Journal of Functional Analysis 280 (2021) 108960

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Gabor single-frame and multi-frame multipliers in any given dimension ☆

Yuanan Diao [a], Deguang Han [b], Zhongyan Li [c,*]

[a] *Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA*
[b] *Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA*
[c] *School of Mathematics and Physics, North China Electric Power University, Beijing, 102206, China*

A R T I C L E I N F O

Article history:
Received 11 November 2020
Accepted 7 February 2021
Available online 20 February 2021
Communicated by T. Schlumprecht

MSC:
primary 42C15, 46C05, 47B10

Keywords:
Gabor family
Gabor multi-frame generators
Functional matrix multipliers

A B S T R A C T

Functional Gabor single-frame or multi-frame generator multipliers are the matrices of function entries that preserve Parseval Gabor single-frame or multi-frame generators. An interesting and natural question is how to characterize all such multipliers. This question has been answered for several special cases including the case of single-frame generators in two dimensions and the case of multi-frame generators in one-dimension. In this paper we completely characterize multipliers for Gabor single-frame and multi-frame generators with respect to separable time-frequency lattices in any given dimension. Our approach is general and applies to the previously known cases as well.

☆ Deguang Han acknowledges the support from NSF under the grant DMS-1712602; Zhongyan Li acknowledges the support from The National Natural Science Foundation of China (Grant No. 11571107).

* Corresponding author.
E-mail addresses: ydiao@uncc.edu (Y. Diao), deguang.han@ucf.edu (D. Han), lzhongy@ncepu.edu.cn (Z. Li).

https://doi.org/10.1016/j.jfa.2021.108960
0022-1236/

1. Introduction

This paper is a continuation of a project on characterizing the frame generator multipliers under various settings, a topic that was initially motivated by the work of Dai and Larson [4] on wandering vector multipliers and the WUTAM paper on basic properties of wavelets [22]. Representative publications resulted from this project include [9,10,17–19]. Let A and B be two nonsingular real matrices so that $A\mathbb{Z}^d$ and $B\mathbb{Z}^d$ are both full-rank lattices in $\mathbb{R}^d$. We say that $G(\mathbf{x}) = (g_1(\mathbf{x}), ..., g_\gamma(\mathbf{x}))^\tau$, with τ being the transpose and $g_j(\mathbf{x}) \in L^2(\mathbb{R}^d)$ for each j, is a *Parseval (or normalized tight) Gabor multi-frame generator* of length γ for $L^2(\mathbb{R}^d)$ (for the separable time-frequency lattice $A\mathbb{Z}^d \times B\mathbb{Z}^d$) if $\{e^{2\pi i\langle B\mathbf{m},\mathbf{x}\rangle} g_j(\mathbf{x} - A\mathbf{n}) : \mathbf{m}, \mathbf{n} \in \mathbb{Z}^d, 1 \le j \le \gamma\}$ is a normalized tight frame, *i.e.*,

$$\sum_{1\le j\le\gamma} \sum_{\mathbf{m},\mathbf{n}\in\mathbb{Z}^d} |\langle f, e^{2\pi i\langle B\mathbf{m},\mathbf{x}\rangle} g_j(\mathbf{x} - A\mathbf{n})\rangle|^2 = \|f\|^2 \tag{1.1}$$

for all $f(\mathbf{x}) \in L^2(\mathbb{R}^d)$. In the special case that $\gamma = 1$, $G(\mathbf{x}) = g_1(\mathbf{x})$ is also called a *Parseval Gabor single-frame generator* or simply a *Parseval Gabor frame generator*. However for the sake of convenience in this paper a Parseval Gabor single-frame generator will simply be regarded as a Parseval Gabor multi-frame generator with length $\gamma = 1$. Gabor multi-frames in higher dimensions play important roles in many applications ([5,6,14,15,23,24]).

A functional matrix $M(\mathbf{x}) = (f_{ij}(\mathbf{x}))_{\gamma\times\gamma}$ with $f_{ij}(\mathbf{x}) \in L^\infty(\mathbb{R}^d)$ is called a functional (matrix) Gabor multi-frame multiplier if $H(\mathbf{x}) = M(\mathbf{x})G(\mathbf{x})$ is a Parseval Gabor multi-frame generator for $L^2(\mathbb{R}^d)$ whenever $G = (g_1, g_2, \cdots, g_\gamma)^\tau$ is. Functional (matrix) Gabor multi-frame multipliers provide a useful tool in the study of Parseval Gabor multi-frames. As such, it is an interesting and important question to ask how they can be characterized. This question has been answered for the special cases of $d = 1$ with any $\gamma \ge 1$ ([9,20]), and $d = 2$ with $\gamma = 1$ [16]. In this paper, we provide a complete characterization for functional (matrix) Gabor multi-frame multipliers with any number $\gamma \ge 1$ of generators at any dimension $d \ge 1$. Our result shall contain all the previously obtained results in [9,16,20] with a unified approach. More specifically, we have proven the following theorem.

Theorem 1.1. *Let A and B be nonsingular real valued $d \times d$ matrices, and γ be the integer satisfying $|\det(AB)| \le \gamma < |\det(AB)| + 1$. Let $M(\mathbf{x}) = (f_{ij}(\mathbf{x}))_{\gamma\times\gamma}$ with $f_{ij}(\mathbf{x}) \in L^\infty(\mathbb{R}^d)$. Then $M(\mathbf{x})$ is a functional matrix Gabor multi-frame multiplier for the time-frequency lattice $A\mathbb{Z}^d \times B\mathbb{Z}^d$ if and only if the following three conditions are satisfied:*

(1) $M(\mathbf{x})$ is unitary for a.e. $\mathbf{x} \in \mathbb{R}^d$;

(2) For any $\mathbf{n} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, $M^*(\mathbf{x})M(\mathbf{x} + (B^\tau)^{-1}\mathbf{n})$ equals $\lambda_\mathbf{n}(\mathbf{x})I$ (a.e. $\mathbf{x} \in \mathbb{R}^d$) for some unimodular scalar-valued function (that depends only on $\mathbf{n}$) $\lambda_\mathbf{n}(\mathbf{x})$, where I is the identity matrix and M^* denotes the conjugate transpose of M.

(3) $\lambda_\mathbf{n}(\mathbf{x})$ is $A\mathbb{Z}^d$-periodic (as a function of $\mathbf{x}$).

Notice that in the case of $\gamma = 1$, a functional matrix Gabor multi-frame multiplier is a scalar function $h(\mathbf{x})$ hence Theorem 1.1 has a simpler form:

Theorem 1.2. *A scalar function $h(\mathbf{x}) \in L^\infty(\mathbb{R}^d)$ is a functional Gabor multiplier for the time-frequency lattice $A\mathbb{Z}^d \times B\mathbb{Z}^d$ if and only if the following two conditions hold:*

(1) $h(\mathbf{x})$ is unimodular for a.e. $\mathbf{x} \in \mathbb{R}^d$;

(2) For any $\mathbf{n} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, $h(\mathbf{x})\overline{h(\mathbf{x} + (B^\tau)^{-1}\mathbf{n})}$ is $A\mathbb{Z}^d$-periodic.

Remark 1.3. The well-known density theorem in Gabor analysis tells us that if a time-frequency lattice $A\mathbb{Z}^d \times B\mathbb{Z}^d$ admits a Gabor multi-frame generator of length γ but does not admit a Gabor multi-frame generator of length $\gamma - 1$ (i.e., γ is the minimal length), then we necessarily have $|\det(AB)| \le \gamma < |\det(AB)| + 1$ (cf. [7]). The converse is trivially true when $d = 1$, and it is also proved to be true for higher dimensions when $\gamma = 1$ by D. Han and Y. Wang [13], and B. Bekka [1] with two completely different approaches. The general case $\gamma > 1$ for any dimension is implied by our proof of Theorem 1.1 (this may be known in the literature already but we failed to find a reference). Our approach can be considered as a refinement of the tiling approach from [13].

Remark 1.4. We shall point out that Theorem 1.1 still holds when $\gamma \ge |\det(AB)| + 1$. It is only that $|\det(AB)| \le \gamma < |\det(AB)| + 1$ is the critical case that we need to focus on hence it is more convenient for us to state the theorem this way. The generalization to the non-minimum γ is trivial as we shall see in Remark 6.1.

The following characterization is well known especially for one dimensional case (cf. [2,8,21]). For high dimension and arbitrary time-frequency lattices, by using the characterization of Parseval Gabor multi-frame generators in frequency domain from [3], we can easily translate it into the following characterization in terms of time domain.

Proposition 1.5. *Let A, B be nonsingular matrices with $|\det(A)| = a$, $|\det(B)| = b$, and $g_1, g_2, \cdots, g_\gamma \in L^2(\mathbb{R}^d)$. Then $G = (g_1, ..., g_\gamma)^\tau$ is a Parseval Gabor multi-frame generator for $L^2(\mathbb{R}^d)$ if and only if the following identities hold (for a.e. $\mathbf{x} \in \mathbb{R}^d$):*

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} \langle G(\mathbf{x} - A\mathbf{n}), G(\mathbf{x} - A\mathbf{n})\rangle = b; \tag{1.2}$$

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} \langle G(\mathbf{x} - A\mathbf{n}), G(\mathbf{x} + (B^\tau)^{-1}\mathbf{l} - A\mathbf{n})\rangle = 0, \ \forall\, \mathbf{l} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}. \tag{1.3}$$

Using Proposition 1.5, the sufficient part of Theorem 1.1 can be easily proven. Indeed, let $M(\mathbf{x})$ be a functional matrix satisfying the conditions (1)–(3) in Theorem 1.1, and $G(\mathbf{x}) = (g_1, \cdots, g_\gamma)^\tau$ be an arbitrary Parseval Gabor multi-frame generator. Denote $H(\mathbf{x}) = M(\mathbf{x})G(\mathbf{x}) = (\eta_1(\mathbf{x}), \cdots, \eta_\gamma(\mathbf{x}))^\tau$. Since $M(\mathbf{x})$ is unitary for any $\mathbf{x} \in \mathbb{R}^d$ *a.e.*, it is obvious that

$$\sum_{\mathbf{n}\in\mathbb{Z}^d}\langle H(\mathbf{x}-A\mathbf{n}),H(\mathbf{x}-A\mathbf{n})\rangle=\sum_{\mathbf{n}\in\mathbb{Z}^d}\langle G(\mathbf{x}-\mathbf{n}),G(\mathbf{x}-\mathbf{n})\rangle,$$

hence (1.2) holds. Furthermore, by conditions (2) and (3) we have

$$\begin{aligned}&\sum_{\mathbf{n}\in\mathbb{Z}^d}\langle H(\mathbf{x}-A\mathbf{n}),H(\mathbf{x}-A\mathbf{n}+(B^\tau)^{-1}\mathbf{l})\rangle\\ &=\sum_{\mathbf{n}\in\mathbb{Z}^d}\langle G(\mathbf{x}-A\mathbf{n}),M^*(\mathbf{x}-A\mathbf{n})M(\mathbf{x}-A\mathbf{n}+(B^\tau)^{-1}\mathbf{l})G(\mathbf{x}-A\mathbf{n}+(B^\tau)^{-1}\mathbf{l})\rangle\\ &=\sum_{\mathbf{n}\in\mathbb{Z}^d}\langle G(\mathbf{x}-A\mathbf{n}),M^*(\mathbf{x})M(\mathbf{x}+(B^\tau)^{-1}\mathbf{l})G(\mathbf{x}-A\mathbf{n}+(B^\tau)^{-1}\mathbf{l})\rangle\\ &=\overline{\lambda_{\mathbf{l}}(\mathbf{x})}\sum_{\mathbf{n}\in\mathbb{Z}^d}\langle G(\mathbf{x}-A\mathbf{n}),G(\mathbf{x}-A\mathbf{n}+(B^\tau)^{-1}\mathbf{l})\rangle=0\end{aligned}$$

for any $\mathbf{x}\in\mathbb{R}^d$ *a.e.* and any $\mathbf{l}\neq\mathbf{0}$. Hence $H(\mathbf{x})=M(\mathbf{x})G(\mathbf{x})$ is a Parseval Gabor multi-frame generator. Thus the rest of the paper is devoted to the proof of the necessary part of Theorem 1.1.

The rest of the paper is organized as follows. In the next section, we show that Theorem 1.1 can be proven under a simplified setting. In Section 3 we provide some necessary background knowledge regarding the lattice tiling and packing of $\mathbb{R}^d$. In Section 4 we prove Theorem 1.1 for the case of $\gamma=1$ with any $d\geq 1$. In Section 5 we prove Theorem 1.1 for the case of $\gamma>1$ with any $d\geq 1$.

2. Auxiliary simplifications

Let A and B be nonsingular real valued $d\times d$ matrices, and γ be the integer satisfying $|\det(AB)|\leq\gamma<|\det(AB)|+1$. Let P, Q be any two $d\times d$ matrices with integer entries and $|\det(P)|=|\det(Q)|=1$. (Note that we will be making specific choices later for P, Q later depending on our needs, but the statement here holds for any such P, Q.) Denote the matrix $(PB^\tau AQ)^{-1}$ by D. This implies that $AQD=(B^\tau)^{-1}P^{-1}$. For any set of functions $\{\tilde g_1(\mathbf{x}),...,\tilde g_\gamma(\mathbf{x})\}$ such that $\tilde g_j\in L^2(\mathbb{R}^d)$, define $g_j(\mathbf{x})$ by $g_j(\mathbf{x})=\tilde g_j((AQ)^{-1}\mathbf{x})=\tilde g_j(\mathbf{z})$ where $\mathbf{z}=(AQ)^{-1}\mathbf{x}$.

Lemma 2.1. *The following two statements hold:*
(1) $\tilde G(\mathbf{x})=(\tilde g_1(\mathbf{x}),...,\tilde g_\gamma(\mathbf{x}))^\tau$ is a Parseval Gabor multi-frame generator for the time-frequency lattice $\mathbb{Z}^d\times(D^\tau)^{-1}\mathbb{Z}^d$ if and only if $G(\mathbf{z})=(g_1(\mathbf{z}),...,g_\gamma(\mathbf{z}))^\tau$ is a Parseval Gabor multi-frame generator for the time-frequency lattice $A\mathbb{Z}^d\times B\mathbb{Z}^d$.
(2) Let $\tilde M(\mathbf{z})$ be a $\gamma\times\gamma$ functional matrix multiplier and define $M(\mathbf{x})=\tilde M(\mathbf{z})$ with $\mathbf{z}=(AQ)^{-1}\mathbf{x}$. If $\tilde M^*(\mathbf{z})\tilde M(\mathbf{z}+D\mathbf{k})$ is $\mathbb{Z}^d$-periodic for any $\mathbf{z}\in\mathbb{R}^d$ and $\mathbf{k}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}$, then $M^*(\mathbf{x})M(\mathbf{x}+(B^\tau)^{-1}\mathbf{k})$ is $A\mathbb{Z}^d$-periodic for any $\mathbf{x}\in\mathbb{R}^d$ and $\mathbf{k}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}$. Moreover, if $\tilde M^*(\mathbf{z})\tilde M(\mathbf{z}+D\mathbf{k})=\tilde\lambda_{\mathbf{k}}(\mathbf{z})I$ for some scalar function $\tilde\lambda_{\mathbf{k}}(\mathbf{z})$, then $M^*(\mathbf{x})M(\mathbf{x}+(B^\tau)^{-1}\mathbf{k}')=\lambda_{\mathbf{k}'}(\mathbf{x})I$ with $\lambda_{\mathbf{k}'}(\mathbf{x})=\tilde\lambda_{\mathbf{k}}(\mathbf{z})$.

Lemma 2.1 implies that in order to prove Theorem 1.1, we only need to consider a special case of it, namely when $A = I_{d\times d}$ and $(B^T)^{-1} = D$, where D is of the form $(PB^T AQ)^{-1}$ for any P, Q with integer entries and $|\det(P)| = |\det(Q)| = 1$. Under this setting, it is necessary that $0 < |\det(D^{-1})| = d_0 \le \gamma < d_0 + 1$ and equations (1.2) and (1.3) become

$$\sum_{\mathbf{l}\in\mathcal{L}} \langle G(\mathbf{x}-\mathbf{l}), G(\mathbf{x}-\mathbf{l})\rangle = d_0, \tag{2.1}$$

$$\sum_{\mathbf{l}\in\mathcal{L}} \langle G(\mathbf{x}-\mathbf{l}), G(\mathbf{x}-\mathbf{l}-\mathbf{k})\rangle = 0,\ \forall\ \mathbf{k}\in\mathcal{K}\setminus\{\mathbf{0}\}. \tag{2.2}$$

3. The tiling and packing of $\mathbb{R}^d$ by $\mathcal{L}$ and $\mathcal{K}$

Let us introduce a few key concepts first. Let $\mathcal{F}$ be any full rank lattice of $\mathbb{R}^d$. A measurable set E is said to *pack* $\mathbb{R}^d$ by $\mathcal{F}$ if $E\cap(E+\mathbf{f}) = \emptyset$ for any nontrivial $\mathbf{f}\in\mathcal{F}$. If E packs $\mathbb{R}^d$ by $\mathcal{F}$ and also satisfies the condition $\mathbb{R}^d = \cup_{\mathbf{f}\in\mathcal{F}}(E+\mathbf{f})$, then we say that E *tiles* $\mathbb{R}^d$ by $\mathcal{F}$. In this case E is called a *tile* or a *fundamental domain* of $\mathcal{F}$. For two measurable sets S_1 and S_2 that pack $\mathbb{R}^d$ by $\mathcal{F}$, we say that S_1 and S_2 are *$\mathcal{F}$-equivalent* if $\cup_{\mathbf{f}\in\mathcal{F}}(S_1+\mathbf{f}) = \cup_{\mathbf{f}\in\mathcal{F}}(S_2+\mathbf{f})$, and we say that S_1 and S_2 are *$\mathcal{F}$-disjoint* if $S_1\cap(S_2+\mathbf{f}) = \emptyset$ for any nontrivial $\mathbf{f}\in\mathcal{F}$.

The materials in this section heavily rely on the work [13], more specifically the proofs in the sequence of lemmas there that lead to the proof of [13, Theorem 1.2], which states that if $|\det(D)| \le 1$, then there exists a measurable set that tiles $\mathbb{R}^d$ by $\mathcal{K} = D\mathbb{Z}^d$ and packs $\mathbb{R}^d$ by $\mathcal{L} = \mathbb{Z}^d$. From this point on, the lattices $\mathcal{L}$, $\mathcal{K}$ always mean $\mathbb{Z}^d$, $D\mathbb{Z}^d$ respectively unless otherwise noted. The following long remark summarizes the results (with slight modifications) extracted from [13] that are necessary for us to prove Theorem 1.1.

Remark 3.1. Consider the group $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ with $\Omega = [0,1)^d$ a representative set of the group. Let $\pi:\ \mathbb{R}^d \longrightarrow \mathbb{T}^d$ be the projection map and consider $\pi((B^\tau A)^{-1}\mathbb{Z}^d)$. $\overline{\pi((B^\tau A)^{-1}\mathbb{Z}^d)}$ is a closed subgroup of $\mathbb{T}^d$, hence $\overline{\pi((B^\tau A)^{-1}\mathbb{Z}^d)} = S\oplus F$ for some rational subspace S and finite set F [13, Lemma 2.1]. The proof of [13, Theorem 1.2] is divided into three cases: **Case 1**: $S = \mathbb{T}^d$; **Case 2**: $S = \{\emptyset\}$; and **Case 3**: $S \ne \mathbb{T}^d$ and $S \ne \{\emptyset\}$. We will follow these cases to make the choices for P, Q and to extract the information we need. Let $\mathbf{k}_j = D\mathbf{l}_j$ where $\{\mathbf{l}_1, \mathbf{l}_2, ..., \mathbf{l}_d\}$ is the standard basis for $\mathcal{L}$. Let γ be the unique integer satisfying $d_0 \le \gamma < d_0 + 1$.

Case 1. In this case we can simply choose $P = Q = I_{d\times d}$ in Lemma 2.1. There are two sub cases to consider here: (i) $d_0 = 1/|\det(D)| = |\det(AB)|$ is rational and (ii) d_0 is irrational.

(i) $d_0 = p/q$ with $(p,q) = 1$. In this case we can partition $D\Omega$ into M_2 parallelepipeds of the same volume μ_0 where M_2 can be any integer multiple of q. We have $M_2\mu_0 =$

$q/p = 1/d_0$ hence $\mu_0 = q/(M_2 p) = 1/(Np)$ where $Nq = M_2$. On the other hand, we can partition Ω into $M_1 = Np$ rectangles such that each rectangle also has volume μ_0. Denote these partitions of Ω and $D\Omega$ by $\mathfrak{P}$ and $\mathfrak{P}'$ respectively, and arbitrarily order and name the ones in $\mathfrak{P}$ as C_1, C_2, ..., C_{M_1}, and the ones in $\mathfrak{P}'$ as C'_1, C'_2, ..., C'_{M_2}. [13, Corollary 2.3] assures that for any pair of C_i and C'_j, there exists a measurable set $J(C_i, C'_j)$ that is $\mathcal{L}$-equivalent to C_i and $\mathcal{K}$-equivalent to C'_j (we say $J(C_i, C'_j)$ is a *matching* of C_i and C'_j). In particular, if there exists a rectangle C such that $C \subset C_i$ and $-\mathbf{l}_0 + C \subset -\mathbf{k}_0 + C'_j$ for some $\mathbf{l}_0 \in \mathcal{L}$ and $\mathbf{k}_0 \in \mathcal{K}$, then $-\mathbf{l}_0 + C$ can be selected as part of $J(C_i, C'_j)$. Recall that $M_1 = pN$ and $M_2 = qN$. If $q = 1$, then $d_0 = p = \gamma$ is an integer and $M_1 = \gamma M_2$. This means in this case we can divide the rectangles in $\mathfrak{P}$ into γ groups F_1, F_2, ..., F_γ such that each group contains M_2 rectangles. If $p > q > 1$ (this happens when $d_0 > 1$), then $p = (\gamma - 1)q + r$ for some positive integer $r < q$ (otherwise $p = \gamma q$ contradicts the condition that $(p, q) = 1$). It follows that $M_1 = Np = (\gamma-1)Nq + Nr = (\gamma-1)M_2 + Nr$. Thus in this case we can divide the rectangles in $\mathfrak{P}$ into γ groups F_1, F_2, ..., F_γ such that each group F_j with $j \geq 2$ contains M_2 rectangles, and F_1 contains the remaining $Nr < Nq = M_2$ rectangles. Finally, if $q > p \geq 1$ (that is, $d_0 < 1$), then we have $M_1 < M_2$.

(ii) d_0 is irrational. Here we need to consider the cases $d_0 > 1$ and $d_0 < 1$ separately.

First consider the case $d_0 > 1$. We have $d_0 = \gamma - \delta$ for some positive constant $\delta < 1$. In this case we can still partition $D\Omega$ into M_2 parallelepipeds (denoted by C'_j's) of the same volume μ_0 where M_2 can be any arbitrarily chosen large positive integer, in particular, we will choose it large enough so that $(1-\delta)M_2 > 1$. This time it is not possible to partition Ω into rectangles such that each rectangle also has the same volume μ_0 since $\mu_0 = 1/(M_2 d_0)$ is irrational, however this can be done if we allow one of these rectangles to have volume less than μ_0. Thus if M_1 is the total number of rectangles in $\mathfrak{P}$ named and ordered as C_i's as before, we can assume that all C_i's have volume μ_0 except that C_{M_1} has volume μ' which is less than μ_0. We leave it to our reader to verify that in this case $M_1 - (\gamma-1)M_2 = (1-\delta)M_2 + 1 - (\mu'/\mu_0) > (1-\delta)M_2 > 1$ by the choice of M_2. This means that we can again divide the rectangles in $\mathfrak{P}$ into γ groups F_1, F_2, ..., F_γ such that each group F_j with $j \geq 2$ contains M_2 rectangles, and F_1 contains the remaining rectangles including C_{M_1} which has volume μ'. By the above inequality we see that F_1 contain at least two rectangles, hence it also contains at least one rectangle that has volume μ_0. The statement in (i) about $J(C_i, C'_j)$ applies if $i \neq M_1$. For $i = M_1$, C_{M_1} can be matched to any parallelepiped of volume μ' that is a subset of any of the C'_j's.

Now consider the case $d_0 < 1$. In this case we first partition Ω into M_1 rectangular parallelepipeds of the same volume μ_0 where M_1 can be arbitrarily large. For example we can partition Ω into N^d small cubes of side length $1/N$ where $N > 0$ can be any arbitrarily chosen integer, obtaining $M_1 = N^d$ cubes, each with volume $\mu_0 = 1/N^d$. On the other hand, we can partition $D\Omega$ into $M_2 = N^d$ parallelepipeds such that each parallelepiped has volume $1/(d_0 N^d) > \mu_0$. Denote these partitions of Ω and $D\Omega$ by $\mathfrak{P}$

and $\mathfrak{P}'$ respectively, and arbitrarily order and name the ones in $\mathfrak{P}$ as $C_1, C_2, ..., C_{M_1}$, and the ones in $\mathfrak{P}'$ as $C'_1, C'_2, ..., C'_{M_1}$. [13, Corollary 2.3] assures that for any pair of C_i and C'_j $(1 \le i, j \le M_1)$, there exists a measurable set $J(C_i, C'_j)$ that is $\mathcal{L}$-equivalent to C_i and $\mathcal{K}$-equivalent to a subset of C'_j. We call $J(C_i, C'_j)$ a *matching* of C_i and C'_j. In particular, if there exists a rectangular parallelepiped C such that $C \subset C_i$ and $-\mathbf{l}_0 + C \subset -\mathbf{k}_0 + C'_j$ for some $\mathbf{l}_0 \in \mathcal{L}$ and $\mathbf{k}_0 \in \mathcal{K}$, then $-\mathbf{l}_0 + C$ can be selected as part of $J(C_i, C'_j)$.

Case 2. In this case P and Q can be chosen so that D is a diagonal matrix with positive rational entries $\{p_1/q_1, p_2/q_2, ..., p_d/q_d\}$ (with $\gcd(p_i, q_i) = 1$). Here we need to choose $\mathfrak{P}$ and $\mathfrak{P}'$ specifically so that the elements in them are rectangular parallelepipeds whose sides are parallel to $\mathbf{l}_1$, ..., $\mathbf{l}_d$ with corresponding side lengths of $1/q_1$, $1/q_2$, ..., $1/q_d$. Again, name and order (arbitrarily) the rectangles in Ω and $D\Omega$ by C_i's and C'_j's with $1 \le i \le M_1 = q_1 q_2 \cdots q_d$ and $1 \le j \le M_2 = p_1 p_2 \cdots p_d$. Then for any pair C_i and C'_j, there exists a rectangle $J(C_i, C'_j)$ such that $J(C_i, C'_j) = \mathbf{l}_0 + C_i = \mathbf{k}_0 + C$ for some $\mathbf{l}_0 \in \mathcal{L}$ and $\mathbf{k}_0 \in \mathcal{K}$. In particular, if C_i is paired with C'_j and $-\mathbf{l}_0 + C_i = -\mathbf{k}_0 + C'_j$ for some $\mathbf{l}_0 \in \mathcal{L}$ and $\mathbf{k}_0 \in K$, then $J(C_i, C'_j) = -\mathbf{l} + C_i$. Since d_0 is rational, by a discussion similar to Case 1 (i), if $d_0 > 1$ then the rectangles in $\mathfrak{P}$ can be divided into γ groups F_1, F_2, ..., F_γ such that each group F_j with $j \ge 2$ contains M_2 rectangles, and F_1 contains at least one and at most M_2 rectangles. On the other hand, if $d_0 \le 1$, then $M_1 \le M_2$.

Case 3. In this case P and Q can be chosen so that $D = \begin{bmatrix} D_1 & B_0 \\ 0 & D_2 \end{bmatrix}$, where D_2 is a $d_2 \times d_2$ diagonal matrix with positive rational entries $\{p_1/q_1, ..., p_{d_2}/q_{d_2}\}$ ($\gcd(p_i, q_i) = 1$ for each i) and $[\,D_1 \quad B_0\,]\,\mathbb{Z}^d \pmod 1$ is dense in $[0,1)^{d_1}$, $d_1 + d_2 = d$. For the sake of convenience let $\{\mathbf{l}'_1, \mathbf{l}'_2, ..., \mathbf{l}'_{d_1}\}$ be the standard basis of $\mathbb{R}^{d_1}$, $\mathcal{K}_1$ be the lattice spanned by $\{\mathbf{k}'_1, \mathbf{k}'_2, ..., \mathbf{k}'_{d_1}\}$ where $\mathbf{k}'_j = D_1 \mathbf{l}'_j$ $(1 \le j \le d_1)$. Similarly, let $\{\mathbf{l}''_1, \mathbf{l}''_2, ..., \mathbf{l}''_{d_2}\}$ be the standard basis of $\mathbb{R}^{d_2}$, $\mathcal{K}_2$ be the lattice spanned by $\{\mathbf{k}''_1, \mathbf{k}''_2, ..., \mathbf{k}''_{d_2}\}$ where $\mathbf{k}''_j = D_2 \mathbf{l}''_j = (p_j/q_j)\mathbf{l}''_j$ $(1 \le j \le d_2)$. Let $\Omega_1 = [0,1)^{d_1}$, $\Omega_2 = [0,1)^{d_2}$ so that $\Omega = \Omega_1 \times \Omega_2$. It is apparent that $D_1\Omega_1$ is a fundamental domain of $\mathcal{K}_1$ and $D_2\Omega_2$ is a fundamental domain of $\mathcal{K}_2$. It is less apparent that $\tilde{\Omega} = (D_1\Omega_1) \times (D_2\Omega_2)$ is also a fundamental domain of $\mathcal{K}$ [13]. So instead of using $D\Omega$, we will use $\tilde{\Omega} = (D_1\Omega_1) \times (D_2\Omega_2)$ instead. The advantage of this is that it allows us to obtain a partition of $\tilde{\Omega}$ by partitioning $D_1\Omega_1$ and $D_2\Omega_2$ separately as described below.

(i) $|\det(D_1)|$ is rational. Similar to Case 1 (i), we can partition $D_1\Omega_1$ into parallelepipeds Δ'_j's $(1 \le j \le M'_2)$ of the same volume μ_0 and partition Ω_1 into rectangular parallelepipeds Δ_i's $(1 \le i \le M'_1)$ with volume μ_0 (μ_0 can be chosen to be as small as we want). On the other hand, we partition Ω_2 and $D_2\Omega_2$ into rectangles R_i's $(1 \le i \le q_1 q_2 \cdots q_{d_2})$ and R'_j's $(1 \le j \le p_1 p_2 \cdots p_{d_2})$ whose sides are parallel to $\mathbf{l}''_1$, ..., $\mathbf{l}''_{d_2}$ with corresponding side lengths of $1/q_1$, $1/q_2$, ..., $1/q_{d_2}$. Then the set $\{\Delta'_j \times R'_i :\ 1 \le j \le M'_2, 1 \le i \le p_1 p_2 \cdots p_{d_2}\}$ is a partition $\mathfrak{P}'$ of $\tilde{\Omega}$ whose total number of elements is $M_2 = M'_2 p_1 p_2 \cdots p_{d_2}$, and the set $\{\Delta_j \times R_i :\ 1 \le j \le M'_1, 1 \le i \le q_1 q_2 \cdots q_{d_2}\}$ is a partition $\mathfrak{P}$ of Ω whose total number of elements is $M_1 = M'_1 q_1 q_2 \cdots q_{d_2}$. By a slightly modified version of [13, Sub Lemma 5], for any $C_{ij} = \Delta_i \times R_j \in \mathfrak{P}$ and any

$C'_{i'j'} = \Delta'_{i'} \times R'_{j'} \in \mathfrak{P}'$, there exists a measurable set $J(C_{ij}, C'_{i'j'})$ that is $\mathcal{L}$-equivalent to C_{ij} and $\mathcal{K}$-equivalent to $C'_{i'j'}$. In particular, if Δ_i contains a small rectangle Δ_0 such that $-\mathbf{l}_0 + \Delta_0 \times R_j \subset -\mathbf{k}_0 + \Delta'_{i'} \times R'_{j'} = -\mathbf{k}_0 + C'_{i'j'}$, then $-\mathbf{l}_0 + \Delta_0 \times R_j$ can be selected as part of $J(C_{ij}, C'_{i'j'})$. In the case that $d_0 > 1$, the discussion in Case 1 (i) applies to $M_1 - (\gamma - 1)M_2$ here (since d_0 is rational). That is, we can divide the elements in $\mathfrak{P}$ into γ groups F_1, F_2, ..., F_γ such that each group F_j with $j \geq 2$ contains M_2 elements, and F_1 contains at least one and at most M_2 elements. On the other hand, if $d_0 \leq 1$, then $M_1 \leq M_2$.

(ii) $|\det(D_1)|$ is irrational hence d_0 is also irrational. Similar to Case 1(ii), we also need to consider the cases $d_0 > 1$ and $d_0 < 1$ separately.

If $d_0 > 1$, let $\delta = \gamma - d_0 > 0$. Similar to (i) above, we partition Ω_2 and $D_2\Omega_2$ into rectangles R_i's ($1 \leq i \leq q_1 q_2 \cdots q_{d_2}$) and R'_j's ($1 \leq j \leq p_1 p_2 \cdots p_{d_2}$) whose sides are parallel to the $\mathbf{l}''_1$, ..., $\mathbf{l}''_{d_2}$ coordinates with corresponding side lengths of $1/q_1$, $1/q_2$, ..., $1/q_{d_2}$. Similar to Case 1 (ii), we can partition $D_1\Omega_1$ into parallelepipeds Δ'_j's ($1 \leq j \leq M'_2$) of the same volume μ'_0 and partition Ω_1 into rectangles Δ_i's ($1 \leq i \leq M'_1$) with volume μ'_0 (μ'_0 can be chosen to be as small as we want since we can choose M'_2 as large as we want), with the exception that $0 < \mu'' = \mu(\Delta_{M'_1}) < \mu'_0$. Similar to (i) above, the set $\{\Delta'_j \times R'_i : 1 \leq j \leq M'_2, 1 \leq i \leq p_1 p_2 \cdots p_{d_2}\}$ is a partition $\mathfrak{P}'$ of $\tilde{\Omega}$ whose total number of elements is $M_2 = M'_2 p_1 p_2 \cdots p_{d_2}$, and the set $\{\Delta_j \times R_i : 1 \leq j \leq M'_1, 1 \leq i \leq q_1 q_2 \cdots q_{d_2}\}$ is a partition $\mathfrak{P}$ of Ω whose total number of elements is $M_1 = M'_1 q_1 q_2 \cdots q_{d_2}$. The difference here is that all the elements in these partitions have measure $\mu_0 = \mu'_0/(q_1 q_2 \cdots q_{d_2})$, except that the elements $\Delta_{M'_1} \times R_i$ (for any $1 \leq i \leq q_1 q_2 \cdots q_{d_2}$) have measure $\mu''/(q_1 q_2 \cdots q_{d_2})$. However, the inequality $M_1 - (\gamma - 1)M_2 > (1 - \delta)M_2$ still holds in this case as one can check, hence we will have $(1 - \delta)M_2 > q_1 q_2 \cdots q_{d_2} + 1$ if M'_2 is large enough. This ensures that we can place all the elements $\Delta_{M'_1} \times R_i$ into the group F_1 as described in Case 1 (ii), and F_1 will still contain at least one element whose measure is μ_0. The statement in (i) about the matching of two elements with measure μ_0 applies. A set of the form $\Delta_{M'_1} \times R_i$, on the other hand, can be matched with a set $\Delta \times R'_j$, where Δ is a properly chosen parallelepiped contained in a Δ'_i with a measure μ''.

In the case that $d_0 < 1$, we first partition Ω_2 into rectangular parallelepipeds whose sides are parallel to $\mathbf{l}''_1$, ..., $\mathbf{l}''_{d_2}$ with corresponding side lengths of $1/q_1$, $1/q_2$, ..., $1/q_{d_2}$, and partition Ω_1 into $M'_1 = N^{d_1} p_1 p_2 \cdots p_{d_2}$ (congruent) rectangular parallelepipeds whose sides are parallel to $\mathbf{l}'_1$, ..., $\mathbf{l}'_{d_1}$ with corresponding side lengths of $1/N$, $1/N$, ..., $1/N$, $1/(N p_1 p_2 \cdots p_{d_2})$, where N can be any arbitrarily chosen positive integer. Combining these partitions yields a partition $\mathfrak{P}$ of $\Omega = \Omega_1 \times \Omega_2$. If we name and number the rectangular parallelepipeds in the partition of Ω_1 as Δ_1, Δ_2, ..., $\Delta_{M'_1}$, name and number the rectangular parallelepipeds in the partition of Ω_2 as R_1, R_2, ..., $R_{M''_1}$ where $M''_1 = q_1 q_2 \cdots q_{d_2}$, then $\mathfrak{P} = \{C_{ij} = \Delta_i \times R_j : 1 \leq i \leq M'_1, 1 \leq j \leq M''_1\}$, and $M_1 = M'_1 M''_1 = N^{d_1}(p_1 p_2 \cdots p_{d_2})(q_1 q_2 \cdots q_{d_2})$ is the total number of partition elements in $\mathfrak{P}$. Notice that the volume of each Δ_j is $\mu_0' = 1/M_1' = 1/(N^{d_1}p_1p_2\cdots p_{d_2})$ and the volume of each R_i is $\mu_0'' = 1/(q_1q_2\cdots q_{d_2})$.

Next, we partition $D_1\Omega_1$ into $M_2' = N^{d_1}q_1q_2\cdots q_{d_2}$ congruent parallelepipeds. The volume of each of these parallelepipeds is $\mu(D_1\Omega_1)/M_2' = |\det(D_1)|/M_2' \geq 1/(N^{d_1}p_1p_2\cdots p_{d_2}) = \mu_0'$. We name and order these parallelepipeds as Δ_i''s $(1 \leq i \leq M_2')$. We now partition $D_2\Omega_2$ into rectangular parallelepipeds that are congruent to the R_i's and name/order them as R_j''s $(1 \leq j \leq p_1p_2\cdots p_{d_2})$. Combining these two partitions yields a partition $\mathfrak{P}'$ of $\tilde{\Omega} = D\Omega_1 \times D\Omega_2$: $\mathfrak{P}' = \{C'_{i'j'} = \Delta'_{i'} \times R'_{j'} : 1 \leq i' \leq M_2', 1 \leq j' \leq M_2''\}$ where $M_2'' = p_1p_2\cdots p_{d_2}$. Notice that the total number of partition elements in $\mathfrak{P}'$ is $M_2 = M_2'M_2'' = N^{d_1}(p_1p_2\cdots p_{d_2})(q_1q_2\cdots q_{d_2}) = M_1$ but $\mu(\Delta'_{i'} \times R'_{j'}) \geq \mu_0'/(q_1q_2\cdots q_{d_2}) = \mu(\Delta_i \times R_j)$. Again by a slightly modified version of [13, Sub Lemma 5], for any $C_{ij} = \Delta_i \times R_j \in \mathfrak{P}$ and any $C'_{i'j'} = \Delta'_{i'} \times R'_{j'} \in \mathfrak{P}'$, there exists a measurable set $J(C_{ij}, C'_{i'j'})$ that is $\mathcal{L}$-equivalent to C_{ij} and $\mathcal{K}$-equivalent to a subset of $C'_{i'j'}$. In particular, if Δ_i contains a small rectangular parallelepiped Δ_0 such that $-\mathbf{l}_0 + \Delta_0 \times R_j \subset -\mathbf{k}_0 + \Delta'_{i'} \times R'_{j'} = -\mathbf{k}_0 + C'_{i'j'}$, then $-\mathbf{l}_0 + \Delta_0 \times R_j$ can be selected as part of $J(C_{ij}, C'_{i'j'})$.

This ends Remark 3.1.

Notice that in Remark 3.1, we used $D\Omega$ as the fundamental domain of $\mathcal{K}$ in Cases 1 and 2, but used $\tilde{\Omega}$ as the fundamental domain of $\mathcal{K}$ in Case 3. For the sake of simplicity, let us denote them by Ω' with the understanding that it means either $D\Omega$ or $\tilde{\Omega}$ depending on which case $\mathcal{K}$ belongs to. Let us call the pair of partitions $\mathfrak{P}$, $\mathfrak{P}'$ of Ω and Ω' with the properties discussed in Remark 3.1 *nice partition pair*.

Remark 3.2. We note that in the above discussion, Ω' can be replaced by any $\mathcal{K}$-translation of Ω'. Similarly, Ω can be replaced by any $\mathcal{L}$-translation of Ω. Thus without loss of generality, one can always assume that $\Omega \cap \Omega'$ is non-empty and contain interior points.

4. The proof of Theorem 1.1, part 1

We now proceed to prove Theorem 1.2, namely the special case $\gamma = 1$ of Theorem 1.1. Here $d_0 = |\det(D^{-1})| \leq 1$ so $|\det(D)| = 1/d_0 \geq 1$. In this case, the functional multiplier is a scalar function $h(\mathbf{x})$ and we need to prove that $h(\mathbf{x})\overline{h(\mathbf{x}-\mathbf{k})} = h(\mathbf{x}-\mathbf{l})\overline{h(\mathbf{x}-\mathbf{l}-\mathbf{k})}$ for any $\mathbf{x} \in \mathbb{R}^d$ *a.e.*, and for any $\mathbf{l} \in \mathcal{L}$ and $\mathbf{k} \in \mathcal{K} \setminus \{\mathbf{0}\}$.

4.1. A few useful lemmas

First, for the very special case of $D = I$, we have the following lemma, which follows by a generalized version of the proof of [16, 4.3 Type III Case].

Lemma 4.1. *If $D = I$ is the identity matrix, then Theorem 1.1 holds.*

We also have the following lemma, whose proof can be found in [16].

Lemma 4.2. *[16, Lemma 2.4] Let h be a functional Gabor frame multiplier and let $\mathbf{l} \in \mathcal{L}$, $\mathbf{k} \in \mathcal{K} \setminus \{\mathbf{0}\}$ be any given pair of vectors, then $h(\mathbf{x})\overline{h(\mathbf{x}-\mathbf{k})} = h(\mathbf{x}-\mathbf{l})\overline{h(\mathbf{x}-\mathbf{l}-\mathbf{k})}$ for any $\mathbf{x} \in \mathbb{R}^d$ if one of the following conditions holds:*
(i) There exist disjoint and measurable sets E_1, E_2, E_3, E_4 and E_5, with E_2 being a rectangular parallelepiped, such that $E_3 = -\mathbf{k} + E_2$, $E_4 = -\mathbf{l} + E_2$, $E_5 = -\mathbf{l} - \mathbf{k} + E_2$, and $E_1 \cup E_2 \cup E_3$ tiles $\mathbb{R}^d$ by $\mathcal{L}$ while $E_1 \cup E_2 \cup E_4$ packs $\mathbb{R}^d$ by $\mathcal{K}$;
(ii) $\mathbf{k} \in \mathcal{L}$ and there exist disjoint and measurable sets E_1, E_2, E_3, E_4 and E_5, with E_2 being a rectangular parallelepiped, such that $E_3 = -\mathbf{k} + E_2$, $E_4 = -\mathbf{l} + E_2$, $E_5 = -\mathbf{l} - \mathbf{k} + E_2$, $E_1 \cup E_2$ tiles $\mathbb{R}^d$ by $\mathcal{L}$ and $E_1 \cup E_2 \cup E_4$ packs $\mathbb{R}^d$ by $\mathcal{K}$.

Notice that the given condition in (i) above implies that $\mathbf{k} \notin \mathcal{L}$ and $\mathbf{l} \notin \mathcal{K}$, while the given condition in (ii) implies that $\mathbf{l} \notin \mathcal{K}$.

Lemma 4.3. *Let h be a functional Gabor frame multiplier and let $\mathbf{l}$, $\mathbf{l}' \in \mathcal{L}$, $\mathbf{k}$, $\mathbf{k}' \in \mathcal{K}\setminus\{\mathbf{0}\}$, then the following statements hold:*
(i) If $h(\mathbf{x})\overline{h(\mathbf{x}-\mathbf{k})} = h(\mathbf{x}-\mathbf{l}')\overline{h(\mathbf{x}-\mathbf{l}'-\mathbf{k})}$ and $h(\mathbf{x})\overline{h(\mathbf{x}-\mathbf{k})} = h(\mathbf{x}+\mathbf{l}'-\mathbf{l}) \times \overline{h(\mathbf{x}+\mathbf{l}'-\mathbf{l}-\mathbf{k})}$ for any $\mathbf{x} \in \mathbb{R}^d$, then $h(\mathbf{x})\overline{h(\mathbf{x}-\mathbf{k})} = h(\mathbf{x}-\mathbf{l})\overline{h(\mathbf{x}-\mathbf{l}-\mathbf{k})}$;
(ii) If $h(\mathbf{x})\overline{h(\mathbf{x}-\mathbf{k}')} = h(\mathbf{x}-\mathbf{l})\overline{h(\mathbf{x}-\mathbf{l}-\mathbf{k}')}$ and $h(\mathbf{x})\overline{h(\mathbf{x}-\mathbf{k}'-\mathbf{k})} = h(\mathbf{x}-\mathbf{l}) \times \overline{h(\mathbf{x}-\mathbf{l}-\mathbf{k}'-\mathbf{k})}$ for any $\mathbf{x} \in \mathbb{R}^d$, then $h(\mathbf{x})\overline{h(\mathbf{x}-\mathbf{k})} = h(\mathbf{x}-\mathbf{l})\overline{h(\mathbf{x}-\mathbf{l}-\mathbf{k})}$.

Proof. (i) Substituting $\mathbf{x}+\mathbf{l}'$ by $\mathbf{x}$ (by a slight abuse of notation) on both sides of the equation

$$h(\mathbf{x})\overline{h(\mathbf{x}-\mathbf{k})} = h(\mathbf{x}+\mathbf{l}'-\mathbf{l})\overline{h(\mathbf{x}+\mathbf{l}'-\mathbf{l}-\mathbf{k})},$$

we obtain

$$h(\mathbf{x}-\mathbf{l}')\overline{h(\mathbf{x}-\mathbf{l}'-\mathbf{k})} = h(\mathbf{x}-\mathbf{l})\overline{h(\mathbf{x}-\mathbf{l}-\mathbf{k})}.$$

But the left side of the above equation is $h(\mathbf{x})\overline{h(\mathbf{x}-\mathbf{k})}$ by the given condition.

(ii) Since h is unimodular, the given condition leads to

$$h(\mathbf{x})\overline{h(\mathbf{x}-\mathbf{l})} = h(\mathbf{x}-\mathbf{k}')\overline{h(\mathbf{x}-\mathbf{l}-\mathbf{k}')}$$

and

$$h(\mathbf{x})\overline{h(\mathbf{x}-\mathbf{l})} = h(\mathbf{x}-\mathbf{k}'-\mathbf{k})\overline{h(\mathbf{x}-\mathbf{l}-\mathbf{k}'-\mathbf{k})}.$$

Thus we have

$$h(\mathbf{x}-\mathbf{k}')\overline{h(\mathbf{x}-\mathbf{l}-\mathbf{k}')} = h(\mathbf{x}-\mathbf{k}'-\mathbf{k})\overline{h(\mathbf{x}-\mathbf{l}-\mathbf{k}'-\mathbf{k})}.$$

Substituting $\mathbf{x}-\mathbf{k}'$ by $\mathbf{x}$ on both sides of the above equation (again by a slight abuse of notation), we obtain

$$h(\mathbf{x})\overline{h(\mathbf{x}-\mathbf{l})} = h(\mathbf{x}-\mathbf{k})\overline{h(\mathbf{x}-\mathbf{l}-\mathbf{k})}$$

for any $\mathbf{x} \in \mathbb{R}^d$ *a.e.* □

Lemma 4.4. *For any given* $\mathbf{k}$ *and* $\mathbf{l}$*, if* $\mathbf{k} \notin \mathcal{L}$ *and* $\mathbf{l} \notin \mathcal{K}$*, then there exist disjoint and measurable sets* E_1*,* E_2*,* E_3*,* E_4 *and* E_5*, with* E_2 *being a rectangular parallelepiped, such that* $E_3 = -\mathbf{k} + E_2$*,* $E_4 = -\mathbf{l} + E_2$*,* $E_5 = -\mathbf{l} - \mathbf{k} + E_2$*, and* $E_1 \cup E_2 \cup E_3$ *tiles* $\mathbb{R}^d$ *by* $\mathcal{L}$ *while* $E_1 \cup E_2 \cup E_4$ *packs* $\mathbb{R}^d$ *by* $\mathcal{K}$*. It follows that* $h(\mathbf{x})\overline{h(\mathbf{x}-\mathbf{l})} = h(\mathbf{x}-\mathbf{k})\overline{h(\mathbf{x}-\mathbf{l}-\mathbf{k})}$ *by Lemma 4.2.*

Proof. We will prove the given statement by discussing the three different cases of D given in Remark 3.1. The general strategy in each case is to choose a suitable nice partition pair $\mathfrak{P}$, $\mathfrak{P}'$ of Ω and Ω' as described in Remark 3.1 with the following additional properties: (a) there exists a rectangular parallelepiped (which is chosen as our set E_2) such that $E_2 \subset C_0 \cap C_0'$ for some $C_0 \in \mathfrak{P}$ and $C_0' \in \mathfrak{P}'$; (b) $-\mathbf{k} - \mathbf{l}_0 + E_2 \subset C_1 \in \mathfrak{P}$, for some $\mathbf{l}_0 \in \mathcal{L}$ and $C_1 \neq C_0$; (c) $-\mathbf{l} - \mathbf{k}_0 + E_2 \subset C_1' \in \mathfrak{P}'$, for some $\mathbf{k}_0 \in \mathcal{K}$ and $C_1' \neq C_0'$. We then assign a one to one correspondence between the elements of $\mathfrak{P}$ and $\mathfrak{P}'$ in an arbitrary way to be used to define the matchings, except that C_0 is matched with C_0' with the choice of $E_2 \subset J(C_0, C_0')$ and C_1 is matched with C_1' with the choice that $\mathbf{k} - \mathbf{l}_0 + C$ is matched with $\mathbf{l} - \mathbf{k}_0 + C$ in $J(C_1, C_1')$. Denote the union of the matchings by S. S tiles $\mathbb{R}^d$ by $\mathcal{L}$ and packs $\mathbb{R}^d$ by $\mathcal{K}$ by our construction. Let the portion of $J(C_1, C_1')$ that is $\mathcal{L}$-equivalent to $-\mathbf{k} - \mathbf{l}_0 + C$ and $\mathcal{K}$-equivalent to $-\mathbf{l} - \mathbf{k}_0 + C$ by J_0, and define $E_1 = S \setminus (E_2 \cup J_0)$, $E_3 = -\mathbf{k} + E_2$, $E_4 = -\mathbf{l} + E_2$ and $E_5 = -\mathbf{l} - \mathbf{k} + E_2$. Since $-\mathbf{l}_0 + E_3 = -\mathbf{k} - \mathbf{l}_0 + E_2$, E_3 is $\mathcal{L}$-equivalent to J_0 hence $E_1 \cup E_2 \cup E_3$ is $\mathcal{L}$-equivalent to S. Thus $E_1 \cup E_2 \cup E_3$ tiles $\mathbb{R}^d$ by $\mathcal{L}$. Similarly, E_4 is $\mathcal{K}$-equivalent to J_0 hence $E_1 \cup E_2 \cup E_4$ is $\mathcal{K}$-equivalent to S, so $E_1 \cup E_2 \cup E_4$ packs $\mathbb{R}^d$ by $\mathcal{K}$. Thus in the following cases we only need to show that a nice partition pair $\mathfrak{P}$, $\mathfrak{P}'$ of Ω and Ω' that satisfies conditions (a) to (c) exists.

Case 1. Let $\mathbf{x}_0$ be an interior point of $\Omega \cap D\Omega$ near the origin. Since Ω is a fundamental domain of $\mathcal{L}$, there exists $\mathbf{l}_0 \in \mathcal{L}$ such that $\mathbf{x}_0 - \mathbf{k} - \mathbf{l}_0 \neq \mathbf{x}_0$ is also an interior point of Ω (since $-\mathbf{k} \notin \mathcal{L}$). Similarly, there exists $\mathbf{k}_0 \in \mathcal{K}$ such that $\mathbf{x}_0 - \mathbf{l} - \mathbf{k}_0 \neq \mathbf{x}_0$ is an interior point of $D\Omega$. It follows that we can choose a nice partition pair $\mathfrak{P}$, $\mathfrak{P}'$ of Ω and $D\Omega$ such that the diameters of the polytopes in them are smaller than $\min\{|-\mathbf{k}-\mathbf{l}_0|, |-\mathbf{l}-\mathbf{k}_0|\}$. WLOG we can assume that there exist $C_{i_0} \in \mathfrak{P}$ and $C_{j_0}' \in \mathfrak{P}'$ such that $\mathbf{x}_0$ is an interior point of $C_{i_0} \cap C_{j_0}'$, $\mathbf{x}_0 - \mathbf{k} - \mathbf{l}_0$ is an interior point of another C_{i_1} in $\mathfrak{P}$, and $\mathbf{x}_0 - \mathbf{l} - \mathbf{k}_0$ is an interior point of another C_{j_1}' in $\mathfrak{P}'$ (otherwise we can replace $\mathbf{x}_0$ by a suitably chosen point very close to it). It follows that there exists a small rectangular parallelepiped E_2 such that $E_2 \subset C_{i_0} \cap C_{j_0}'$, $\mathbf{k} - \mathbf{l}_0 + C \subset C_{i_1}$ and $\mathbf{l} - \mathbf{k}_0 + C \subset C_{j_1}'$.

Case 2. In this case $\mathfrak{P}$ and $\mathfrak{P}'$ are as described in Case 2 of Remark 3.1 with the rectangular parallelepipeds named and ordered in an arbitrary way. Let E_2 be the common rectangular parallelepiped of $\mathfrak{P}$ and $\mathfrak{P}'$ containing the origin (say $E_2 = C_{i_0} = C'_{j_0}$). Similar to the proof of Case 1, there exist $\mathbf{l}_0 \in \mathcal{L}$ and $\mathbf{k}_0 \in \mathcal{K}$ such that $-\mathbf{k}-\mathbf{l}_0+E_2 = C_{i_1} \in \mathfrak{P}$ for some $C_{i_1} \in \mathfrak{P}$, and $-\mathbf{l} - \mathbf{k}_0 + E_2 = C'_{j_1}$ for some $C'_{j_1} \in \mathfrak{P}'$. $i_1 \neq i_0$ and $j_1 \neq j_0$ since $-\mathbf{l} \notin \mathcal{K}$ and $-\mathbf{k} \notin \mathcal{L}$.

Case 3. Recall that in this case $D = \begin{bmatrix} D_1 & B_0 \\ 0 & D_2 \end{bmatrix}$, where D_2 is a $d_2 \times d_2$ diagonal matrix with positive rational entries, $[D_1 \quad B_0]\,\mathbb{Z}^d \pmod 1$ is dense in $[0,1)^{d_1}$, and $\tilde{\Omega} = D_1\Omega_1 \times D_2\Omega_2$ is a fundamental domain of $\mathcal{K}$. There exist $\mathbf{l}_0 \in \mathcal{L}$ and $\mathbf{k}_0 \in \mathcal{K}$ such that $-\mathbf{k} - \mathbf{l}_0 = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}$ with $\mathbf{y}_1 \in \Omega_1$, $\mathbf{y}_2 \in \Omega_2$ and at least one of them is not $\mathbf{0}$, $-\mathbf{l} - \mathbf{k}_0 = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix}$ with $\mathbf{z}_1 \in D_1\Omega_1$, $\mathbf{z}_2 \in D_2\Omega_2$ and at least one of them is not $\mathbf{0}$. Thus the nice partition pair $\mathfrak{P}$, $\mathfrak{P}'$ of Ω and $\tilde{O}$ obtained by combining a suitably chosen nice partition pair $\mathfrak{P}_1$, $\mathfrak{P}'_1$ of Ω_1 and $D_1\Omega_1$, and the nice partition pair $\mathfrak{P}_2$, $\mathfrak{P}'_2$ of Ω_2 and $D_2\Omega_2$ as described in Case 2 of Remark 3.1, we will have the following: there exists a small rectangular parallelepiped $C \subset \Omega_1 \cap D_1\Omega_1$ such that $E_2 = C \times R$, where $R = R_{i'_0} = R'_{j'_0}$ is the common rectangular parallelepiped of $\mathfrak{P}_2$ and $\mathfrak{P}'_2$ containing the origin, satisfies the condition that $E_2 \subset C_{i_0,i'_0} \cap C'_{j_0,j'_0}$, $C_{i_0,i'_0} = \Delta_{i_0} \times R_{i'_0} \in \mathfrak{P}$ and $C'_{j_0,j'_0} = \Delta'_{j_0} \times R'_{j'_0} \in \mathfrak{P}'$, $-\mathbf{k} - \mathbf{l}_0 + E_2 \subset C_{i_1,i'_1} = \Delta_{i_1} \times R_{i'_1} \in \mathfrak{P}$ ($C_{i_1,i'_1} \neq C_{i_0,i'_0}$), $-\mathbf{l} - \mathbf{k}_0 + E_2 \subset C'_{j_1,j'_1} = \Delta'_{j_1} \times R'_{j'_1} \in \mathfrak{P}'$ ($C'_{j_1,j'_1} \neq C'_{j_0,j'_0}$). $\square$

4.2. The main proof

We shall consider the following three cases: I. $\mathcal{L} \not\subset \mathcal{K}$ and $\mathcal{K} \not\subset \mathcal{L}$; II. $\mathcal{L} \subset \mathcal{K}$; III. $\mathcal{K} \subset \mathcal{L}$ but $\mathcal{L} \not\subset \mathcal{K}$.

I. $\mathcal{L} \not\subset \mathcal{K}$ **and** $\mathcal{K} \not\subset \mathcal{L}$. Let $\mathbf{l} \in \mathcal{L}$ and $\mathbf{k} \in \mathcal{K} \setminus \{\mathbf{0}\}$ be any two vectors. If $\mathbf{l} \notin \mathcal{K}$ and $\mathbf{k} \notin \mathcal{L}$, then by Lemma 4.4 we have $h(\mathbf{x})\overline{h(\mathbf{x}-\mathbf{l})} = h(\mathbf{x}-\mathbf{k})\overline{h(\mathbf{x}-\mathbf{l}-\mathbf{k})}$. If $\mathbf{l} \notin \mathcal{L}$ and $\mathbf{k} \in \mathcal{K} \setminus \{\mathbf{0}\}$, then choose any $\mathbf{k}' \in \mathcal{K} \setminus \{\mathbf{0}\}$ such that $\mathbf{k}' \notin \mathcal{L}$, and apply Lemma 4.4 to the pairs $(\mathbf{l}, \mathbf{k}')$ and $(\mathbf{l}, \mathbf{k}' + \mathbf{k})$ (note that $\mathbf{k}' + \mathbf{k} \notin \mathcal{L}$). Similarly, if $\mathbf{l} \in \mathcal{L}$ and $\mathbf{k} \notin \mathcal{K} \setminus \{\mathbf{0}\}$, then choose any $\mathbf{l}' \in \mathcal{L}$ such that $\mathbf{l}' \notin \mathcal{K}$, and apply Lemma 4.4 to the pairs $(\mathbf{l}', \mathbf{k})$ and $(-\mathbf{l}'+\mathbf{l}, \mathbf{k})$ (note that $-\mathbf{l}'+\mathbf{l} \notin \mathcal{K}$). Finally, if $\mathbf{l} \in \mathcal{K}$ and $\mathbf{k} \in \mathcal{L}$, then choose any $\mathbf{l}' \in \mathcal{L}$ such that $\mathbf{l}' \notin \mathcal{K}$ and $\mathbf{k}' \in \mathcal{K} \setminus \{\mathbf{0}\}$ such that $\mathbf{k}' \notin \mathcal{L}$. The application of Lemma 4.4 to the pairs $(\mathbf{l}', \mathbf{k}')$ and $(-\mathbf{l}' + \mathbf{l}, \mathbf{k}')$ leads to $h(\mathbf{x})\overline{h(\mathbf{x}-\mathbf{l})} = h(\mathbf{x}-\mathbf{k}')\overline{h(\mathbf{x}-\mathbf{l}-\mathbf{k}')}$, and to the pairs $(\mathbf{l}', \mathbf{k}' + \mathbf{k})$ and $(-\mathbf{l}' + \mathbf{l}, \mathbf{k}' + \mathbf{k})$ leads to $h(\mathbf{x})\overline{h(\mathbf{x}-\mathbf{l})} = h(\mathbf{x}-\mathbf{k}'-\mathbf{k})\overline{h(\mathbf{x}-\mathbf{l}-\mathbf{k}'-\mathbf{k})}$. The result now follows by Lemma 4.3 (ii).

II. $\mathcal{L} \subset \mathcal{K}$. Notice that $[\mathbf{l}_1 \quad \mathbf{l}_2 \quad \cdots \quad \mathbf{l}_d] = I$ is the identity matrix, so we have $I = DU$ for some $d \times d$ matrix U with integer entries since each $\mathbf{l}_j$ is a linear combination of $\mathbf{k}_1 = D\mathbf{l}_1$, $\mathbf{k}_2 = D\mathbf{l}_2$, ..., $\mathbf{k}_d = D\mathbf{l}_d$ with integer coefficients. It follows that $D^{-1} = U$ is a matrix with integer entries hence $|\det(D^{-1})| \geq 1$ since $|\det(D^{-1})|$ is positive and is an integer. However we also have $|\det(D^{-1})| = d_0 \leq 1$. Thus $|\det(D^{-1})| = 1$ and it follows that D itself is a matrix with integer entries and $|\det(D)| = 1$. By Case 2 of Remark 3.1, we must have $D = I$ and the result follows from Lemma 4.1.

III. $\mathcal{K} \subset \mathcal{L}$ **but** $\mathcal{L} \not\subset \mathcal{K}$. Since $\mathcal{L} \not\subset \mathcal{K}$, there exists $\mathbf{l}' \in \mathcal{L}$ such that $\mathbf{l}' \notin \mathcal{K}$. In this case Ω tiles $\mathbb{R}^d$ by $\mathcal{L}$ and packs $\mathbb{R}^d$ by $\mathcal{K}$, and $E_1 = \emptyset$, $E_2 = \Omega$, $E_3 = -\mathbf{k} + E_2$, $E_4 = -\mathbf{l}' + E_2$, $E_5 = -\mathbf{l}' - \mathbf{k} + E_2$ satisfy condition (ii) of Lemma 4.2 since $-\mathbf{l}' + E_2$ is $\mathcal{K}$-disjoint from E_2. It follows that

$$h(\mathbf{x})\overline{h(\mathbf{x}-\mathbf{k})} = h(\mathbf{x}-\mathbf{l}')\overline{h(\mathbf{x}-\mathbf{l}'-\mathbf{k})}$$

for any $\mathbf{x} \in \mathbb{R}^d$ *a.e.* Similarly, the sets $E'_1 = \emptyset$, $E'_2 = \Omega$, $E'_3 = -\mathbf{k}+E'_2$, $E'_4 = -\mathbf{l}+\mathbf{l}'+E'_2$, $E'_5 = -\mathbf{l}+\mathbf{l}'-\mathbf{k}+E'_2$ also satisfy condition (ii) of Lemma 4.2 since $-\mathbf{l}+\mathbf{l}'+E_2$ is also $\mathcal{K}$-disjoint from E_2. Thus we have

$$h(\mathbf{x})\overline{h(\mathbf{x}-\mathbf{k})} = h(\mathbf{x}+\mathbf{l}'-\mathbf{l})\overline{h(\mathbf{x}+\mathbf{l}'-\mathbf{l}-\mathbf{k})}.$$

The result now follows from Lemma 4.3 (i).

5. The proof of Theorem 1.1, part 2

We now prove the remaining case of Theorem 1.1, namely the case $\gamma > 1$, which is uniquely determined by $1 < d_0 \le \gamma < d_0 + 1$.

5.1. A few more lemmas

Lemma 5.1. *Let D be as given in Remark 3.1, γ be the unique integer determined by $d_0 \le \gamma < d_0 + 1$ where $d_0 = 1/|\det(D)| > 1$. Then there exist measurable sets E_1, E_2, ..., E_γ such that*
(1) $\mu(E_j) = |\det(D)| = 1/d_0$ for $2 \le j \le \gamma$ and $0 < \mu(E_1) \le 1/d_0$;
(2) E_i and E_j are $\mathcal{L}$-disjoint if $i \ne j$ and $\cup_{1\le j\le\gamma} E_j$ tiles $\mathbb{R}^d$ by $\mathcal{L}$;
(3) E_j tiles $\mathbb{R}^d$ by $\mathcal{K}$ for each $j \ge 2$ and E_1 packs $\mathbb{R}^d$ by $\mathcal{K}$.

Proof. Let $\mathfrak{P}$ and $\mathfrak{P}'$ be as defined in Remark 3.1 and F_1, F_2, ..., F_γ be the groups defined there. Assign an arbitrary one to one correspondence between the elements of F_j and the elements of $\mathfrak{P}'$ ($j \ge 2$) so they are paired according to this one to one correspondence. The union of the matchings of these pairs is a measurable set E_j that is $\mathcal{L}$-equivalent to the union of the sets in F_j and $\mathcal{K}$-equivalent to Ω'. For F_1 we assign an arbitrary one to one mapping from F_1 to $\mathfrak{P}'$, and each pair leads to a matching of an element in F_1 and a subset of its corresponding element in $\mathfrak{P}'$. The union of these matchings is a measurable set E_1 that is $\mathcal{L}$-equivalent to the union of the sets in F_1 and $\mathcal{K}$-equivalent to a subset of Ω'. E_i and E_j are $\mathcal{L}$-disjoint since F_i and F_j contain disjoint subsets of Ω if $i \ne j$, and $\cup_{1\le j\le\gamma} E_j$ is $\mathcal{L}$-equivalent to Ω hence tiles $\mathbb{R}^d$ by $\mathcal{L}$. □

Lemma 5.2. *For any given* $\mathbf{k} \in \mathcal{K} \setminus \{\mathbf{0}\}$, *the following statements hold.*

(1) If $\mathbf{k} \notin \mathcal{L}$, *then there exists a choice of* E_1, ..., E_γ *such that* E_1 *contains a rectangular parallelepiped* C *such that* $-\mathbf{k} + C \subset E_2$;

(2) If $\mathbf{k} \in \mathcal{L}$, *then there exists a choice of* E_1, ..., E_γ *such that* E_1 *contains a rectangular parallelepiped* C *such that* $C' = -\mathbf{k}_{i_0} + C \subset E_2$ *for some* $\mathbf{k}_{i_0} = D\mathbf{l}_{i_0}$.

Proof. (1) We will prove this by discussing the three different cases of D given in Remark 3.1.

Case 1. Let $\mathbf{l}_0 \in \mathcal{L}$ be such that $-\mathbf{k} - \mathbf{l}_0 = \pi(-\mathbf{k}) \in \Omega$. Since $\mathbf{k} \notin \mathcal{L}$, $\pi(-\mathbf{k}) \neq \mathbf{0}$. Let $\mathbf{x}_0$ be an interior point of Ω such that $\mathbf{x}_0 + \pi(-\mathbf{k})$ is also an interior point of Ω. Choose a nice partition pair $\mathfrak{P}$, $\mathfrak{P}'$ of Ω and $D\Omega$ such that the diameters of the polytopes in them are smaller than $|\pi(-\mathbf{k})|$. WLOG we can assume that $\mathbf{x}_0$ is an interior point of $D\Omega$ as well by Remark 3.2. Furthermore, we can assume that there exist $C_{i_0} \in \mathfrak{P}$ and $C'_{j_0} \in \mathfrak{P}'$ such that $\mathbf{x}_0$ is an interior point of $C_{i_0} \cap C'_{j_0}$, and $\mathbf{x}_0 + \pi(-\mathbf{k})$ is also an interior point of another C_{i_1} (otherwise we can replace $\mathbf{x}_0$ by a point very close to it). It follows that there exists a small rectangle C such that $C \subset C_{i_0} \cap C'_{j_0}$ and $\pi(-\mathbf{k}) + C \subset C_{i_1}$. We now place C_{i_0} in the group F_1 and C_{i_1} in the group F_2 (F_1 and F_2 are as defined in Remark 3.1), and match both C_{i_0} and C_{i_1} to C'_{j_0}. By Remark 3.1, we can choose C to be a subset of the matching of C_{i_0} and C'_{j_0} since it is both $\mathcal{L}$-equivalent and $\mathcal{K}$-equivalent to itself. Similarly, we can choose $-\mathbf{k} + C$ to be a subset of the matching of C_{i_1} and C'_{j_0}, since it is $\mathcal{K}$-equivalent to $C \subset C'_{j_0}$ and is $\mathcal{L}$-equivalent to $\pi(-\mathbf{k}) + C \subset C_{i_1}$ (since $-\mathbf{k} + C = \mathbf{l}_0 + (\pi(-\mathbf{k}) + C) \subset \mathbf{l}_0 + C_{i_1}$). Thus $C \subset E_1$ and $-\mathbf{k} + C \subset E_2$.

Case 2. Let $\mathbf{l}_0 \in \mathcal{L}$ be such that $-\mathbf{k} - \mathbf{l}_0 = \pi(-\mathbf{k}) \in \Omega$. Let C' be the common rectangle of $\mathfrak{P}$ and $\mathfrak{P}'$ containing the origin. In this case $\pi(-\mathbf{k}) + C'$ is another rectangle of $\mathfrak{P}$. Similar to the discussion in Case 1, we can choose to have $C \subset E_1$ and $-\mathbf{k} + C \subset E_2$.

Case 3. Let $\mathbf{l}_0 \in \mathcal{L}$ be such that $-\mathbf{k} - \mathbf{l}_0 = \pi(-\mathbf{k}) \in \Omega$. Similar to the discussion of Case 1, if we choose the diameters of the parallelepipeds in $\mathfrak{P}'$ small enough, then there exist $C_{ij} = \Delta_i \times R_j \in \mathfrak{P}$ and $C'_{i'j'} = \Delta'_{i'} \times R'_{j'} \in \mathfrak{P}'$ such that $R_j = R'_{j'}$, $\Delta_i \cap \Delta'_{i'}$ contains a small rectangle C_0, and $\pi(-\mathbf{k}) + C_0 \times R_j \subset C_{i_1 j_1} = \Delta_{i_1} \times R_{j_1} \in \mathfrak{P}$. We place C_{ij} in the group F_1 and $C_{i_1 j_1}$ in the group F_2, and match both C_{ij} and $C_{i_1 j_1}$ to $C'_{i'j'}$. Again by Remark 3.1, we can choose $C = C_0 \times R_i$ to be a subset of the matching of C_{ij} and $C'_{i'j'}$ since it is both $\mathcal{L}$-equivalent and $\mathcal{K}$-equivalent to itself. Similarly, we can choose $-\mathbf{k} + C$ to be a subset of the matching of $C_{i_1 j_1}$ and $C'_{i'j'}$, since it is $\mathcal{K}$-equivalent to $C \subset C'_{i'j'}$ and is $\mathcal{L}$-equivalent to $\pi(-\mathbf{k}) + C \subset C_{i_1 j_1}$. Thus $C \subset E_1$ and $-\mathbf{k} + C \subset E_2$.

(2) Notice that at least one of the $\mathbf{k}_j$'s has length less than 1. Let $\mathbf{k}_{j_0}$ be one such. The discussions in the above apply here with $\pi(-\mathbf{k})$ replaced by $\mathbf{k}_{j_0}$. It follows that in each case there exists a small rectangle C such that $C \subset E_1$ and $\mathbf{k}_{j_0} + C \subset E_2$. $\square$

5.2. The main proof

We now proceed to prove Theorem 1.1 for the case of $\gamma \geq 2$, under the simplified setting of $A = I_{d\times d}$ and $(B^T)^{-1} = D$, where D is of the form $(PB^T AQ)^{-1}$ as described in Remark 3.1. The proof is divided into three parts. In Part 1 we show that if $M(\mathbf{x})$ is a Parseval Gabor multi-frame multiplier, then $M(\mathbf{x})$ is unitary. In Part 2 we show that $M^*(\mathbf{x})M^*(\mathbf{x}-\mathbf{k}) = \lambda_{\mathbf{k}}(\mathbf{x})I$ for any $\mathbf{k} \in \mathcal{K} \setminus \{\mathbf{0}\}$, where $\lambda_{\mathbf{k}}(\mathbf{x})$ is a scalar function that depends only on $\mathbf{k}$ and $\mathbf{x}$. In the last part we show that $M^*(\mathbf{x})M^*(\mathbf{x}-\mathbf{k})$ is $\mathcal{L}$-periodic.

Remark 5.3. Notice that the discussions in the last section can be applied to any translation of Ω (together with the set Ω', of course). Thus in order to verify that equations (2.1) and (2.2) hold for any $\mathbf{x} \in \mathbb{R}^d$ *a.e.*, we only need to verify them for any $\mathbf{x} \in \Omega$ *a.e.* We should stress that the statement here is different from the statement of Remark 3.2.

Part 1. Let $M(\mathbf{x})$ be a $\gamma \times \gamma$ functional matrix Gabor multi-frame multiplier for the time-frequency lattice $\mathbb{Z}^d \times (D^\tau)^{-1}\mathbb{Z}^d$. First, if $G(\mathbf{x})$ is a Parseval Gabor multi-frame generator for $L^2(\mathbb{R}^d)$ and $H(\mathbf{x}) = M(\mathbf{x})G(\mathbf{x})$, then $H(\mathbf{x})$ satisfies equation (2.1), that is:

$$\begin{aligned} d_0 &= \sum_{\mathbf{n}\in\mathbb{Z}^d} \langle M(\mathbf{x}-\mathbf{n})G(\mathbf{x}-\mathbf{n}), M(\mathbf{x}-\mathbf{n})G(\mathbf{x}-\mathbf{n})\rangle \\ &= \sum_{\mathbf{n}\in\mathbb{Z}^d} \langle G(\mathbf{x}-\mathbf{n}), M^*(\mathbf{x}-\mathbf{n})M(\mathbf{x}-\mathbf{n})G(\mathbf{x}-\mathbf{n})\rangle. \end{aligned}$$

Combining the above with

$$d_0 = \sum_{\mathbf{n}\in\mathbb{Z}^d} \langle G(\mathbf{x}-\mathbf{n}), G(\mathbf{x}-\mathbf{n})\rangle,$$

we obtain

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} \langle G(\mathbf{x}-\mathbf{n}), (I - M^*(\mathbf{x}-\mathbf{n})M(\mathbf{x}-\mathbf{n}))G(\mathbf{x}-\mathbf{n})\rangle = 0. \tag{5.1}$$

Now let $\{\xi_1, \xi_2, ..., \xi_\gamma\}$ be any orthonormal basis for $\mathbb{C}^\gamma$, and define

$$G(\mathbf{x}) = \sum_{1\leq j\leq\gamma} \sqrt{d_0}\chi_{E_j}(\mathbf{x})\xi_j,$$

where E_1, E_2, ..., E_γ are as defined in Lemma 5.1. For any $\mathbf{x} \in \Omega$, $\mathbf{x} \in E_j$ for some j, and equations (2.1) and (2.2) hold trivially for $G(\mathbf{x})$ so it is a Parseval Gabor multi-frame generator for $L^2(\mathbb{R}^d)$ by Remark 5.3. Furthermore, we have $G(\mathbf{x}-\mathbf{n}) = \mathbf{0}$ for any $\mathbf{n} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ and $G(\mathbf{x}) = \xi_j$. Thus (5.1) becomes

$$d_0\langle \xi_j, (I - M^*(\mathbf{x})M(\mathbf{x}))\xi_j\rangle = 0.$$

Since ξ_j can be any unit vector in $\mathbb{C}^\gamma$, this implies that $M^*(\mathbf{x})M(\mathbf{x}) = I$ for any $x \in \Omega$ *a.e.*, hence for $x \in \mathbb{R}^d$ *a.e.* by Remark 5.3.

Part 2. Let $\mathbf{k}_0 \in \mathcal{K} \setminus \{\mathbf{0}\}$ be any vector. There are two cases to consider: $\mathbf{k}_0 \notin \mathcal{L}$ or $\mathbf{k}_0 \in \mathcal{L}$.

Case 1. $\mathbf{k}_0 \notin \mathcal{L}$. Let C be a small rectangle with the property described in and guaranteed by Lemma 5.2. For a.e. $x_0 \in \mathbb{R}^d$, we can perform a translation so that $\mathbf{x}_0 \in C'$ where C' is the translation of C. Let Ω^t be the corresponding translation of Ω, then the previous discussions apply to Ω^t and C'. Thus WLOG we can assume that $\mathbf{x}_0 \in C$. We can choose E_1, ..., E_γ such that $C \subset E_1$ and $-\mathbf{k}_0 + C \subset E_2$ by Lemma 5.2. Define

$$G(\mathbf{x}) = \sum_{1\le q\le \gamma} \sqrt{d_0}\chi_{E_q}(\mathbf{x})\xi_q,$$

where $\{\xi_1, \xi_2, ..., \xi_\gamma\}$ is any orthonormal basis for $\mathbb{R}^\gamma$. (2.1) and (2.2) hold trivially, hence $G(\mathbf{x})$ is a Parseval Gabor multi-frame generator for $L^2(\mathbb{R}^d)$. For $M(\mathbf{x})G(\mathbf{x})$ at $\mathbf{x}_0$ and $\mathbf{k}_0$, equation (2.2) contains only one term (since $\mathbf{x}_0 - \mathbf{k}_0 \in E_2$):

$$\langle M(\mathbf{x}_0)G(\mathbf{x}_0), M(\mathbf{x}_0 - \mathbf{k}_0)G(\mathbf{x}_0 - \mathbf{k}_0)\rangle = d_0\langle M(\mathbf{x}_0)\xi_1, M(\mathbf{x}_0 - \mathbf{k}_0)\xi_2\rangle = 0. \tag{5.2}$$

That is, $\xi_1^\tau M^*(\mathbf{x}_0)M(\mathbf{x}_0 - \mathbf{k}_0)\xi_2 = 0$. Since $\{\xi_1, \xi_2, ..., \xi_\gamma\}$ is arbitrary, we can replace ξ_1 and ξ_2 by $\mathbf{e}_i$ and $\mathbf{e}_j$ for any distinct i, j between 1 and γ where $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_d\}$ is the standard basis for $\mathbb{R}^d$. That is, $\mathbf{e}_i^\tau M^*(\mathbf{x}_0)M(\mathbf{x}_0 - \mathbf{k}_0)\mathbf{e}_j = 0$ for any $i \neq j$. This implies that $M^*(\mathbf{x}_0)M(\mathbf{x}_0 - \mathbf{k}_0)$ is a diagonal matrix. On the other hand, if we replace ξ_1 and ξ_2 by $(\mathbf{e}_i + \mathbf{e}_j)/\sqrt{2}$ and $(\mathbf{e}_i - \mathbf{e}_j)/\sqrt{2}$ respectively, then equation (5.2) leads to $\mathbf{e}_i^\tau M^*(\mathbf{x}_0)M(\mathbf{x}_0 - \mathbf{k}_0)\mathbf{e}_i = \mathbf{e}_j^\tau M^*(\mathbf{x}_0)M(\mathbf{x}_0 - \mathbf{k}_0)\mathbf{e}_j$ for any i and j, proving that $M^*(\mathbf{x}_0)M(\mathbf{x}_0 - \mathbf{k}_0) = \lambda_{\mathbf{k}_0}(\mathbf{x}_0)I$ with $\lambda_{\mathbf{k}_0}(\mathbf{x}_0)$ being a unimodular scalar function.

Case 2. $\mathbf{k}_0 \in \mathcal{L}$. Let C and $\mathbf{k}_{i_0}$ be as given in Lemma 5.2 (2) so that $C \subset E_1$ and $-\mathbf{k}_{i_0} + C \subset E_2$. For any orthonormal basis $\{\xi_1, \xi_2, ..., \xi_\gamma\}$ for $\mathbb{R}^\gamma$, define

$$\begin{aligned} G(\mathbf{x}) = {} & \sqrt{d_0/2}\,(\chi_{E_1}(\mathbf{x})\xi_1 + \chi_{-\mathbf{k}_0+E_1}(\mathbf{x})\xi_2 + \chi_{E_2}(\mathbf{x})\xi_1 - \chi_{-\mathbf{k}_0+E_2}(\mathbf{x})\xi_2) \\ & + \sum_{3\le q\le\gamma} \sqrt{d_0}\chi_{E_q}(\mathbf{x})\xi_q. \end{aligned}$$

Equations (2.1) and (2.2) hold trivially for any $\mathbf{x} \in E_j$, $j \ge 3$, and (2.1) also holds trivially for $\mathbf{x} \in E_1 \cup E_2$. For $\mathbf{x} \in E_1$, and any $\mathbf{k} \in \mathcal{K} \setminus \{\mathbf{0}\}$, (2.2) contains only two non-trivial terms corresponding to $\mathbf{l} = \mathbf{0}$ and $\mathbf{l} = \mathbf{k}_0$, that is:

$$\begin{aligned} & \sum_{\mathbf{l}\in\mathcal{L}} \langle G(\mathbf{x} - \mathbf{l}), G(\mathbf{x} - \mathbf{l} - \mathbf{k})\rangle \\ & = \langle G(\mathbf{x}), G(\mathbf{x} - \mathbf{k})\rangle + \langle G(\mathbf{x} - \mathbf{k}_0), G(\mathbf{x} - \mathbf{k}_0 - \mathbf{k})\rangle. \end{aligned} \tag{5.3}$$

If $\mathbf{x}-\mathbf{k} \notin E_2$, then $\mathbf{x}-\mathbf{k}_0-\mathbf{k} \notin -\mathbf{k}_0+E_2$ and both terms in (5.3) equal to zero. If $\mathbf{x}-\mathbf{k} \in E_2$, then $\mathbf{x}-\mathbf{k}_0-\mathbf{k} \in -\mathbf{k}_0+E_2$ and (5.3) becomes:

$$\begin{aligned}&\langle G(\mathbf{x}), G(\mathbf{x}-\mathbf{k})\rangle + \langle G(\mathbf{x}-\mathbf{k}_0), G(\mathbf{x}-\mathbf{k}_0-\mathbf{k})\rangle\\ &= (d_0/2)\left(\langle \xi_1, \xi_1\rangle + \langle \xi_2, -\xi_2\rangle\right) = 0.\end{aligned}$$

Thus (2.2) holds for any $\mathbf{x} \in E_1$ and any $\mathbf{k} \in \mathcal{K}\setminus\{\mathbf{0}\}$. Similarly, (2.2) holds for any $\mathbf{x} \in E_2$ and any $\mathbf{k} \in \mathcal{K}\setminus\{\mathbf{0}\}$. This proves that $G(\mathbf{x})$ is a Parseval Gabor multi-frame generator for $L^2(\mathbb{R}^d)$. Now consider $M(\mathbf{x})G(\mathbf{x})$ at $\mathbf{x}_0$ and $\mathbf{k}_0 \in \mathcal{K}\setminus\{\mathbf{0}\}$. Equation (2.2) contains only one nontrivial term corresponding to $\mathbf{l}=\mathbf{0}$ (since the other possible non-trivial term corresponds to $\mathbf{l}=\mathbf{k}_0$ but $G(\mathbf{x}_0-\mathbf{k}_0-\mathbf{k}_0)=\mathbf{0}$):

$$\langle M(\mathbf{x}_0)G(\mathbf{x}_0), M(\mathbf{x}_0-\mathbf{k}_0)G(\mathbf{x}_0-\mathbf{k}_0)\rangle = (d_0/2)\langle M(\mathbf{x}_0)\xi_1, M(\mathbf{x}_0-\mathbf{k}_0)\xi_2\rangle = 0.$$

That is, $\xi_1^{\tau} M^*(\mathbf{x}_0)M(\mathbf{x}_0-\mathbf{k}_0)\xi_2 = 0$. Since ξ_1 and ξ_2 are arbitrary, repeating the argument used in Case 1 leads to $M^*(\mathbf{x}_0)M(\mathbf{x}_0-\mathbf{k}_0) = \lambda_{\mathbf{k}_0}(\mathbf{x}_0)I$ with $\lambda_{\mathbf{k}_0}(\mathbf{x}_0)$ being a unimodular scalar function.

Part 3. Continue the discussion from Part 2 under the same setting and consider the two different cases.

Case 1. $\mathbf{k}_0 \notin \mathcal{L}$. Recall that we have $C \subset E_1$ and $-\mathbf{k}_0+C \subset E_2$. For any given $\mathbf{l}_0 \in \mathcal{L}\setminus\{\mathbf{0}\}$, define

$$\begin{aligned}G(\mathbf{x}) = {}&\sqrt{d_0/2}\left(\chi_{E_1}(\mathbf{x})\mathbf{e}_1 + \chi_{-\mathbf{l}_0+E_1}(\mathbf{x})\mathbf{e}_2 + \chi_{E_2}(\mathbf{x})\mathbf{e}_1 - \chi_{-\mathbf{l}_0+E_2}(\mathbf{x})\mathbf{e}_2\right)\\ &+ \sum_{3\le q\le \gamma} \sqrt{d_0}\chi_{E_q}(\mathbf{x})\mathbf{e}_q.\end{aligned}$$

Again, equations (2.1) and (2.2) hold trivially for any $\mathbf{x} \in E_j$, $j \ge 3$, and (2.1) also holds trivially for $\mathbf{x} \in E_1 \cup E_2$. For any $\mathbf{x} \in E_1$, and any $\mathbf{k} \in \mathcal{K}\setminus\{\mathbf{0}\}$, (2.2) contains only two non-trivial terms corresponding to $\mathbf{l}=\mathbf{0}$ and $\mathbf{l}=\mathbf{l}_0$, that is:

$$\begin{aligned}&\sum_{\mathbf{l}\in\mathcal{L}} \langle G(\mathbf{x}-\mathbf{l}), G(\mathbf{x}-\mathbf{l}-\mathbf{k})\rangle\\ &= \langle G(\mathbf{x}), G(\mathbf{x}-\mathbf{k})\rangle + \langle G(\mathbf{x}-\mathbf{l}_0), G(\mathbf{x}-\mathbf{l}_0-\mathbf{k})\rangle. \qquad (5.4)\end{aligned}$$

If $\mathbf{x}-\mathbf{k} \notin E_2$, then $\mathbf{x}-\mathbf{l}_0-\mathbf{k} \notin -\mathbf{l}_0+E_2$ and both terms in (5.4) equal to zero. If $\mathbf{x}-\mathbf{k} \in E_2$, then $\mathbf{x}-\mathbf{l}_0-\mathbf{k} \in -\mathbf{l}_0+E_2$ and (5.3) becomes:

$$\begin{aligned}&\langle G(\mathbf{x}), G(\mathbf{x}-\mathbf{k})\rangle + \langle G(\mathbf{x}-\mathbf{l}_0), G(\mathbf{x}-\mathbf{l}_0-\mathbf{k})\rangle\\ &= (d_0/2)\left(\langle \mathbf{e}_1, \mathbf{e}_1\rangle + \langle \mathbf{e}_2, -\mathbf{e}_2\rangle\right) = 0.\end{aligned}$$

Thus (2.2) holds for any $\mathbf{x} \in E_1$ and any $\mathbf{k} \in \mathcal{K}\setminus\{\mathbf{0}\}$. Similarly, (2.2) holds for any $\mathbf{x} \in E_2$ and any $\mathbf{k} \in \mathcal{K}\setminus\{\mathbf{0}\}$. This proves that $G(\mathbf{x})$ is a Parseval Gabor multi-frame generator for $L^2(\mathbb{R}^d)$. Now consider $M(\mathbf{x})G(\mathbf{x})$ at $\mathbf{x}_0$ and $\mathbf{k}_0 \in \mathcal{K} \setminus \{\mathbf{0}\}$. Equation (2.2) contains two nontrivial terms corresponding to $\mathbf{l} = \mathbf{0}$ and $\mathbf{l} = \mathbf{l}_0$ (since $\mathbf{x}_0 - \mathbf{k}_0 \in E_2$ and $\mathbf{x}_0 - \mathbf{l}_0 - \mathbf{k}_0 \in -\mathbf{l}_0 + E_2$):

$$
\begin{aligned}
&\langle M(\mathbf{x}_0)G(\mathbf{x}_0), M(\mathbf{x}_0 - \mathbf{k}_0)G(\mathbf{x}_0 - \mathbf{k}_0)\rangle \\
&+ \langle M(\mathbf{x}_0 - \mathbf{l}_0)G(\mathbf{x}_0 - \mathbf{l}_0), M(\mathbf{x}_0 - \mathbf{l}_0 - \mathbf{k}_0)G(\mathbf{x}_0 - \mathbf{l}_0 - \mathbf{k}_0)\rangle \\
&= (d_0/2)\left(\langle M(\mathbf{x}_0)\mathbf{e}_1, M(\mathbf{x}_0 - \mathbf{k}_0)\mathbf{e}_1\rangle - \langle M(\mathbf{x}_0 - \mathbf{l}_0)\mathbf{e}_2, M(\mathbf{x}_0 - \mathbf{l}_0 - \mathbf{k}_0)\mathbf{e}_2\rangle\right) \\
&= 0.
\end{aligned}
$$

This implies that $\mathbf{e}_1^\tau M^*(\mathbf{x}_0)M(\mathbf{x}_0 - \mathbf{k}_0)\mathbf{e}_1 = \mathbf{e}_2^\tau M^*(\mathbf{x}_0 - \mathbf{l}_0)M(\mathbf{x}_0 - \mathbf{l}_0 - \mathbf{k}_0)\mathbf{e}_2$. Since $M^*(\mathbf{x}_0)M(\mathbf{x}_0 - \mathbf{k}_0)$ and $M^*(\mathbf{x}_0 - \mathbf{l}_0)M(\mathbf{x}_0 - \mathbf{l}_0 - \mathbf{k}_0)$ are both scalar multiples of the identity matrix $I_{d\times d}$, this means that $M^*(\mathbf{x}_0)M(\mathbf{x}_0 - \mathbf{k}_0) = M^*(\mathbf{x}_0 - \mathbf{l}_0)M(\mathbf{x}_0 - \mathbf{l}_0 - \mathbf{k}_0)$ as desired.

Case 2. $\mathbf{k}_0 \in \mathcal{L}$. Recall that we have $C \subset E_1$ and $-\mathbf{k}_{i_0} + C \subset E_2$ for some $\mathbf{k}_{i_0}$ with $1 \le i_0 \le d$. For any given $\mathbf{l}_0 \in \mathcal{L} \setminus \{\mathbf{0}\}$, we need to consider several different cases.

Subcase 1. $\mathbf{l}_0 \ne \pm\mathbf{k}_0$. In this case we define

$$
\begin{aligned}
G(\mathbf{x}) &= \sqrt{d_0}/2\left(\chi_{E_1}(\mathbf{x})\mathbf{e}_1 + \chi_{-\mathbf{k}_0+E_1}(\mathbf{x})\mathbf{e}_1 + \chi_{-\mathbf{l}_0+E_1}(\mathbf{x})\mathbf{e}_2 - \chi_{-\mathbf{l}_0-\mathbf{k}_0+E_1}(\mathbf{x})\mathbf{e}_2\right) \\
&\quad + \sqrt{d_0}/2\left(\chi_{E_2}(\mathbf{x})\mathbf{e}_1 - \chi_{-\mathbf{k}_0+E_2}(\mathbf{x})\mathbf{e}_1 + \chi_{-\mathbf{l}_0+E_2}(\mathbf{x})\mathbf{e}_2 + \chi_{-\mathbf{l}_0-\mathbf{k}_0+E_2}(\mathbf{x})\mathbf{e}_2\right) \\
&\quad + \sum_{3\le q\le\gamma} \sqrt{d_0}\chi_{E_q}(\mathbf{x})\mathbf{e}_q.
\end{aligned}
$$

Equations (2.1) and (2.2) hold trivially for any $\mathbf{x} \in E_j$, $j \ge 3$, and (2.1) also holds trivially for $\mathbf{x} \in E_1 \cup E_2$. For any $\mathbf{x} \in E_1$, and any $\mathbf{k} \in \mathcal{K} \setminus \{\mathbf{0}\}$, (2.2) contains only four non-trivial terms corresponding to $\mathbf{l} = \mathbf{0}$, $\mathbf{l} = \mathbf{k}_0$, $\mathbf{l} = \mathbf{l}_0$ and $\mathbf{l} = \mathbf{l}_0 + \mathbf{k}_0$, that is:

$$
\begin{aligned}
&\sum_{\mathbf{l}\in\mathcal{L}} \langle G(\mathbf{x} - \mathbf{l}), G(\mathbf{x} - \mathbf{l} - \mathbf{k})\rangle \\
&= \langle G(\mathbf{x}), G(\mathbf{x} - \mathbf{k})\rangle + \langle G(\mathbf{x} - \mathbf{k}_0), G(\mathbf{x} - \mathbf{k}_0 - \mathbf{k})\rangle \\
&+ \langle G(\mathbf{x} - \mathbf{l}_0), G(\mathbf{x} - \mathbf{l}_0 - \mathbf{k})\rangle + \langle G(\mathbf{x} - \mathbf{l}_0 - \mathbf{k}_0), G(\mathbf{x} - \mathbf{l}_0 - \mathbf{k}_0 - \mathbf{k})\rangle \\
&= \sqrt{d_0}/2\langle \mathbf{e}_1, G(\mathbf{x} - \mathbf{k})\rangle + \sqrt{d_0}/2\langle \mathbf{e}_1, G(\mathbf{x} - \mathbf{k}_0 - \mathbf{k})\rangle \\
&+ \sqrt{d_0}/2\langle \mathbf{e}_2, G(\mathbf{x} - \mathbf{l}_0 - \mathbf{k})\rangle + \sqrt{d_0}/2\langle -\mathbf{e}_2, G(\mathbf{x} - \mathbf{l}_0 - \mathbf{k}_0 - \mathbf{k})\rangle.
\end{aligned}
\tag{5.5}
$$

If $\mathbf{x} - \mathbf{k} \notin E_2$ and $\mathbf{x} - \mathbf{k} \notin -\mathbf{k}_0 + E_2$, then $\mathbf{x} - \mathbf{l}_0 - \mathbf{k} \notin -\mathbf{l}_0 + E_2$ and $\mathbf{x} - \mathbf{l}_0 - \mathbf{k}_0 - \mathbf{k} \notin -\mathbf{l}_0 - \mathbf{k}_0 + E_2$ and each term in (5.5) equals zero. If $\mathbf{x} - \mathbf{k} \in E_2$, then $\mathbf{x} - \mathbf{k}_0 - \mathbf{k} \in -\mathbf{k}_0 + E_2$, $\mathbf{x} - \mathbf{l}_0 - \mathbf{k} \in -\mathbf{l}_0 + E_2$ and $\mathbf{x} - \mathbf{l}_0 - \mathbf{k}_0 - \mathbf{k} \in -\mathbf{l}_0 - \mathbf{k}_0 + E_2$. Thus (5.5) becomes:

$$
(d_0/4)\left(\langle \mathbf{e}_1, \mathbf{e}_1\rangle + \langle \mathbf{e}_1, -\mathbf{e}_1\rangle + \langle \mathbf{e}_2, \mathbf{e}_2\rangle + \langle -\mathbf{e}_2, \mathbf{e}_2\rangle\right) = 0.
$$

If $\mathbf{x}-\mathbf{k} \in -\mathbf{k}_0 + E_2$, then $\mathbf{x}-\mathbf{k}_0-\mathbf{k} \in -2\mathbf{k}_0 + E_2$, $\mathbf{x}-\mathbf{l}_0-\mathbf{k} \in -\mathbf{l}_0-\mathbf{k}_0+E_2$ and $\mathbf{x}-\mathbf{l}_0-\mathbf{k}_0-\mathbf{k} \in -\mathbf{l}_0-2\mathbf{k}_0+E_2$. Since $-2\mathbf{k}_0+E_2$ is disjoint from E_2 and $-\mathbf{k}_0+E_2$, $G(\mathbf{x}-\mathbf{k}_0-\mathbf{k}) \neq (\sqrt{d_0}/2)\mathbf{e}_1$. Similarly, $-\mathbf{l}_0-2\mathbf{k}_0+E_2$ is disjoint from $-\mathbf{l}_0+E_2$ and $-\mathbf{l}_0-\mathbf{k}_0+E_2$ hence $G(\mathbf{x}-\mathbf{l}_0-\mathbf{k}_0-\mathbf{k}) \neq (\sqrt{d_0}/2)\mathbf{e}_2$. It follows that (5.5) contains only two nontrivial terms corresponding to $G(\mathbf{x}-\mathbf{k})$ and $G(\mathbf{x}-\mathbf{l}_0-\mathbf{k})$, which becomes:

$$(d_0/4)\left(\langle \mathbf{e}_1, -\mathbf{e}_1\rangle + \langle \mathbf{e}_2, \mathbf{e}_2\rangle\right) = 0.$$

The case $\mathbf{x} \in E_2$ can be similarly verified. Thus $G(\mathbf{x})$ is a Parseval Gabor multi-frame generator for $L^2(\mathbb{R}^d)$. Substituting $G(\mathbf{x})$ in (2.2) by $M(\mathbf{x})G(\mathbf{x})$ with $\mathbf{x}=\mathbf{x}_0$ and $\mathbf{k}=\mathbf{k}_0$ then yields (keep in mind that $G(\mathbf{x}_0-2\mathbf{k}_0)=\mathbf{0}$ and $G(\mathbf{x}_0-\mathbf{l}_0-2\mathbf{k}_0)=\mathbf{0}$):

$$\begin{aligned}
&\langle M(\mathbf{x}_0)G(\mathbf{x}_0), M(\mathbf{x}_0-\mathbf{k}_0)G(\mathbf{x}_0-\mathbf{k}_0)\rangle \\
&+ \langle M(\mathbf{x}_0-\mathbf{l}_0)G(\mathbf{x}_0-\mathbf{l}_0), M(\mathbf{x}_0-\mathbf{l}_0-\mathbf{k}_0)G(\mathbf{x}_0-\mathbf{l}_0-\mathbf{k}_0)\rangle \\
&= (d_0/4)\left(\langle M(\mathbf{x}_0)\mathbf{e}_1, M(\mathbf{x}_0-\mathbf{k}_0)\mathbf{e}_1\rangle - \langle M(\mathbf{x}_0-\mathbf{l}_0)\mathbf{e}_2, M(\mathbf{x}_0-\mathbf{l}_0-\mathbf{k}_0)\mathbf{e}_2\rangle\right) \\
&= 0.
\end{aligned}$$

This implies that $\mathbf{e}_1^\tau M^*(\mathbf{x}_0)M(\mathbf{x}_0-\mathbf{k}_0)\mathbf{e}_1 = \mathbf{e}_2^\tau M^*(\mathbf{x}_0-\mathbf{l}_0)M(\mathbf{x}_0-\mathbf{l}_0-\mathbf{k}_0)\mathbf{e}_2$. This means that $M^*(\mathbf{x}_0)M(\mathbf{x}_0-\mathbf{k}_0) = M^*(\mathbf{x}_0-\mathbf{l}_0)M(\mathbf{x}_0-\mathbf{l}_0-\mathbf{k}_0)$ as desired.

Subcase 2. $\mathbf{l}_0=\mathbf{k}_0$. In this case we define

$$\begin{aligned}
G(\mathbf{x}) = &\sqrt{d_0}/2\left(\chi_{E_1}(\mathbf{x})\mathbf{e}_1 + \chi_{-\mathbf{l}_0+E_1}(\mathbf{x})(-\mathbf{e}_1+\mathbf{e}_2) + \chi_{-2\mathbf{l}_0+E_1}(\mathbf{x})\mathbf{e}_2\right) \\
&+ \sqrt{d_0}/2\left(\chi_{E_2}(\mathbf{x})\mathbf{e}_1 + \chi_{-\mathbf{l}_0+E_2}(\mathbf{x})(\mathbf{e}_1+\mathbf{e}_2) - \chi_{-2\mathbf{l}_0+E_2}(\mathbf{x})\mathbf{e}_2\right) \\
&+ \sum_{3\le q\le\gamma} \sqrt{d_0}\chi_{E_q}(\mathbf{x})\mathbf{e}_q.
\end{aligned}$$

We leave it to our reader to verify that equations (2.1) and (2.2) hold for any $\mathbf{x} \in \Omega$ (hence for any $\mathbf{x} \in \mathbb{R}^d$) and for any $\mathbf{k} \in \mathcal{K} \setminus \{\mathbf{0}\}$, that is, $G(\mathbf{x})$ is a Parseval Gabor multi-frame generator for $L^2(\mathbb{R}^d)$. Substituting $G(\mathbf{x})$ in (2.2) by $M(\mathbf{x})G(\mathbf{x})$ with $\mathbf{x}=\mathbf{x}_0$ and $\mathbf{k}=\mathbf{k}_0$ then yields

$$\begin{aligned}
&\langle M(\mathbf{x}_0)G(\mathbf{x}_0), M(\mathbf{x}_0-\mathbf{k}_0)G(\mathbf{x}_0-\mathbf{k}_0)\rangle \\
&+ \langle M(\mathbf{x}_0-\mathbf{l}_0)G(\mathbf{x}_0-\mathbf{l}_0), M(\mathbf{x}_0-\mathbf{l}_0-\mathbf{k}_0)G(\mathbf{x}_0-\mathbf{l}_0-\mathbf{k}_0)\rangle \\
&= (d_0/4)\left(\langle M(\mathbf{x}_0)\mathbf{e}_1, M(\mathbf{x}_0-\mathbf{k}_0)(-\mathbf{e}_1+\mathbf{e}_2)\rangle\right) \\
&+ (d_0/4)\left(\langle M(\mathbf{x}_0-\mathbf{l}_0)(-\mathbf{e}_1+\mathbf{e}_2), M(\mathbf{x}_0-\mathbf{l}_0-\mathbf{k}_0)\mathbf{e}_2\rangle\right) \\
&= 0.
\end{aligned}$$

That is,

$$\begin{aligned}&\mathbf{e}_1^{\tau} M^*(\mathbf{x}_0)M(\mathbf{x}_0-\mathbf{k}_0)(-\mathbf{e}_1+\mathbf{e}_2)\\&=-(-\mathbf{e}_1+\mathbf{e}_2)^{\tau} M^*(\mathbf{x}_0-\mathbf{l}_0)M(\mathbf{x}_0-\mathbf{l}_0-\mathbf{k}_0)\mathbf{e}_2.\end{aligned}$$

Since $M^*(\mathbf{x}_0)M(\mathbf{x}_0-\mathbf{k}_0)$ and $M^*(\mathbf{x}_0-\mathbf{l}_0)M(\mathbf{x}_0-\mathbf{l}_0-\mathbf{k}_0)$ are both scalar multiples of the identity matrix $I_{d\times d}$, the above simplifies to

$$\mathbf{e}_1^{\tau} M^*(\mathbf{x}_0)M(\mathbf{x}_0-\mathbf{k}_0)\mathbf{e}_1=\mathbf{e}_2^{\tau} M^*(\mathbf{x}_0-\mathbf{l}_0)M(\mathbf{x}_0-\mathbf{l}_0-\mathbf{k}_0)\mathbf{e}_2.$$

This means that $M^*(\mathbf{x}_0)M(\mathbf{x}_0-\mathbf{k}_0)=M^*(\mathbf{x}_0-\mathbf{l}_0)M(\mathbf{x}_0-\mathbf{l}_0-\mathbf{k}_0)$ as desired.

Subcase 3. $\mathbf{l}_0=-\mathbf{k}_0$. In this case we define $G(\mathbf{x})$ by

$$\begin{aligned}G(\mathbf{x})=&\sqrt{d_0}/2\,(\chi_{\mathbf{l}_0+E_1}(\mathbf{x})\mathbf{e}_1+\chi_{E_1}(\mathbf{x})(-\mathbf{e}_1+\mathbf{e}_2)+\chi_{-\mathbf{l}_0+E_1}(\mathbf{x})\mathbf{e}_2)\\&+\sqrt{d_0}/2\,(\chi_{\mathbf{l}_0+E_2}(\mathbf{x})\mathbf{e}_1+\chi_{E_2}(\mathbf{x})(\mathbf{e}_1+\mathbf{e}_2)-\chi_{-\mathbf{l}_0+E_2}(\mathbf{x})\mathbf{e}_2)\\&+\sum_{3\le q\le\gamma}\sqrt{d_0}\chi_{E_q}(\mathbf{x})\mathbf{e}_q.\end{aligned}$$

The rest of the proof is similar to Subcase 2 and is left to the reader.

Since $\mathbf{k}_0\in\mathcal{K}$, $\mathbf{l}_0\in\mathcal{L}$ are arbitrary and $\mathbf{x}_0$ is any point in $\mathbb{R}^d$ (in the *a.e.* sense), we have shown that $M^*(\mathbf{x})M(\mathbf{x}-\mathbf{k})$ is $\mathcal{L}$-periodic for any $\mathbf{x}\in\mathbb{R}^d$ *a.e.* and any $\mathbf{k}\in\mathcal{K}\setminus\{\mathbf{0}\}$. This concludes the proof of Theorem 1.1.

6. Final remarks

Remark 6.1. While Theorem 1.1 is only stated and proved for the case when γ is the *minimal length* of all Gabor multi-frame generators, it holds for any $\gamma\ge|\det(AB)|$. More specifically, if $1<|\det(AB)|\le\gamma<|\det(AB)|+1$, then for any $\gamma'>\gamma$, the proof is almost identical to the one given in Section 5.2 by defining $E_{\gamma+1}=\cdots=E_{\gamma'}=\emptyset$. On the other hand, if $\gamma=1$ (that is, $|\det(AB)|\le 1$), then there exists a measurable set E such that E tiles $\mathbb{R}^d$ by $\mathcal{L}$ and packs $\mathbb{R}^d$ by $\mathcal{K}$. In this case the proof in Section 5.2 can be modified by defining $E_1=E$, $E_2=\cdots=E_{\gamma'}=\emptyset$ in Part 1, and by defining $E_1=E$, $E_2=-\mathbf{k}_0+E$, $E_3=\cdots=E_{\gamma'}=\emptyset$ in Part 2 and Part 3. The verification is straightforward and is left to the reader.

Remark 6.2. Suppose that $G=(g_1,g_2,\cdots,g_\gamma)^{\tau}$ and $\tilde{G}=(\tilde{g}_1,\tilde{g}_2,\cdots,\tilde{g}_\gamma)^{\tau}$ form a Gabor multi-frame generator dual pair in the sense that

$$f=\sum_{1\le j\le\gamma}\sum_{\mathbf{m},\mathbf{n}\in\mathbb{Z}^d}\langle f,e^{2\pi i\langle B\mathbf{m},\mathbf{x}\rangle}g_j(\mathbf{x}-A\mathbf{n})\rangle e^{2\pi i\langle B\mathbf{m},\mathbf{x}\rangle}\tilde{g}_j(\mathbf{x}-A\mathbf{n})$$

for any $f\in L^2(\mathbb{R}^d)$. By a similar characterization as Proposition 1.5, this is equivalent to the conditions that

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} \langle \tilde{G}(\mathbf{x}-A\mathbf{n}), G(\mathbf{x}-A\mathbf{n})\rangle = b;$$

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} \langle \tilde{G}(\mathbf{x}-A\mathbf{n}), G(\mathbf{x}+(B^\tau)^{-1}\mathbf{l}-A\mathbf{n})\rangle = 0, \forall\, \mathbf{l}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}.$$

Let M be a functional (matrix) Gabor multi-frame multiplier. Then M satisfies all the conditions in Theorem 1.1. By replacing one of G's from the argument following Proposition 1.5, we immediately get that $(MG, M\tilde{G})$ is also a dual pair, which implies that MG is a Gabor multi-frame generator. Therefore a multiplier that preserves Parseval Gabor multi-frame generators also preserves any Gabor multi-frame generators. It would be interesting to know how to characterize all the M that preserves (not necessarily Parseval) Gabor multi-frame generators. More generally, let $M = [T_{ij}]_{m\times m}$ with T_{ij} being bounded linear operators on $L^2(\mathbb{R}^d)$. We say that M is an *operator matrix Gabor multi-frame multiplier* if it maps any Gabor multi-frame generator $G = (g_1, ..., g_m)^\tau$ to a multi-frame generator $H = (h_1, ...h_m)^\tau$, where $h_i = \sum_{j=1}^m T_{ij}g_j$. It may not be an easy task to obtain a complete characterization for all such multipliers even for the case $m = \gamma = 1$. In this case, the set of all the multipliers is a very rich class of operators. In fact, by the operator parametrization theorem for Gabor frame generators [11,12] for Gabor frame generators, any invertible operator either in the von Neumann algebra $\mathcal{A}$ or in its commutant $\mathcal{A}'$ is an operator Gabor frame multiplier and hence so are their products, where $\mathcal{A}$ is the von Neumann algebra generated by the translation and modulation operators associated with the time-frequency lattice $A\mathbb{Z}^d \times B\mathbb{Z}^d$. However, it remains unclear how to characterize all the multipliers.

Acknowledgment

The authors would like to thank the referee for carefully reading the manuscript and for providing many constructive comments and suggestions.

References

[1] B. Bekka, Square integrable representations, von Neumann algebras and an application to Gabor analysis, J. Fourier Anal. Appl. 10 (2004) 325–349.
[2] P. Casazza, Modern tools for Weyl-Heisenberg frame theory, Adv. Imaging Electron Phys. 115 (2001) 1–127.
[3] W. Czaja, Characterizations of Gabor systems via Fourier transform, Collect. Math. 51 (2000) 205–224.
[4] X. Dai, D. Larson, Wandering Vectors for Unitary Systems and Orthogonal Wavelets, Memoirs Amer. Math. Soc., vol. 640, 1998.
[5] H.G. Feichtinger, D.M. Onchis, Constructive reconstruction from irregular sampling in multi-window spline-type spaces, in: Progress in Analysis and Its Applications., World Sci Publ, Hackensack, NJ, 2010, pp. 257–265.
[6] H.G. Feichtinger, D.M. Onchis, Constructive realization of dual systems for generators of multi-window spline-type spaces, J. Comput. Appl. Math. 234 (2010) 3467–3479.
[7] J. Gabardo, D. Han, Frame representations for group-like unitary operator systems, J. Oper. Theory 49 (2003) 223–244.

[8] K. Gröchenig, Foundations of Time-Frequency Analysis, Appl. Numer. Harmon. Anal., Birkhäuser, Boston, MA, 2001.
[9] Q. Gu, D. Han, Functional Gabor frame multipliers, J. Geom. Anal. 13 (3) (2003) 467–478.
[10] D. Han, Wandering vectors for irrational unitary systems, Trans. Am. Math. Soc. 350 (1998) 309–320.
[11] D. Han, D. Larson, Frames, Bases and Group Representations, Memoirs Amer. Math. Soc., vol. 697, 2000.
[12] D. Han, D. Larson, Wandering vector multipliers for unitary groups, Trans. Am. Math. Soc. 353 (2001) 3347–3370.
[13] D. Han, Y. Wang, Lattice tiling and the Weyl-Heisenberg frames, Geom. Funct. Anal. 11 (2001) 742–758.
[14] F. Jaillet, B. Torresani, Time-frequency jigsaw puzzle: adaptive multiwindow and multilayered Gabor expansions, Int. J. Wavelets Multiresolut. Inf. Process. 5 (2007) 293–315.
[15] S. Li, Discrete multi-Gabor expansions, IEEE Trans. Inf. Theory 45 (11) (1999) 1954–1967.
[16] Z.Y. Li, Y. Diao, Gabor functional multiplier in the higher dimensions, preprint, http://arxiv.org/abs/2007.13623.
[17] Z.Y. Li, X. Dai, Y. Diao, J. Xin, Multipliers, phases and connectivity of wavelets in $L^2(\mathbb{R}^2)$, J. Fourier Anal. Appl. 16 (2010) 155–176.
[18] Z.Y. Li, D. Han, Matrix Fourier multipliers for Parseval multi-wavelet frames, Appl. Comput. Harmon. Anal. 35 (2013) 407–418.
[19] Z.Y. Li, D. Han, Frame vector multipliers for finite group representations, Linear Algebra Appl. 519 (2017) 191–207.
[20] Z.Y. Li, D. Han, Functional matrix multipliers for Parseval Gabor multi-frame generators, Acta Appl. Math. 160 (1) (2019) 53–65.
[21] D. Walnut, Continuity properties of the Gabor frame operator, J. Math. Anal. Appl. 165 (1992) 479–504.
[22] The Wutam Consortium, Basic properties of wavelets, J. Fourier Anal. Appl. 4 (4) (1998) 575–594.
[23] Y.Y. Zeevi, M. Zibulski, M. Porat, Multi-window Gabor schemes in signal and image representations, in: Gabor Analysis and Algorithms, in: Appl. Numer. Harmon. Anal., Birkhäuser, Boston, MA, 1998, pp. 381–407.
[24] M. Zibulski, Y.Y. Zeevi, Analysis of multiwindow Gabor-type schemes by frame methods, Appl. Comput. Harmon. Anal. 4 (1997) 188–221.