(u-\frac{1}{2}\right)^2 \\ &+ \frac{1}{6}h^{(3)}\left(\frac{1}{2}\right)\left(u-\frac{1}{2}\right)^3 + \frac{1}{24}h^{(4)}\left(\frac{1}{2}\right)\left(u-\frac{1}{2}\right)^4 \\ &\left.+ \frac{1}{120}h^{(5)}(u^*(u))\left(u-\frac{1}{2}\right)^5\right) du \end{aligned} \tag{6}$$

for some $u^*(u)$ between $1/2$ and u. By definition of $B(a,b)$,

$$\begin{aligned} &\int_0^1 \frac{u^{\frac{n-1}{2}-r}(1-u)^{\frac{n-1}{2}-r}}{B(\frac{n+1}{2},\frac{n+1}{2})}\left(\frac{1}{2}\right)^{2r}\omega du \\ &= \omega\frac{n(n-2)(n-4)\cdots(n+2-2r)}{(n-1)(n-3)(n-5)\cdots(n+1-2r)} \\ &= \omega\left(1+\frac{1}{n-1}\right)\left(1+\frac{1}{n-3}\right)\cdots\left(1+\frac{1}{n+1-2r}\right) \\ &= \omega\left(1+\frac{1}{n}+\frac{1}{n^2}+O\left(n^{-3}\right)\right)\left(1+\frac{1}{n}+\frac{3}{n^2}+O\left(n^{-3}\right)\right)\cdots \\ &\left(1+\frac{1}{n}+\frac{2r-1}{n^2}+O\left(n^{-3}\right)\right) \\ &= \omega + \frac{r\omega}{n} + \frac{(\frac{3}{2}r^2-\frac{1}{2}r)\omega}{n^2} + O\left(n^{-3}\right). \end{aligned}$$

Since $(u-1/2)^{2k+1}$ is symmetric with respect to $(1/2, 0)$,

$$\int_0^1 \frac{u^{\frac{n-1}{2}-r}(1-u)^{\frac{n-1}{2}-r}}{B(\frac{n+1}{2},\frac{n+1}{2})} h'\left(\frac{1}{2}\right)\left(u-\frac{1}{2}\right) du = 0$$

and

$$\int_0^1 \frac{u^{\frac{n-1}{2}-r}(1-u)^{\frac{n-1}{2}-r}}{B(\frac{n+1}{2},\frac{n+1}{2})} h^{(3)}\left(\frac{1}{2}\right)\left(u-\frac{1}{2}\right)^3 du = 0.$$

On the other hand, even terms in $u - 1/2$ integrate to nonnegative contributions. Note that $h^{(2)}(1/2)$ can be expressed in terms of ω and $g^{(2)}(1/2)$. Then

$$\begin{aligned} &\int_0^1 \frac{u^{\frac{n-1}{2}-r}(1-u)^{\frac{n-1}{2}-r}}{B(\frac{n+1}{2},\frac{n+1}{2})} \frac{h^{(2)}\left(\frac{1}{2}\right)}{2}\left(u-\frac{1}{2}\right)^2 du \\ &= \int_0^1 \frac{u^{\frac{n-1}{2}-r}(1-u)^{\frac{n-1}{2}-r}}{B(\frac{n+1}{2},\frac{n+1}{2})} \frac{h^{(2)}\left(\frac{1}{2}\right)}{2}\left(u(u-1)+\frac{1}{4}\right) du \\ &= \frac{1}{2}\frac{h^{(2)}\left(\frac{1}{2}\right)}{B\left(\frac{n+1}{2},\frac{n+1}{2}\right)}\left(-B\left(\frac{n+3}{2}-r,\frac{n+3}{2}-r\right)\right. \\ &\left.+\frac{1}{4}B\left(\frac{n+1}{2}-r,\frac{n+1}{2}-r\right)\right) \\ &= \frac{1}{2}\frac{h^{(2)}\left(\frac{1}{2}\right)}{B\left(\frac{n+1}{2},\frac{n+1}{2}\right)}\frac{((\frac{n-1}{2}-r)!)^2}{(n-2r)!} \\ &\left(\frac{1}{4}-\frac{\left(\frac{n+1}{2}-r\right)^2}{(n+2-2r)(n+1-2r)}\right) \\ &= \left(-r\omega+\frac{1}{8}g^{(2)}\left(\frac{1}{2}\right)\right)\left(1+\frac{1}{n-1}\right)\left(1+\frac{1}{n-3}\right)\cdots \\ &\left(1+\frac{1}{n+1-2r}\right)\frac{1}{n+2-2r} \\ &= \left(-r\omega+\frac{1}{8}g^{(2)}\left(\frac{1}{2}\right)\right)\left(1+\frac{1}{n}+O\left(n^{-2}\right)\right) \\ &\left(1+\frac{1}{n}+O\left(n^{-2}\right)\right)\cdots\left(1+\frac{1}{n}+O\left(n^{-2}\right)\right) \\ &\left(\frac{1}{n}+\frac{2r-2}{n^2}+O\left(n^{-3}\right)\right) \\ &= \frac{-r\omega+\frac{1}{8}g^{(2)}(\frac{1}{2})}{n} + \frac{(3r-2)(-r\omega+\frac{1}{8}g^{(2)}(\frac{1}{2}))}{n^2} + O\left(n^{-3}\right). \end{aligned}$$

Similarly,

$$\int_0^1 \frac{u^{\frac{n-1}{2}-r}(1-u)^{\frac{n-1}{2}-r}}{B(\frac{n+1}{2},\frac{n+1}{2})}\frac{h^{(4)}(\frac{1}{2})}{24}\left(u-\frac{1}{2}\right)^4 du = \frac{(\frac{3}{2}r^2-\frac{3}{2}r)\omega - \frac{3}{8}rg^{(2)}(\frac{1}{2}) + \frac{g^{(4)}(\frac{1}{2})}{128}}{n^2} + O\left(n^{-3}\right).$$

Throughout the above calculation, the term involving $(u-1/2)^k$ corresponds to $O(n^{-k/2})$.

For calculating the term involving $(u-1/2)^5$, we use a trick here, as follows:

Suppose that $c(u)$ and $d(u)$ are nonnegative integrable function on $(0,1)$. Then

$$\left(\int_0^1 cd du\right)^2 = \left(\int_0^1 \sqrt{c}\sqrt{c}d du\right)^2 \leq \left(\int_0^1 c du\right)\left(\int_0^1 cd^2 du\right),$$

by the Cauchy–Schwarz inequality. Let $c(u) = u^{(n-1)/2}(1-u)^{(n-1)/2}/B((n+1)/2,(n+1)/2)$ and $d(u) = h^{(5)}(u^*(u))|u-1/2|^5/(u^r(1-u)^r)$. Condition 3 guarantees that h has bounded fifth derivative on $(0,1)$. Then there exists $M > 0$ such that $|h^{(5)}(u)| \leq M$ on $(0,1)$.

$$\left(\int_0^1 \frac{u^{\frac{n-1}{2}-r}(1-u)^{\frac{n-1}{2}-r}}{B(\frac{n+1}{2},\frac{n+1}{2})}\frac{h^{(5)}(u^*(u))}{120}\left(u-\frac{1}{2}\right)^5 du\right)^2 \leq \left(\int_0^1 \frac{u^{\frac{n-1}{2}}(1-u)^{\frac{n-1}{2}}}{B\left(\frac{n+1}{2},\frac{n+1}{2}\right)}du\right)\left(\int_0^1 \frac{u^{\frac{n-1}{2}}(1-u)^{\frac{n-1}{2}}}{B\left(\frac{n+1}{2},\frac{n+1}{2}\right)}\left(\frac{h^{(5)}(u^*(u))\left|u-\frac{1}{2}\right|^5}{u^r(1-u)^r}\right)^2 du\right) \leq M^2\int_0^1 \frac{u^{\frac{n-1}{2}-2r}(1-u)^{\frac{n-1}{2}-2r}}{B(\frac{n+1}{2},\frac{n+1}{2})}\left(u-\frac{1}{2}\right)^{10} du = O\left(n^{-5}\right).$$

All in all,

$$E\left(X_{(\frac{n+1}{2})}\right) - \omega = \frac{a}{n} + \frac{b_1}{n^2} + O\left(n^{-\frac{5}{2}}\right), \tag{7}$$

where $a = g^{(2)}(1/2)/8$ and $b_1 = -g^{(2)}(1/2)/4 + g^{(4)}(1/2)/128$.

2.2. Case 2: n Is Even

The sample median is $(X_{(n/2)}+X_{(n/2+1)})/2$. By a Taylor formula with Lagrange residual term about $u = 1/2$,

$$E\left(X_{(\frac{n}{2})}\right) = \int_0^1 \frac{u^{\frac{n}{2}-1}(1-u)^{\frac{n}{2}}}{B(\frac{n}{2},\frac{n}{2}+1)}F^{-1}(u)du = \int_0^1 \frac{u^{\frac{n}{2}-1-r}(1-u)^{\frac{n}{2}-r}}{B(\frac{n}{2},\frac{n}{2}+1)}h(u)du = \int_0^1 \frac{u^{\frac{n}{2}-1-r}(1-u)^{\frac{n}{2}-r}}{B(\frac{n}{2},\frac{n}{2}+1)}\left(h\left(\frac{1}{2}\right) + h'\left(\frac{1}{2}\right)\left(u-\frac{1}{2}\right) + \frac{1}{2}h^{(2)}\left(\frac{1}{2}\right)\left(u-\frac{1}{2}\right)^2 + \frac{1}{6}h^{(3)}\left(\frac{1}{2}\right)\left(u-\frac{1}{2}\right)^3 + \frac{1}{24}h^{(4)}\left(\frac{1}{2}\right)\left(u-\frac{1}{2}\right)^4 + \frac{1}{120}h^{(5)}(u^*(u))\left(u-\frac{1}{2}\right)^5\right)du, \tag{8}$$

for u^* as in Section 2.1. To evaluate the second term, denote the sample by $X := (X_1, X_2, \ldots, X_n)$. Let $Y = -X$. Then

$$E\left(X_{(\frac{n}{2}+1)}\right) = -E\left(Y_{(\frac{n}{2})}\right) = \int_0^1 \frac{u^{\frac{n}{2}-1-r}(1-u)^{\frac{n}{2}-r}}{B(\frac{n}{2},\frac{n}{2}+1)}h(1-u)du = \int_0^1 \frac{u^{\frac{n}{2}-1-r}(1-u)^{\frac{n}{2}-r}}{B(\frac{n}{2},\frac{n}{2}+1)}\left(h\left(\frac{1}{2}\right) - h'\left(\frac{1}{2}\right)\left(u-\frac{1}{2}\right) + \frac{1}{2}h^{(2)}\left(\frac{1}{2}\right)\left(u-\frac{1}{2}\right)^2 - \frac{1}{6}h^{(3)}\left(\frac{1}{2}\right)\left(u-\frac{1}{2}\right)^3 + \frac{1}{24}h^{(4)}\left(\frac{1}{2}\right)\left(u-\frac{1}{2}\right)^4 - \frac{1}{120}h^{(5)}(u^*(1-u))\left(u-\frac{1}{2}\right)^5\right)du. \tag{9}$$

Add the above two equations and obtain

$$E\left(\frac{X_{(\frac{n}{2})}+X_{(\frac{n}{2}+1)}}{2}\right) = \frac{1}{B(\frac{n}{2},\frac{n}{2}+1)}\int_0^1 u^{\frac{n}{2}-1-r}(1-u)^{\frac{n}{2}-r}\left(h\left(\frac{1}{2}\right) + \frac{1}{2}h^{(2)}\left(\frac{1}{2}\right)\left(u-\frac{1}{2}\right)^2 + \frac{1}{24}h^{(4)}\left(\frac{1}{2}\right)\left(u-\frac{1}{2}\right)^4 + \frac{h^{(5)}(u^*(u)) - h^{(5)}(u^*(1-u))}{240}\left(u-\frac{1}{2}\right)^5\right)du. \tag{10}$$

Similar to the case when n is odd,

$$\frac{1}{B(\frac{n}{2},\frac{n}{2}+1)}\int_0^1 u^{\frac{n}{2}-1-r}(1-u)^{\frac{n}{2}-r}h\left(\frac{1}{2}\right)du = \omega + \frac{r\omega}{n} + \frac{(\frac{3}{2}r^2+\frac{1}{2}r)\omega}{n^2} + O\left(n^{-3}\right).$$

$$\int_0^1 \frac{u^{\frac{n}{2}-1-r}(1-u)^{\frac{n}{2}-r}}{B(\frac{n}{2},\frac{n}{2}+1)}\frac{h^{(2)}(\frac{1}{2})}{2}\left(u-\frac{1}{2}\right)^2 du = \frac{-r\omega + \frac{g^{(2)}(\frac{1}{2})}{8}}{n} + \frac{(3r-1)(-r\omega + \frac{g^{(2)}(\frac{1}{2})}{8})}{n^2} + O\left(n^{-3}\right).$$

$$\int_0^1 \frac{u^{\frac{n}{2}-1-r}(1-u)^{\frac{n}{2}-r}}{B(\frac{n}{2},\frac{n}{2}+1)}\frac{h^{(4)}(\frac{1}{2})}{24}\left(u-\frac{1}{2}\right)^4 du = \frac{(\frac{3}{2}r^2-\frac{3}{2}r)\omega - \frac{3}{8}rg^{(2)}(\frac{1}{2}) + \frac{1}{128}g^{(4)}(\frac{1}{2})}{n^2} + O\left(n^{-3}\right),$$

and

$$\left(\frac{1}{B(\frac{n}{2},\frac{n}{2}+1)}\int_0^1 u^{\frac{n}{2}-1-r}(1-u)^{\frac{n}{2}-r}\frac{h^{(5)}(u^*(u)) - h^{(5)}(u^*(1-u))}{240}\left(u-\frac{1}{2}\right)^5 du\right)^2 \leq \frac{M^2}{B(\frac{n}{2},\frac{n}{2}+1)}\int_0^1 u^{\frac{n}{2}-1-2r}(1-u)^{\frac{n}{2}-2r}\left(u-\frac{1}{2}\right)^{10} du = O\left(n^{-5}\right).$$

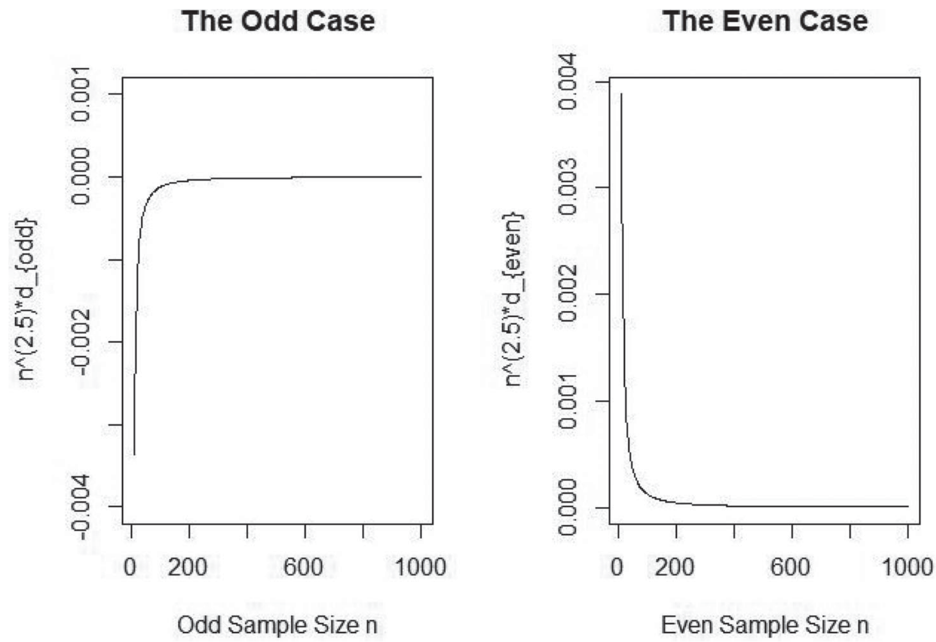

Figure 1. The plots of $n^{5/2} * d_{\text{odd}}$ and $n^{5/2} * d_{\text{even}}$ versus n.

Hence

$$E\left(\frac{X_{(\frac{n}{2})}+X_{(\frac{n}{2}+1)}}{2}\right)-\omega=\frac{a}{n}+\frac{b_2}{n^2}+O\left(n^{-\frac{5}{2}}\right), \tag{11}$$

where the constant a is exactly the same as the constant a in the odd case and $b_2 = -g^{(2)}(1/2)/8 + g^{(4)}(1/2)/128$.

2.3. The Jackknife Does Not Adequately Reduce the Bias of the Sample Median

Recall the sample median T_n and the other notations $T^*_{n-1,i}, \bar{T}^*_n$ and B in the introduction section and note that $b_1 - b_2 = -g^{(2)}(1/2)/8 = -a$.

When n is odd,

$$E(B)=(n-1)\left(\omega+\frac{a}{n-1}+\frac{b_2}{(n-1)^2}-\omega-\frac{a}{n}-\frac{b_1}{n^2}\right)+O\left(n^{-\frac{3}{2}}\right)=\frac{2a}{n}+O\left(n^{-\frac{3}{2}}\right).$$

When n is even,

$$E(B)=0$$

as illustrated in the introduction.

Note that $g^{(2)}(u) = -f'(F^{-1}(u))/(f(F^{-1}(u)))^3$. Recall that ω is the median of the population.

When the density function $f(x)$ satisfies Conditions 1, 2, and 3 at the beginning of Section 2, the impact of application of the jackknife to the sample median can be summarized below.

1. When $f'(\omega) = 0$, the jackknife increases the bias of the sample median since $a = 0$ and $E(B) = O\left(n^{-3/2}\right)$ and bias of T_n is $O\left(n^{-2}\right)$, but the bias of $T_n - B$ is $O\left(n^{-3/2}\right)$.
2. When $f'(\omega) \neq 0$, the jackknife fails for the sample median since $a \neq 0$ and the bias of $T_n - B$ is still $O\left(n^{-1}\right)$.

All in all, the delete-1 jackknife is not conducive to reducing the bias of the sample median. However, from Equations (7) and (11), one can deduce that a delete-2 jackknife will achieve the bias reduction.

3. Verification of the Theoretical Results in This Article in the Case of the Standard Exponential Distribution

This section verifies Equations (7) and (11) in the case of the exponential distribution with $rate = 1$. We rely on the following result (David and Nagaraja 2003, chap.3).

$$E\left(X_{(r)}\right)=\sum_{i=n-r+1}^{n} i^{-1}. \tag{12}$$

Denote the difference between the result from Equation (12) and $\omega + a/n + b_1/n^2$ in the odd case according to Equation (7) or $\omega + a/n + b_2/n^2$ in the even case according to Equation (11) by d_{odd} and d_{even}, respectively. Note that by direct calculation the median of standard exponential distribution is $\omega = \ln 2$, $a = 1/2$, $b_1 = -1/4$ and $b_2 = 1/4$. Specifically,

$$d_{\text{odd}}=\ln 2+\frac{1}{2n}-\frac{1}{4n^2}-\sum_{i=\frac{n+1}{2}}^{n} i^{-1}; \tag{13}$$

$$d_{\text{even}}=\ln 2+\frac{1}{2n}+\frac{1}{4n^2}-\frac{1}{n}-\sum_{i=\frac{n}{2}+1}^{n} i^{-1}. \tag{14}$$

According to Equations (7) and (11), both d_{odd} and d_{even} should be $O\left(n^{-5/2}\right)$. Figure 1 presents $n^{5/2} * d_{\text{odd}}$ and $n^{5/2} * d_{\text{even}}$ versus n, where $n = 11, 13, 15, \ldots, 1001$ and $n = 10, 12, 14, \ldots, 1000$, respectively. In accordance with this figure, $n^{5/2} * d_{\text{odd}}$ and $n^{5/2} * d_{\text{even}}$ are $O(1)$, which advocates the correctness of Equations (7) and (11).

Supplementary Materials

Supplementary material includes additional supporting mathematical calculation.

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