

FROG-measurement based phase retrieval for analytic signals[☆]Youfa Li^a, Yaoshuai Ma^a, Deguang Han^{b,*}^a College of Mathematics and Information Science, Guangxi University, Nanning, 530004, China^b University of Central Florida, Orlando, FL 32816, United States of America

ARTICLE INFO

Article history:

Received 28 February 2021

Received in revised form 9 May 2021

Accepted 20 May 2021

Available online 25 May 2021

Communicated by Radu Balan

MSC:

42C40

65T60

94A20

Keywords:

Phase retrieval

FROG measurement

Analytic signal

Determination approach

ABSTRACT

While frequency-resolved optical gating (FROG) is widely used in characterizing the ultrafast pulse in optics, analytic signals are often considered in time-frequency analysis and signal processing, especially when extracting instantaneous features of events. In this paper we examine the phase retrieval (PR) problem of analytic signals in $\mathbb{C}^N$ by their FROG measurements. After establishing the ambiguity of the FROG-PR of analytic signals, we found that the FROG-PR of analytic signals of even lengths is different from that of analytic signals of odd lengths, and it is also different from the case of B -bandlimited signals with $B \leq N/2$. The existing approach to bandlimited signals can be applied to analytic signals of odd lengths, but it does not apply to the even length case. With the help of two relaxed FROG-PR problems and a translation technique, we develop an approach to FROG-PR for the analytic signals of even lengths, and prove that in this case the generic analytic signals can be uniquely (up to the ambiguity) determined by their $(3N/2+1)$ FROG measurements. Phase derivative (or instantaneous frequency) is a significant feature for signal analysis. As an application of our main result, an approach is established to determine the phase derivatives of decaying signals by exploiting the ambiguity.

© 2021 Elsevier Inc. All rights reserved.

1. Introduction

The classical phase retrieval (PR) is a nonlinear problem that seeks to reconstruct a signal $\mathbf{z} = (\mathbf{z}_0, \dots, \mathbf{z}_{N-1}) \in \mathbb{C}^N$ (up to the ambiguities) from the intensities of its Fourier measurements (cf. [8,21–24, 42])

$$b_k := \left| \sum_{n=0}^{N-1} \mathbf{z}_n e^{-2\pi i k n / N} \right|, \quad k \in \Gamma.$$

[☆] Youfa Li is partially supported by Natural Science Foundation of China (Nos: 61961003, 61561006, 11501132), Natural Science Foundation of Guangxi Province (Nos: 2018JJA110110, 2016GXNSFAA380049) and the talent project of Education Department of Guangxi Government for Young-Middle-Aged backbone teachers. Deguang Han is partially supported by the NSF grant DMS-1712602 and DMS-2105038.

* Corresponding author.

E-mail addresses: youfalee@hotmail.com (Y. Li), MYShuai2021@163.com (Y. Ma), Deguang.Han@ucf.edu (D. Han).

Here the ambiguity (cf. [6]) means that, there exist other signals in $\mathbb{C}^N$ such that they have the same intensity measurements as $\mathbf{z}$.

PR has been widely applied to engineering problems such as the coherent diffraction imaging ([42]) and quantum tomography ([28]). To be adapted to more applications, it has been investigated by non Fourier measurements such as frame intensity measurements (e.g. [2–5,10,9,11,12,29,32,33,47,48]) and phaseless sampling (e.g. [14,15,34]). In this paper we examine the PR problem related to the FROG (frequency-resolved optical gating) measurements (cf. [7,44,45]) for a special type of signals.

Given a pair $[N, L] \in \mathbb{N}^2$ such that $L \leq N$, denote $r := \lceil N/L \rceil$. For $\mathbf{z} = (\mathbf{z}_0, \dots, \mathbf{z}_{N-1}) \in \mathbb{C}^N$, $k = 0, 1, \dots, N-1$ and $m = 0, 1, \dots, r-1$, define $y_{k,m} := \mathbf{z}_k \mathbf{z}_{k+mL}$. Then the $[N, L]$ -**FROG measurement** of $\mathbf{z}$ at (k, m) is defined as

$$\begin{aligned} |\hat{y}_{k,m}|^2 &= \left| \sum_{n=0}^{N-1} y_{n,m} e^{-i2\pi kn/N} \right|^2 \\ &= \left| \sum_{n=0}^{N-1} \mathbf{z}_n \mathbf{z}_{n+mL} e^{-i2\pi kn/N} \right|^2. \end{aligned} \quad (1.1)$$

It follows from [7, (3.1)] that

$$\hat{y}_{k,m} = \frac{1}{N} \sum_{l=0}^{N-1} \widehat{\mathbf{z}}_l \widehat{\mathbf{z}}_{k-l} w^{lm}, \quad (1.2)$$

where $w = e^{i2\pi/r}$. Now the FROG-PR asks to determine $\mathbf{z}$ by its FROG measurements $\{|\hat{y}_{k,m}|^2\}$, up to the ambiguity which might be different depending on the type of signals we are working with. This problem was recently investigated by T. Bendory, D. Edidin and Y.C. Eldar [7] for B -bandlimited signals in $\mathbb{C}^N$. When $B \leq N/2$, the ambiguity arises from arbitrary rotation, arbitrary translation and reflection.

Theorem 1.1. [7] Assume that $N/L \geq 4$ and $B \leq N/2$. Then generic B -bandlimited signals in $\mathbb{C}^N$ are uniquely (up to their ambiguity) determined from $3B$ FROG measurements.

One of the key steps in the proof in [7, section 3.2] for the FROG-PR is to assign 0 or π to both the two phases $\arg(\widehat{\mathbf{z}}_0)$ and $\arg(\widehat{\mathbf{z}}_1)$. Such an assignment holds due to the above ambiguity. In this paper we are interested in investigating analytic signals whose ambiguity varies depending on the lengths of the signals.

Let $\mathbf{x} \in \mathbb{R}^N$ be a real-valued signal with **discrete Fourier transform (DFT)** $\widehat{\mathbf{x}} = (\widehat{\mathbf{x}}_0, \dots, \widehat{\mathbf{x}}_{N-1})$. By L. Marple [36], the analytic signal $A(\mathbf{x}) = ((A(\mathbf{x}))_0, \dots, (A(\mathbf{x}))_{N-1})$ corresponding to $\mathbf{x}$ is defined through its DFT $\widehat{A(\mathbf{x})} = ((\widehat{A(\mathbf{x})})_0, \dots, (\widehat{A(\mathbf{x})})_{N-1})$, where for even length N ,

$$(\widehat{A(\mathbf{x})})_k = \begin{cases} \widehat{\mathbf{x}}_0, & k = 0, \\ 2\widehat{\mathbf{x}}_k, & 1 \leq k \leq N/2 - 1, \\ \widehat{\mathbf{x}}_{N/2}, & k = N/2, \\ 0, & N/2 + 1 \leq k \leq N - 1, \end{cases} \quad (1.3)$$

and for odd length N ,

$$(\widehat{A(\mathbf{x})})_k = \begin{cases} \widehat{\mathbf{x}}_0, & k = 0, \\ 2\widehat{\mathbf{x}}_k, & 1 \leq k \leq (N-1)/2, \\ 0, & (N+1)/2 \leq k \leq N-1. \end{cases} \quad (1.4)$$

By [36], the real part $\Re(A(\mathbf{x})) = \mathbf{x}$ and the imaginary part $\Im(A(\mathbf{x}))$ is the discrete Hilbert transform of $\mathbf{x}$. Moreover, the inner product $\langle \Re(A(\mathbf{x})), \Im(A(\mathbf{x})) \rangle = 0$.

Analytic signals form an important class of signals that have been widely used in time-frequency analysis and signal processing, especially in extracting instantaneous features (e.g. [20,35,41,46]). In order to examine the phase retrieval (up to ambiguity) problem for analytic signals from their FROG measurements, we need first to identify their ambiguity of FROG-PR measurements, which will be described in Proposition 2.3 and Proposition 2.4. A key point is that the FROG-PR ambiguity of analytic signals of even lengths is different from that of analytic signals of odd lengths, and consequently they are different from that of B -bandlimited signals in $\mathbb{C}^N$ [7] with $B \leq N/2$. In the next section we will explain while the approach from [7] can be applied to the signals of odd lengths, even length analytic signals are quite different from the odd length ones and it requires a different approach for such signals. So we only need to focus on analytic signals of even lengths. The following is the main result (the proof follows from Theorem 3.8) of this paper.

Theorem 1.2. *Assume that N is even, L is odd and $r = \lceil N/L \rceil \geq 5$. Then generic analytic signals in $\mathbb{C}^N$ can be determined uniquely up to the ambiguity in Proposition 2.3 by their $(3N/2 + 1)$ $[N, L]$ -FROG measurements.*

Here by **up to ambiguity** of an analytic signal $\mathbf{z} \in \mathbb{C}^N$ we mean that the signal is uniquely determined up to its π -rotation $-\mathbf{z}$, integer-translation $\mathbf{z}_{tr}^l$ with $l \in \mathbb{Z}$, and reflection $\mathbf{z}_{ref}$. Note that for **generic** analytic signals in $\mathbb{C}^N$ of even lengths, the bandlimit is $B = N/2 + 1$. The above theorem tells us that such an analytic signal can be uniquely (up to the ambiguity) determined by $3B - 2$ number of FROG measurements. Moreover, the procedures for such a determination will be provided by Approach 3.3. In what follows, we introduce an application of the main result.

Phase derivative (or instantaneous frequency) of a signal is a significant feature for signal analysis (e.g. [16–19,25]). It is also widely applied in optics such as wave-front reconstruction (e.g. [39]), holography (e.g. [26]), phase unwrapping (e.g. [40]) and the approximation to the effect of a changing pulse frequency (e.g. [27]). Like phase, it is very difficult to measure phase derivative ([38]), and therefore its recovery is of great importance. There are some recovery methods in the literature such as that in [30] from the fringe pattern and that in [31] from holographic interference field. Recovery of phase derivative from FROG measurement is highly desirable, albeit a hitherto unexplored topic, to the best of our knowledge. Since FROG is commonly used for measuring the ultrashort laser pulse (ULP), such a recovery is helpful for approximating the effect of the changing frequency of ULP ([27]). By examining the effect of the above mentioned ambiguity on the phase distribution, we also establish the determination of phase derivatives of decaying signals.

Outline of the paper. In Section 2 we first present a characterization of analytic signals, and then establish the ambiguity of FROG-PR based on the parity of the lengths of the analytic signals. A rationale is provided at the end of this section to explain why we need a different approach for the even length case. Section 3 is devoted to presenting the step-by-step approach that eventually led to the main theorem. The technical proofs for most of the theoretical preparations that led to the main result (Theorem 3.8) are presented in Section 5. At the end of the paper, we point out that our approach fails for the case when L is even, and further investigation is needed to address this case. As an application of the main result, in Section 4 a determination approach (Approach 4.1) is established for the phase derivatives of decaying signals.

2. The ambiguity of FROG-PR of analytic signals

We first fix a few standard notations. We use lower-case letters, e.g., z , to denote scalars, and use boldface letters, e.g., $\mathbf{z}$ to denote a vector. For the vector $\mathbf{z} \in \mathbb{C}^N$, it is represented as $\mathbf{z} = (\mathbf{z}_0, \dots, \mathbf{z}_{N-1})$. A complex number $0 \neq z = \Re(z) + \mathbf{i}\Im(z) \in \mathbb{C}$ can be denoted by $|z|e^{\mathbf{i}\arg(z)}$, where $\mathbf{i}$, $|z|$ and $\arg(z)$ are the imaginary unit,

modulus and phase, respectively. The conjugation of a complex vector $\mathbf{z} = (z_0, \dots, z_{N-1}) \in \mathbb{C}^N$ is denoted by $\bar{\mathbf{z}} := (\bar{z}_0, \dots, \bar{z}_{N-1})$, where $\bar{z}_k$ is the complex conjugation of z_k . Throughout this paper, the signal $\mathbf{z}$ is N -periodic, namely, $\mathbf{z}_\ell = \mathbf{z}_{\ell+N}$ for all $\ell \in \mathbb{Z}$. From now on, its DFT is defined by $\hat{\mathbf{z}} := (\hat{z}_0, \dots, \hat{z}_{N-1})$ with $\hat{z}_k = \sum_{n=0}^{N-1} z_n e^{-2\pi i k n / N}$. And through the inverse discrete Fourier transform (IDFT), $\mathbf{z}$ can be reconstructed by $\mathbf{z}_k = \frac{1}{N} \sum_{n=0}^{N-1} \hat{z}_n e^{2\pi i k n / N}$. For any $\mathbf{z} \in \mathbb{C}^N$, its θ -rotation $\mathbf{z}_{ro}^\theta := e^{i\theta} \mathbf{z}$, reflection $\mathbf{z}_{ref} := (\bar{z}_0, \bar{z}_{-1}, \dots, \bar{z}_{-(N-1)})$ and for any $\gamma \in \mathbb{R}$, its γ -translation $\mathbf{z}_{tr}^\gamma$ is defined through $\hat{\mathbf{z}}_{tr}^\gamma = (\hat{z}_0, \hat{z}_1 e^{i2\pi\gamma/N}, \dots, \hat{z}_{N-1} e^{i2\pi(N-1)\gamma/N})$. Denote by $\#\Lambda$ the cardinality of a set Λ , and by $\lceil x \rceil$ the smallest integer that is not smaller than $x \in \mathbb{R}$.

Definition 2.1. We say that $\mathbf{z}$ is B -bandlimited if $\hat{\mathbf{z}}$ contains $N - B$ consecutive zeros (cf. [7]).

The following characterizes analytic signals, and its proof will be presented in section 5.1.

Proposition 2.1. Suppose that $\mathbf{z} \in \mathbb{C}^N$. Denote the Cartesian product of sets by $\times$. Then $\mathbf{z}$ is analytic if and only if one of the following two items holds:

- (i) for even length N , $\hat{\mathbf{z}} \in \mathbb{R} \times \mathbb{C}^{N/2-1} \times \mathbb{R} \times \overbrace{\{0\} \times \dots \times \{0\}}^{(N/2-1) \text{ copies}};$
- (ii) for odd length N , $\hat{\mathbf{z}} \in \mathbb{R} \times \mathbb{C}^{(N-1)/2} \times \overbrace{\{0\} \times \dots \times \{0\}}^{[(N-1)/2] \text{ copies}}.$

As a consequence of Proposition 2.1, we get

Proposition 2.2. Suppose that $\mathbf{z} \in \mathbb{C}^N$ is analytic. Then the following two items hold.

- (i) If N is even and $\hat{\mathbf{z}} = (\hat{z}_0, \dots, \hat{z}_{N/2}, 0, \dots, 0)$ satisfies $\hat{z}_{N/2} \neq 0$, then the γ -translation $\mathbf{z}_{tr}^\gamma$ is still analytic if and only if $\gamma \in \mathbb{Z}$.
- (ii) If $\hat{z}_0 \neq 0$, then its θ -rotation $e^{i\theta} \mathbf{z}$ is still analytic if and only if $\theta = k\pi$ for $k \in \mathbb{Z}$.

Proposition 2.2 (i) implies that in the even length case, the non-integer translation does not inherit the analytic property. Moreover, the following example shows that if $\gamma \notin \mathbb{Z}$, then the γ -translation $\mathbf{z}_{tr}^\gamma$ does not necessarily have the same FROG measurements as $\mathbf{z}$.

Example 2.1. Let $\mathbf{x} = (0.3252, -0.7549, 1.3703, -1.7115) \in \mathbb{R}^4$. Then its analytic signal is given by $\mathbf{z} = A(\mathbf{x}) = (0.3252 - 0.4783i, -0.7549 - 0.5226i, 1.3703 + 0.4783i, -1.7115 + 0.5226i)$. By DFT, we have $\hat{\mathbf{z}} = (-0.7710, -2.0902 - 1.9132i, 4.1619, 0)$. Choose $\gamma = \frac{2}{\pi}$. Then we have $\hat{\mathbf{z}}_{tr}^\gamma = (-0.7710, 0.4805 - 2.7925i, -1.7320 + 3.7844i, 0)$ and $\mathbf{z}_{tr}^\gamma = (-0.5056 + 0.2480i, 0.9384 - 0.8260i, -0.7459 + 1.6442i, -0.4579 - 1.0662i)$. Take $L = 1$ for example. By direct calculation, the FROG measurement of $\mathbf{z}$ at $(0, 0)$ is $|\hat{y}_{0,0}|^2 = 20.0614$ while that of $\mathbf{z}_{tr}^\gamma$ is $|\hat{y}_{0,0}^\gamma|^2 = 17.9335$. Thus the $\mathbf{z}$ and $\mathbf{z}_{tr}^\gamma$ do not have the same FROG measurements.

The following two propositions establish the ambiguity of FROG-PR of analytic signals.

Proposition 2.3. Suppose that $\mathbf{z} \in \mathbb{C}^N$ is analytic such that N is even. Then its π -rotation $-\mathbf{z}$, integer-translation $\mathbf{z}_{tr}^l$ with $l \in \mathbb{Z}$, and reflection $\mathbf{z}_{ref}$ are all analytic. Moreover, they have the same FROG measurements as $\mathbf{z}$.

Proof. By Proposition 2.2, both $-\mathbf{z}$ and $\mathbf{z}_{tr}^l$ are analytic. By Proposition 2.1 (i), $\mathbf{z}_{ref}$ is also analytic. Moreover, by [7, Proposition 2.2] they have the same FROG measurements as $\mathbf{z}$. $\square$

Proposition 2.4. Suppose that $\mathbf{z} \in \mathbb{C}^N$ is analytic such that N is odd. Then its π -rotation $-\mathbf{z}$, any translation $\mathbf{z}_{tr}^\gamma$ with $\gamma \in \mathbb{R}$, and reflection $\mathbf{z}_{ref}$ are all analytic. Moreover, they have the same FROG measurements as $\mathbf{z}$.

By Propositions 2.3 and 2.4, the FROG-PR ambiguity of analytic signals of odd lengths is essentially different from that of analytic signals of even lengths, and the ambiguity of analytic signals for the even length case is also different from ambiguity of signals investigated in [7, Proposition 2.2]. Therefore it is expected that different approaches might be needed for analytic signals of different lengths. Next we explain that while the FROG-PR of analytic signals of odd lengths can be achieved through the similar procedures as those in [7, section 3.2], such procedures do not hold for analytic signals of even lengths.

Case I: N is odd. Suppose that $\mathbf{z} \in \mathbb{C}^N$ is analytic and N is odd. Through the direct calculation, the equation system (1.2) enjoys the following “pyramid” structure w.r.t. variables $\widehat{\mathbf{z}}_0, \dots, \widehat{\mathbf{z}}_{(N-1)/2}$:

$$\begin{aligned}
 & \widehat{\mathbf{z}}_0^2, 0, 0, \dots, 0, 0, \dots, 0 \\
 & \widehat{\mathbf{z}}_0 \widehat{\mathbf{z}}_1, \widehat{\mathbf{z}}_1 \widehat{\mathbf{z}}_0, 0, \dots, 0, 0, \dots, 0 \\
 & \widehat{\mathbf{z}}_0 \widehat{\mathbf{z}}_2, \widehat{\mathbf{z}}_1^2, \widehat{\mathbf{z}}_2 \widehat{\mathbf{z}}_0, \dots, 0, 0, \dots, 0 \\
 & \vdots \\
 & \widehat{\mathbf{z}}_0 \widehat{\mathbf{z}}_{(N-1)/2}, \widehat{\mathbf{z}}_1 \widehat{\mathbf{z}}_{(N-1)/2-1}, \widehat{\mathbf{z}}_2 \widehat{\mathbf{z}}_{(N-1)/2-2}, \dots, \widehat{\mathbf{z}}_{(N-1)/2} \widehat{\mathbf{z}}_0, 0, \dots, 0 \\
 & 0, \widehat{\mathbf{z}}_1 \widehat{\mathbf{z}}_{(N-1)/2}, \widehat{\mathbf{z}}_2 \widehat{\mathbf{z}}_{(N-1)/2-1}, \dots, \widehat{\mathbf{z}}_{(N-1)/2} \widehat{\mathbf{z}}_1, 0, \dots, 0 \\
 & \vdots \\
 & 0, 0, 0, \dots, \widehat{\mathbf{z}}_{(N-1)/2-1} \widehat{\mathbf{z}}_{(N-1)/2}, \widehat{\mathbf{z}}_{(N-1)/2} \widehat{\mathbf{z}}_{(N-1)/2-1}, 0, \dots, 0 \\
 & 0, 0, 0, \dots, 0, \widehat{\mathbf{z}}_{(N-1)/2}^2, 0, \dots, 0.
 \end{aligned} \tag{2.1}$$

It takes the identical form as that in [7, (3.4)] for B -bandlimited signals where $B \leq N/2$. Therefore, by letting $\arg(\widehat{\mathbf{z}}_0) \in \{0, \pi\}$ and using the similar procedures in [7, section 3.2], we can determine $\mathbf{z}$ up to the ambiguity in Proposition 2.4 by $(3(N+1)/2-6)$ FROG measurements. Thus this paper will be only focused the case when N is even.

Case II: N is even. Suppose that $\mathbf{z} \in \mathbb{C}^N$ such that N is even. Then the corresponding system (1.2) has the following structure w.r.t. variables $\widehat{\mathbf{z}}_0, \dots, \widehat{\mathbf{z}}_{N/2}$:

$$\begin{aligned}
 & \widehat{\mathbf{z}}_0^2, 0, 0, \dots, \widehat{\mathbf{z}}_{N/2}^2, 0, \dots, 0 \\
 & \widehat{\mathbf{z}}_0 \widehat{\mathbf{z}}_1, \widehat{\mathbf{z}}_1 \widehat{\mathbf{z}}_0, 0, \dots, 0, 0, \dots, 0 \\
 & \widehat{\mathbf{z}}_0 \widehat{\mathbf{z}}_2, \widehat{\mathbf{z}}_1^2, \widehat{\mathbf{z}}_2 \widehat{\mathbf{z}}_0, \dots, 0, 0, \dots, 0 \\
 & \vdots \\
 & \widehat{\mathbf{z}}_0 \widehat{\mathbf{z}}_{N/2-1}, \widehat{\mathbf{z}}_1 \widehat{\mathbf{z}}_{N/2-2}, \widehat{\mathbf{z}}_2 \widehat{\mathbf{z}}_{N/2-3}, \dots, 0, 0, \dots, 0 \\
 & \widehat{\mathbf{z}}_0 \widehat{\mathbf{z}}_{N/2}, \widehat{\mathbf{z}}_1 \widehat{\mathbf{z}}_{N/2-1}, \widehat{\mathbf{z}}_2 \widehat{\mathbf{z}}_{N/2-2}, \dots, \widehat{\mathbf{z}}_{N/2} \widehat{\mathbf{z}}_0, 0, \dots, 0 \\
 & 0, \widehat{\mathbf{z}}_1 \widehat{\mathbf{z}}_{N/2}, \widehat{\mathbf{z}}_2 \widehat{\mathbf{z}}_{N/2-1}, \dots, \widehat{\mathbf{z}}_{N/2} \widehat{\mathbf{z}}_1, 0, \dots, 0 \\
 & \vdots \\
 & 0, 0, \dots, 0, \widehat{\mathbf{z}}_{N/2-1} \widehat{\mathbf{z}}_{N/2}, \widehat{\mathbf{z}}_{N/2} \widehat{\mathbf{z}}_{N/2-1}, 0, \dots, 0.
 \end{aligned} \tag{2.2}$$

Note that if $\widehat{\mathbf{z}}_{N/2} \neq 0$, then (2.2) (actually not a pyramid structure) takes the different form from (2.1). The procedures for FROG-PR in [7, section 3.2] do not hold for $\mathbf{z}$. Firstly, in [7, section 3.2], for a B -bandlimited signal $\mathbf{y} \in \mathbb{C}^N$ such that $B \leq N/2$, $\arg(\widehat{\mathbf{y}}_0)$ and $\arg(\widehat{\mathbf{y}}_1)$ can be assigned arbitrarily. Such an assignment does not hold for the analytic signal $\mathbf{z}$ since Proposition 2.3 implies that the arbitrary translation and rotation do not necessarily lead to the ambiguity of FROG-PR. Secondly, unlike those in (2.1) and [7, (3.4)], the first row in (2.2) is involved with the two variables $\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_{N/2} \in \mathbb{R}$. Then $\widehat{\mathbf{z}}_0$ can not be determined by just letting $\arg(\widehat{\mathbf{z}}_0) \in \{0, \pi\}$. Actually, it will be clear in Theorems 3.4 and 3.5 that the determination of $\widehat{\mathbf{z}}_0$ is absolutely not trivial when $|\widehat{\mathbf{z}}_0| \neq |\widehat{\mathbf{z}}_{N/2}|$. On the other hand, from the perspective of L in (1.1), the problem of FROG-PR of analytic signals of even lengths is also different from that of bandlimited signals in [7] since it is required in this paper that L needs to be odd. More details will be included in section 6. In what follows, we state that the FROG-PR of even length can not be achieved through changing the length.

Although the FROG-PR for the odd length case can be achieved by the approach in [7], we do not intend to do the FROG-PR of $A(\mathbf{x}) \in \mathbb{C}^N$ with even N through that of $A(\tilde{\mathbf{x}}) \in \mathbb{C}^{N+1}$ with $\tilde{\mathbf{x}} = (\mathbf{x}, 0)$. There are two main reasons for this. (1) In general, $A(\mathbf{x})$ can not be recovered from $A(\tilde{\mathbf{x}})$ by just exchanging the positions of partial elements since such an operation does not lead to the ambiguity of FROG-PR. (2) It follows from Proposition 2.4 that arbitrary translation can lead to the ambiguity of FROG-PR of $A(\tilde{\mathbf{x}})$. Note that the non-integer translation probably alters the absolute values of the elements of the real part of $A(\mathbf{x})$. Such an alteration will make the recovery of phase derivatives of some types of signals such as the decaying ones (to be addressed in Section 4) much more complicated.

3. The main results

The aim of this section is to establish the uniqueness results for the FROG-PR problem for analytic signals in $\mathbb{C}^N$ by the $[N, L]$ -FROG measurements, where N and L are respectively even and odd such that $r = \lceil N/L \rceil \geq 5$. This will be achieved by introducing a series of approaches/algorithms for the determination of such analytic signals.

From now on the DFT of an analytic signal $\mathbf{z} \in \mathbb{C}^N$ is denoted by $\widehat{\mathbf{z}} = (\widehat{\mathbf{z}}_0, \dots, \widehat{\mathbf{z}}_{N/2}, 0, \dots, 0)$. It follows from (1.2) and $\text{supp}(\widehat{\mathbf{z}}) \subseteq \{0, \dots, N/2\}$ that, the FROG-PR of $\mathbf{z}$ is equivalent to finding an analytic signal $\tilde{\mathbf{z}} \in \mathbb{C}^N$ such that its DFT $\widehat{\tilde{\mathbf{z}}} = (\widehat{\tilde{\mathbf{z}}}_0, \dots, \widehat{\tilde{\mathbf{z}}}_{N/2}, \overbrace{0, \dots, 0}^{N/2-1})$ satisfies the following conditions:

$$\left\{ \begin{array}{l} |\hat{y}_{0,m}| = \frac{1}{N} |\widehat{\mathbf{z}}_0^2 + \widehat{\mathbf{z}}_{N/2}^2 w^{Nm/2}|, \quad (3.1A) \\ |\hat{y}_{k,m}| = \frac{1}{N} \left| \sum_{l=0}^k \widehat{\mathbf{z}}_l \widehat{\mathbf{z}}_{k-l} w^{lm} \right|, k = 1, \dots, N/2, \quad (3.1B) \\ |\hat{y}_{k,m}| = \frac{1}{N} \left| \sum_{l=k-N/2}^{N/2} \widehat{\mathbf{z}}_l \widehat{\mathbf{z}}_{k-l} w^{lm} \right|, k = N/2 + 1, \dots, N-1, \quad (3.1C) \\ m = 0, 1, \dots, r-1, \end{array} \right. \quad (3.1)$$

where $\{|\hat{y}_{k,m}|^2\}$ are the $[N, L]$ -FROG measurements of $\mathbf{z}$. Clearly, $\widehat{\mathbf{z}}$ satisfies (3.1).

We outline below the four key steps that we will use to find such a $\tilde{\mathbf{z}}$.

- (i) It follows from Proposition 2.1 (i) that $\widehat{\mathbf{z}}_0$ is real-valued. Subsection 3.2 will be used to determine $\widehat{\mathbf{z}}_0 \in \mathbb{R}$ up to a sign.

(ii) It follows from (3.1B) that $|\widehat{\mathbf{z}}_1| := \frac{N|\hat{y}_{1,0}|}{2|\widehat{\mathbf{z}}_0|}$. Setting $\widehat{\mathbf{z}}_0 = \epsilon \widehat{\mathbf{z}}_0$ with $\epsilon \in \{1, -1\}$ and $\widehat{\mathbf{z}}_1 = \frac{N|\hat{y}_{1,0}|}{2|\widehat{\mathbf{z}}_0|}$, Subsection

3.3 will be devoted to finding a signal $\tilde{\mathbf{z}} \in \mathbb{C}^N$ such that its DFT $\widehat{\tilde{\mathbf{z}}} = (\widehat{\tilde{\mathbf{z}}}_0, \widehat{\tilde{\mathbf{z}}}_1, \dots, \widehat{\tilde{\mathbf{z}}}_{N/2}, \underbrace{0, \dots, 0}_{N/2-1}) = (\epsilon \widehat{\mathbf{z}}_0, \frac{N|\hat{y}_{1,0}|}{2|\widehat{\mathbf{z}}_0|}, \dots, \widehat{\mathbf{z}}_{N/2}, \underbrace{0, \dots, 0}_{N/2-1})$ satisfies

$$\begin{cases} |\hat{y}_{k,m}| = \frac{1}{N} \left| \sum_{l=0}^k \widehat{\tilde{\mathbf{z}}}_l \widehat{\tilde{\mathbf{z}}}_{k-l} w^{lm} \right|, k = 2, \dots, N/2, \\ |\hat{y}_{k,m}| = \frac{1}{N} \left| \sum_{l=k-N/2}^{N/2} \widehat{\tilde{\mathbf{z}}}_l \widehat{\tilde{\mathbf{z}}}_{k-l} w^{lm} \right|, k = N/2 + 1, \dots, N-1, \\ m = 0, 1, \dots, r-1. \end{cases} \quad (3.2)$$

Clearly, (3.2) is the relaxed form of (3.1).

(iii) Based on the result of (ii), subsection 3.4 concerns on the solution to the following equation system

w.r.t. $\tilde{\mathbf{z}} \in \mathbb{C}^N$ such that its DFT $\widehat{\tilde{\mathbf{z}}} = (\widehat{\tilde{\mathbf{z}}}_0, \dots, \widehat{\tilde{\mathbf{z}}}_{N/2}, \underbrace{0, \dots, 0}_{N/2-1})$:

$$\begin{cases} |\hat{y}_{k,m}| = \frac{1}{N} \left| \sum_{l=0}^k \widehat{\tilde{\mathbf{z}}}_l \widehat{\tilde{\mathbf{z}}}_{k-l} w^{lm} \right|, k = 1, \dots, N/2, \\ |\hat{y}_{k,m}| = \frac{1}{N} \left| \sum_{l=k-N/2}^{N/2} \widehat{\tilde{\mathbf{z}}}_l \widehat{\tilde{\mathbf{z}}}_{k-l} w^{lm} \right|, k = N/2 + 1, \dots, N-1, \\ m = 0, 1, \dots, r-1. \end{cases} \quad (3.3)$$

Clearly, (3.3) is also the relaxed form of (3.1) but less relaxed than (3.2) since it also contains the $k = 1$ case.

(iv) Having $\tilde{\mathbf{z}} \in \mathbb{C}^N$ satisfying (3.3) at hand, the procedure in (translation technique-based) Approach 3.3 in subsection 3.5 will allow us to determine the solution to (3.1).

In summary, the purpose of step (i) is to determine $\widehat{\mathbf{z}}_0 \in \mathbb{R}$ up to a sign. Since (3.2) in step (ii) is the relaxed form of (3.1), the solution to (3.1) is also the one to (3.2). We next consider (3.3) in step (iii) which is less relaxed than (3.2). We will select an appropriate translation of the solution to (3.3) so that such a translation not only satisfies (3.3) but also satisfies (3.3A) with $k = 0$, and consequently we find a solution to (3.1),

3.1. Auxiliary results

Note 3.1. ([7]) Consider the equation system w.r.t. $z \in \mathbb{C}$:

$$\begin{cases} |z + v_1| = n_1, \\ |z + v_2| = n_2, \\ |z + v_3| = n_3, \end{cases} \quad (3.4)$$

where $n_1, n_2, n_3 \geq 0$, and $v_1, v_2, v_3 \in \mathbb{C}$ are pairwise distinct. Suppose that there exists a solution $\hat{z} = a + ib$ to (3.4). If

$$\Im\left\{\frac{v_1 - v_2}{v_1 - v_3}\right\} \neq 0, \quad (3.5)$$

then $\hat{z}$ is the unique solution and it is given through

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{2} \begin{pmatrix} c & -d \\ e & -f \end{pmatrix}^{-1} \begin{pmatrix} n_1^2 - n_2^2 + |v_2|^2 - |v_1|^2 \\ n_1^2 - n_3^2 + |v_3|^2 - |v_1|^2 \end{pmatrix}, \quad (3.6)$$

where $v_1 - v_2 = c - \mathbf{i}d$ and $v_1 - v_3 = e - \mathbf{i}f$.

Comparing with Note 3.1, the following system of two equations has more than one solution.

Note 3.2. Consider the equation system w.r.t. $z \in \mathbb{C}$:

$$\begin{cases} |z + mv_1| = n_1, \\ |z + mv_2| = n_2, \end{cases} \quad (3.7)$$

where $n_1, n_2 \geq 0$, $v_1, v_2 \in \mathbb{C}$ are distinct and $0 \neq m \in \mathbb{C}$. Suppose that there exists a solution $\hat{z} = a + \mathbf{i}b$ to (3.7).

(i) If $m, v_1, v_2 \in \mathbb{R}$, then the other solution is $\bar{\hat{z}}$ and it is given through

$$\begin{cases} a = (n_1^2 - n_2^2)/[2m(v_1 - v_2)], \\ |b| = \sqrt{n_1^2 - (a + mv_1)^2}. \end{cases} \quad (3.8)$$

(ii) If $m \in \mathbb{C} \setminus \mathbb{R}$ and $v_1, v_2 \in \mathbb{R}$, then there exists another solution $z' \neq \bar{\hat{z}}$. Moreover, $\hat{z}$ and z' can be given by

$$\hat{z} = m(\hat{a} + \mathbf{i}\hat{b}), z' = m(\hat{a} - \mathbf{i}\hat{b}) \quad (3.9)$$

where $\hat{a}, \hat{b} \in \mathbb{R}$ are given by

$$\begin{cases} \hat{a} = (|\frac{n_1}{m}|^2 - |\frac{n_2}{m}|^2)/[2(v_1 - v_2)], \\ |\hat{b}| = \sqrt{|\frac{n_1}{m}|^2 - (a + v_1)^2}. \end{cases} \quad (3.10)$$

Proof. Item (i) is derived from the last paragraph of the proof of [7, Lemma 3.2]. We just need to prove item (ii). Denote $\frac{z}{m} = \hat{a} + \mathbf{i}\hat{b}$. Then it follows from $v_1, v_2 \in \mathbb{R}$ that (3.7) is equivalent to

$$\begin{cases} (\hat{a} + v_1)^2 + \hat{b}^2 = |\frac{n_1}{m}|^2, \\ (\hat{a} + v_2)^2 + \hat{b}^2 = |\frac{n_2}{m}|^2. \end{cases} \quad (3.11)$$

Through the direct calculation we have $\hat{a} = (|\frac{n_1}{m}|^2 - |\frac{n_2}{m}|^2)/[2(v_1 - v_2)]$. On the other hand, if $(\hat{a}, \hat{b})$ is the solution to (3.11), then the other solution is $(\hat{a}, -\hat{b})$. Stated another way, the solutions to (3.7) are $\hat{z} = m(\hat{a} + \mathbf{i}\hat{b})$ and $z' = m(\hat{a} - \mathbf{i}\hat{b})$. Since $m \in \mathbb{C} \setminus \mathbb{R}$, then $z' \neq \bar{\hat{z}}$. $\square$

To be clear in Remarks 3.1 and 3.3 that, the following lemma will be needed for the existence of the FROG measurements in Approaches 3.1 and 3.2.

Lemma 3.3. If $5 \leq r \in \mathbb{N}$, then there exist $m_1, \dots, m_5 \subseteq \{0, 1, \dots, r-1\}$ such that (1) $1 + w^{2m_l} \neq 0$ ($l = 1, \dots, 5$) and (2) $\frac{w^{m_1}}{1+w^{2m_1}} \neq \frac{1}{2}$, where $w = e^{\frac{i2\pi}{r}}$. Moreover, (3) there exists $i \in \{1, 2, \dots, r-1\}$ such that $\frac{w^i + w^{2i}}{1+w^{3i}} \neq 1$,

Proof. The proof will be given in section 5.2. $\square$

The following theorem will be used in section 3.4.

Theorem 3.4. Suppose that $\mathbf{z} \in \mathbb{C}^N$ is a generic analytic signal such that N is even, and its DFT

$$\widehat{\mathbf{z}} = (\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_1, \dots, \widehat{\mathbf{z}}_{N/2}, 0, \dots, 0) := (|\widehat{\mathbf{z}}_0|e^{i\theta_0}, |\widehat{\mathbf{z}}_1|e^{i\theta_1}, \dots, |\widehat{\mathbf{z}}_{N/2}|e^{i\theta_{N/2}}, 0, \dots, 0) \quad (3.12)$$

satisfies $|\widehat{\mathbf{z}}_0| \neq 0$ and $|\widehat{\mathbf{z}}_1| \neq 0$. Consider the following equation system w.r.t. $\tilde{\mathbf{z}} \in \mathbb{C}^N$ such that $\widehat{\tilde{\mathbf{z}}} = (\widehat{\tilde{\mathbf{z}}}_0, \dots, \widehat{\tilde{\mathbf{z}}}_{N/2}, \underbrace{0, \dots, 0}_{N/2-1})$:

$$\begin{cases} |\hat{y}_{k,m}| = \frac{1}{N} \left| \sum_{l=0}^k \widehat{\tilde{\mathbf{z}}}_l \widehat{\tilde{\mathbf{z}}}_{k-l} w^{lm} \right|, k = 1, \dots, N/2, \\ |\hat{y}_{k,m}| = \frac{1}{N} \left| \sum_{l=k-N/2}^{N/2} \widehat{\tilde{\mathbf{z}}}_l \widehat{\tilde{\mathbf{z}}}_{k-l} w^{lm} \right|, k = N/2 + 1, \dots, N-1, \\ m \in \Lambda, \end{cases} \quad (3.13)$$

where $|\hat{y}_{k,m}|^2$ is the $[N, L]$ -FROG measurement of $\mathbf{z}$, $r = \lceil N/L \rceil \geq 5$ and $\Lambda \subseteq \{0, 1, \dots, r-1\}$ is arbitrary such that $\#\Lambda = 5$. If $|\widehat{\tilde{\mathbf{z}}}_0| \neq |\widehat{\mathbf{z}}_0|$ then $\tilde{\mathbf{z}}$ does not satisfy (3.13).

Proof. Suppose that $\widehat{\tilde{\mathbf{z}}}_0 = \lambda |\widehat{\mathbf{z}}_0| e^{i\alpha}$ such that $1 \neq \lambda \geq 0$ and $\alpha \in \mathbb{R}$ is arbitrary. If $\widehat{\tilde{\mathbf{z}}}_0 = 0$ or $\widehat{\tilde{\mathbf{z}}}_1 = 0$, then it is easy to check that FROG measurements at $(1, 0)$ of $\tilde{\mathbf{z}}$ and those of $\mathbf{z}$ are not identical, and the proof is concluded. Otherwise, we have $\lambda > 0$ and

$$\frac{2}{N} |\widehat{\tilde{\mathbf{z}}}_0 \widehat{\tilde{\mathbf{z}}}_1| = |\hat{y}_{1,0}| = \frac{2}{N} |\widehat{\mathbf{z}}_0 \widehat{\mathbf{z}}_1|, \quad (3.14)$$

which implies $|\widehat{\tilde{\mathbf{z}}}_1| = |\widehat{\mathbf{z}}_1|$. Now let $\widehat{\tilde{\mathbf{z}}}_1 = |\widehat{\mathbf{z}}_1| e^{i\theta}$ such that $\theta \in \mathbb{R}$ is arbitrary. If the FROG measurements at $(2, s)$ of $\tilde{\mathbf{z}}$ and $\mathbf{z}$ are not identical, then the proof is concluded, where $s \in \Lambda$. Otherwise, we have

$$\frac{1}{N} |\widehat{\tilde{\mathbf{z}}}_0 \widehat{\tilde{\mathbf{z}}}_2 + \widehat{\tilde{\mathbf{z}}}_1^2 w^m + \widehat{\tilde{\mathbf{z}}}_2 \widehat{\tilde{\mathbf{z}}}_0 w^{2m}| = |\hat{y}_{2,m}| = \frac{1}{N} |\widehat{\mathbf{z}}_0 \widehat{\mathbf{z}}_2 + \widehat{\mathbf{z}}_1^2 w^m + \widehat{\mathbf{z}}_2 \widehat{\mathbf{z}}_0 w^{2m}|, m \in \Lambda. \quad (3.15)$$

Since $\widehat{\tilde{\mathbf{z}}}_0 = \lambda |\widehat{\mathbf{z}}_0| e^{i\alpha}$ and $\widehat{\tilde{\mathbf{z}}}_1 = |\widehat{\mathbf{z}}_1| e^{i\theta}$, then (3.15) can be expressed as

$$|\widehat{\tilde{\mathbf{z}}}_0 \widehat{\tilde{\mathbf{z}}}_2 (1 + w^{2m}) + \widehat{\tilde{\mathbf{z}}}_1^2 w^m|^2 = \left| \lambda |\widehat{\mathbf{z}}_0| e^{i\alpha} \widehat{\tilde{\mathbf{z}}}_2 (1 + w^{2m}) + |\widehat{\mathbf{z}}_1|^2 e^{i2\theta} w^m \right|^2. \quad (3.16)$$

It is easy to check that (3.16) is equivalent to

$$\begin{aligned} 0 = & (\lambda^2 |\widehat{\mathbf{z}}_2|^2 - |\widehat{\mathbf{z}}_2|^2) |\widehat{\mathbf{z}}_0|^2 |1 + w^{2m}|^2 + \left(\frac{1}{\lambda^4} - 1 \right) |\widehat{\mathbf{z}}_1|^4 \\ & + 2\Re \left\{ \overline{(1 + w^{2m})} w^m \overline{(\lambda |\widehat{\mathbf{z}}_0| e^{i\alpha} \widehat{\tilde{\mathbf{z}}}_2)} \frac{\widehat{\tilde{\mathbf{z}}}_1^2}{\lambda} |e^{i2\theta}|^2 - \overline{\widehat{\mathbf{z}}_0 \widehat{\mathbf{z}}_2 \widehat{\mathbf{z}}_1^2} \right\}, \end{aligned} \quad (3.17)$$

where $m \in \Lambda$. Multiplying both sides of (3.17) by w^{2m} leads to

$$0 = (\lambda^2 |\widehat{\mathbf{z}}_2|^2 - |\widehat{\mathbf{z}}_2|^2) \widehat{\mathbf{z}}_0^2 (1 + w^{2m})^2 + \left(\frac{1}{\lambda^4} - 1\right) |\widehat{\mathbf{z}}_1|^4 w^{2m} \\ + \left[\overline{(\lambda |\widehat{\mathbf{z}}_0| e^{i\alpha} \widehat{\mathbf{z}}_2)} \frac{\widehat{\mathbf{z}}_1}{\lambda} |^2 e^{i2\theta} - \lambda |\widehat{\mathbf{z}}_0| e^{i\alpha} \widehat{\mathbf{z}}_2 \overline{\left(\frac{\widehat{\mathbf{z}}_1}{\lambda} |^2 e^{i2\theta} \right)} - (\widehat{\mathbf{z}}_0 \widehat{\mathbf{z}}_2 \widehat{\mathbf{z}}_1^2 - \widehat{\mathbf{z}}_0 \widehat{\mathbf{z}}_2 \widehat{\mathbf{z}}_1^2) \right] (1 + w^{2m}) w^m. \quad (3.18)$$

Consider the following equation w.r.t. x :

$$0 = (\lambda^2 |\widehat{\mathbf{z}}_2|^2 - |\widehat{\mathbf{z}}_2|^2) \widehat{\mathbf{z}}_0^2 (1 + x^2)^2 + \left(\frac{1}{\lambda^4} - 1\right) |\widehat{\mathbf{z}}_1|^4 x^2 \\ + \left[\overline{(\lambda |\widehat{\mathbf{z}}_0| e^{i\alpha} \widehat{\mathbf{z}}_2)} \frac{\widehat{\mathbf{z}}_1}{\lambda} |^2 e^{i2\theta} - \lambda |\widehat{\mathbf{z}}_0| e^{i\alpha} \widehat{\mathbf{z}}_2 \overline{\left(\frac{\widehat{\mathbf{z}}_1}{\lambda} |^2 e^{i2\theta} \right)} - (\widehat{\mathbf{z}}_0 \widehat{\mathbf{z}}_2 \widehat{\mathbf{z}}_1^2 - \widehat{\mathbf{z}}_0 \widehat{\mathbf{z}}_2 \widehat{\mathbf{z}}_1^2) \right] (1 + x^2) x. \quad (3.19)$$

Clearly, if the polynomial on the right-hand side of (3.19) is not a zero polynomial, then it has at most 4 solutions. By (3.18), w^m ($m \in \Lambda$) are the $\#\Lambda$ solutions to (3.19). Since $\#\Lambda = 5$, then all the coefficients in (3.19) are zero. That is,

$$\begin{cases} (\lambda^2 |\widehat{\mathbf{z}}_2|^2 - |\widehat{\mathbf{z}}_2|^2) \widehat{\mathbf{z}}_0^2 = 0, \\ \left(\frac{1}{\lambda^4} - 1\right) |\widehat{\mathbf{z}}_1|^4 = 0, \\ \overline{(\lambda |\widehat{\mathbf{z}}_0| e^{i\alpha} \widehat{\mathbf{z}}_2)} \frac{\widehat{\mathbf{z}}_1}{\lambda} |^2 e^{i2\theta} - \lambda |\widehat{\mathbf{z}}_0| e^{i\alpha} \widehat{\mathbf{z}}_2 \overline{\left(\frac{\widehat{\mathbf{z}}_1}{\lambda} |^2 e^{i2\theta} \right)} - (\widehat{\mathbf{z}}_0 \widehat{\mathbf{z}}_2 \widehat{\mathbf{z}}_1^2 - \widehat{\mathbf{z}}_0 \widehat{\mathbf{z}}_2 \widehat{\mathbf{z}}_1^2) = 0. \end{cases} \quad (3.20)$$

Since $|\widehat{\mathbf{z}}_1| \neq 0$ then $\frac{1}{\lambda^4} - 1 = 0$ and $\lambda = 1$, which contradicts with the previous assumption $1 \neq \lambda \geq 0$. This completes the proof. $\square$

3.2. Determination of $\widehat{\mathbf{z}}_0$

In what follows, we establish an approach to determining $\widehat{\mathbf{z}}_0$ in (3.1) up to a sign. Its theoretical guarantee will be presented in Theorem 3.5.

Approach 3.1. Input: $[N, L]$ -FROG measurements $\left\{ |\hat{y}_{0,0}|^2, |\hat{y}_{0,1}|^2, |\hat{y}_{1,0}|^2, |\hat{y}_{2,i_2^{(q)}}|^2 : i_2^{(1)} = 0, w^{2i_2^{(q)}} \neq -1, \frac{w^{i_2^{(2)}}}{1+w^{2i_2^{(2)}}} \neq \frac{1}{2}, 0 \leq i_2^{(q)} \leq r-1, q = 1, 2, \dots, 5 \right\}$. % % $w = e^{i2\pi/r}$ and $r = \lceil N/L \rceil$.

Step 1: If $|\hat{y}_{0,1}| = 0$, then $\widehat{\mathbf{z}}_0 = \sqrt{\frac{N|\hat{y}_{0,0}|}{2}}$ and we terminate the program. If not, then conduct Step 2 and Step 3 to find $\widehat{\mathbf{z}}_0$.

Step 2: $\widehat{\mathbf{z}}_0 \leftarrow \sqrt{\frac{N(|\hat{y}_{0,0}| + |\hat{y}_{0,1}|)}{2}}$; $\widehat{\mathbf{z}}_1 \leftarrow \frac{N|\hat{y}_{1,0}|}{2\widehat{\mathbf{z}}_0}$;

Step 3: If the following equation system w.r.t. $\widehat{\mathbf{z}}_2$:

$$\frac{1}{N} |\widehat{\mathbf{z}}_0 \widehat{\mathbf{z}}_2 (1 + w^{2i_2^{(q)}}) + \widehat{\mathbf{z}}_1 w^{i_2^{(q)}}| = |\hat{y}_{2,i_2^{(q)}}|, q = 1, 2, \dots, 5 \quad (3.21)$$

does not have a solution, then $\widehat{\mathbf{z}}_0 \leftarrow \sqrt{\frac{N(|\hat{y}_{0,0}| - |\hat{y}_{0,1}|)}{2}}$.

Output: $\widehat{\mathbf{z}}_0$.

Remark 3.1. (1) By Lemma 3.3, the requirement: $w^{2i_2^{(q)}} \neq -1, \frac{w^{i_2^{(2)}}}{1+w^{2i_2^{(2)}}} \neq \frac{1}{2}$ in Approach 3.1 can be satisfied.

(2) Clearly, (3.21) is equivalent to the condition that $\widehat{\mathbf{z}}_2$ lies in the 5 circles on $\mathbb{C}$:

$$\left\{ z : \left| z - (-\widehat{\mathbf{z}}_1 w^{i_2^{(q)}} / (\widehat{\mathbf{z}}_0 (1 + w^{2i_2^{(q)}}))) \right| = N |\hat{y}_{2, i_2^{(q)}}| \right\}, \quad q = 1, 2, \dots, 5.$$

By the correlations among the five circles, it is easy to check that whether (3.21) has a solution.

The following theorem states that $\widehat{\mathbf{z}}_0$ can be determined (up to a sign) by Approach 3.1.

Theorem 3.5. Suppose that $\mathbf{z} \in \mathbb{C}^N$ (with N being even) is a generic analytic signal such that its DFT $\widehat{\mathbf{z}} = (\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_1, \dots, \widehat{\mathbf{z}}_{N/2}, \dots, 0)$ satisfies $|\widehat{\mathbf{z}}_0| \neq 0$ and $|\widehat{\mathbf{z}}_1| \neq 0$. If L is odd and $r = \lceil N/L \rceil \geq 5$, then $\widehat{\mathbf{z}}_0$ in (3.1) can be determined up to a sign by Approach 3.1.

Proof. The proof will be presented in section 5.3. $\square$

Remark 3.2. Recall that $w^m (m \in \Lambda)$ in the proof of Theorem 3.4 are distinctive and they are the zeros to the quartic polynomial on the right-hand side of (3.19). If $r < 5$, then $\#\Lambda < 5$ and we can not guarantee that (3.20) holds (or λ in (3.20) equals to 1) or that the conclusion of Theorem 3.4 holds. It will be clear in section 5.3 that Theorem 3.5 is derived from Theorem 3.4. Therefore, if $r < 5$ then we can not guarantee that $\widehat{\mathbf{z}}_0$ (or the unknown $\widehat{\mathbf{z}}_0$ in (3.1)) be determined up to a sign by Approach 3.1.

3.3. Finding $\tilde{\mathbf{z}} \in \mathbb{C}^N$ such that its DFT $\widehat{\tilde{\mathbf{z}}} = (\epsilon \widehat{\mathbf{z}}_0, \frac{N|\hat{y}_{1,0}|}{2|\widehat{\mathbf{z}}_0|}, \widehat{\mathbf{z}}_2, \dots, \widehat{\mathbf{z}}_{N/2}, 0, \dots, 0)$ satisfies (3.2)

Recall that $\widehat{\mathbf{z}}_0$ in (3.1) can be determined (up to a sign) by Approach 3.1, and it follows from 3.1(B) that $|\widehat{\mathbf{z}}_1| := \frac{N|\hat{y}_{1,0}|}{2|\widehat{\mathbf{z}}_0|}$. This subsection concerns on the solution to (3.2) w.r.t. $\tilde{\mathbf{z}}$. For convenience, (3.2) is stated again as follows,

$$\begin{cases} |\hat{y}_{k,m}| = \frac{1}{N} \left| \sum_{l=0}^k \widehat{\mathbf{z}}_l \widehat{\mathbf{z}}_{k-l} w^{lm} \right|, k = 2, \dots, N/2, \\ |\hat{y}_{k,m}| = \frac{1}{N} \left| \sum_{l=k-N/2}^{N/2} \widehat{\mathbf{z}}_l \widehat{\mathbf{z}}_{k-l} w^{lm} \right|, k = N/2 + 1, \dots, N-1, \\ m = 0, 1, \dots, r-1, \end{cases} \quad (3.22)$$

where $\widehat{\mathbf{z}}_0 = \epsilon \widehat{\mathbf{z}}_0$ with $\epsilon \in \{1, -1\}$ and $\widehat{\mathbf{z}}_1 := \frac{N|\hat{y}_{1,0}|}{2|\widehat{\mathbf{z}}_0|}$. In what follows, we establish an approach to find such a $\tilde{\mathbf{z}}$ by the $3N/2 + 1$ measurements:

$$\begin{aligned} & \left\{ |\hat{y}_{0,0}|^2, |\hat{y}_{0,1}|^2, |\hat{y}_{1,0}|^2, |\hat{y}_{2, i_2^{(q)}}|^2, |\hat{y}_{3,0}|^2, |\hat{y}_{3, i_3}|^2, |\hat{y}_{k, i_2^{(p)}}|^2 : i_2^{(1)} = 0, w^{2i_k^{(q)}} \neq -1, \frac{w^{i_2^{(2)}}}{1 + w^{2i_2^{(2)}}} \neq \frac{1}{2}, \right. \\ & \left. \frac{w^{i_3} + w^{2i_3}}{1 + w^{3i_3}} \neq 1, w^{ki_k^{(p)}} \neq -1, i_k^{(1)} = 0, i_k^{(2)} + i_k^{(3)} \neq r, 0 \leq i_2^{(q)} \leq r-1, 1 \leq i_3 \leq r-1, \right. \\ & \left. 0 \leq i_k^{(p)} \leq r-1, 4 \leq k \leq N/2, p = 1, 2, 3, q = 1, 2, 3, 4, 5 \right\}. \end{aligned} \quad (3.23)$$

The theoretical guarantee for such an approach will be given in Theorem 3.6. The existence of the above measurements is addressed in the following remark.

Remark 3.3. (1) Recall that the FROG measurements in (3.23) need to satisfy the requirements:

$$w^{2i_k^{(q)}} \neq -1, \frac{w^{i_2^{(2)}}}{1 + w^{2i_2^{(2)}}} \neq \frac{1}{2}, \frac{w^{i_3} + w^{2i_3}}{1 + w^{3i_3}} \neq 1,$$

which is guaranteed by Lemma 3.3.

(2) According to the analysis in [7, Page 1038], for any $k \in \{4, \dots, N/2\}$ there exist $\{i_k^{(p)} : p = 1, 2, 3\} \subset \{0, 1, \dots, r-1\}$ such that $w^{ki_k^{(p)}} \neq -1$, $i_k^{(1)} = 0$ and $i_k^{(2)} + i_k^{(3)} \neq r$.

Approach 3.2. Input: FROG measurements in (3.23), $\widehat{\mathbf{z}}_0$ (derived from Approach 3.1) and $\widehat{\mathbf{z}}_1 = \frac{N|\widehat{y}_{1,0}|}{2|\widehat{\mathbf{z}}_0|}$.

Step 1: By Note 3.2 (i), choose a solution $\widehat{\mathbf{z}}_2$ to

$$\begin{cases} |\widehat{y}_{2,0}| = \frac{1}{N} |2\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_2 + \widehat{\mathbf{z}}_1^2|, \\ |\widehat{y}_{2,i_2^{(2)}}| = \frac{1}{N} |\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_2 + \widehat{\mathbf{z}}_1^2 w^{i_2^{(2)}} + \widehat{\mathbf{z}}_2\widehat{\mathbf{z}}_0 w^{2i_2^{(2)}}|. \end{cases}$$

Step 2: Given $(\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_1, \widehat{\mathbf{z}}_2)$. Use Note 3.2 (ii) and Note 3.1 to find the solution to the system w.r.t. $\widehat{\mathbf{z}}_3, \widehat{\mathbf{z}}_4$:

$$\begin{cases} |\widehat{y}_{3,0}| = \frac{1}{N} |2\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_3 + 2\widehat{\mathbf{z}}_1\widehat{\mathbf{z}}_2|, \\ |\widehat{y}_{3,i_3}| = \frac{1}{N} |\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_3 + \widehat{\mathbf{z}}_1\widehat{\mathbf{z}}_2 w^{i_3} + \widehat{\mathbf{z}}_1\widehat{\mathbf{z}}_2 w^{2i_3} + \widehat{\mathbf{z}}_3\widehat{\mathbf{z}}_0 w^{3i_3}|, \\ |\widehat{y}_{4,i_4^{(p)}}| = \frac{1}{N} |\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_4 + \widehat{\mathbf{z}}_1\widehat{\mathbf{z}}_3 w^{i_4^{(p)}} + \widehat{\mathbf{z}}_2^2 w^{2i_4^{(p)}} + \widehat{\mathbf{z}}_3\widehat{\mathbf{z}}_1 w^{3i_4^{(p)}} + \widehat{\mathbf{z}}_4\widehat{\mathbf{z}}_0 w^{4i_4^{(p)}}|, p = 1, 2, 3. \end{cases} \quad (3.24)$$

Step 3: Given $(\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_1, \widehat{\mathbf{z}}_2, \widehat{\mathbf{z}}_3, \widehat{\mathbf{z}}_4)$. Use Note 3.1 to find iteratively the solution to the system w.r.t. $\widehat{\mathbf{z}}_j, j \geq 5$:

$$|\widehat{y}_{j,i_j^{(p)}}| = \frac{1}{N} |\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_j + \widehat{\mathbf{z}}_1\widehat{\mathbf{z}}_{j-1} w^{i_j^{(p)}} + \dots + \widehat{\mathbf{z}}_j\widehat{\mathbf{z}}_0 w^{ji_j^{(p)}}|, p = 1, 2, 3. \quad (3.25)$$

Output: $\widehat{\mathbf{z}} := (\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_1, \dots, \widehat{\mathbf{z}}_{N/2}, 0, \dots, 0)$ and its IDFT $\tilde{\mathbf{z}}$.

The following theorem guarantees that $\tilde{\mathbf{z}}$ derived from Approach 3.2 is the solution to (3.22).

Theorem 3.6. Suppose that $\mathbf{z} \in \mathbb{C}^N$ is a generic analytic signal such that its DFT

$$\widehat{\mathbf{z}} = (\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_1, \dots, \widehat{\mathbf{z}}_{N/2}, 0, \dots, 0) := (|\widehat{\mathbf{z}}_0|e^{i\theta_0}, |\widehat{\mathbf{z}}_1|e^{i\theta_1}, \dots, |\widehat{\mathbf{z}}_{N/2}|e^{i\theta_{N/2}}, 0, \dots, 0). \quad (3.26)$$

Let $\widehat{\mathbf{z}}_0 = \epsilon \widehat{\mathbf{z}}_0$ and $\widehat{\mathbf{z}}_1 = \frac{N|\widehat{y}_{1,0}|}{2|\widehat{\mathbf{z}}_0|}$ with $\epsilon \in \{1, -1\}$. Then the solutions to (3.22) are the vector $(\mathfrak{E}(2)|\widehat{\mathbf{z}}_2|e^{i(\theta_2-2\theta_1)}, \mathfrak{E}(3)|\widehat{\mathbf{z}}_3|e^{i(\theta_3-3\theta_1)}, \dots, \mathfrak{E}(N/2)|\widehat{\mathbf{z}}_{N/2}|e^{i(\theta_{N/2}-(N/2)\theta_1)})$ and its complex conjugate, where $\mathfrak{E}(k)$ takes ϵ and 1 for k being even and odd, respectively. Moreover, one of the two solutions can be determined through Approach 3.2.

Proof. The proof will be presented in subsection 5.4. $\square$

3.4. On the solution to (3.3)

As noted at the beginning of section 3, our final approach for FROG-PR is derived from the two relaxed FROG-PR problems in (3.2) ((3.22)) and (3.3). Recall that (3.2) has been addressed in Theorem 3.6. We now address (3.3) with the following theorem.

Theorem 3.7. Suppose that $\mathbf{z} \in \mathbb{C}^N$ is a generic analytic signal such that N is even, and its DFT

$$\widehat{\mathbf{z}} = (\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_1, \dots, \widehat{\mathbf{z}}_{N/2}, 0, \dots, 0) := (|\widehat{\mathbf{z}}_0|e^{i\theta_0}, |\widehat{\mathbf{z}}_1|e^{i\theta_1}, \dots, |\widehat{\mathbf{z}}_{N/2}|e^{i\theta_{N/2}}, 0, \dots, 0) \quad (3.27)$$

satisfies $|\widehat{\mathbf{z}}_0| \neq 0$ and $|\widehat{\mathbf{z}}_1| \neq 0$. Moreover, consider the system w.r.t. $\tilde{\mathbf{z}} \in \mathbb{C}^N$ such that $\text{supp}(\widehat{\tilde{\mathbf{z}}}) \subseteq \{0, \dots, N/2\}$:

$$\begin{cases} |\hat{y}_{k,m}| = \frac{1}{N} \left| \sum_{l=0}^k \widehat{\tilde{\mathbf{z}}}_l \widehat{\tilde{\mathbf{z}}}_{k-l} w^{lm} \right|, k = 1, \dots, N/2, \\ |\hat{y}_{k,m}| = \frac{1}{N} \left| \sum_{l=k-N/2}^{N/2} \widehat{\tilde{\mathbf{z}}}_l \widehat{\tilde{\mathbf{z}}}_{k-l} w^{lm} \right|, k = N/2 + 1, \dots, N-1, \\ m = 0, 1, \dots, r-1, \end{cases} \quad (3.28)$$

where $\{|\hat{y}_{k,m}|^2\}$ are the $[N, L]$ -FROG measurements of $\mathbf{z}$ and $r = \lceil N/L \rceil \geq 5$. Then $\tilde{\mathbf{z}}$ is uniquely determined up to the arbitrary rotation, arbitrary translation and reflection. Moreover, if requiring that the phases of the first two components of $\widehat{\tilde{\mathbf{z}}}$ be zeros, then $\tilde{\mathbf{z}}$ can be determined (up to the reflection) such that its DFT is

$$\widehat{\tilde{\mathbf{z}}} = (|\widehat{\mathbf{z}}_0|, |\widehat{\mathbf{z}}_1|, \mathfrak{E}(2)|\widehat{\mathbf{z}}_2|e^{i(\theta_2-2\theta_1)}, \dots, \mathfrak{E}(N/2)|\widehat{\mathbf{z}}_{N/2}|e^{i[\theta_{N/2}-(N/2)\theta_1]}, 0, \dots, 0), \quad (3.29)$$

where $\mathfrak{E}(k)$ takes $\text{sgn}(\widehat{\mathbf{z}}_0)$ and 1 for k being even and odd, respectively.

Proof. Note that the DFT of the γ -translation of $\tilde{\mathbf{z}}$ is $(\widehat{\tilde{\mathbf{z}}}_0, \widehat{\tilde{\mathbf{z}}}_1 e^{i2\pi\gamma/N}, \dots, \widehat{\tilde{\mathbf{z}}}_{N/2} e^{i2\pi(N/2)\gamma/N}, 0, \dots, 0)$. We first prove that $(\widehat{\tilde{\mathbf{z}}}_0, \widehat{\tilde{\mathbf{z}}}_1 e^{i2\pi\gamma/N}, \dots, \widehat{\tilde{\mathbf{z}}}_{N/2} e^{i2\pi(N/2)\gamma/N}, 0, \dots, 0)$ satisfies (3.28). By the direct calculation we obtain

$$\begin{aligned} & \frac{1}{N} \left| \sum_{l=0}^k \widehat{\tilde{\mathbf{z}}}_l e^{-i2\pi l\gamma/N} \widehat{\tilde{\mathbf{z}}}_{k-l} e^{-i2\pi(k-l)\gamma/N} w^{lm} \right| \\ &= \frac{1}{N} |e^{-i\frac{2k\pi}{N}\gamma}| \left| \sum_{l=0}^{N-1} \widehat{\tilde{\mathbf{z}}}_l \widehat{\tilde{\mathbf{z}}}_{k-l} w^{lm} \right| \\ &= |\hat{y}_{k,m}|, m = 0, 1, \dots, r-1, \end{aligned} \quad (3.30)$$

and

$$\begin{aligned} & \frac{1}{N} \left| \sum_{l=k-N/2}^{N/2} \widehat{\tilde{\mathbf{z}}}_l e^{-i2\pi l\gamma/N} \widehat{\tilde{\mathbf{z}}}_{k-l} e^{-i2\pi(k-l)\gamma/N} w^{lm} \right| \\ &= \frac{1}{N} |e^{-i\frac{2k\pi}{N}\gamma}| \left| \sum_{l=k-N/2}^{N/2} \widehat{\tilde{\mathbf{z}}}_l \widehat{\tilde{\mathbf{z}}}_{k-l} w^{lm} \right| \\ &= |\hat{y}_{k,m}|, m = 0, 1, \dots, r-1. \end{aligned} \quad (3.31)$$

Thus (3.28) is satisfied. By [7, Proposition 2.2], the DFT of the reflection and arbitrary rotation of $\tilde{\mathbf{z}}$ satisfies (3.28), respectively. Since the arbitrary rotation can lead to the ambiguity, we assign that $\widehat{\tilde{\mathbf{z}}}_0 > 0$. By Theorem 3.4, if $|\widehat{\tilde{\mathbf{z}}}_0| \neq |\widehat{\mathbf{z}}_0|$, then $\tilde{\mathbf{z}}$ does not satisfy (3.28). Now we choose $\widehat{\tilde{\mathbf{z}}}_0 = |\widehat{\mathbf{z}}_0|$. On the other hand,

$$\frac{2}{N} |\widehat{\tilde{\mathbf{z}}}_0 \widehat{\tilde{\mathbf{z}}}_1| = |\hat{y}_{1,0}| = \frac{2}{N} |\widehat{\mathbf{z}}_0 \widehat{\mathbf{z}}_1|, \quad (3.32)$$

from which we derive $|\widehat{\tilde{\mathbf{z}}}_1| = \frac{N|\hat{y}_{1,0}|}{2|\widehat{\mathbf{z}}_0|} = |\widehat{\mathbf{z}}_1|$. As proved above, any translation of $\tilde{\mathbf{z}}$ is also a solution, then we assign $\widehat{\tilde{\mathbf{z}}}_1$ by $\frac{N|\hat{y}_{1,0}|}{2|\widehat{\mathbf{z}}_0|}$. Now it follows from Theorem 3.6 that

$$(|\widehat{\mathbf{z}}_0|, |\widehat{\mathbf{z}}_1|, \mathfrak{E}(2)|\widehat{\mathbf{z}}_2|e^{i(\theta_2-2\theta_1)}, \mathfrak{E}(3)|\widehat{\mathbf{z}}_3|e^{i(\theta_3-3\theta_1)}, \dots, \mathfrak{E}(N/2)|\widehat{\mathbf{z}}_{N/2}|e^{i[\theta_{N/2}-(N/2)\theta_1]}, 0, \dots, 0)$$

is the solution to (3.28), up to the arbitrary rotation, arbitrary translation and reflection, where $\mathfrak{E}(k)$ takes $\text{sgn}(\widehat{\mathbf{z}}_0)$ and 1 for k being even and odd, respectively. Naturally, if requiring the phases of the first two components of $\widehat{\mathbf{z}}$ to be zeros, then $\widehat{\mathbf{z}}$ is the solution (up to the reflection) to (3.28). $\square$

3.5. Determination of generic analytic signals of even lengths by FROG measurements

In what follows we establish the approach for the FROG-PR of generic analytic signals of even lengths. This is the key approach which leads to the main result of this paper.

Approach 3.3. Input: $(3N/2 + 1)$ FROG measurements in (3.23) of $\mathbf{z}$.

Step 1: Conduct Approach 3.1 to obtain the corresponding output $\widehat{\mathbf{z}}_0$.

Step 2: Conduct Approach 3.2 to obtain the output $\widehat{\mathbf{z}} := (\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_1, \dots, \widehat{\mathbf{z}}_{N/2}, 0, \dots, 0)$.

Step 3: Construct $\widehat{\mathbf{z}} = (\widehat{\mathbf{z}}_0, \dots, \widehat{\mathbf{z}}_{N/2}, 0, \dots, 0) \in \mathbb{C}^N$, where $\widehat{\mathbf{z}}_k = \widehat{\mathbf{z}}_k e^{-i\frac{2k}{N} \arg(\widehat{\mathbf{z}}_{N/2})}$, $k = 0, \dots, N/2$.

Output: $\widehat{\mathbf{z}} = \text{IDFT}(\widehat{\mathbf{z}})$. %% IDFT is the inverse discrete Fourier transform.

Now comes to our main theorem.

Theorem 3.8. Let $\mathbf{z} \in \mathbb{C}^N$ be a generic analytic signal such that N is even and its DFT

$$\widehat{\mathbf{z}} = (\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_1, \dots, \widehat{\mathbf{z}}_{N/2}, 0, \dots, 0) \quad (3.33)$$

satisfies $|\widehat{\mathbf{z}}_0| \neq 0$ and $|\widehat{\mathbf{z}}_1| \neq 0$. Suppose that $\widehat{\mathbf{z}}$ is the output of Approach 3.3 conducted by the $3N/2 + 1$ $[N, L]$ -FROG measurements in (3.23) of $\mathbf{z}$, where L is odd and $\lceil N/L \rceil \geq 5$. Then $\widehat{\mathbf{z}}$ is the solution to (3.1) up to the π -rotation, integer-translation and reflection.

Proof. Note that $\widehat{\mathbf{z}}$ is the $(-\frac{\arg(\widehat{\mathbf{z}}_{N/2})}{\pi})$ -translation of $\widehat{\mathbf{z}}$ in Theorem 3.7. Then, by Theorem 3.7, $\widehat{\mathbf{z}}$ is a solution to (3.28), up to the arbitrary rotation, arbitrary translation and reflection. So any solution to (3.28) can be derived from the composition of the rotation, translation and reflection of $\widehat{\mathbf{z}}$. We next prove that $\widehat{\mathbf{z}}$ is a (analytic) solution to (3.1). Note that both $\widehat{\mathbf{z}}_0$ and $\widehat{\mathbf{z}}_{N/2}$ are real-valued. Then, by Proposition 2.1 (i), $\widehat{\mathbf{z}}$ is analytic. Comparing (3.1) and (3.28), we only need to check (3.1A). Indeed,

$$\frac{1}{N} |\widehat{\mathbf{z}}_0^2 + \widehat{\mathbf{z}}_{N/2}^2 w^{Nm/2}| = \frac{1}{N} |\widehat{\mathbf{z}}_0|^2 + |\widehat{\mathbf{z}}_{N/2}|^2 w^{Nm/2} = |\widehat{y}_{0,m}|, m = 0, 1, \dots, r-1. \quad (3.34)$$

Note that any solution to (3.1) is a solution to (3.28). Thus any solution to (3.1) is the composition of the rotation, translation and reflection of $\widehat{\mathbf{z}}$, and hence, by Proposition 2.2, we complete the proof. $\square$

4. An application: recovery of phase derivative of decaying signals

In this section, we apply Theorem 3.8 to the recovery of the phase derivative for an analytic signal $A(\mathbf{x})$ provided that $\mathbf{x}$, the real part of $A(\mathbf{x})$, is of decaying energy, a natural condition that have been addressed in applications (e.g. [13,43]).

4.1. Recovery of phase derivative of decaying signals

We start with the definition of phase derivative for analytic signals. For a real-valued signal $\mathbf{x} = (\mathbf{x}_0, \dots, \mathbf{x}_{N-1}) \in \mathbb{R}^N$, its associated analytic signal $A(\mathbf{x}) = ((A(\mathbf{x}))_0, \dots, (A(\mathbf{x}))_{N-1})$ is denoted by $(|(A(\mathbf{x}))_0|e^{i\arg((A(\mathbf{x}))_0)}, \dots, |(A(\mathbf{x}))_{N-1}|e^{i\arg((A(\mathbf{x}))_{N-1})})$. Denote $\varphi(k) = \arg((A(\mathbf{x}))_k)$ and

$$\varphi^*(k) := (\varphi(k) - \varphi(k-1)) \bmod 2\pi. \quad (4.1)$$

Then $\Psi_{A(\mathbf{x})}^* := (\varphi^*(0), \dots, \varphi^*(N-1))$ is referred to as the **phase derivative** of $A(\mathbf{x})$ (cf. [19]). Following [19], $\Psi_{A(\mathbf{x})}^*$ is also referred to as the (analytic) phase derivative of $\mathbf{x}$. In what follows, based on Theorem 3.8, we establish an approach for the recovery of $\Psi_{A(\mathbf{x})}^*$ provided that $|\mathbf{x}_0| > |\mathbf{x}_1| > \dots > |\mathbf{x}_{N-1}|$. Its theoretical guarantee will be established in Proposition 4.1.

Approach 4.1. Input: $(3N/2 + 1)$ FROG measurements in (3.23) of $\mathbf{z}$.

Step 1: Conduct Approach 3.3 and the output is $\hat{\mathbf{z}} = (\hat{\mathbf{z}}_0, \dots, \hat{\mathbf{z}}_{N-1})$. Denote $\Re(\hat{\mathbf{z}}) = (\Re(\hat{\mathbf{z}}_0), \dots, \Re(\hat{\mathbf{z}}_{N-1}))$ and $k_{\max} = \arg \max_k \{|\Re(\hat{\mathbf{z}}_k)| : k = 0, \dots, N-1\}$.

Step 2: If $|\Re(\hat{\mathbf{z}}_{k_{\max}-1})| > |\Re(\hat{\mathbf{z}}_{k_{\max}+1})|$, then, through rearranging the order of the elements of $\bar{\mathbf{z}} = (\bar{\mathbf{z}}_0, \dots, \bar{\mathbf{z}}_{N-1})$, construct $\tilde{\mathbf{z}} = (\tilde{\mathbf{z}}_0, \dots, \tilde{\mathbf{z}}_{N-1})$ such that $|\Re(\tilde{\mathbf{z}}_0)| > |\Re(\tilde{\mathbf{z}}_1)| > \dots > |\Re(\tilde{\mathbf{z}}_{N-1})|$.

Step 3: If $|\Re(\hat{\mathbf{z}}_{k_{\max}-1})| < |\Re(\hat{\mathbf{z}}_{k_{\max}+1})|$, then, through rearranging the order of the elements of $\hat{\mathbf{z}} = (\hat{\mathbf{z}}_0, \dots, \hat{\mathbf{z}}_{N-1})$, construct $\tilde{\mathbf{z}} = (\tilde{\mathbf{z}}_0, \dots, \tilde{\mathbf{z}}_{N-1})$ such that $|\Re(\tilde{\mathbf{z}}_0)| > |\Re(\tilde{\mathbf{z}}_1)| > \dots > |\Re(\tilde{\mathbf{z}}_{N-1})|$.

Output: Compute and output the phase derivative of $\tilde{\mathbf{z}}$.

The following proposition states that the output of Approach 4.1 is the phase derivative of $\mathbf{z}$. As will be explained in (4.4) and (4.7) that the key ingredient for the recovery of phase derivative is the effect of the ambiguity in Theorem 3.8 on the phase distribution.

Proposition 4.1. *Suppose that the analytic signal $A(\mathbf{x})$ is as in Theorem 3.8 such that its real part $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{N-1})$ is decaying, namely, $|\mathbf{x}_0| > |\mathbf{x}_1| > \dots > |\mathbf{x}_{N-1}|$. Then the phase derivative derived from Approach 4.1 is identical to that of $A(\mathbf{x})$.*

Proof. Suppose that $\hat{\mathbf{z}}$ is the output of Approach 3.3. Then it follows from Theorem 3.8 that $\hat{\mathbf{z}}$ equals to $A(\mathbf{x})$ up to the π -rotation, integer-translation and reflection. Denote $A(\mathbf{x})$ by $\mathbf{z} = (\mathbf{z}_0, \mathbf{z}_1, \dots, \mathbf{z}_{N-1})$. It is easy to check that the π -rotation does not alter the phase derivative of any analytic signal. The reflection of $\mathbf{z}$ is expressed as

$$\mathbf{z}_{ref} = (\mathbf{z}_{ref,0}, \dots, \mathbf{z}_{ref,N-1}) = (\bar{\mathbf{z}}_0, \bar{\mathbf{z}}_{N-1}, \dots, \bar{\mathbf{z}}_1). \quad (4.2)$$

Moreover, for any fixed $l \in \{1, \dots, N-1\}$, the l -translation $\mathbf{z}_{tr}^l$ of $\mathbf{z}_{ref}$ is

$$(\mathbf{z}_{tr,0}^l, \dots, \mathbf{z}_{tr,N-1}^l) = (\bar{\mathbf{z}}_{N-l}, \bar{\mathbf{z}}_{N-l-1}, \dots, \bar{\mathbf{z}}_1, \bar{\mathbf{z}}_0, \bar{\mathbf{z}}_{N-1}, \dots, \bar{\mathbf{z}}_{N-l+1}). \quad (4.3)$$

A common thing shared by (4.2) and (4.3) is that

$$|\Re(\mathbf{z}_{ref,k_{\max}-1})| > |\Re(\mathbf{z}_{ref,k_{\max}+1})|, \quad |\Re(\mathbf{z}_{tr,k_{\max}-1}^l)| > |\Re(\mathbf{z}_{tr,k_{\max}+1}^l)|, \quad (4.4)$$

where $k_{\max} = \arg \max_k \{|\Re(\mathbf{z}_{ref,k})| : k = 0, \dots, N-1\}$ and $\hat{k}_{\max} = \arg \max_k \{|\Re(\mathbf{z}_{tr,k}^l)| : k = 0, \dots, N-1\}$. On the other hand, the l -translation of $\mathbf{z}$ is

$$\mathbf{z}_{tr}^l = (\mathbf{z}_l, \dots, \mathbf{z}_{N-1}, \mathbf{z}_0, \mathbf{z}_1, \dots, \mathbf{z}_{l-1}). \quad (4.5)$$

And the reflection of $\mathbf{z}_{tr}^l$ is

$$\mathbf{z}_{tr,ref}^l = (\mathbf{z}_{tr,ref,0}^l, \dots, \mathbf{z}_{tr,ref,N-1}^l) = (\bar{\mathbf{z}}_l, \bar{\mathbf{z}}_{l-1}, \dots, \bar{\mathbf{z}}_1, \bar{\mathbf{z}}_0, \bar{\mathbf{z}}_{N-1}, \dots, \bar{\mathbf{z}}_{l+1}). \quad (4.6)$$

Clearly, $\mathbf{z}_{tr,ref}^l$ still satisfies the property (4.4) as $\mathbf{z}_{ref}$ and $\mathbf{z}_{tr}^l$. But

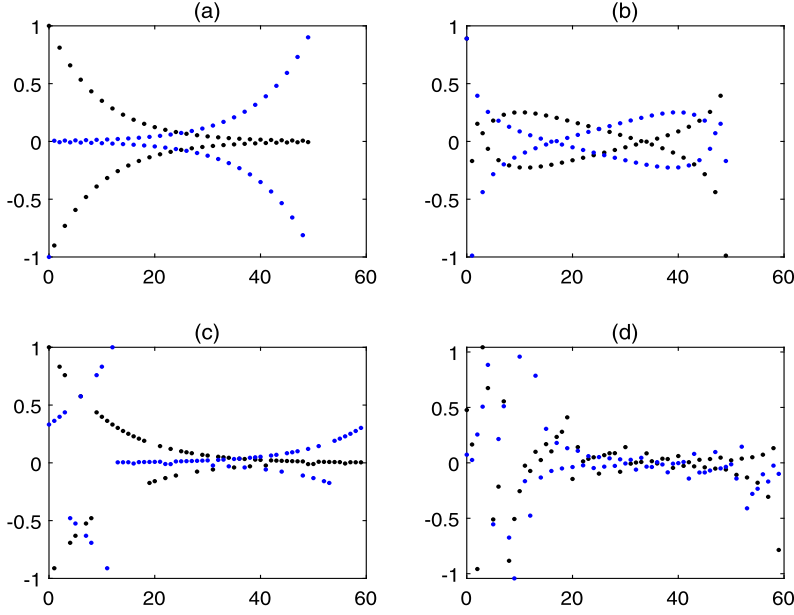


Fig. 4.1. (a) Black dot-line: the real part $\mathbf{x}_{-0.900742,50}^\epsilon$ of the analytic signal $A(\mathbf{x}_{-0.900742,50}^\epsilon)$; Blue dot-line: the real part of the reconstructed version (by Approach 3.3) of $A(\mathbf{x}_{-0.900742,50}^\epsilon)$. (b) Black dot-line: the imaginary part of $A(\mathbf{x}_{-0.900742,50}^\epsilon)$; Blue dot-line: the imaginary part of the reconstructed version of $A(\mathbf{x}_{-0.900742,50}^\epsilon)$. (c) Black dot-line: the real part $\mathbf{x}_{0.9120,60}^\epsilon$ of the analytic signal $A(\mathbf{x}_{0.9120,60}^\epsilon)$; Blue dot-line: the real part of the reconstructed version (by Approach 3.3) of $A(\mathbf{x}_{0.9120,60}^\epsilon)$. (d) Black dot-line: the imaginary part of $A(\mathbf{x}_{0.9120,60}^\epsilon)$; Blue dot-line: the imaginary part of the reconstructed version of $A(\mathbf{x}_{0.9120,60}^\epsilon)$. (For interpretation of the colors in the figure, the reader is referred to the web version of this article.)

$$|\Re(\mathbf{z}_{tr,ref,k_{\max}-1}^l)| < |\Re(\mathbf{z}_{tr,ref,k_{\max}+1}^l)|, \quad (4.7)$$

where $k_{\max} = \arg \max_k \{|\Re(\mathbf{z}_{tr,ref,k}^l)| : k = 0, \dots, N-1\}$. Now it follows from (4.4) and (4.7) that $\tilde{\mathbf{z}}$ constructed in Approach 4.1 satisfies $\tilde{\mathbf{z}} = \mathbf{z}$ or $\tilde{\mathbf{z}} = -\mathbf{z}$. The proof is concluded. $\square$

4.2. Simulation

As an example, we performed simulation to recover the phase derivatives of the analytic signals of exponentially decaying ones defined by

$$\mathbf{x}_{\alpha,N}^\epsilon = (\epsilon_0, \epsilon_1 \alpha, \dots, \epsilon_{N-1} \alpha^{N-1}) \in \mathbb{R}^N, \quad (4.8)$$

where $\alpha \in (-1, 1)$ and $\epsilon = (\epsilon_0, \dots, \epsilon_{N-1})$ with $\epsilon_i \in \{1, -1\}$. Note that $\mathbf{x}_{\alpha,N}^\epsilon$ is such a signal example examined in [37, Page 290], where $\hat{\epsilon} = (1, \dots, 1)$. We choose $A(\mathbf{x}_{-0.900742,50}^\epsilon)$ and $A(\mathbf{x}_{0.9120,60}^\epsilon)$ as two examples to check Approach 4.1. See Fig. 4.1 for the result of FROG-PR of Approach 3.3 by the FROG measurements in (3.23) with $L = 5$. As illustrated in Fig. 4.1, we found that the recovery results are the π -rotation of the reflection of $A(\mathbf{x}_{-0.900742,50}^\epsilon)$, and the 12-translation of the reflection of $A(\mathbf{x}_{0.9120,60}^\epsilon)$.

For the phase derivative $\Psi_{A(\mathbf{x})}^* := (\varphi^*(0), \dots, \varphi^*(N-1))$ of an analytic signal $A(\mathbf{x})$, its recovery version is denoted by $\tilde{\Psi}_{A(\mathbf{x})}^* := (\tilde{\varphi}^*(0), \dots, \tilde{\varphi}^*(N-1))$. And the recovery error function is defined as $(|\varphi^*(0) - \tilde{\varphi}^*(0)|, \dots, |\varphi^*(N-1) - \tilde{\varphi}^*(N-1)|)$. The phase derivatives of $A(\mathbf{x}_{-0.900742,50}^\epsilon)$ and $A(\mathbf{x}_{0.9120,60}^\epsilon)$ are plotted in Fig. 4.2. See Fig. 4.3 for the recovery error functions of the phase derivatives of $A(\mathbf{x}_{-0.900742,50}^\epsilon)$ and $A(\mathbf{x}_{0.9120,60}^\epsilon)$. Clearly, the two phase derivatives are recovered efficiently. The error is mainly derived from the roundoff error of FROG-PR. In general, the efficiency of phase retrieval deteriorates as the dimension increases ([1]). However, the error ($O(10^{-6})$) (corresponding to the signal length 60) in Fig. 4.3 (b) is much

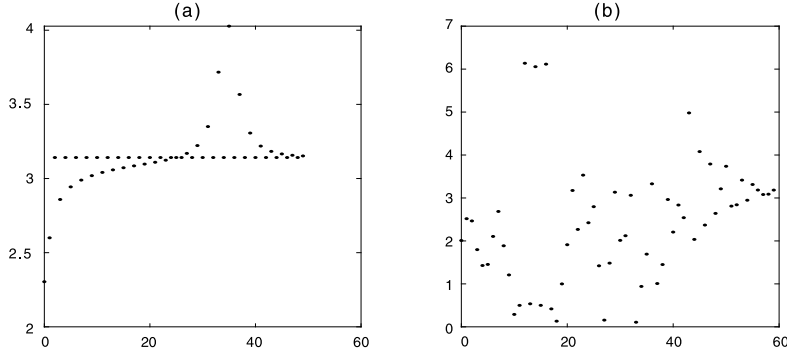


Fig. 4.2. (a) The phase derivative of $A(\mathbf{x}_{-0.900742,50}^\epsilon)$. (b) The phase derivative of $A(\mathbf{x}_{0.9120,60}^\epsilon)$.

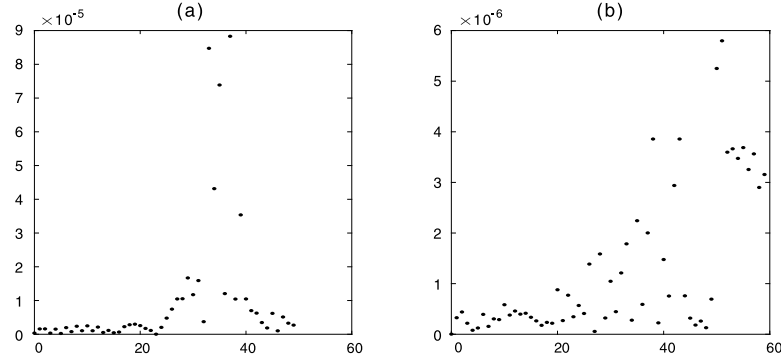


Fig. 4.3. (a) Recovery error function of the phase derivative of $A(\mathbf{x}_{-0.900742,50}^\epsilon)$; (b) Recovery error function of the phase derivative of $A(\mathbf{x}_{0.9120,60}^\epsilon)$.

smaller than that ($O(10^{-5})$) in Fig. 4.3 (a) (corresponding to the signal length 50). Therefore, the simulation result implies that the efficiency of FROG-PR is also related with the signal itself.

5. The proofs

5.1. Proof of Proposition 2.1

We just need to prove (i). Item (ii) can be proved similarly. By (1.3), we only need to prove the sufficiency. Denote $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ such that where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, the DFTs of $\mathbf{x}$ and $\mathbf{y}$, can be expressed by

$$\hat{\mathbf{x}}_k = \begin{cases} \hat{\mathbf{z}}_0 & k = 0, \\ \frac{1}{2}\hat{\mathbf{z}}_k & 1 \leq k \leq N/2 - 1, \\ \hat{\mathbf{z}}_{N/2} & k = N/2, \\ \frac{1}{2}\hat{\mathbf{z}}_{N-k} & N/2 + 1 \leq k \leq N - 1, \end{cases} \quad (5.1)$$

and

$$i\hat{\mathbf{y}}_k = \begin{cases} 0, & k = 0, \\ \frac{1}{2}\hat{\mathbf{z}}_k, & 1 \leq k \leq N/2 - 1, \\ 0, & k = N/2, \\ -\frac{1}{2}\hat{\mathbf{z}}_{N-k}, & N/2 + 1 \leq k \leq N - 1. \end{cases} \quad (5.2)$$

We claim that $\bar{\mathbf{x}}_k = \mathbf{x}_k$ and $\bar{\mathbf{y}}_k = \mathbf{y}_k$. Indeed, it follows from

$$\mathbf{x}_k = \frac{1}{N} \sum_{n=0}^{N-1} \hat{\mathbf{x}}_n e^{2\pi i k n / N}, \quad \mathbf{y}_k = \frac{1}{N} \sum_{n=0}^{N-1} \hat{\mathbf{y}}_n e^{2\pi i k n / N}$$

that

$$\begin{aligned} \bar{\mathbf{x}}_k &= \frac{1}{N} \overline{\sum_{n=0}^{N-1} \hat{\mathbf{x}}_n e^{2\pi i k n / N}} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \widehat{\bar{\mathbf{x}}}_n e^{2\pi i k (N-n) / N} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \hat{\mathbf{x}}_{N-n} e^{2\pi i k (N-n) / N} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \widehat{\mathbf{x}}_n e^{2\pi i k n / N} \\ &= \mathbf{x}_k, \end{aligned} \tag{5.3}$$

and similarly,

$$\begin{aligned} \overline{\mathbf{i}\mathbf{y}_k} &= \frac{1}{N} \overline{\sum_{n=0}^{N-1} \hat{\mathbf{y}}_n e^{2\pi i k n / N}} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \widehat{\bar{\mathbf{y}}}_n e^{2\pi i k (N-n) / N} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} -\mathbf{i}\hat{\mathbf{y}}_{N-n} e^{2\pi i k (N-n) / N} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} -\mathbf{i}\hat{\mathbf{y}}_n e^{2\pi i k n / N} \\ &= -\mathbf{i}\mathbf{y}_k, \end{aligned} \tag{5.4}$$

where the third identities in (5.3) and (5.4) are derived from (5.1) and (5.2). $\square$

5.2. Proof of Lemma 3.3

(1) If r is odd, then for any $m \in \{0, 1, 2, \dots, r-1\}$, $\frac{4m}{r}$ is not an odd integer and consequently, $1 + w^{2m} = 1 + e^{\frac{i4m\pi}{r}} \neq 0$. Therefore, for the odd integer $r \geq 5$, there exist at least 5 numbers, denoted by $m_1, \dots, m_5 \subseteq \{0, 1, 2, \dots, r-1\}$ such that $1 + w^{2m_l} \neq 0, l = 1, \dots, 5$. Now assume that r is even. If $r = 6$, then $1 + w^{2m} = 1 + e^{\frac{i2m\pi}{3}}$. Note that $\frac{2m}{3}$ is not an odd integer. Thus for any $m \in \{0, 1, 2, \dots, 5\}$, we have $1 + w^{2m} \neq 0$. If $r > 6$ and even (or $r \geq 8$), then $4m \leq 4(r-1) < 4r$. If $1 + w^{2m} = 0$, then $4m = r$ or $4m = 3r$. Thus there exist at most two numbers, denoted by $\bar{m}_1$ and $\bar{m}_2$, in $\{0, 1, 2, \dots, r-1\}$ such that $1 + w^{2\bar{m}_k} = 0, k = 1, 2$. Therefore there exist at least 5 numbers, denoted by $m_1, \dots, m_5 \subseteq \{0, 1, 2, \dots, r-1\}$ such that $1 + w^{2m_l} \neq 0 (l = 1, \dots, 5)$.

(2) Let $m \in \{1, 2, \dots, r-1\}$. If $\frac{w^m}{1+w^{2m}} = \frac{1}{2}$, then and so $(w^m - 1)^2 = 0$, i.e., $w^m = 1$. Since $2 \leq 2m \leq 2(r-1) < 2r$, we have $w^m \neq 1$, and hence there exists $m_1 \in \{1, 2, \dots, r-1\}$ such that $\frac{w^{m_1}}{1+w^{2m_1}} \neq \frac{1}{2}$.

(3) We first prove that if $r \geq 5$, then there exist at least three numbers in $\{1, 2, \dots, r-1\}$, denoted by m_1, m_2, m_3 , such that $1 + w^{3m_k} = 1 + e^{\frac{i6m_k\pi}{r}} \neq 0$.

For $r = 5$ and any $m \in \{1, 2, \dots, 4\}$, $\frac{6m}{r}$ is not an odd integer. Therefore $1 + w^{3m} = 1 + e^{\frac{i6m\pi}{5}} \neq 0$. Let $r \geq 6$ and $m \in \{1, 2, \dots, r-1\}$. Suppose that $1 + w^{3m} = 1 + e^{\frac{i6m\pi}{r}} = 0$. Since $\frac{6}{r} \leq \frac{6m}{r} \leq \frac{6(r-1)}{r} < 6$, we have $1 + e^{\frac{i6m\pi}{r}} = 0$ which implies that $\frac{6m}{r} \in \{1, 3, 5\}$. Thus there exist at most three numbers $\bar{m}_1, \bar{m}_2, \bar{m}_3 \in \{1, 2, \dots, r-1\}$ such that $1 + w^{3\bar{m}_k} = 0$, and therefore there exist at least three numbers $m_1, m_2, m_3 \in \{1, 2, \dots, r-1\}$ such that $1 + w^{3m_k} \neq 0$.

Now we prove that there exists at least a number $i \in \{m_1, m_2, m_3\}$ such that $\frac{w^i + w^{2i}}{1 + w^{3i}} \neq 1$. Note that for any $m \in \{m_1, m_2, m_3\}$, $\frac{w^m + w^{2m}}{1 + w^{3m}} = 1$ is equivalent to $(w^m - 1)(w^{2m} - 1) = 0$. Thus we have $w^m = 1$ or $w^{2m} = 1$. Since $2 \leq 2m \leq 2(r-1) < 2r$, there does not exist $m \in \{m_1, m_2, m_3\}$ such that $w^m = 1$. If $w^{2m} = e^{\frac{i4m\pi}{r}} = 1$, then from $4 \leq 4m \leq 4(r-1) < 4r$ we have that $4m = 2r$. Thus there exists at most one number, e.g. $m_1 \in \{m_1, m_2, m_3\}$ such that $\frac{w^{m_1} + w^{2m_1}}{1 + w^{3m_1}} = 1$, and consequently there exists $i \in \{m_2, m_3\}$ such that $\frac{w^i + w^{2i}}{1 + w^{3i}} \neq 1$. $\square$

5.3. Proof of Theorem 3.5

Since $\mathbf{z}$ is analytic, we have that both $\hat{\mathbf{z}}_0$ and $\hat{\mathbf{z}}_{N/2}$ are real-valued. We first consider the equation system w.r.t. the real-valued variables $\hat{\mathbf{z}}_0, \hat{\mathbf{z}}_{N/2} \in \mathbb{R}$:

$$\begin{cases} \frac{1}{N}|\hat{\mathbf{z}}_0^2 + \hat{\mathbf{z}}_{N/2}^2| = |\hat{y}_{0,0}| = \frac{1}{N}|\hat{\mathbf{z}}_0^2 + \hat{\mathbf{z}}_{N/2}^2|, & (5.5A) \\ \frac{1}{N}|\hat{\mathbf{z}}_0^2 - \hat{\mathbf{z}}_{N/2}^2| = |\hat{y}_{0,1}| = \frac{1}{N}|\hat{\mathbf{z}}_0^2 - \hat{\mathbf{z}}_{N/2}^2|. & (5.5B) \end{cases} \quad (5.5)$$

It is easy to check that the solutions to (5.5) are

$$(\hat{\mathbf{z}}_0, \hat{\mathbf{z}}_{N/2}) = \left(\pm \sqrt{\frac{N(|\hat{y}_{0,0}| + |\hat{y}_{0,1}|)}{2}}, \pm \sqrt{\frac{N(|\hat{y}_{0,0}| - |\hat{y}_{0,1}|)}{2}} \right) \quad (5.6)$$

and

$$(\hat{\mathbf{z}}_0, \hat{\mathbf{z}}_{N/2}) = \left(\pm \sqrt{\frac{N(|\hat{y}_{0,0}| - |\hat{y}_{0,1}|)}{2}}, \pm \sqrt{\frac{N(|\hat{y}_{0,0}| + |\hat{y}_{0,1}|)}{2}} \right). \quad (5.7)$$

If $|\hat{y}_{0,1}| = 0$, then $|\hat{\mathbf{z}}_0| = |\hat{\mathbf{z}}_{N/2}|$ and consequently $\hat{\mathbf{z}}_0$ can be determined (up to a sign) by Approach 3.1.

If $|\hat{y}_{0,1}| \neq 0$ (equivalently $|\hat{\mathbf{z}}_0| \neq |\hat{\mathbf{z}}_{N/2}|$), then we need to prove that if a signal $\tilde{\mathbf{z}} \in \mathbb{C}^N$ whose DFT $(\hat{\tilde{\mathbf{z}}}_0, \dots, \hat{\tilde{\mathbf{z}}}_{N/2}, 0, \dots, 0)$ satisfies

$$\hat{\tilde{\mathbf{z}}}_0 \in \{\hat{\mathbf{z}}_{N/2}, -\hat{\mathbf{z}}_{N/2}\} \text{ and } \hat{\tilde{\mathbf{z}}}_{N/2} \in \{\hat{\mathbf{z}}_0, -\hat{\mathbf{z}}_0\}, \quad (5.8)$$

then it does not have the same $[N, L]$ -FROG measurements as $\mathbf{z}$. This follows from Theorem 3.4. $\square$

5.4. Proof of Theorem 3.6

We divide the proof into three parts based on the three steps from Approach 3.2.

5.4.1. Determination of $\hat{\mathbf{z}}_2$ in (3.22)

Proposition 5.1. Suppose that N is even and $\mathbf{z} \in \mathbb{C}^N$ is a generic analytic signal such that its DFT

$$\hat{\mathbf{z}} = (\hat{\mathbf{z}}_0, \hat{\mathbf{z}}_1, \dots, \hat{\mathbf{z}}_{N/2}, 0, \dots, 0) := (|\hat{\mathbf{z}}_0|e^{i\theta_0}, |\hat{\mathbf{z}}_1|e^{i\theta_1}, \dots, |\hat{\mathbf{z}}_{N/2}|e^{i\theta_{N/2}}, 0, \dots, 0) \quad (5.9)$$

satisfies $|\widehat{\mathbf{z}}_0|, |\widehat{\mathbf{z}}_1| \neq 0$. The $[N, L]$ -FROG measurements of $\mathbf{z}$ are $\{|\hat{y}_{k,m}|^2 : k = 0, 1, \dots, N-1; m = 0, \dots, r-1\}$. Denote by $\widehat{\mathbf{z}}_0$ the output of Approach 3.1 such that $\widehat{\mathbf{z}}_0 = \epsilon \widehat{\mathbf{z}}_0$ with $\epsilon \in \{1, -1\}$. Define $\widehat{\mathbf{z}}_1 := \frac{N|\hat{y}_{1,0}|}{2|\widehat{\mathbf{z}}_0|}$ such that $\frac{2}{N}|\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_1| = \frac{2}{N}|\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_1| = |\hat{y}_{1,0}|$. Then the solutions to the system of equations w.r.t. $\widehat{\mathbf{z}}_2$:

$$\frac{1}{N}|\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_2(1+w^{2m}) + \widehat{\mathbf{z}}_1^2 w^m| = |\hat{y}_{2,m}|, m = 0, \dots, r-1, \quad (5.10)$$

are $\epsilon|\widehat{\mathbf{z}}_2|e^{i(\theta_2-2\theta_1)}$ and $\epsilon|\widehat{\mathbf{z}}_2|e^{i(-\theta_2+2\theta_1)}$. Moreover, one of the two solutions can be determined by **Step 1** in Approach 3.2.

Proof. Clearly, $|\widehat{\mathbf{z}}_1| = |\widehat{\mathbf{z}}_1|$ and $\widehat{\mathbf{z}}_1 = |\widehat{\mathbf{z}}_1|e^{i(\theta_1-\theta_1)}$. For $\widehat{\mathbf{z}}_2 = \epsilon|\widehat{\mathbf{z}}_2|e^{i(\theta_2-2\theta_1)}$ and $m = 0, 1, \dots, r-1$, compute

$$\begin{aligned} & \frac{1}{N}|\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_2 + \widehat{\mathbf{z}}_1^2 w^m + \widehat{\mathbf{z}}_2\widehat{\mathbf{z}}_0 w^{2m}| \\ &= \frac{1}{N}|\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_2|e^{i(\theta_2-2\theta_1)} + (|\widehat{\mathbf{z}}_1|e^{i(\theta_1-\theta_1)})^2 w^m + |\widehat{\mathbf{z}}_2|e^{i(\theta_2-2\theta_1)}\widehat{\mathbf{z}}_0 w^{2m}| \\ &= \frac{1}{N}|e^{i(2\theta_1)}||\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_2 + \widehat{\mathbf{z}}_1^2 w^m + \widehat{\mathbf{z}}_2\widehat{\mathbf{z}}_0 w^{2m}| \\ &= \frac{1}{N}|\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_2 + \widehat{\mathbf{z}}_1^2 w^m + \widehat{\mathbf{z}}_2\widehat{\mathbf{z}}_0 w^{2m}| \\ &= |\hat{y}_{2,m}|. \end{aligned} \quad (5.11)$$

Therefore, $\widehat{\mathbf{z}}_2 = \epsilon|\widehat{\mathbf{z}}_2|e^{i(\theta_2-2\theta_1)}$ is the solution to (5.10). Moreover, it follows from Lemma 3.3 (2) that there exists $m_1 \in \{1, 2, \dots, r-1\}$ such that $\frac{w^{m_1}}{1+w^{2m_1}} \neq \frac{1}{2}$. For $m = 0$ and m_1 , (5.10) is equivalent to

$$\frac{|\hat{y}_{2,m}|}{|(1+w^{2m})\widehat{\mathbf{z}}_0|} = \frac{1}{N}|\widehat{\mathbf{z}}_2 + \frac{w^m}{1+w^{2m}}\frac{\widehat{\mathbf{z}}_1^2}{\widehat{\mathbf{z}}_0}|. \quad (5.12)$$

It is easy to check that $\frac{w^m}{1+w^{2m}} \in \mathbb{R}$ for any $m \in \{0, m_1\}$. Thus, by Note 3.2 (i) and (5.11), the solutions to (5.12) are $\epsilon|\widehat{\mathbf{z}}_2|e^{i(\theta_2-2\theta_1)}$ and $\epsilon|\widehat{\mathbf{z}}_2|e^{i(-\theta_2+2\theta_1)}$. By the similar calculation as (5.11), $\epsilon|\widehat{\mathbf{z}}_2|e^{i(-\theta_2+2\theta_1)}$ is the other solution to (5.10). The proof is concluded. $\square$

5.4.2. Determination of $\widehat{\mathbf{z}}_3, \widehat{\mathbf{z}}_4$ in (3.22)

Let $\mathbf{z} \in \mathbb{C}^N$ be a generic analytic signal such that its $[N, L]$ -FROG measurements are $\{|\hat{y}_{k,m}|^2 : k = 0, 1, \dots, N-1; m = 0, \dots, r-1\}$. Denote its DFT by

$$\widehat{\mathbf{z}} = (\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_1, \dots, \widehat{\mathbf{z}}_{N/2}, 0, \dots, 0) := (|\widehat{\mathbf{z}}_0|e^{i\theta_0}, |\widehat{\mathbf{z}}_1|e^{i\theta_1}, \dots, |\widehat{\mathbf{z}}_{N/2}|e^{i\theta_{N/2}}, 0, \dots, 0). \quad (5.13)$$

Recall that in (3.22), $\widehat{\mathbf{z}}_0 = \epsilon\widehat{\mathbf{z}}_0$ with $\epsilon \in \{1, -1\}$ is the output of Approach 3.1, and $\widehat{\mathbf{z}}_1 = |\widehat{\mathbf{z}}_1|$. Consequently, by the similar analysis in the proof of Proposition 5.1, $\epsilon|\widehat{\mathbf{z}}_2|e^{i(\theta_2-2\theta_1)}$ and $\epsilon|\widehat{\mathbf{z}}_2|e^{i(-\theta_2+2\theta_1)}$ are the solutions to (5.10) w.r.t. $\widehat{\mathbf{z}}_2$.

Proposition 5.2. Let the generic analytic signal $\mathbf{z} \in \mathbb{C}^N$ be as above. If we choose $\widehat{\mathbf{z}}_2 = \epsilon|\widehat{\mathbf{z}}_2|e^{i(\theta_2-2\theta_1)}$, then the solution to the system of equations w.r.t. $\widehat{\mathbf{z}}_3$ and $\widehat{\mathbf{z}}_4$:

$$\begin{cases} |\hat{y}_{3,m}| = \frac{1}{N}|\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_3 + \widehat{\mathbf{z}}_1\widehat{\mathbf{z}}_2 w^m + \widehat{\mathbf{z}}_2\widehat{\mathbf{z}}_1 w^{2m} + \widehat{\mathbf{z}}_3\widehat{\mathbf{z}}_0 w^{3m}|, \\ |\hat{y}_{4,m}| = \frac{1}{N}|\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_4 + \widehat{\mathbf{z}}_1\widehat{\mathbf{z}}_3 w^m + (\widehat{\mathbf{z}}_2)^2 w^{2m} + \widehat{\mathbf{z}}_3\widehat{\mathbf{z}}_1 w^{3m} + \widehat{\mathbf{z}}_4\widehat{\mathbf{z}}_0 w^{4m}|, \\ m = 0, 1, \dots, r-1, \end{cases} \quad (5.14)$$

is $(|\widehat{\mathbf{z}}_3|e^{i(\theta_3-3\theta_1)}, \epsilon|\widehat{\mathbf{z}}_4|e^{i(\theta_4-4\theta_1)})$. Similarly, if $\widehat{\mathbf{z}}_2 = \epsilon|\widehat{\mathbf{z}}_2|e^{i(-\theta_2+2\theta_1)}$, then the solution to (5.14) is $(|\widehat{\mathbf{z}}_3|e^{i(-\theta_3+3\theta_1)}, \epsilon|\widehat{\mathbf{z}}_4|e^{i(-\theta_4+4\theta_1)})$. Moreover, the solution can be determined by **Step 2** in Approach 3.2.

Proof. Clearly, $\widehat{\mathbf{z}}_1 = |\widehat{\mathbf{z}}_1|e^{i(\theta_1-\theta_1)}$. Suppose that we choose $\widehat{\mathbf{z}}_2 = \epsilon|\widehat{\mathbf{z}}_2|e^{i(\theta_2-2\theta_1)}$. Then it follows from

$$\begin{aligned} & \frac{1}{N} |\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_3 + \widehat{\mathbf{z}}_1\widehat{\mathbf{z}}_2w^m + \widehat{\mathbf{z}}_2\widehat{\mathbf{z}}_1w^{2m} + \widehat{\mathbf{z}}_3\widehat{\mathbf{z}}_0w^{3m}| \\ &= \frac{1}{N} |\epsilon\widehat{\mathbf{z}}_0|\widehat{\mathbf{z}}_3|e^{i(\theta_3-3\theta_1)} + |\widehat{\mathbf{z}}_1|e^{i(\theta_1-\theta_1)}\epsilon|\widehat{\mathbf{z}}_2|e^{i(\theta_2-2\theta_1)}w^m + \epsilon|\widehat{\mathbf{z}}_2|e^{i(\theta_2-2\theta_1)}|\widehat{\mathbf{z}}_1|e^{i(\theta_1-\theta_1)}w^{2m} \\ & \quad + |\widehat{\mathbf{z}}_3|e^{i(\theta_3-3\theta_1)}\epsilon\widehat{\mathbf{z}}_0w^{3m}| \\ &= \frac{1}{N} |e^{i3(-\theta_1)}| |\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_3 + \widehat{\mathbf{z}}_1\widehat{\mathbf{z}}_2w^m + \widehat{\mathbf{z}}_2\widehat{\mathbf{z}}_1w^{2m} + \widehat{\mathbf{z}}_3\widehat{\mathbf{z}}_0w^{3m}| \\ &= \frac{1}{N} |\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_3 + \widehat{\mathbf{z}}_1\widehat{\mathbf{z}}_2w^m + \widehat{\mathbf{z}}_2\widehat{\mathbf{z}}_1w^{2m} + \widehat{\mathbf{z}}_3\widehat{\mathbf{z}}_0w^{3m}| \\ &= |\hat{y}_{3,m}| \end{aligned}$$

that $\widehat{\mathbf{z}}_3 = |\widehat{\mathbf{z}}_3|e^{i(\theta_3-3\theta_1)}$ is a solution to

$$|\hat{y}_{3,m}| = \frac{1}{N} |\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_3 + \widehat{\mathbf{z}}_1\widehat{\mathbf{z}}_2w^m + \widehat{\mathbf{z}}_2\widehat{\mathbf{z}}_1w^{2m} + \widehat{\mathbf{z}}_3\widehat{\mathbf{z}}_0w^{3m}|, m = 0, 1, \dots, r-1. \quad (5.15)$$

As for the above system, it follows from the analysis in [7, Page 1037] that $\widehat{\mathbf{z}}_3' = |\widehat{\mathbf{z}}_3|e^{i(-\theta_3+3\theta_1+2\bar{\theta}_2)}$ with $\bar{\theta}_2 = \arg(\widehat{\mathbf{z}}_2)$ is the other solution to (5.15). By [7, Lemma 4.2], however, if we choose $\widehat{\mathbf{z}}_3'$ as the solution, then there does not exist any other solution to

$$|\hat{y}_{4,m}| = \frac{1}{N} |\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_4 + \widehat{\mathbf{z}}_1\widehat{\mathbf{z}}_3w^m + (\widehat{\mathbf{z}}_2)^2w^{2m} + \widehat{\mathbf{z}}_3\widehat{\mathbf{z}}_1w^{3m} + \widehat{\mathbf{z}}_4\widehat{\mathbf{z}}_0w^{4m}|, m = 0, 1, \dots, r-1. \quad (5.16)$$

Therefore, we just need to consider $\widehat{\mathbf{z}}_3 = |\widehat{\mathbf{z}}_3|e^{i(\theta_3-3\theta_1)}$.

Having $\widehat{\mathbf{z}}_2 = \epsilon|\widehat{\mathbf{z}}_2|e^{i(\theta_2-2\theta_1)}$ and $\widehat{\mathbf{z}}_3 = |\widehat{\mathbf{z}}_3|e^{i(\theta_3-3\theta_1)}$, we can verify that $\widehat{\mathbf{z}}_4 = \epsilon|\widehat{\mathbf{z}}_4|e^{i(\theta_4-4\theta_1)}$ is a solution to (5.16). Indeed this follows from:

$$\begin{aligned} & \frac{1}{N} |\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_4 + \widehat{\mathbf{z}}_1\widehat{\mathbf{z}}_3w^m + (\widehat{\mathbf{z}}_2)^2w^{2m} + \widehat{\mathbf{z}}_3\widehat{\mathbf{z}}_1w^{3m} + \widehat{\mathbf{z}}_4\widehat{\mathbf{z}}_0w^{4m}| \\ &= \frac{1}{N} |\epsilon^2\widehat{\mathbf{z}}_0|\widehat{\mathbf{z}}_4|e^{i(\theta_4-4\theta_1)} + |\widehat{\mathbf{z}}_1|e^{i(\theta_1-\theta_1)}|\widehat{\mathbf{z}}_3|e^{i(\theta_3-3\theta_1)}w^m + \epsilon^2[|\widehat{\mathbf{z}}_2|e^{i(\theta_2-2\theta_1)}]^2w^{2m} \\ & \quad + |\widehat{\mathbf{z}}_3|e^{i(\theta_3-3\theta_1)}|\widehat{\mathbf{z}}_1|e^{i(\theta_1-\theta_1)}w^{3m} + \epsilon^2|\widehat{\mathbf{z}}_4|e^{i(\theta_4-4\theta_1)}\widehat{\mathbf{z}}_0w^{4m}| \\ &= \frac{1}{N} |e^{i4(-\theta_1)}| |\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_4 + \widehat{\mathbf{z}}_1\widehat{\mathbf{z}}_3w^m + \widehat{\mathbf{z}}_2^2w^{2m} + \widehat{\mathbf{z}}_3\widehat{\mathbf{z}}_1w^{3m} + \widehat{\mathbf{z}}_4\widehat{\mathbf{z}}_0w^{4m}| \\ &= \frac{1}{N} |\widehat{\mathbf{z}}_0\widehat{\mathbf{z}}_4 + \widehat{\mathbf{z}}_1\widehat{\mathbf{z}}_3w^m + \widehat{\mathbf{z}}_2^2w^{2m} + \widehat{\mathbf{z}}_3\widehat{\mathbf{z}}_1w^{3m} + \widehat{\mathbf{z}}_4\widehat{\mathbf{z}}_0w^{4m}| \\ &= |\hat{y}_{4,m}|. \end{aligned}$$

We next prove that $\widehat{\mathbf{z}}_4 = \epsilon|\widehat{\mathbf{z}}_4|e^{i(\theta_4-4\theta_1)}$ is the unique solution to (5.16) for the generic analytic signal $\mathbf{z}$. Suppose that $\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_1$ and $\widehat{\mathbf{z}}_2$ are fixed. Since $\widehat{\mathbf{z}}_3$ is generic then $\widehat{\mathbf{z}}_3 = |\widehat{\mathbf{z}}_3|e^{i(\theta_3-3\theta_1)} = \widehat{\mathbf{z}}_3e^{i3\theta_1}$ is also generic. We next use Note 3.1 to determine $\widehat{\mathbf{z}}_4$. For the generic $\widehat{\mathbf{z}}_3$, recall that the corresponding $\frac{v_1-v_2}{v_1-v_3}$ in Note 3.1 (3.5) is a rational polynomial w.r.t. $\widehat{\mathbf{z}}_3$. Therefore it is easy to check that (3.5) holds, and the solution to (5.16) is unique for the choice of $(\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_1, \widehat{\mathbf{z}}_2, \widehat{\mathbf{z}}_3) = (\epsilon\widehat{\mathbf{z}}_0, |\widehat{\mathbf{z}}_1|, \epsilon|\widehat{\mathbf{z}}_2|e^{i(\theta_2-2\theta_1)}, |\widehat{\mathbf{z}}_3|e^{i(\theta_3-3\theta_1)})$. And $(\widehat{\mathbf{z}}_3, \widehat{\mathbf{z}}_4)$ can be determined by **Step 2** in Approach 3.2. Now we concluded the proof of the first part, and the second part can be proved similarly. $\square$

5.4.3. Determination of $\widehat{\mathbf{z}}_k$ in (3.22) for $k \geq 5$

We first prove that if $(\widehat{\mathbf{z}}_0, \dots, \widehat{\mathbf{z}}_{k-1}) = (\epsilon \widehat{\mathbf{z}}_0, |\widehat{\mathbf{z}}_1|, \dots, \mathfrak{E}(k-1)|\widehat{\mathbf{z}}_{k-1}|e^{i[\theta_{k-1}-(k-1)\theta_1]})$, then $\widehat{\mathbf{z}}_k = \mathfrak{E}(k)|\widehat{\mathbf{z}}_k|e^{i(\theta_k-k\theta_1)}$ satisfies

$$|\hat{y}_{k,m}| = \frac{1}{N} \left| \sum_{l=0}^k \widehat{\mathbf{z}}_l \widehat{\mathbf{z}}_{k-l} w^{lm} \right|, m = 0, 1, \dots, r-1. \quad (5.17)$$

Indeed, from

$$\begin{aligned} & \frac{1}{N} \left| \sum_{l=0}^k \widehat{\mathbf{z}}_l \widehat{\mathbf{z}}_{k-l} w^{lm} \right| \\ &= \frac{1}{N} \left| \sum_{l=0}^k |\widehat{\mathbf{z}}_l| e^{i(\theta_l-l\theta_1)} |\widehat{\mathbf{z}}_{k-l}| e^{i[\theta_{k-l}-(k-l)\theta_1]} w^{lm} \right| \\ &= \frac{1}{N} |e^{i(k\alpha)}| \left| \sum_{l=0}^k \widehat{\mathbf{z}}_l \widehat{\mathbf{z}}_{k-l} w^{lm} \right| \\ &= |\hat{y}_{k,m}|, m = 0, 1, \dots, r-1, \end{aligned} \quad (5.18)$$

we get that $\widehat{\mathbf{z}}_k = \mathfrak{E}(k)|\widehat{\mathbf{z}}_k|e^{i(\theta_k-k\theta_1)}$ satisfies (5.17). Through the similar analysis in the proof of Proposition 5.2 that $\widehat{\mathbf{z}}_k = \mathfrak{E}(k)|\widehat{\mathbf{z}}_k|e^{i(\theta_k-k\theta_1)}$ is the unique solution to (5.17). Similarly, we can prove that if $(\widehat{\mathbf{z}}_0, \dots, \widehat{\mathbf{z}}_{k-1}) = (\epsilon \widehat{\mathbf{z}}_0, |\widehat{\mathbf{z}}_1|, \dots, \mathfrak{E}(k-1)|\widehat{\mathbf{z}}_{k-1}|e^{i[-\theta_{k-1}+(k-1)\theta_1]})$, then $\widehat{\mathbf{z}}_k = \mathfrak{E}(k)|\widehat{\mathbf{z}}_k|e^{i(-\theta_k+k\theta_1)}$ is the unique solution to (5.17). Now by Note 3.1, $\widehat{\mathbf{z}}_k$ can be determined through **Step 3** in Approach 3.2.

5.4.4. (3.22) has two complex conjugation solutions

According to sections 5.4.1, 5.4.2 and 5.4.3, the solutions to (3.22) are the vector $(\mathfrak{E}(2)|\widehat{\mathbf{z}}_2|e^{i(\theta_2-2\theta_1)}, \mathfrak{E}(3)|\widehat{\mathbf{z}}_3|e^{i(\theta_3-3\theta_1)}, \dots, \mathfrak{E}(N/2)|\widehat{\mathbf{z}}_{N/2}|e^{i[\theta_{N/2}-(N/2)\theta_1]})$ and its complex conjugate.

6. A final remark

If L is even and $r = \lceil N/L \rceil \geq 5$, then

$$\frac{1}{N} (\widehat{\mathbf{z}}_0^2 + \widehat{\mathbf{z}}_{N/2}^2) = |\hat{y}_{0,m}|, m = 0, 1, \dots, r-1. \quad (6.1)$$

Therefore in this case $|\hat{y}_{0,m}|$ can not be expressed by the form of (5.5B). And **Step 3** in Approach 3.1 does not hold for the case of L being even. Clearly, $\widehat{\mathbf{z}}_0 \in [-\sqrt{N|\hat{y}_{0,0}|}, \sqrt{N|\hat{y}_{0,0}|}]$. Assume that we assign $\widehat{\mathbf{z}}_0$ by an arbitrary value $\alpha \in [-\sqrt{N|\hat{y}_{0,0}|}, \sqrt{N|\hat{y}_{0,0}|}]$. Then it holds that $\alpha \neq \pm \widehat{\mathbf{z}}_0$ with probability 1. Now it follows from $\frac{2}{N} |\widehat{\mathbf{z}}_0 \widehat{\mathbf{z}}_1| = |\hat{y}_{1,0}|$ that $|\widehat{\mathbf{z}}_1| = \frac{N|\hat{y}_{1,0}|}{2|\alpha|}$. For any $\theta \in [0, 2\pi)$, assign the phase θ to $\widehat{\mathbf{z}}_1$ as $\frac{N|\hat{y}_{1,0}|}{2|\alpha|} e^{i\theta}$. Then, by the proof of Theorem 3.4, the system w.r.t. $\widehat{\mathbf{z}}_2$

$$\frac{1}{N} |\widehat{\mathbf{z}}_0 \widehat{\mathbf{z}}_2 (1 + w^{2m}) + \widehat{\mathbf{z}}_1^2 w^m| = |\hat{y}_{2,m}|, m = 0, \dots, r-1 \quad (6.2)$$

does not have a solution. This implies that, with probability 1, Approach 3.3 does not hold for the FROG-PR problem with even L and $r = \lceil N/L \rceil \geq 5$, and so a different approach need to be developed for this case in the future work.

Acknowledgment

The authors would like to thank the referees for carefully reading the manuscript and for providing many constructive comments and suggestions that have significantly improved the content and the presentation of this paper. We also thank Dr. Shengli Fan at CREOL College of Optics & Photonics of University of Central Florida for some helpful discussions related to the motivation of this investigation and its applications in optical sciences.

References

- [1] R. Alaifari, I. Daubechies, P. Grohs, R. Yin, Stable phase retrieval in infinite dimensions, *Found. Comput. Math.* 19 (2019) 869–900.
- [2] R. Balan, P. Casazza, D. Edidin, On signal reconstruction without noisy phase, *Appl. Comput. Harmon. Anal.* 20 (2006) 345–356.
- [3] R. Balan, Y. Wang, Invertibility and robustness of phaseless reconstruction, *Appl. Comput. Harmon. Anal.* 38 (3) (2015) 469–488.
- [4] R. Balan, D. Zou, On Lipschitz analysis and Lipschitz synthesis for the phase retrieval problem, *Linear Algebra Appl.* 496 (2016) 152–181.
- [5] A.S. Bandeira, J. Cahill, D.G. Mixon, A.A. Nelsonc, Saving phase: injectivity and stability for phase retrieval, *Appl. Comput. Harmon. Anal.* 37 (1) (2014) 106–125.
- [6] R. Beinert, G. Plonka, Ambiguities in one-dimensional discrete phase retrieval from Fourier magnitudes, *J. Fourier Anal. Appl.* 21 (2015) 1169–1198.
- [7] T. Bendory, D. Edidin, Y.C. Eldar, On signal reconstruction from frog measurements, *Appl. Comput. Harmon. Anal.* 48 (3) (2020) 1030–1044.
- [8] T. Bendory, Y.C. Eldar, A least squares approach for stable phase retrieval from short-time Fourier transform magnitude, preprint, arXiv:1510.00920, 2015.
- [9] B.G. Bodmann, N. Hammen, Stable phase retrieval with low-redundancy frames, *Adv. Comput. Math.* 41 (2015) 317–331.
- [10] I. Bojarovska, A. Flinth, Phase retrieval from Gabor measurements, *J. Fourier Anal. Appl.* 22 (2016) 542–567.
- [11] J. Cahill, P.G. Casazza, I. Daubechies, Phase retrieval in infinite-dimensional Hilbert spaces, *Trans. Am. Math. Soc. Ser. B* 3 (2016) 63–76.
- [12] E.J. Candès, Y.C. Eldar, T. Strohmer, V. Voroninski, Phase retrieval via matrix completion, *SIAM Rev.* 57 (2) (2015) 225–251.
- [13] C.M. Chan, S.K. Tang, H. Wong, On analysis of exponentially decaying pulse signals using stochastic volatility model, *J. Acoust. Soc. Am.* 119 (2006) 1519.
- [14] Y. Chen, C. Cheng, Q. Sun, H. Wang, Phase retrieval of real-valued signals in a shift-invariant space, *Appl. Comput. Harmon. Anal.* 49 (2020) 56–73.
- [15] C. Cheng, J. Jiang, Q. Sun, Phaseless sampling and reconstruction of real-valued signals in shift-invariant spaces, *J. Fourier Anal. Appl.* 25 (2019) 1361–1394.
- [16] L. Cohen, *Time-Frequency Analysis*, Prentice Hall, 1995.
- [17] P. Dang, T. Qian, Z. You, Hardy-Sobolev spaces decomposition in signal analysis, *J. Fourier Anal. Appl.* 17 (2011) 36–64.
- [18] P. Dang, T. Qian, Analytic phase derivatives, all-pass filters and signals of minimum phase, *IEEE Trans. Signal Process.* 59 (2011) 4708–4718.
- [19] P. Dang, T. Qian, Y. Yang, Hardy-Sobolev derivatives of phase and amplitude, and their applications, *Math. Methods Appl. Sci.* 35 (2012) 2017–2030.
- [20] J.K. Nielsen, M.G. Christensen, S.H. Jensen, Default Bayesian estimation of the fundamental frequency, *IEEE Trans. Audio Speech Lang. Process.* 21 (2013) 598–610.
- [21] J.R. Fienup, Phase retrieval algorithms: a comparison, *Appl. Opt.* 21 (15) (1982) 2758–2769.
- [22] J.R. Fienup, Reconstruction of an object from the modulus of its Fourier transform, *Opt. Lett.* 3 (1) (1978) 27–29.
- [23] J.R. Fienup, Phase retrieval algorithms: a personal tour, *Appl. Opt.* 52 (1) (2013) 45–56.
- [24] R.W. Gerchberg, W.O. Saxton, A practical algorithm for the determination of phase from image and diffraction plane pictures, *Optik* 35 (1972) 237–246.
- [25] D. Gabor, Theory of communication, *J. Inst. Electr. Eng., Part 3, Radio Commun. Eng.* 93 (26) (1946) 429–441.
- [26] C. Guo, B. Wang, B. Sha, Y. Lu, M. Xu, Phase derivative method for reconstruction of slightly off-axis digital holograms, *Opt. Express* 22 (2014) 30553–30558.
- [27] J.R. Gulley, T.E. Lanier, Model for ultrashort laser pulse-induced ionization dynamics in transparent solids, *Phys. Rev. B* 90 (2014) 155119.
- [28] T. Heinosaari, L. Mazzarella, M.M. Wolf, Quantum tomography under prior information, *Commun. Math. Phys.* 318 (2) (2013) 355–374.
- [29] M. Huang, Y. Rong, Y. Wang, Z. Xu, Almost everywhere generalized phase retrieval, *Appl. Comput. Harmon. Anal.* 50 (2021) 16–33.
- [30] R. Kulkarni, P. Rastogi, Simultaneous estimation of unwrapped phase and phase derivative from a closed fringe pattern, *Opt. Lasers Eng.* 87 (2016) 168–175.
- [31] R. Kulkarni, P. Rastogi, Matrix pencil based phase derivative estimation in digital holographic interferometry, *J. Opt.* 17 (2015) 045604.

- [32] L. Li, C. Cheng, D. Han, Q. Sun, G. Shi, Phase retrieval from multiple-window short-time Fourier measurements, *IEEE Signal Process. Lett.* 24 (4) (2017) 372–376.
- [33] L. Li, T. Juste, J. Brennan, C. Cheng, D. Han, Phase retrievable projective representation frames for finite Abelian groups, *J. Fourier Anal. Appl.* 25 (1) (2019) 86–100.
- [34] Y. Li, W. Sun, Random phaseless sampling for causal signals in shift-invariant spaces: a zero distribution perspective, *IEEE Trans. Signal Process.* 68 (2020) 5473–5486.
- [35] Y.T. Li, L. Zhang, T. Qian, 2D partial unwinding—a novel non-linear phase decomposition of images, *IEEE Trans. Image Process.* 28 (10) (2019) 4762–4773.
- [36] L. Marple, Computing the discrete-time ‘analytic signal’ via FFT, *IEEE Trans. Signal Process.* 47 (9) (1999) 2600–2603.
- [37] S.K. Mitra, *Digital Signal Processing: A Computer-Based Approach*, 3rd ed., McGraw Hill, 2006.
- [38] L.A. Poyneer, Scene-based Shack-Hartmann wave-front sensing: analysis and simulation, *Appl. Opt.* 42 (2003) 5807–5815.
- [39] S. Velghe, J. Primot, N. Guérineau, M. Cohen, B. Wattellier, Wave-front reconstruction from multidirectional phase derivatives generated by multilateral shearing interferometers, *Opt. Lett.* 30 (2005) 245–247.
- [40] H.Y.H. Huang, L. Tian, Z. Zhang, Y. Liu, G. Barbastathis, Path-independent phase unwrapping using phase derivative and total-variation (TV) denoising, *IEEE Trans. Acoust. Speech Signal Process.* 30 (2012) 245–247.
- [41] S. Rudresh, S. Nagesh, C. Seelamantula, Asymmetric pulse modeling for FRI sampling, *IEEE Trans. Signal Process.* 66 (2018) 2027–2040.
- [42] Y. Shechtman, Y.C. Eldar, O. Cohen, H.N. Chapman, J. Miao, M. Segev, Phase retrieval with application to optical imaging, *IEEE Signal Process. Mag.* 32 (3) (2015) 87–109.
- [43] J. Swärdn, S.I. Adalbjörnsson, A. Jakobsson, High resolution sparse estimation of exponentially decaying N -dimensional signals, *Signal Process.* 128 (2016) 309–317.
- [44] R. Trebino, *Frequency-Resolved Optical Gating: The Measurement of Ultrashort Laser Pulses*, Springer Science & Business Media, 2012.
- [45] R. Trebino, K.W. DeLong, D.N. Fittinghoff, J.N. Sweetser, M.A. Krumbigel, B.A. Richman, D.J. Kane, Measuring ultra-short laser pulses in the time frequency domain using frequency-resolved optical gating, *Rev. Sci. Instrum.* 68 (9) (1997) 3277–3295.
- [46] J. Toole, M. Mesbah, B. Boashash, A new discrete analytic signal for reducing aliasing in the discrete Wigner-Ville distribution, *IEEE Trans. Audio Speech Lang. Process.* 56 (2008) 5427–5434.
- [47] Y. Xia, Z. Xu, The recovery of complex sparse signals from few phaseless measurements, *Appl. Comput. Harmon. Anal.* 50 (2021) 1–15.
- [48] Y. Wang, Z. Xu, Generalized phase retrieval: measurement number, matrix recovery and beyond, *Appl. Comput. Harmon. Anal.* 47 (2019) 423–446.