INJECTIVE CONTINUOUS FRAMES AND QUANTUM DETECTIONS

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Abstract. A quantum injective frame is a frame whose frame measurements for density operators can be used to distinguish them in a quantum system, and the frame quantum detection problem asks to characterize all such frames. This problem was recently settled in [5] mainly for finite or infinite but discrete frames. In this paper, we consider the continuous frame version of the quantum detection problem. Instead of using the frame element itself, we use discrete representations of continuous frames to obtain several versions of characterizations for quantum injective continuous frames. With the help of these characterizations, we also examine the issues involving constructions and stability of continuous quantum injective frames.

1. Introduction

The quantum detection problem by using discrete frame measurements was recently settled by Botelho-Andrade, S., Casazza, P. G., Cheng, D., et al. for both finite and infinite dimensional Hilbert spaces in [5], where the characterization was given in terms the spanning properties of some derived sequences from the frame vectors. Naturally we would wonder how much of the results from [5] is still valid for other type of frames, for instance, continuous frames. While it is possible to have a similar type of characterizations in terms of the range space of a continuous frame, it also seems unpractical to verify the injectivity by performing uncountably many number of operations. The purpose of this paper is to present a similar type of characterization for injective continuous frames in terms of their discrete representations that were introduced in [12]. Constructions of injective continuous frames and their stability will also be discussed.

Let us first recall some backgrounds and basics about frames and the quantum detection problem. The notion of discrete frames was first introduced by Duffin and shaeffer [10], and it allows (like basis) stable but not necessarilly unique decomposition of arbitrary element into expression of the frame element. Later, motivated by the theory of coherent states in mathematical physics [18], this concept was generalized to continuous frames whose families is indexed by some locally compact space endowed with a Radon measure [1]. This type of frames are also called frames associated with measurable spaces, or generalized frames in the literature (c.f. [12, 2]. Continuous frames and coherent states are widely used in

2010 *Mathematics Subject Classification.* Primary ; Secondary .

Key words and phrases. Operator-valued measure;

Deguang Han is partially supported by NSF DMS-1712602, and Rui Liu is supported by the National Science Foundation of China (11671214), "the Hundred Young Academia Leaders Program of Nankai University" (91923104, 91823003, 63174012), and "the Fundamental Research Funds for the Central Universities" (63191503, 63171225).

mathematical physics and harmonic analysis and they appear prominently in quantum mechanics and quantum optics (c.f. [8, 11]).

In quantum theory, quantum state tomography asks to recover a state from the probability of observing outcomes from a collection of measurements of the system on this state, and retrieving data from quantum systems is carried out according to quantum measurement theory [16, 6]. In this process *positive operator-valued measure* (POVM) [14, 17] plays an important role. In this paper, we are mostly interested in POVMs that are derived from the Hilbert space frame theory [13, 15], and we will call them *Frame POVMs* in short.

Recall that a POVM ν is called *informationally complete* if it can uniquely determine density operators ρ (i.e., positive trace-one operators on a separable Hilbert space). More precisely, ν is informationally complete if $\mathrm{tr}\,(\rho_1\nu(E)) = \mathrm{tr}\,(\rho_2\nu(E))$ for every measurable set E implies that $\rho_1 = \rho_2$, where $\rho_1, \rho_2 \in B(\mathcal{H})$ are density operators. The quantum detection problem asks to "characterize" the OPVMs that are informationally complete. (A much more subtle question is to characterize those that only distinguishes pure states. This is often referred to as the phase-retrieval problem). In the case that a POVM ν is derived from a discrete (Parseval) frame $\{x_n\}_{n\in I}$ (purely atomic case, where I is finite or countable), Botelho-Andrade, S., Casazza, P. G., Cheng, D., et al. settled the quantum detection problem, in addition to many other results, for both the real and complex cases and in both the finite dimensional and infinite dimension case [5, 7]. Due to the possible peculiarities of underlying measure spaces, the continuous frame do not behave quite as well as discrete counterparts. We propose to approach the quantum detection problem with a discrete representation system of the involved continuous frame.

Let $\mathcal{H}$ is a separable Hilbert space. Here is a list of notations that will be used in this paper.

- $B(\mathcal{H})$ – the linear bounded operators on $\mathcal{H}$;
- $B_{sa}(\mathcal{H})$ – the the real linear space of self-adjoint bounded operators on $\mathcal{H}$;
- $B(\mathcal{H})_+$ – for the real cone of positive operators on $\mathcal{H}$;
- $S_1(\mathcal{H})$ – the space of trace class operators on $\mathcal{H}$;
- $S_2(\mathcal{H})$ – the Hilbert space of Hilbert-Schmidt operators endowed with the inner product$\langle T, S\rangle_{HS} = \mathrm{tr}\,(TS^*)$;
- $\mathcal{S}(\mathcal{H})$ – the set of *states* or *density operators* on $\mathcal{H}$ consisting of $\rho : \rho \in S_1(\mathcal{H})$ such that $\rho \geq 0$ and $\mathrm{tr}(\rho) = 1$.

Let Ω will be a locally compact Hausdorff space and Σ be the σ-algebra of Borel sets of Ω. Following [13, 17, 14], we use the following definition for operator-valued measures.

Definition 1.1. A map $\nu : \Sigma \to B(\mathcal{H})$ is an operator-valued measure (OVM) if it is weakly countable additive, meaning that for every countable collection $\{E_k\}_{k\in\mathbb{N}} \subseteq \Sigma$ with $E_i \cap E_j = \emptyset$ for $i \neq j$ we have

$$\nu\left(\bigcup_{k\in\mathbb{N}} E_k\right) = \sum_{k\in\mathbb{N}} \nu\left(E_k\right).$$

where the convergence on the right side of the equation above is with respect to the ultraweak topology of $B(\mathcal{H})$. We say ν is

(1) bounded if $\sup\{\|\nu(E)\| : E \in \Sigma\} < \infty$.

(2) self-adjoint if $\nu(E)^* = \nu(E)$, for all $E \in \Sigma$.
(3) positive if $\nu(E) \in B(\mathcal{H})_+$, for all $E \in \Sigma$.
(4) spectral if $\nu(E_1 \cap E_2) = \nu(E_1)\nu(E_2)$ for all $E_1, E_2 \in \Sigma$.

Moreover, ν is call a positive operator-valued measure (POVM) if it is positive and $\nu(\Omega) = I_{\mathcal{H}}$, and is called a projection value measure (PVM) if it is self-adjoint and spectral.

Remark 1.2. The ultraweak topology and the weak operator topology coincides in the bounded set of $B(\mathcal{H})$, thus we simplify the definition of POVM.

Definition 1.3. Let Σ denote a σ-algebra of subsets of Ω, a positive operator-valued measure (POVM) is a function $\mu : \Sigma \to B(\mathcal{H})_+$ satisfying

(1) $\mu(\emptyset) = 0$ (the zero operator).
(2) For every disjoint family $\{E_i\}_{i\in I} \subset \Sigma$, we have

$$\langle \mu(\cup_{i\in I} E_i)x, y\rangle = \sum_{i\in I} \langle \mu(E_i)x, y\rangle \ \forall x, y \in \mathcal{H}.$$

(3) $\mu(\Omega) = I$ (the identity operator).

Given a state ρ, the *quantum measurement* is the map p performed by the POVM ν. Given a POVM $\nu : \Sigma \to B(\mathcal{H})_+$ then $p : \Sigma \to \mathbb{R}$

$$p(E) = \operatorname{tr}(\rho\nu(E)), \ \forall E \in \Sigma.$$

Let $B(\Sigma, \mathbb{R})$ denoted the set of bounded function on Σ. Given a quantum system $\mathcal{H}$ and a POVM $\nu : \Sigma \to B(\mathcal{H})_+$, the **quantum detection problem** asks the following question: Is there a "prefect" quantum measurement or underlying POVM ν for the following map to be injective, and how to characterize such a POVM?

$$\mathbb{P} : \mathcal{S}(\mathcal{H}) \to B(\Sigma, \mathbb{R}), \ \mathbb{P}(\rho)(E) = \operatorname{tr}(\rho\nu(E)), \forall E \in \Sigma.$$

In other words, the above question is about the existence of informationally complete POVM (IC-POVM) which can separate states. That is if for $\rho_1, \rho_2 \in \mathcal{S}(\mathcal{H})$,

$$\operatorname{tr}(\rho_1\nu(E)) = \operatorname{tr}(\rho_2\nu(E)), \ \forall E \in \Sigma.$$

it follows $\rho_1 = \rho_2$.

Let $\mathcal{H}$ be a complex Hilbert spaces and Ω a measure spaces with a positive measure μ. Recall [8, 13] that a mapping $\mathcal{F} : \Omega \to \mathcal{H}$ is called a *continuous frame* with respect to (Ω, μ) or (Ω, μ)-frame if

(i) for all $f \in \mathcal{H}, \omega \mapsto \langle f, \mathcal{F}(\omega)\rangle$ is a measurable function on Ω.
(ii) there exists constants $A, B > 0$ such that

$$A\|f\|^2 \le \int_\Omega |\langle f, \mathcal{F}(\omega)\rangle|^2 d\mu(\omega) \le B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

The constants A, B are the lower and upper bounds of the frame, respectively. If $A = B$, then the continuous frame is called *tight* and if $A = B = 1$, then the continuous frame is called *Parseval* or a *coherent state*. If Ω is at most countable with counting measure, then it becomes a discrete frame. Associated to $\mathcal{F}$ is the frame operator $S_{\mathcal{F}}$ defined in the weak sense by

$$S_{\mathcal{F}} : \mathcal{H} \to \mathcal{H}, \quad \langle S_{\mathcal{F}}(x), y\rangle := \int_\Omega \langle x, \mathcal{F}(\omega)\rangle\langle \mathcal{F}(\omega), y\rangle d\mu(\omega)$$

It follows from the definition that $S_{\mathcal{F}}$ is bounded, positive and invertible operator. We define the following transform associated to $\mathcal{F}$,

$$V_{\mathcal{F}}:\mathcal{H}\to L^2(\Omega,\mu),\quad V_{\mathcal{F}}(x)(\omega):=\langle x,\mathcal{F}(\omega)\rangle.$$

This operator is called the analysis operator and its adjoint operator is given by

$$V_{\mathcal{F}}^*:L^2(\Omega,\mu)\to\mathcal{H},\quad \langle V_{\mathcal{F}}^*(f),x\rangle:=\int_\Omega f(\omega)\langle\mathcal{F}(\omega),x\rangle d\mu(\omega).$$

Then we have $S_{\mathcal{F}}=V_{\mathcal{F}}^*V_{\mathcal{F}}$, and

$$\langle x,y\rangle=\int_\Omega\langle x,\mathcal{F}(\omega)\rangle\langle\mathcal{G}(\omega),y\rangle d\mu(\omega).$$

where $\mathcal{G}(\omega):=S_{\mathcal{F}}^{-1}\mathcal{F}(\omega)$ is the standard dual of $\mathcal{F}$.

For (Ω,μ)-frame $\mathcal{F}$, we define

$$\nu:\Sigma\to B(\mathcal{H})_+,\quad \nu(E)=\int_E\mathcal{F}(\omega)\otimes\mathcal{F}(\omega)\mathrm{d}\mu(\omega)$$

in the sense of

$$\langle\nu(E)x,y\rangle=\int_E\langle x,\mathcal{F}(\omega)\rangle\langle\mathcal{F}(\omega),y\rangle\mathrm{d}\mu(\omega),\forall x,y\in\mathcal{H}$$

which naturally induces an operator-valued measure (OVP). In the case that $\mathcal{F}$ is a Parseval continuous frame, then we also have $\nu(\Omega)=I_{\mathcal{H}}$, and so it induces a POVM. In what follows we will use the term *frame POVMs* to refer to such operator-valued measures.

Let ν be a frame OVM associated with a continuous frame $\mathcal{F}$. Then quantum measurement for a state $\rho\in\mathcal{S}(\mathcal{H})$ is given by the map $\mathbb{P}:\mathcal{S}(\mathcal{H})\to B(\Sigma,\mathbb{R})$

$$\mathbb{P}(\rho)(E)=\mathrm{tr}(\rho\nu(E))=\mathrm{tr}\left(\rho(\int_E\mathcal{F}(\omega)\otimes\mathcal{F}(\omega)\mathrm{d}\mu(\omega))\right)\ \forall E\in\Sigma. \tag{1.1}$$

It is easy to verify that

$$\mathrm{tr}(\rho\nu(E))=\int_E\langle\rho\mathcal{F}(\omega),\mathcal{F}(\omega)\rangle\mathrm{d}\mu(\omega)\ \forall E\in\Sigma,\rho\in S_1(\mathcal{H}). \tag{1.2}$$

Clearly, the quantum detection problem ask for the injectivity of the map $\mathbb{P}$ on $\mathcal{S}(\mathcal{H})$, in which case we say that $\mathcal{F}$ is *quantum injective*. We are interested in the characterizations of injective continuous frames and the existence problem of such frames for a given triple $(\Omega,\nu,\mathcal{H})$.

2. Characterizations of Quantum Injective Frames

2.1. **Injective Frames.** Let (Ω,Σ,μ) be a measure space, where Σ is the σ-algebra over Ω and μ is σ-finite positive measure. It is not hard to see (Proposition 2.2) that quantum injectivity of a continuous (Ω,μ)-frame $\mathcal{F}$ is equivalent to the condition that if $\langle T\mathcal{F}(\omega),\mathcal{F}(\omega)\rangle=0$, for a.e $\omega\in\Omega$ for a self-adjoint trace class operator T with trace zero, then $T=0$. Similarly, we say that $\mathcal{F}$ is $\mathcal{S}_2$-*injective* (respectively, $\mathcal{S}_2$-*injective*) if whenever a self-adjoint Hilbert-Schmidt (respectively, self-adjoint trace-class) operator T satisfies

$$\langle T\mathcal{F}(\omega),\mathcal{F}(\omega)\rangle=0,\ \text{for a.e. }\omega\in\Omega.$$

then $T = 0$.

Clearly, we have "$\mathcal{S}_2$-injectivity $\Rightarrow$ $\mathcal{S}_1$-injectivity $\Rightarrow$ quantum injectivity." In the case that $\mathcal{F}$ is a Parseval frame, quantum injectivity also implies the $\mathcal{S}_1$-injectivity (see Corollary 2.3). We point out that the Parseval frame requirement it not too much to ask since every frame is similar to a Parseval frame, and similar frames preserve $\mathcal{S}_j$-injectivity, here we say that two continuous frames $\mathcal{F}$ and $\mathcal{G}$ are *similar* if there is bounded invertible operator S such that $\mathcal{F}(\omega) = S\mathcal{G}(\omega)$ (a.e. $\omega \in \Omega$). We first point out the following two elementary facts:

Proposition 2.1. *Given a measure space (Ω, Σ, μ) and a (Ω, μ)-frame $\mathcal{F}$ for $\mathcal{H}$. For $j = 1, 2$, the following are equivalent:*

(1) *If* $\mathrm{T}, \mathrm{S} \in \mathcal{S}_j$ *are positive operators, and*
$$\langle T\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle = \langle S\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle, \text{ for a.e. } \omega \in \Omega$$
then $T = S$.

(2) *If* $\mathrm{T}, \mathrm{S} \in \mathcal{S}_j$ *are self-adjoint operators, and*
$$\langle T\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle = \langle S\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle, \text{ for a.e. } \omega \in \Omega$$
then $T = S$

(3) *$\mathcal{F}$ is $\mathcal{S}_j$-injective.*

(4) *For any $T \in \mathcal{S}_j$, the condition $\langle T\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle = 0$ $(a.e. \omega \in \Omega)$ implies that $T = 0$.*

Proof. Clearly, we have (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). For (1) $\Rightarrow$ (4), let $T \in \mathcal{S}_j$ be such that $\langle T\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle = 0$ $(a.e. \omega \in \Omega)$. Write $T = (T_1^+ - T_1^-) + i(T_2^+ - T_2^-)$, where $T_1^+, T_1^-, T_2^+, T_2^-$ are positive operators in $\mathcal{S}_j$. The $\langle T\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle = 0$ implies that $\langle (T_1^+ - T_1^-)\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle = 0$ and $\langle (T_2^+ - T_2^-)\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle = 0$, which in turn implies that $\langle T_1^+\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle = \langle T_1^-\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle$ and $\langle T_2^+\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle = \langle T_2^-\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle$. Thus, by (1), we get $T_1^+ = T_1^-$, $T_2^+ = T_2^-$ and so $T = 0$. □

The proof of the following fact is identical to that of Proposition 2.1 after taking the trace condition into the consideration

Proposition 2.2. *Given a (Ω, μ)-frame $\mathcal{F}$ for Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(1) *$\mathcal{F}$ is quantum injective.*

(2) *If T, S are self-adjoint trace class operators and trace one, and*
$$\langle T\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle = \langle S\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle, \text{ for a.e } \omega \in \Omega$$
then $T = S$.

(3) *If T are self-adjoint trace class operator and trace zero, and*
$$\langle T\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle = 0, \text{ for a.e } \omega \in \Omega$$
then $T = 0$.

(4) *If T are self-adjoint trace class operator and trace zero, and*
$$\langle T\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle = 0, \text{ for a.e } \omega \in \Omega$$
then $T = 0$.

Corollary 2.3. *If $\mathcal{F}$ is a Parseval frame, then it is $\mathcal{S}_1$-injective if and only if quantum injective.*

Proof. Let $\mathcal{F}$ be a Parseval frame which is quantum injective. Then

$$I=\int_{\Omega}\mathcal{F}(\omega)\otimes\mathcal{F}(\omega)d\mu(\omega).$$

and thus

$$tr(T)=\int_{\Omega}\langle T\mathcal{F}(\omega),\mathcal{F}(\omega)\rangle d\mu(\omega).$$

Now assume that

$$\langle T\mathcal{F}(\omega),\mathcal{F}(\omega)\rangle=0,\ \text{for a.e. }\omega\in\Omega.$$

for some self-adjoint trace-class T. Then $tr(T)=0$. By the equivalence of (1) and (3) in Proposition 2.2, we get that $T=0$. Hence $\mathcal{F}$ is $\mathcal{S}_1$-injective. □

2.2. **Characterizations of Injective Frames.** In order to obtain characterizations of quantum injective frames in terms of discrete frame sequences, we use the following continuous frame representations proposed in [12].

Proposition 2.4. *Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. Then the following are equivalent:*

(1) *$\mathcal{F}$ is a Parseval (Ω,μ)-frame for $\mathcal{H}$.*
(2) *There exists an orthonormal set $\{\varphi_i\}_{i\in\mathbb{I}}$ in $L^2(\Omega,\mu)$ having the property that $\sum_{i\in\mathbb{I}}|\varphi_i(\omega)|^2<\infty$ for a.e. $\omega\in\Omega$ and $\mathcal{F}(\omega)=\sum_{i\in\mathbb{I}}\varphi_i(\omega)e_i$ holds for a.e. $\omega\in\Omega$.*

For general continuous frames, we also have

Proposition 2.5. *The following are equivalent:*

(1) *$\mathcal{F}$ is a (Ω,μ)-frame for $\mathcal{H}$.*
(2) *$\mathcal{F}(\omega)=\sum_{i\in\mathbb{I}}\varphi_i(\omega)e_i$ for some orthonormal basis $\{e_i\}_{i\in\mathbb{I}}$ of $\mathcal{H}$ and some family $\{\varphi_i\}_{i\in\mathbb{I}}$ in $L^2(\Omega,\mu)$ with the properties that $\{\varphi_i\}_{i\in\mathbb{I}}$ is a Riesz basis for $\overline{span}\,\{\varphi_i\}_{i\in\mathbb{I}}$ and that $\sum_{i\in\mathbb{I}}|\varphi_i(\omega)|^2<\infty$ for a.e. $\omega\in\Omega$.*
(3) *$\mathcal{F}(\omega)=\sum_{i\in\mathbb{I}}\varphi_i(\omega)e_i$ for some Riesz basis $\{e_i\}_{i\in\mathbb{I}}$ of $\mathcal{H}$ and some set $\{\varphi_i\}_{i\in\mathbb{I}}$ orthonormal in $L^2(\Omega,\mu)$ with the property that, for a.e. $\omega\in\Omega,\sum_{i\in\mathbb{I}}|\varphi_i(\omega)|^2<\infty$.*
(4) *$\mathcal{F}(\omega)=\sum_{i\in\mathbb{I}}\varphi_i(\omega)e_i$ for some Riesz basis $\{e_i\}_{i\in\mathbb{I}}$ of $\mathcal{H}$ and some family $\{\varphi_i\}_{i\in\mathbb{I}}$ in $L^2(\Omega,\mu)$ with the properties that $\{\varphi_i\}_{i\in\mathbb{I}}$ is a Riesz basis for $\overline{span}\,\{\varphi_i\}_{i\in\mathbb{I}}$ and that $\sum_{i\in\mathbb{I}}|\varphi_i(\omega)|^2<\infty$ for a.e. $\omega\in\Omega$.*

We first presents several characterizations for $\mathcal{S}_2$-injective frames.

Theorem 2.6. *Let $\mathcal{F}$ be a (Ω,μ)-frame for the $\mathcal{H}$ with representation $\mathcal{F}(\omega)=\sum_{i\in\mathbb{I}}\varphi_i(\omega)e_i$ for some orthonormal basis $\{e_i\}_{i\in\mathbb{I}}$ of $\mathcal{H}$. Then the following are equivalent:*

(1) *$\mathcal{F}$ is $\mathcal{S}_2$-injective.*
(2) *The operators $(\varphi_i(\omega)\overline{\varphi_j(\omega)})_{i,j}$ spans the self-adjoint Hilbert-Schmidt operators on the Hilbert space $\ell_2(\mathbb{I})$ i.e. $S_2(\ell_2(\mathbb{I}))_{sa}$ for a.e. $\omega\in\Omega$.*

Proof. Our proof starts with two observation:

First, let $\mathcal{F}$ be a (Ω,μ)-frame for $\mathcal{H}$ and $\mathcal{F}(\omega)=\sum_{i\in\mathbb{I}}\varphi_i(\omega)e_i$ for some orthonormal basis $\{e_i\}_{i\in\mathbb{I}}$ of $\mathcal{H}$ and some family $\{\varphi_i\}_{i\in\mathbb{I}}$ in $L^2(\Omega,\mu)$ with the properties that $\{\varphi_i\}_{i\in\mathbb{I}}$ is a Riesz basis for $\overline{span}\,\{\varphi_i\}_{i\in\mathbb{I}}$ and that $\sum_{i\in\mathbb{I}}|\varphi_i(\omega)|^2<\infty$ for a.e. $\omega\in\Omega$. We define the operator

$$\left(\varphi_i(\omega)\overline{\varphi_j(\omega)}\right)_{i,j}.$$

We claim it is self-adjoint Hilbert-Schmidt operator on $\ell_2(\mathbb{I})$ for a.e. $\omega\in\Omega$. Obviously it is self-joint. Meanwhile, its Hilbert-Schmidt norm is finite for a.e. $\omega\in\Omega$. based on the following computation

$$\left\|\left(\varphi_i(\omega)\overline{\varphi_j(\omega)}\right)_{i,j}\right\|_2=\left(\sum_{i\in\mathbb{I},j\in\mathbb{I}}|\varphi_i(\omega)\overline{\varphi_j(\omega)}|^2\right)^{1/2}=\sum_{i\in\mathbb{I}}|\varphi_i(\omega)|^2<\infty.$$

Secondly, we set $a_{i,j}=\langle Te_i,e_j\rangle$ and define the operator $\left(a_{i,j}\right)_{i,j}$ on $\ell_2(\mathbb{I})$. We propose to prove an analogous results for $\left(a_{i,j}\right)_{i,j}$ namely, it is self-adjoint Hilbert-Schmidt operator on Hilbert space. Since T is self-adjoint, then

$$a_{i,j}=\langle Te_i,e_j\rangle=\langle e_i,Te_j\rangle=\overline{\langle Te_j,e_i\rangle}=\overline{a_{j,i}}$$

hence the matrix (may be infinite) $\left(a_{i,j}\right)_{i,j}$ is self-adjoint. Furthurmore

$$\|(a_{i,j})_{i,j}\|_2=\left(\sum_{i\in\mathbb{I},j\in\mathbb{I}}|a_{i,j}|^2\right)^{1/2}=\left(\sum_{i\in\mathbb{I},j\in\mathbb{I}}|\langle Te_i,e_j\rangle|^2\right)^{1/2}=\left(\sum_{i\in\mathbb{I}}\|Te_i\|^2\right)^{1/2}=\|T\|_2$$

Now we use the representation of (Ω,μ)-frame and we obtain

$$\begin{aligned}\langle T\mathcal{F}(\omega),\mathcal{F}(\omega)\rangle&=\left\langle T(\sum_{i\in\mathbb{I}}\varphi_i(\omega)e_i),(\sum_{j\in\mathbb{I}}\varphi_j(\omega)e_j)\right\rangle\\&=\sum_{i\in\mathbb{I}}\sum_{j\in\mathbb{I}}\varphi_i(\omega)\overline{\varphi_j(\omega)}\langle Te_i,e_j\rangle\\&=\sum_{i\in\mathbb{I}}\sum_{j\in\mathbb{I}}\varphi_i(\omega)\overline{\varphi_j(\omega)}a_{ij}\\&=\left\langle\left(\varphi_i(\omega)\overline{\varphi_j(\omega)}\right)_{i,j},(\overline{a_{i,j}})_{i,j}\right\rangle_{HS}\end{aligned}$$

Recall that all self-adjoint Hilbert-Schmidt operators under the inner product $\langle T,S\rangle_{HS}=\operatorname{tr}(TS^*)$ is real Hilbert space. By orthogonality, we conclude that if $\mathcal{F}$ is $\mathcal{S}_2$-injective, then orthogonal complement space of $\operatorname{span}\{(\varphi_i(\omega)\overline{\varphi_j(\omega)})_{i,j}\}$ for a.e. $\omega\in\Omega$ is 0, hence that the operators $(\varphi_i(\omega)\overline{\varphi_j(\omega)})_{i,j}$ spans $S_2(\ell_2(\mathbb{I}))_{sa}$ for a.e. $\omega\in\Omega$.

Conversely, for any Self-adjoint Hilbert-Schmidt operator T satisfies
$\langle T\mathcal{F}(\omega),\mathcal{F}(\omega)\rangle=0$, for a.e $\omega\in\Omega$, then

$$\left\langle\left(\varphi_i(\omega)\overline{\varphi_j(\omega)}\right)_{i,j},(\overline{a_{i,j}})_{i,j}\right\rangle_{HS}=0,\quad\text{for a.e }\omega\in\Omega$$

Since $\overline{\operatorname{span}}\{((\varphi_i(\omega)\overline{\varphi_j(\omega)})_{i,j})_{a.e\ \omega\in\Omega}\}$ is $S_2(\ell_2(\mathbb{I}))_{sa}$, we have $(\overline{a_{i,j}})_{i,j}=0$, hence $(a_{i,j})_{i,j}=0$ as well as $T=0$. Thus $\mathcal{F}$ is $\mathcal{S}_2$-injective. ☐

Definition 2.7. Let $\mathbb{H}$ denote the direct sum of real Hilbert spaces ℓ_2 in the sense

$$\mathbb{H} = \left(\sum_{i \in \mathbb{I}} \oplus \ell_2 \right)_{\ell_2}$$

Remark 2.8. we can define the inner product $\langle X, Y \rangle_{\mathbb{H}} = \sum_{i \in \mathbb{I}} \langle x_i, y_i \rangle$. where $X = (x_i)_{i \in \mathbb{I}}, Y = (y_i)_{i \in \mathbb{I}}$ are in $\mathbb{H}$. Under the conditions stated above, it follows that $\mathbb{H}$ is a Hilbert space.

Theorem 2.9. *Let $\mathcal{F}$ be a (Ω, μ)-frame for Hilbert space $\mathcal{H}$ and $\mathcal{F}(\omega) = \sum_{i \in \mathbb{I}} \varphi_i(\omega) e_i$ for some orthonormal basis $\{e_i\}_{i \in \mathbb{I}}$ of $\mathcal{H}$, then the following are equivalent:*

(1) *$\mathcal{F}$ is $\mathcal{S}_2$-injective,*

(2) *the sequence*

$$\begin{aligned} (\ & |\varphi_1(\omega)|^2, \mathrm{Re}(\varphi_1(\omega)\overline{\varphi_2(\omega)}), \mathrm{Im}(\varphi_1(\omega)\overline{\varphi_2(\omega)}) \cdots , \\ & |\varphi_2(\omega)|^2, \mathrm{Re}(\varphi_2(\omega)\overline{\varphi_3(\omega)}), \mathrm{Im}(\varphi_2(\omega)\overline{\varphi_3(\omega)}) \cdots , \\ & |\varphi_3(\omega)|^2, \mathrm{Re}(\varphi_3(\omega)\overline{\varphi_4(\omega)}), \mathrm{Im}(\varphi_3(\omega)\overline{\varphi_4(\omega)}) \cdots , \cdots) \end{aligned}$$

spans $\mathbb{H}$ for a.e. $\omega \in \Omega$.

Proof. Let T be self-adjoint Hilbert-Schmidt operator and $\{e_i\}_{i \in \mathbb{I}}$ be the orthonormal basis for $\mathcal{H}$ and the representation of $\mathcal{F}$ is $\mathcal{F}(\omega) = \sum_{i \in \mathbb{I}} \varphi_i(\omega) e_i$. We set

$$a_{i,j} = \langle Te_i, e_j \rangle \quad b_{i,j} = \varphi_i(\omega)\overline{\varphi_j(\omega)}.$$

Define $A = (A_1, A_2, A_3 \cdots A_i \cdots)_{i \in \mathbb{I}}$ where

$$A_i = (a_{i,i}, 2\,\mathrm{Re}(a_{i,i+1}), -2\,\mathrm{Im}(a_{i,i+1}), 2\,\mathrm{Re}(a_{i,i+2}), -2\,\mathrm{Im}(a_{i,i+2}), \cdots)$$

and $B = (B_1, B_2, B_3 \cdots B_i \cdots)_{i \in \mathbb{I}}$ where

$$B_i = (b_{i,i}, \mathrm{Re}(b_{i,i+1}), \mathrm{Im}(b_{i,i+1}), \mathrm{Re}(b_{i,i+2}), \mathrm{Im}(b_{i,i+2}), \cdots)$$

We can check at once that A is in $\mathbb{H}$ as

$$\|A_i\| \leq \left(\sum_{j \geq i, j \in \mathbb{I}} |2a_{i,j}|^2 \right)^{1/2} = \left(\sum_{j \geq i, j \in \mathbb{I}} |2\langle Te_i, e_j \rangle|^2 \right)^{1/2} \leq 2\|Te_i\|$$

$$\|A\|_{\mathbb{H}} = \left(\sum_{i \in \mathbb{I}} \|A_i\|^2 \right)^{1/2} \leq \left(\sum_{i \in \mathbb{I}} (2\|Te_i\|)^2 \right)^{1/2} = 2\|T\|_{HS}$$

Similarly we can verify B is in $\mathbb{H}$ too.

$$\begin{aligned} \langle T\mathcal{F}(\omega), \mathcal{F}(\omega) \rangle &= \left\langle T(\sum_{i \in \mathbb{I}} \varphi_i(\omega) e_i), (\sum_{j \in \mathbb{I}} \varphi_j(\omega) e_j) \right\rangle \\ &= \sum_{i \in \mathbb{I}} \sum_{j \in \mathbb{I}} \varphi_i(\omega)\overline{\varphi_j(\omega)} \langle Te_i, e_j \rangle \\ &= \sum_{i \in \mathbb{J}} \sum_{j \in \mathbb{I}} a_{i,j} b_{i,j} \\ &= \langle A, B \rangle_{\mathbb{H}} \end{aligned}$$

By the same method as in the previous theorem, then the theorem follows. □

Theorem 2.10. *Let $\mathcal{F}$ be a (Ω,μ)-frame for the Hilbert spaces $\mathcal{H}$ then the following are equivalent:*

(1) *$\mathcal{F}$ is $\mathcal{S}_2$-injective.*
(2) *For every orthonormal basis $\mathcal{E}=\{e_i : i\in\mathbb{I}\}$ and the corresponding representation $\mathcal{F}(\omega)=\sum_{I\in\mathbb{I}}\varphi_i(\omega)e_i$, then the sequence*

$$\big(|\varphi_1(\omega)|^2, |\varphi_2(\omega)|^2, |\varphi_3(\omega)|^2, \cdots, |\varphi_i(\omega)|^2, \cdots\big)$$

spans the real Hilbert spaces $\ell_2(\mathbb{I})$ for a.e. $\omega\in\Omega$.

Proof. We set

$$H(\mathcal{E}) := \overline{\text{span}}\{\big(|\varphi_1(\omega)|^2, |\varphi_2(\omega)|^2, |\varphi_3(\omega)|^2, \cdots, |\varphi_i(\omega)|^2, \cdots\big)\}$$

(1)⇒(2) If the statement was not true. Given an orthonormal basis $\mathcal{E}=\{e_i : i\in\mathbb{I}\}$ and (Ω,μ)-frame $\mathcal{F}$, then we have the representation $\mathcal{F}(\omega)=\sum_{i\in\mathbb{I}}\varphi_i(\omega)e_i$ such that $H(\mathcal{E})\neq \ell(\mathbb{I})$. Therefore there exist non-zero vector $\lambda=(\lambda_1,\lambda_2\cdots\lambda_i\cdots)\in\ell(\mathbb{I})$ such that $\lambda\perp H(\mathcal{E})$.

Define

$$Te_i=\lambda_i e_i \text{ for } i\in\mathbb{I}$$

It is easy to verify T is non-zero self-adjoint Hilbert operator. Furthermore

$$\langle T\mathcal{F}(\omega),\mathcal{F}(\omega)\rangle = \sum_{i\in\mathbb{I}}\lambda_i|\varphi_i(\omega)|^2=0.$$

This is contradicts the fact that $\mathcal{F}$ is $\mathcal{S}_2$-injective.

(2)⇒(1) Let T be a self-adjoint Hilbert-Schmidt operator such that

$$\langle T\mathcal{F}(\omega),\mathcal{F}(\omega)\rangle = 0, \quad \text{for a.e. } \omega\in\Omega.$$

As T is a Hilbert-Schmidt operator and hence compact operator, so there is an eigenbasis $\mathcal{E}=\{e_i : i\in\mathbb{I}\}$ (Schmidt-orthogonalized, normalized and completed as an orthonormal basis) for T with respect to the eigenvalue $\{\lambda_i\}_{i\in\mathbb{I}}$, further T is self-adjoint so that $\lambda_i\in\mathbb{R}$, $\forall\ i\in\mathbb{I}$. For the continuous frame $\mathcal{F}$ and an orthonormal basis $\mathcal{E}=\{e_i : i\in\mathbb{I}\}$, we have representation $\mathcal{F}(\omega)=\sum_{i\in\mathbb{I}}\varphi_i(\omega)e_i$, then

$$\langle T\mathcal{F}(\omega),\mathcal{F}(\omega)\rangle = \sum_{i\in\mathbb{I}}\lambda_i|\varphi_i(\omega)|^2=0.$$

Meanwhile since T is Hilbert-Schmidt operator, we know $(\lambda_1,\lambda_2\cdots\lambda_i\cdots)\in\ell_2(\mathbb{I})$ as

$$\sum_{i\in\mathbb{I}}|\lambda_i|^2=\sum_{i\in\mathbb{I}}\|Te_i\|^2<\infty$$

By assumption $H(\mathcal{E})=\ell_2(\mathbb{I})$, moreover

$$(\lambda_1,\lambda_2\cdots\lambda_i\cdots)\perp H(\mathcal{E})$$

Therefore, $(\lambda_1,\lambda_2\cdots\lambda_i\cdots)=0$ and hence $T=0$. Thus $\mathcal{F}$ is $\mathcal{S}_2$-injective □

Now we give a characterization for quantum injective frames.

Theorem 2.11. *Given a (Ω,μ)-frame for Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(1) *$\mathcal{F}$ is quantum injective.*
(2) *For every orthonormal basis $\{e_i : i \in \mathbb{I}\}$ for $\mathcal{H}$, and $\mathcal{F}(\omega) = \sum_{I\in\mathbb{I}} \varphi_i(\omega)e_i$. then the closed span of*

$$\big(|\varphi_1(\omega)|^2, |\varphi_2(\omega)|^2, |\varphi_3(\omega)|^2, \cdots, |\varphi_i(\omega)|^2, \cdots\big) \ \omega \in \Omega.$$

is real space $c_0(\mathbb{I})$

Proof. By Proposition 2.2, we know that $\mathcal{F}$ is quantum injective is equivalent to the condition that $\langle T\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle = 0$ (a.e. $\omega \in \Omega$) for some trace class self-adjoint operator T of trace zero implies $T = 0$. Set

$$H(\mathcal{E}) := \overline{\operatorname{span}}^{c_0}\{\big(|\varphi_1(\omega)|^2, |\varphi_2(\omega)|^2, |\varphi_3(\omega)|^2, \cdots, |\varphi_i(\omega)|^2, \cdots, \omega \in \Omega\big)\}$$

(1)⇒(2): If not, $H(\mathcal{E}) \neq c_0(\mathbb{I})$, then by Hahn Banach separation theorem, there exists non-zero vector $\lambda = (\lambda_i) \in \ell_1(\mathbb{I})$ and we take λ as linear functional on $c_0(\mathbb{I})$ $(c_0(\mathbb{I})^* = \ell_1(\mathbb{I}))$ such that $H(\mathcal{E}) \in \operatorname{Ker}\lambda$. Then we define $Te_i = \lambda_i e_i$, then

$$\langle T\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle = \sum_{i\in\mathcal{I}} \lambda_i |\varphi_i(\omega)|^2 = 0,$$

which is a contradiction. Hence $H(\mathcal{E}) \neq c_0(\mathbb{I})$.

(2)⇒(1): For any self-adjoint trace class operator, then from [19], there is some orthonormal basis $\{e_i\}_{i\in\mathbb{I}}$ such that $Tx = \sum_{i\in\mathbb{I}} \lambda_i \langle x, e_i\rangle e_i, \forall x \in \mathcal{H}$, where $(\lambda_i) \in \ell_1(\mathbb{I})$. Then

$$\langle T\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle = \sum_{i\in\mathbb{I}} \lambda_i |\varphi_i(\omega)|^2$$

If for a.e.$\Omega \in \Omega$, $\langle T\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle = 0$, since $H(\mathcal{E}) = c_0(\mathbb{I})$, then we have $(\lambda_i) = 0$ hence we conclude that $T = 0$ □

3. Existence and perturbation theory of the injective frame

3.1. **Method to construct the injective frame.** We are now in a position to show the existence of such continuous frames and construct some concrete examples. For any continuous Parseval frame $\mathcal{F}$, we write its representation $\mathcal{F}(\omega) = \sum_{i\in\mathbb{I}} \varphi_i(\omega)e_i$. Now emphasis is put on the index set $\mathbb{I}$, from which we can conclude the following *dimension formula* and distinguish whether a Hilbert space is a finite-dimensional or an infinite-dimensional space.

Corollary 3.1. *For every Parseval (Ω, μ)-frame $\mathcal{F}$ for a Hilbert space $\mathcal{H}$, we have*

$$\dim \mathcal{H} = \int_\Omega \|\mathcal{F}(\omega)\|^2 \mathrm{d}\mu(\omega).$$

Proof. By preceding proposition, we can write $\mathcal{F}(\omega) = \sum_{i\in\mathbb{I}} \varphi_i(\omega)e_i$ for some orthonormal basis $\{e_i\}_{i\in\mathbb{I}}$ of $\mathcal{H}$ and an orthonormal set $\{\varphi_i\}_{i\in\mathbb{I}}$ in $L^2(\Omega, \mu)$. Thus

$$\int_\Omega \|F(\omega)\|^2 \mathrm{d}\mu(\omega) = \int_\Omega \sum_{i\in\mathbb{I}} |\varphi_i(\omega)|^2 \, \mathrm{d}\mu(\omega) = \sum_{i\in\mathbb{I}} \|\varphi_i\|^2 = \operatorname{card}(\mathbb{I}) = \dim \mathcal{H}.$$

□

First, we may assume card($\mathbb{I}$) is finite, that is $\mathcal{H}$ is a finite-dimensional Hilbert space. Even if $\mathcal{H}$ is finite-dimensional, we still can consider the continuous frame for $\mathcal{H}$.

Suppose we have an n-dimensional Hilbert space $\mathcal{H}$, and we have continuous Parseval frame $\mathcal{F}$. Its representation is $\mathcal{F}(\omega) = \sum_{i=1}^{n} \varphi_i(\omega)e_i$ for some orthonormal basis $\{e_i\}_{i=1}^{n}$ and orthonormal set $\{\varphi_i(\omega)\}_{i=1}^{n}$ in $L_2(\Omega, \mu)$.

We want to construct a continuous frame to give injectivity, from theorem 2.9, it is equivalent to find orthonormal set $\{\varphi_i\}_i^n$ such that the sequence

$$\begin{aligned}&(|\varphi_1(\omega)|^2, \operatorname{Re}(\varphi_1(\omega)\overline{\varphi_2(\omega)}), \operatorname{Im}(\varphi_1(\omega)\overline{\varphi_2(\omega)}) \cdots, \operatorname{Re}(\varphi_1(\omega)\overline{\varphi_n}(\omega)), \operatorname{Im}(\varphi_1(\omega)\overline{\varphi_n}(\omega));\\ &|\varphi_2(\omega)|^2, \operatorname{Re}(\varphi_2(\omega)\overline{\varphi_3(\omega)}), \operatorname{Im}(\varphi_2(\omega)\overline{\varphi_3(\omega)}) \cdots, \operatorname{Re}(\varphi_2(\omega)\overline{\varphi_n}(\omega)), \operatorname{Im}(\varphi_2(\omega)\overline{\varphi_n}(\omega));\\ &\cdots; |\varphi_n(\omega)|^2).\end{aligned}$$

span the real Hilbert space $\ell_2^{n^2}$. This is equivalent to that pointwise multiplication vector $\{\varphi_i\overline{\varphi_i}\}$, and $\{\operatorname{Re}(\varphi_i\overline{\varphi_j}), \operatorname{Im}(\varphi_i\overline{\varphi_j})\}_{1\le i<j\le n}$ are linear independent.

Base on the above analysis, we give a way to construct continuous frames to give injectivity. We consider compactly supported wavelet basis. Let

$$R_i := \overline{\{\omega \in \Omega : \operatorname{Re}\varphi_i(\omega) \neq 0\}} \quad I_i := \overline{\{\omega \in \Omega : \operatorname{Im}\varphi_i(\omega) \neq 0\}}$$

Theorem 3.2. *Suppose R_i is the support of $\operatorname{Re}\varphi_i$ (the real part of φ_i) and I_i is the support of $\operatorname{Im}\varphi_i$. Let $I_1 = \emptyset$ and in the interval R_1, let $\{\varphi_1, \operatorname{Re}\varphi_2, \operatorname{Im}\varphi_2 \cdots \operatorname{Re}\varphi_n, \operatorname{Im}\varphi_n\}$ be linearly independent ; Let $I_2\backslash R_1 = \emptyset$ and in In the interval $R_2\backslash R_1$, let $\{\varphi_2, \operatorname{Re}\varphi_3, \operatorname{Im}\varphi_3 \cdots \operatorname{Re}\varphi_n, \operatorname{Im}\varphi_n\}$ linearly independent; Let $I_3\backslash(R_1 \cup R_2) = \emptyset$ and in the interval $R_3\backslash(R_1 \cup R_2)$, let $\{\varphi_3, \operatorname{Re}\varphi_4, \operatorname{Im}\varphi_4 \cdots \operatorname{Re}\varphi_n, \operatorname{Im}\varphi_n\}$ linearly independent; Continuing this procedure until in the interval $R_n\backslash \cup_{i=1}^{n-1}(R_i)$, $\varphi_n \neq 0$, then we get a continuous frame $\mathcal{F}(\omega) = \sum_{i=1}^{n} \varphi_i(\omega)e_i$ which gives injectivity.*

Proof. Suppose $\{e_i\}_{i=1}^{n}$ is orthonormal basis for $\mathcal{H}$, $\mathcal{F}(\omega)$ is continuous frame and its representation is $\mathcal{F}(\omega) = \sum_{i=1}^{n} \varphi_i(\omega)e_i$. We set $\langle Te_i, e_j\rangle = a_{i,j} = \alpha_{i,j} + \mathrm{i}\beta_{i,j}$, then

$$\begin{aligned}\langle T\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle &= \langle T(\sum_{i=1}^{n} \varphi_i(\omega)e_i), (\sum_{j=1}^{n} \varphi_j(\omega)e_j)\rangle\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \varphi_i(\omega)\overline{\varphi_j(\omega)}\langle Te_i, e_j\rangle\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}\varphi_i(\omega)\overline{\varphi_j(\omega)}\\ &= \sum_{1\le i\le j\le n} 2\alpha_{i,j}\operatorname{Re}(\varphi_i(\omega)\overline{\varphi_j(\omega)}) - 2\beta_{i,j}\operatorname{Im}(\varphi_i(\omega)\overline{\varphi_j(\omega)})\end{aligned}$$

If $\langle T\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle = 0$, we first take $\omega \in R_n\backslash \cup_{i=1}^{n-1}(R_i)$, then $\varphi_i(\omega) = 0$ $(1 \le i \le n-1)$, it follows that $\alpha_{n,n} = 0$,namely $a_{n,n} = 0$. Second we take $\omega \in R_{n-1}\backslash \cup_{i=1}^{n-2}(R_i)$, then $\varphi_i(\omega) = 0 (1 \le i \le n-1)$ and $\{\varphi_{n-1}, \operatorname{Re}\varphi_n, \operatorname{Im}\varphi_n\}$ are linearly independent, it is equivalent to that $\{\varphi_{n-1}^2, \operatorname{Re}(\varphi_{n-1}\overline{\varphi_n}), \operatorname{Im}(\varphi_{n-1}\overline{\varphi_n})\}$ are linearly independent, it follows that $\alpha_{n-1,n-1} = 0, \alpha_{n-1,n} = 0, \beta_{n-1,n} = 0$, that is $a_{n-1,n-1} = 0, a_{n-1,n} = 0, a_{n,n-1} = 0$.

Continuing to choose ω in different intervals, we conclude that $a_{i,j} = 0$ $(1 \leq i, j \leq n)$. It follows that $T = 0$. ☐

As a byproduct of this results, if we consider the countable set Ω and counting measure μ then $\mathrm{card}(\Omega) \geq n^2$ is necessary because if (Ω, μ)-frame $\mathcal{F}$ gives injectivity then $\{\varphi_i \overline{\varphi_i}\}$, and $\{\mathrm{Re}(\varphi_i \overline{\varphi_j}), \mathrm{Im}(\varphi_i \overline{\varphi_j})\}_{1 \leq i < j \leq n}$ are linear independent. Directly from the above construction method, we can get a simple example.

Example 3.3. Let φ_i be the i-th column vector of the following matrix. Then we define $\mathcal{F}(\omega) = \sum_{i=1}^{n} \varphi_i(\omega) e_i$

$$
\begin{array}{r}
2n-1 \left\{ \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array} \right. \\
2n-3 \left\{ \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array} \right. \\
\vdots \\
\\
\end{array}
\left(
\begin{array}{ccccc}
1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 0 \\
1 & \mathrm{i} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & \mathrm{i} \\
\hline
0 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & \cdots & 0 \\
0 & 1 & \mathrm{i} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 1 & 0 & \cdots & 1 \\
0 & 1 & 0 & \cdots & \mathrm{i} \\
\hline
 & & \vdots & & \\
\hline
0 & 0 & 0 & \cdots & 1
\end{array}
\right)
$$

$$\underbrace{\qquad\qquad}_{n}$$

Obviously $\mathcal{F}$ is continuous frame. Besides we can write it as a discrete frame, namely $\{e_i\}_{i=1}^{n} \cup \{e_i + e_j : i < j\}_{i,j=1}^{n} \cup \{e_i + \mathrm{i}e_j : i < j\}_{i,j=1}^{n}$. An trivial to verification show that $\mathcal{F}$ gives injectivity of the self-adjoint matrix on ℓ_2^n.

So far we have not constructed general examples even a Parseval frame though we can turn it into a Parseval frame by applying $S^{-1/2}$, where S is the frame operator of the frame. While there are some methods to construct a specific frame based on a given frame operator[9], however we will investigate the representation of continuous frame on a countable set and this yield an alternative approach to construct injective frames directly for finite dimensional Hilbert space. Furthermore we will give a method to construct a general injective frame even injective Parseval frame by the way of induction. The following Lemma will be useful.

Lemma 3.4. *Let* $A \in M_{n_1+n_2, n_1+n_2}$ *be partitioned as* $A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$, *If* $\det(A_{1,1}) \neq 0$, *then*

$$\det(A) = \det(A_{1,1}) \det\left(A_{2,2} - A_{2,1} A_{1,1}^{-1} A_{1,2}\right)$$

If $\det(A_{2,2}) \neq 0$, *then*

$$\det(A) = \det(A_{2,2}) \det\left(A_{1,1} - A_{1,2} A_{2,2}^{-1} A_{2,1}\right)$$

Lemma 3.4 is the result of classical matrix theory and the proof is omitted.

For n=1, we take $a_1 \neq 0$, then it is done. For $n = 2$, our goal is to find 4-dimensional vector φ_1, φ_2 such that pointwise multiplication vector $\{\varphi_1\overline{\varphi_1}, \mathrm{Re}(\varphi_1\overline{\varphi_2}), \mathrm{Im}(\varphi_1\overline{\varphi_2}), \varphi_2\overline{\varphi_2}\}$ are linearly independent, if we suppose φ_2 is real vector, then it is equivalent to that $\{\varphi_1\overline{\varphi_1}, \varphi_1\overline{\varphi_2}, \varphi_2\overline{\varphi_1}, \varphi_2^2\}$ are linearly independent, that is the determinant of the matrix $(\varphi_i\overline{\varphi_j})_{1\leq i\leq j\leq 2}$ is not zero.

TABLE 1. From Vector to Matirx

$$
\begin{pmatrix} a_1 \end{pmatrix} \to
\begin{matrix} 1\,\{ \\ 3\left\{\begin{matrix}\\ \\ \\ \end{matrix}\right. \end{matrix}
\underbrace{\left(\begin{array}{cc} a_1 & b_1 \\ \hline a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \end{array}\right)}_{2}
\to \cdots \Rightarrow \begin{pmatrix} a_1\overline{a_1} \end{pmatrix} \to
\begin{matrix} 1\,\{ \\ 3\left\{\begin{matrix}\\ \\ \\ \end{matrix}\right. \end{matrix}
\left(\begin{array}{c|ccc} a_1\overline{a_1} & a_1\overline{b_1} & b_1\overline{a_1} & b_1\overline{b_1} \\ \hline a_2\overline{a_2} & a_2\overline{b_2} & b_2\overline{a_2} & b_2\overline{b_2} \\ a_3\overline{a_3} & a_3\overline{b_3} & b_3\overline{a_3} & b_3\overline{b_3} \\ \underbrace{a_4\overline{a_4}}_{1} & \underbrace{a_3\overline{b_4} \quad b_4\overline{a_4} \quad b_4\overline{b_4}}_{3} \end{array}\right) \to \cdots
$$

From the above Table 1 we can see that from $n = 1$ to $n = 2$, the vectors becomes matrix. For $n = 2$ we can partition $M_{4,4}$ as $M_{4,4} = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$, where $A_{1,1}$ is the corresponding matrix of $M_{1,1} = (a_1\overline{a_1})$ whose determinant is not 0. By the Lemma 3.4, if we can find suitable b_1 and $B = \begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \end{pmatrix}$ such that $\det\left(A_{2,2} - A_{2,1}A_{1,1}^{-1}A_{1,2}\right) \neq 0$, we attain our goal.

If we set b_1 then $A_{1,2} = 0$,

$$\det\left(A_{2,2} - A_{2,1}A_{1,1}^{-1}A_{1,2}\right) = \det(A_{2,2}) = b_2b_3b_4 \det D$$

where D is $\begin{pmatrix} a_2 & \overline{a_2} & b_2 \\ a_3 & \overline{a_3} & b_3 \\ a_4 & \overline{a_4} & b_4 \end{pmatrix}$. The determinant of $A_{2,2}$ is not 0 is equivalent to that the column vectors of matrix D are linearly independent and all coordinates of $\begin{pmatrix} b_2 \\ b_3 \\ b_4 \end{pmatrix}$ are non-zero.

Suppose when the dimension is n, we have already got vectors $\{\varphi_1, \varphi_2, \cdots \varphi_n\}$ such that $\{\varphi_i\overline{\varphi_j}\}_{1\leq i,j\leq n}$ are linearly independent, then the determinant of the corresponding matrix is not 0, denoted as $B_{1,1}$, then for $n+1$, our aim is to find a $(2n+1)\times(n+1)$ matrix A and a column vector φ_{n+1}, and we set

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} & a_{1,n+1} \\ a_{2,1} & \cdots & a_{2,n} & a_{2,n+1} \\ \vdots & \ddots & \vdots & \vdots \\ a_{2n,1} & \cdots & a_{2n,n} & a_{2n,n+1} \\ a_{2n+1,1} & \cdots & a_{2n+1,n} & a_{2n+1,n+1} \end{pmatrix} \quad \text{and} \quad \varphi_{n+1} = \begin{pmatrix} \varphi_{1,n+1} \\ \varphi_{2,n+1} \\ \vdots \\ \vdots \\ \varphi_{n^2,n+1} \end{pmatrix}$$

then the process of vector transformation can be shown by the following table.

TABLE 2. From n to $n+1$

$$
n^2\left\{\underbrace{\begin{pmatrix} \varphi_{1,1} & \cdots & \varphi_{1,n} \\ \varphi_{2,1} & \cdots & \varphi_{2,n} \\ \vdots & \vdots & \vdots \\ \vdots & \ddots & \vdots \\ \varphi_{n^2,1} & \cdots & \varphi_{n^2,n} \end{pmatrix}}_{n}\right. \to \begin{array}{c} n^2\left\{\begin{array}{c}\\ \\ \\ \\ \\ \end{array}\right. \\ 2n+1\left\{\begin{array}{c}\\ \\ \\ \\ \\ \end{array}\right. \end{array} \underbrace{\left(\begin{array}{cccc} \varphi_{1,1} & \cdots & \varphi_{1,n} & \varphi_{1,n+1} \\ \varphi_{2,1} & \cdots & \varphi_{2,n} & \varphi_{2,n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \vdots & \vdots \\ \varphi_{n^2,1} & \cdots & \varphi_{n^2,n} & \varphi_{n^2,n+1} \\ \hline a_{1,1} & \cdots & a_{1,n} & a_{1,n+1} \\ a_{2,1} & \cdots & a_{2,n} & a_{2,n+1} \\ \vdots & \ddots & \vdots & \vdots \\ a_{2n,1} & \cdots & a_{2n,n} & a_{2n,n+1} \\ a_{2n+1,1} & \cdots & a_{2n+1,n} & a_{2n+1,n+1} \end{array}\right)}_{n+1}
$$

Meanwhile the corresponding matrix an be written as

$$
B = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}
$$

where $B_{1,1}$ is the matrix corresponding to $\{\varphi_i\}_{1\le i\le n}$, if we suppose $\varphi_{n+1}=0$, then

$$
\begin{aligned}
\det(B) &= \det(B_{1,1})\det\left(B_{2,2} - B_{2,1}B_{1,1}^{-1}B_{1,2}\right) \\
&= \det(B_{1,1})\det(B_{2,2}) \\
&= \det(B_{1,1})\cdot a_{1,n+1}a_{2,n+1}\cdots a_{2n+1,n+1}\det(C).
\end{aligned}
$$

where

$$
C = \begin{pmatrix} a_{1,1} & \overline{a_{1,1}} & \cdots & a_{1,n} & \overline{a_{1,n}} & a_{1,n+1} \\ a_{2,1} & \overline{a_{2,1}} & \cdots & a_{2,n} & \overline{a_{2,n}} & a_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{2n,1} & \overline{a_{2n,1}} & \cdots & a_{n,n} & \overline{a_{2n,n}} & a_{n,n+1} \\ a_{2n+1,1} & \overline{a_{2n+1,1}} & \cdots & a_{n+1,n} & \overline{a_{2n+1,n}} & a_{n+1,n+1} \end{pmatrix}
$$

Thus if we set $\varphi_{n+1}=0$ and find the matrix C whose the column vectors are linearly independent and all coordinates of the last column are non-zero, then we get $n+1$ column vector $\{\phi_i\}_{1\le i\le n+1}$ such that $\mathcal{F}=\sum_{i=1}^{n+1}\phi_i e_i$ gives injectivity.

Based on the above analysis, we can give an way to construct injective frames for n-dimension Hilbert space.

Theorem 3.5. *We define the matrix*

$$\begin{pmatrix} W_{1,1} & 0 & 0 & \cdots & 0 \\ W_{2,1} & W_{2,2} & \mathbf{0} & \cdots & \mathbf{0} \\ W_{3,1} & W_{3,2} & W_{3,3} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_{n,1} & W_{n,2} & W_{n,3} & \cdots & W_{n,n} \end{pmatrix}$$

where for any $1 \leq l \leq n$, we choose $2l-1$ linearly independent column vectors $\{V_{l,k}\}_{1\leq k\leq 2l-1}$ in $\mathbb{R}^{2l-1}$, and all coordinates of $V_{l,2l-1}(1 \leq l \leq n)$ are not zero. We suppose that i will be used to denote the complex unit and set

$$W_{l,k} = \begin{cases} V_{l,2k-1} + i\, V_{l,2k}, & k¡l; \\ \\ V_{l,2k-1}, & k{=}l. \end{cases}$$

then we take the column vector $\{\varphi_i\}$ and define $\mathcal{F} = \sum_{i=1}^{n+1} \varphi_i e_i$ is the frame that gives injectivity for the self-adjoint operator on the $\mathbb{C}^n$.

Due to the representation of the continuous frame, if we want to construct a Parseval frame, we only need to choose orthonormal set $\{\varphi_i\}$. Thus we have the following corollary.

Corollary 3.6. *we define the matrix*

$$\begin{pmatrix} \lambda_{1,1}U_{1,1} & 0 & 0 & \cdots & 0 \\ \lambda_{2,1}U_{2,1} & \lambda_{2,2}U_{2,2} & 0 & \cdots & 0 \\ \lambda_{3,1}U_{3,1} & \lambda_{3,2}U_{3,2} & \lambda_{3,3}U_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_{n,1}U_{n,1} & \lambda_{n,2}U_{n,2} & \lambda_{n,3}U_{n,3} & \cdots & \lambda_{n,n}U_{n,n} \end{pmatrix}$$

where

$$U_{l,k} = \begin{cases} \frac{\sqrt{2}}{2}(V_{l,2k-1} + i\, V_{l,2k}), & k¡l; \\ \\ V_{l,2k-1}, & k{=}l. \end{cases}$$

the coefficient $\{\lambda_{i,j}\}_{j\leq i}$ satisfies

$$\lambda_{i,j} \neq 0 \quad \text{for} \quad 1 \leq j \leq i \leq n, \qquad \text{and} \sum_{j\leq i} |\lambda_{i,j}|^2 = 1, \forall\ 1 \leq j \leq n$$

Besides column vectors$\{V_{i,j}\}_{j\leq i}$ are orthonormal basis for $\mathbb{R}^i$ and all coordinates of $V_{i,i}$ are not zero for all $1 \leq i \leq n$. We take the column vector $\widetilde{\{\varphi_i\}}_{1\leq i\leq n}$ and define $\widetilde{\mathcal{F}} = \sum_{i=1}^{n} \widetilde{\varphi}_i e_i$. Then $\mathcal{F}$ is the Parseval frame that gives injectivity.

Remark 3.7. Obviously $\mathcal{F} = \sum_{i=1}^{n} \widetilde{\varphi}_i e_i$ gives injective from the theorem (2.9). Meanwhile from the equivalent characterization of continuous frame, it is easily seen that is Parseval frame. The representation of continuous frame on countable set provides a different perspective on the frame, while if we take the row vectors, it actually is the ordinary discrete frame. It is not difficult to verify its frame operator is I, equivalent to being the Parseval frame.

For the infinite dimensional Hilbert space, the situation is different. Given a continuous frame $\mathcal{F}$, unlike finite dimensional cases, from the Theorem 2.9, $\{\varphi_i\overline{\varphi_i}\}$, and $\{\mathrm{Re}(\varphi_i\overline{\varphi_j}), \mathrm{Im}(\varphi_i\overline{\varphi_j})\}_{1\leq i<j}$ may be finite linearly independent, but they may ω-dependent which implies it can not give injectivity. A typical counterexample is as follows

Example 3.8. For a infinite dimensional Hilbert space $\mathcal{H}$, let $\mathcal{F}$ be the canonical Parseval continuous frame with respect to a σ-finite measure space (Ω, μ), which is

$$\mathcal{F}(\omega) := \sum_{i=1}^{\infty} \frac{\chi_{\Omega_i}}{\sqrt{\mu(\Omega_i)}} e_i, \quad \text{for } \omega \in \Omega$$

where $\{\Omega_k\}$ is a countable sequence of mutually disjoint, finite, positive measure subsets of Ω with union Ω.

Proof. For simplicity of notation we set $\varphi_i(\omega) = \dfrac{\chi_{\Omega_i}}{\sqrt{\mu(\Omega_i)}}$, then $\mathcal{F}(\omega) := \sum_{i=1}^{\infty} \varphi_i(\omega)e_i$, for $\omega \in \Omega$. Therefore $\mathcal{F}$ is not injective because by the equivalent characterization of theorem 2.6 the operators

$$\left(\varphi_i(\omega)\overline{\varphi_j(\omega)}\right)_{i,j} = \begin{pmatrix} 0 & \vdots & 0 \\ \cdots & \frac{1}{\mu(\Omega_k)} & \cdots \\ 0 & \vdots & 0 \end{pmatrix}$$

can not span the $S_2(\ell_2)_{sa}$. Or we can compute directly. For $\omega \in \Omega$, then $\omega \in \Omega_k$ for some k it follows that

$$\langle T\mathcal{F}(\omega), \mathcal{F}(\omega)\rangle = \frac{1}{\mu(\Omega)}\langle Te_k, e_k\rangle = 0$$

from which we can not derive $T = 0$. $\square$

However we give a concrete example that is injective.

Example 3.9. Given Hilbert space $\mathcal{H}$ and its orthonormal basis $\{e_i\}_{i=1}^{\infty}$. Suppose $\chi = \chi_{[0,1)}$, $\psi = \chi_{[0,1/2)} - \mathrm{i}\chi_{[1/2,1)}$. For integer $n, \in \mathbb{N}^+$, define

$$\begin{aligned}
\varphi_1(t) =& \chi \\
\varphi_2(t) =& \chi(t-1) + \psi(4(t-\frac{1}{2}) \\
\varphi_3(t) =& \chi(t-2) + \psi(8(t-\frac{1}{2}-\frac{1}{4})) + \psi(8(t-1-\frac{1}{2}-\frac{1}{4})) \\
\varphi_4(t) =& \chi(t-3) + \psi(16(t-\frac{1}{2}-\frac{1}{4}-\frac{1}{8})) + \psi(16(t-1-\frac{1}{2}-\frac{1}{4}-\frac{1}{8})) \\
&+ \psi(16(t-2-\frac{1}{2}-\frac{1}{4}-\frac{1}{8})) \\
&\vdots \\
\varphi_n(t) =& \chi(t-n+1) + \psi(2^n(t-1+2^{\frac{1}{n\;1})} + \cdots + \psi(2^n(t-n+1+2^{\frac{1}{n\;1})}) \\
&\vdots
\end{aligned}$$

Define $\mathcal{F} : \mathbb{R}^+ \to \mathcal{H}$ by $\mathcal{F}(t) = \sum_{i=1}^{\infty} \varphi_i(t)e_i$. Let m be the Lebegue measure on $\mathbb{R}^+$, then $(\mathbb{R}^+, m)$-frame $\mathcal{F}$ is the continuous frame for $\mathcal{H}$ which gives injectivity.

Proof. : First, we verify $(\mathbb{R}^+, m)$-frame $\mathcal{F}$ is the continuous frame for $\mathcal{H}$. We observe $\forall \ \{c_n\} \in \ell_2$, the value of $\sum_{n=1}^{\infty} c_n\varphi_i(t)$ can only be $c_i, 2c_i$, $(1-\mathrm{i})c_i$ for $i \leq t < i+1$, thus

$$\sum_{n=1}^{\infty} |c_n|^2 \leq \| \sum_{n=1}^{\infty} c_n\varphi_i\|^2 \leq 4\sum_{n=1}^{\infty} |c_n|^2, \quad \forall \ \{c_n\} \in \ell_2$$

By the equivalent characterization of Riesz basis from [8], we conclude that $\{\varphi_i\}$ is Riesz basis for $\overline{span}\,\{\varphi_i\}$ hence that $\mathcal{F}(t) = \sum_{i=1}^{\infty} \varphi_i(t)e_i$ is continuous frame for $\mathcal{H}$ by Corollary 2.5.

By the definition of $\mathcal{F}$,we get

$$\mathcal{F}(t) = \begin{cases} e_1, & t \in [0, \frac{1}{2}); \\ e_1 + e_2, & t \in [\frac{1}{2}, \frac{5}{8}); \\ e_1 - \mathrm{i}e_2, & t \in [\frac{5}{8}, \frac{3}{4}); \\ \vdots & \vdots \\ e_2, & t \in [1, \frac{7}{4}); \\ e_2 + e_3, & t \in [\frac{7}{4}, \frac{29}{16}); \\ e_2 - \mathrm{i}e_3, & t \in [\frac{29}{16}, \frac{15}{8}); \\ \vdots & \vdots \end{cases}$$

For any self-adjoint Hilbert-Schmidt operator T if

$$\langle T\mathcal{F}(t), \mathcal{F}(t)\rangle = 0, \ a.e \ t \in \mathbb{R}^+$$

then if we choose $t \in [0, \frac{3}{4}) \cup [1, \frac{7}{4})$ we get

$$\langle Te_1, e_1\rangle = 0; \langle T(e_1 + e_2), (e_1 + e_2)\rangle = 0; \langle T(e_1 - \mathrm{i}e_2), (e_1 - \mathrm{i}e_2)\rangle = 0; \langle Te_2, e_2\rangle = 0;$$

it follows that

$$\langle Te_1, e_1\rangle = 0; \quad \langle Te_1, e_2\rangle = \langle Te_2, e_1\rangle = 0; \quad \langle Te_2, e_2\rangle = 0;$$

Repeating the previous argument in all intervals in $\mathbb{R}^+$ leads to $\langle Te_i, e_j\rangle = 0$ for all $i, j \in \mathbb{N}^+$which imply $T = 0$ hence $\mathcal{F}$ is injective. □

3.2. **Small perturbation of the injective frame.** As stated in the previous theorem, we have constructed injective continuous frames. Now we proceed to describe the property of the injective frame after a small perturbation. To do this, we need to define a metrics between frames. Intuitively there is a standard metric measuring the distance between frames.

Definition 3.10. Given two (Ω, μ)-frames $\mathcal{F}, \mathcal{G}$ for Hilbert space $\mathcal{H}$, the distance between $\mathcal{F}, \mathcal{G}$ is

$$d^2(\mathcal{F}, \mathcal{G}) = \int_{\Omega} \|\mathcal{F}(\omega) - \mathcal{G}(\omega)\|^2 \mathrm{d}\mu(\omega).$$

If we suppose that $\{e_i\}_{I\in\mathbb{I}}$ is an orthonormal basis for $\mathcal{H}$ and the representation of $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{F}(\omega)=\sum_{i\in\mathbb{I}}\varphi_i(\omega)e_i,\ \mathcal{G}(\omega)=\sum_{i\in\mathbb{I}}\phi_i(\omega)e_i$ respectively, then we can compute that

$$\begin{aligned}d^2(\mathcal{F},\mathcal{G})&=\int_\Omega\|\mathcal{F}(\omega)-\mathcal{G}(\omega)\|^2\mathrm{d}\mu(\omega).\\&=\int_\Omega\|\sum_{i\in\mathbb{I}}(\varphi_i(\omega)-\phi_i(\omega))e_i\|^2\mathrm{d}\mu(\omega).\\&=\int_\Omega\sum_{i\in\mathbb{I}}|\varphi_i(\omega)-\phi_i(\omega)|^2\mathrm{d}\mu(\omega).\\&=\sum_{i\in\mathbb{I}}\|\varphi_i-\phi_i\|_2^2.\end{aligned}$$

In the case of the finite dimension Hilbert space, we shall have established the following perturbation theorem.

Theorem 3.11. *Given the continuous frame $\mathcal{F}$ gives injectivity, if for all $\varepsilon\geq 0$,*

$$d(\mathcal{F},\mathcal{G})<\varepsilon.$$

then the continuous frame $\mathcal{G}$ also gives injectivity.

Proof. Let the representation of $\mathcal{F}$ be $\mathcal{F}(\omega)=\sum_{i=1}^n\varphi_ie_i,\omega\in\Omega$. and another continuous frame $\mathcal{G}(\omega)=\sum_{i=1}^\infty\phi_i(\omega)e_i$ such that $d(\mathcal{F},\mathcal{G})<\varepsilon$. then we claim that $\mathcal{G}$ is injective.

We prove the result by way of contradiction. If not, by Theorem 2.9, this is equivalent to that $\{\varphi_i\overline{\varphi_i}\}$, and $\{\mathrm{Re}(\varphi_i\overline{\varphi_j}),\mathrm{Im}(\varphi_i\overline{\varphi_j})\}_{1\leq i<j\leq n}$ are linearly independent, then there exist not all zero scalar $c_{i,j}$, where $c_{i,i}\in\mathbb{R}$, and $c_{i,j}=\overline{c_{j,i}}$ (without loss of generality we can assume $\max|c_{i,j}|\leq 1$ for $1\leq i,j\leq n$) such that

$$\sum_{1\leq i,j\leq n}c_{i,j}\phi_i(\omega)\overline{\phi_j(\omega)}=0\quad a.e\ \omega\in\Omega$$

However we can compute

$$\begin{aligned}&\int_\Omega|\sum_{1\leq i,j\leq n}c_{i,j}\varphi_i\overline{\varphi_j}|d\mu(\omega)\\&=\int_\Omega|\sum_{1\leq i,j\leq n}c_{i,j}(\varphi_i(\omega)\overline{\varphi_j(\omega)}-\phi_i(\omega)\overline{\phi_j(\omega)})|d\mu(\omega)\\&\leq\sum_{1\leq i,j\leq n}|c_{i,j}|\int_\Omega|\varphi_i(\omega)\overline{\varphi_j(\omega)}-\phi_i(\omega)\overline{\phi_j(\omega)}|d\mu(\omega)\\&=\sum_{1\leq i,j\leq n}|c_{i,j}|\int_\Omega|\varphi_i(\omega)\overline{\varphi_j(\omega)}-\varphi_i(\omega)\overline{\phi_j(\omega)}+\varphi_i(\omega)\overline{\phi_j(\omega)}-\phi_i(\omega)\overline{\phi_j(\omega)}|d\mu(\omega)\\&=\sum_{1\leq i,j\leq n}|c_{i,j}|\left(\int_\Omega|\varphi_i(\omega)||\varphi_j(\omega)-\phi_j(\omega)|d\mu(\omega)+\int_\Omega|\varphi_i(\omega)-\phi_i(\omega)||\phi_j(\omega)|d\mu(\omega)\right)\\&<\sum_{1\leq i,j\leq n}|c_{i,j}|\left(\|\varphi_i\|_2\|\varphi_j-\phi_j\|_2+\|\phi_j\|_2\|\varphi_i-\phi_i\|_2\right)\end{aligned}$$

By the definition of continuous frame, φ_i are Riesz basis for $[\varphi_i]_{1\le i\le n}$, thus we get a constant C_1 such that $\|\varphi_i\| \le C_1 \ \forall\ 1 \le i \le n$. Same for ϕ_j, we have $\|\phi_j\| \le C_2 \ \forall\ 1 \le j \le n$. Thus we conclude that

$$\| \sum_{1\le i,j\le n} c_{i,j}\varphi_i\overline{\varphi_j}\|_1 \le 2C\varepsilon.$$

where $C = \max\{C_1, C_2\}$.

Meanwhile $\mathcal{F}$ gives injectivity, so $\sum_{1\le i,j\le n} c_{i,j}\varphi_i\overline{\varphi_j} \ne 0$, thus its L_1-norm can not small enough. However we can take ε sufficiently small. It is a contradiction. Thus we have proved the claim. □

However in infinite dimension, the situation becomes quite different. The following example shows that the property that the continuous frame gives injectivity will not be preserved after a small perturbation.

Example 3.12. Let $\mathcal{F}(t) = \sum_{i=1}^{\infty} \varphi_i(t)e_i$ be the injective frame for Hilbert space $\mathcal{H}$ as in example 3.9, Then for any $\varepsilon > 0$, there is a frame $\mathcal{G}$ such that $d(\mathcal{F},\mathcal{G}) < \varepsilon$, but $\mathcal{G}$ is not injective.

Proof. Let the representation of $\mathcal{G}$ be the $\mathcal{G}(t) = \sum_{n=1}^{\infty} \phi_n(t)e_n$. Let any $\varepsilon > 0$. since the series $\sum_{n=1}^{\infty} \dfrac{n}{2^n}$ converges, $\forall \varepsilon > 0$ there exists integer N such that

$$\sum_{n=N}^{\infty} \frac{n}{2^n} \le \varepsilon^2$$

we can set

$$\phi_n(t) = \begin{cases} \varphi_n(t), & n \le N; \\ \\ \chi(t-n), & n > N. \end{cases}$$

it is easy to check that $\mathcal{G}(t) = \sum_{n=1}^{\infty} \phi_n(t)e_i$ is continuous frame for $\mathcal{H}$ and

$$d(\mathcal{F},\mathcal{G}) = \left(\sum_{n=1}^{\infty} \|\varphi_n - \phi_n\|\right)^{1/2} = \left(\sum_{n=N}^{\infty} \|\varphi_n - \chi(\cdot - n)\|\right)^{1/2} = \left(\sum_{n=N}^{\infty} \frac{n}{2^n}\right)^{1/2} \le \varepsilon.$$

However $\mathcal{G}$ can not give injectivity since if we set

$$T = \sum_{i=N+1}^{\infty} c_i(e_i \otimes e_{i+1} + e_{i+1} \otimes e_i)$$

where real number sequence $\{c_i\}$ is in ℓ_2, which means for $x, y \in \mathcal{H}$

$$Tx = \sum_{i=N+1}^{\infty} c_i(\langle x, e_{i+1}\rangle e_i, + \langle x, e_i\rangle e_{i+1})$$

It is easily seen that T is self-adjoint Hilbert-Schmidt operator and $t \in [0, N)$ then $\mathcal{G}(t) = e_m$ or $\mathcal{G}(t) = e_m + e_n$ for some $m, n \le N$. Moreover

$$\langle e_m, e_i\rangle\langle e_{i+1}, e_n\rangle + \langle e_m, e_{i+1}\rangle\langle e_i, e_n\rangle = 0, \forall m, n \le N, i > N$$

hence

$$\langle T\mathcal{G}(t), \mathcal{G}t\rangle = 0$$

and $t \in [N, \infty)$ $\mathcal{G}(t) = e_j$ for some $j > N$, and

$$\langle T\mathcal{G}(t), \mathcal{G}t \rangle = \langle Te_j, e_j \rangle = \sum_{i=N+1}^{\infty} c_i \langle e_j, e_i \rangle \langle e_{i+1}, e_j \rangle + \langle e_j, e_{i+1} \rangle \langle e_i, e_j \rangle = 0.$$

Therefore

$$\langle T\mathcal{G}(t), \mathcal{G}t \rangle = 0 \quad \forall\, t \in \mathbb{R}^+.$$

which implies $\mathcal{G}$ is not injective. □

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