

SATELLITE RULING POLYNOMIALS, DGA REPRESENTATIONS, AND THE COLORED HOMFLY-PT POLYNOMIAL

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ABSTRACT. We establish relationships between two classes of invariants of Legendrian knots in \mathbb{R}^3 : Representation numbers of the Chekanov-Eliashberg DGA and satellite ruling polynomials. For positive permutation braids, $\beta \in J^1 S^1$, we give a precise formula in terms of representation numbers for the m -graded ruling polynomial $R_{S(K,\beta)}^m(z)$ of the satellite of K with β specialized at $z = q^{1/2} - q^{-1/2}$ with q a prime power, and we use this formula to prove that arbitrary m -graded satellite ruling polynomials, $R_{S(K,L)}^m$, are determined by the Chekanov-Eliashberg DGA of K . Conversely, for $m \neq 1$, we introduce an n -colored m -graded ruling polynomial, $R_{n,K}^m(q)$, in strict analogy with the n -colored HOMFLY-PT polynomial, and show that the total n -dimensional m -graded representation number of K to \mathbb{F}_q^n , $\text{Rep}_m(K, \mathbb{F}_q^n)$, is exactly equal to $R_{n,K}^m(q)$. In the case of 2-graded representations, we show that $R_{n,K}^2 = \text{Rep}_2(K, \mathbb{F}_q^n)$ arises as a specialization of the n -colored HOMFLY-PT polynomial.

1. INTRODUCTION

The study of Legendrian knots in the standard contact \mathbb{R}^3 benefits from interactions with both symplectic topology and classical knot theory. For instance, the symplectic theory of holomorphic curves applied to a Legendrian knot $K \subset \mathbb{R}^3$ produces an invariant known as the Chekanov-Eliashberg DGA (differential graded algebra), $(\mathcal{A}(K), \partial)$, that is capable of distinguishing knots with the same underlying framed knot type. On the other hand, a connection with topological knot theory is made via the m -graded ruling polynomial invariants, $R_K^m(z)$ (see [6]). As they are characterized by skein relations, the ruling polynomials can be regarded as Legendrian cousins of well known topological knot invariants. This is especially natural since, when $m = 1$ or 2 , $R_K^m(z)$ arises as a specialization of the 2-variable Kauffman or HOMFLY-PT polynomial (see [25, 24]).

Ruling polynomials are defined via a count of certain combinatorial objects called normal rulings, and there is a well studied correspondence between augmentations of $(\mathcal{A}(K), \partial)$, i.e. DGA maps into a field $(\mathbb{F}, 0)$, and normal rulings of K , cf. [10, 11, 14, 16, 20, 23, 26, 28]. In [21], this was strengthened to an equivalence between the existence of

- (a) finite dimensional representations of $(\mathcal{A}(K), \partial)$ over $\mathbb{Z}/2$, and
- (b) normal rulings of links obtained from K via the Legendrian satellite construction.

Both of these items may be refined to arrive at whole collections of Legendrian knot invariants. From (a), normalized counts of representations of (\mathcal{A}, ∂) on vector spaces over finite fields produce Legendrian invariant *representation numbers*; see Section 3. From (b), for any fixed $L \subset J^1 S^1$, considering the satellite link $S(K, L)$ provides an invariant of a Legendrian knot $K \subset \mathbb{R}^3$ via the assignment $K \mapsto R_{S(K,L)}^m(z)$. We refer to the latter invariants as *satellite ruling polynomials*. The results of the present article show that satellite ruling polynomials provide information about representation counts of the Chekanov-Eliashberg algebra and at the same time are determined by such counts. In addition, we strengthen the connection with topological knot invariants by showing that a natural specialization of the n -colored HOMFLY-PT polynomial recovers certain representation numbers of K .

We now give a more detailed overview of our results, under the simplifying assumption that the Legendrian knot $K \subset J^1 \mathbb{R}$ has rotation number $r(K) = 0$. In the following, $m \geq 0$ is a non-negative integer and \mathbb{F}_q is a finite field of order q . When m is odd, we assume $\text{char}(\mathbb{F}_q) = 2$. In Section 3, we introduce Legendrian isotopy invariant m -graded *representation numbers* denoted $\text{Rep}_m(K, (V, d), B)$ that count representations of (\mathcal{A}, ∂) on a differential graded vector space (V, d) over \mathbb{F}_q where certain distinguished generators, t_i , associated to basepoints on K have their images restricted by a choice of subset $B \subset GL(V)$. We also consider *reduced representation numbers*, $\text{Rep}_m(K, (V, d), B)$, that are renormalized so the unknot takes the value 1, and a *total n -dimensional representation number*,

$\text{Rep}_m(K, \mathbb{F}_q^n)$, that counts all m -graded representations to \mathbb{F}_q^n (with grading concentrated in degree 0) without restriction on the image of the t_i .

In Theorem 6.3, we provide a precise formula for a certain class of satellite ruling polynomials in terms of reduced representation numbers. A special case (see Proposition 4.13) is the following:

Theorem A. *Let $\beta \in J^1 S^1$ be an n -stranded positive permutation braid $\beta \in J^1 S^1$. Then,*

$$R_{S(K, \beta)}^m(z) \Big|_{z=q^{1/2}-q^{-1/2}} = q^{-\lambda_m(\beta)/2} (q^{1/2} - q^{-1/2})^{-n} \sum_d \widetilde{\text{Rep}}_m(K, (V_\beta, d), B_\beta^m),$$

where V_β is a graded vector space over \mathbb{F}_q associated to β (in Section 6); $B_\beta \subset GL(V_\beta)$ is the path subset of β (defined in Section 4); $\lambda_m(\beta)$ is a certain signed count of the crossings of β (defined in Section 6); and the summation is over all strictly upper triangular differentials, $d : V_\beta \rightarrow V_\beta$, with $\deg(d) = +1$.

Combining Theorem 6.3 with a skein relation argument, we establish in Corollary 6.8 that for any fixed $L \subset J^1 S^1$ the satellite ruling polynomial $R_{S(K, L)}^m$ is determined by the Chekanov-Eliashberg DGA of K .

We also obtain in Theorem 7.4 a converse-type formula for representation numbers in terms of ruling polynomials that we state here as:

Theorem B. *Assume $m \neq 1$. The total n -dimensional m -graded representation number satisfies*

$$\text{Rep}_m(K, \mathbb{F}_q^n) = R_{n, K}^m(q),$$

where $R_{n, K}^m(q)$ denotes the n -colored m -graded ruling polynomial (defined in Section 7).

Here, the n -colored ruling polynomial is a specific linear combination of satellite ruling polynomials that is so named because of a strong formal relation with, $P_{n, K}(a, q)$, the n -colored HOMFLY-PT polynomial: The linear combination that defines $R_{n, K}^m$ and obtains the total n -dimensional representation number is *exactly* the same linear combination of braids that defines the n -colored HOMFLY-PT polynomial!

The case $m = 2$ is of special interest since the results of [24] show the 2-graded ruling polynomial is a specialization of the HOMFLY-PT polynomial. In Theorem 7.10, we extend this result to see that $R_{n, K}^2(q) = \text{Rep}_2(K, \mathbb{F}_q^n)$ arises as a specialization of $P_{n, K}(a, q)$,

$$P_{n, K}(a, q) \Big|_{a^{-1}=0} = \text{Rep}_2(K, \mathbb{F}_q^n).$$

In the process, we generalize the HOMFLY-PT estimate for $tb(K) + |r(K)|$ from [12] to an estimate

$$(1.1) \quad tb(K) + |r(K)| \leq \frac{1}{n} \deg_a \widehat{P}_{n, K}(a, q),$$

where $\widehat{P}_{n, K}$ is the framing independent version of $P_{n, K}$. Moreover, in Corollary 7.11 we observe additional results parallel to [24]; for example, the inequality (1.1) is sharp as an estimate for $tb(K)$ if and only if K has an n -dimensional 2-graded representation.

The colored HOMFLY-PT polynomials (where in general the color is by an arbitrary partition) compute the quantum $U_q(\mathfrak{sl}_n)$ -invariants of a knot associated to arbitrary irreducible representations of \mathfrak{sl}_n , cf. [2, 1]. This relationship between representations of the Chekanov-Eliashberg DGA and quantum invariants looks tantalizing, and it would be interesting to find a more direct connection between the representation theories of (\mathcal{A}, ∂) and $U_q(\mathfrak{sl}_n)$.

1.1. Organization. The remainder of the article is organized as follows. In Section 2 we collect relevant background material concerning the Chekanov-Eliashberg DGA and ruling polynomials. In particular, the theorem of Henry and the second author from [14] that relates augmentation numbers with ordinary ruling polynomials is fundamental to our approach. Section 3 presents definitions of representation numbers and establishes their invariance under Legendrian isotopy. In Section 4, we obtain several results about the path matrices, originally introduced by Kálmán [15], of positive braid Legendrians $\beta \in J^1 S^1$. The path matrix of β appears prominently in the calculation of the Chekanov-Eliashberg DGA of the satellite $S(K, \beta)$ that is given in Section 5 from the point of view of the Lagrangian (xy) -projection. We note that the exposition in the current article is independent of that from [21] where computations of $S(K, \beta)$ were made using the front (xz) -projection. Overall,

the Lagrangian approach is more efficient for producing precise bijections between augmentations of $S(K, \beta)$ and higher dimensional representations of K . In Theorem 6.1 we obtain such a bijection, and the remainder of Section 6 establishes Theorem 6.3 and Corollary 6.8. In the concluding Section 7 we introduce the n -colored ruling polynomials and then prove the formula from Theorem 7.4. The connection with the colored HOMFLY-PT polynomial, in the case $m = 2$, is then established in Theorem 7.10 and Corollary 7.11.

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2. BACKGROUND

We work with Legendrian links in the 1-jet spaces, $J^1\mathbb{R}^1 \cong \mathbb{R}^3$ and $J^1S^1 \cong S^1 \times \mathbb{R}^2$ coordinatized as (x, y, z) , with their standard contact structures given by the contact form $dz - ydx$. We assume familiarity with some standard notions including the **front** (xz)-**projection** and the **Lagrangian** (xy)-**projection**; in J^1S^1 these projections are to $S^1 \times \mathbb{R}$ which we illustrate as $[0, 1] \times \mathbb{R}$ with left and right boundaries identified. When K is connected we denote the **rotation number** of K by $r(K)$, and we adopt the convention that when K has multiple components, $K = \sqcup_{i=1}^c K_i$, $r(K)$ denotes the greatest common divisor of $r(K_i)$. For $m \geq 0$ a non-negative integer, \mathbb{Z}/m -valued **Maslov potential**, μ , for a Legendrian K in $J^1\mathbb{R}^1$ or J^1S^1 is an \mathbb{Z}/m -valued function on K that is locally constant except at cusp points where the value at the upper branch of the cusp is 1 larger than at the lower branch. A connected Legendrian K has a \mathbb{Z}/m -valued Maslov potential iff $m \mid 2r(K)$. When m is even, we always assume that Maslov potentials are chosen to be compatible with the orientation of K so that μ takes even (resp. odd) values on strands that are oriented in the increasing (resp. decreasing) x -direction. For further background on Legendrian knots in \mathbb{R}^3 and J^1S^1 see [8, 22].

2.1. Chekanov-Eliashberg DGA. In this article, **differential graded algebras**, abbrev. DGA, are unital; graded by \mathbb{Z}/m for some $m \geq 0$; and have differentials of degree -1 satisfying the graded signed Leibniz rule, $\partial(xy) = (\partial x)y + (-1)^{|x|}x(\partial y)$, where if m is odd we assume the coefficient ring has characteristic 2.

Let $K \subset J^1\mathbb{R}$ or J^1S^1 be a Legendrian link equipped with some collection of basepoints (allowing for multiple basepoints on each component of K). We will use the *fully non-commutative* version of the **Chekanov-Eliashberg DGA** (also called the Legendrian contact homology DGA), $(\mathcal{A}(K), \partial)$, as defined via the xy -projection of K . To establish conventions, we briefly review $(\mathcal{A}(K), \partial)$ here. More details can be found in [6, 9]; see also [14] for an exposition matching our sign conventions (which follow [19]), and [21] for the case of the fully non-commutative DGA with multiple basepoints.

Using \mathbb{Z} -coefficients, $\mathcal{A}(K)$ is a unital associative algebra with non-invertible generators, a_1, \dots, a_r , in bijection with Reeb chords of K , i.e. the double points of $\pi_{xy}(K)$, and invertible generators $t_1^{\pm 1}, \dots, t_\ell^{\pm 1}$ in bijection with basepoints of K . There are no relations other than $t_i t_i^{-1} = t_i^{-1} t_i = 1$. A choice of $2r(K)$ -valued Maslov potential, μ , for K leads to a $\mathbb{Z}/2r(K)$ -grading, $\mathcal{A}(K) = \bigoplus_{k \in \mathbb{Z}/2r(K)} \mathcal{A}_k$, where $\mathcal{A}_k = \{a \in \mathcal{A}(K) : |a| = k \pmod{2r(K)}\}$. In particular, if all components of K have $r(K_i) = 0$, then $\mathcal{A}(K)$ is \mathbb{Z} -graded. Degrees of generators (mod $2r(K)$) are given by $|t_i| = 0$ and $|a_i| = \mu(U_i) - \mu(L_i) + \text{ind}(a_i) - 1$ where U_i and L_i are the upper and lower strands (with respect to the z -coordinate) of K that cross in $\pi_{xy}(K)$ and $\text{ind}(a_i)$ is the Morse index of the critical point of $z|_{U_i}(x) - z|_{L_i}(x)$ at a_i .

The differential $\partial : \mathcal{A}(K) \rightarrow \mathcal{A}(K)$ satisfies the signed Leibniz rule and has $\partial t_i = 0$. For Reeb chords a and b_1, \dots, b_n , let $\mathcal{M}(a; b_1, \dots, b_n)$ denote the set modulo reparametrization of boundary punctured holomorphic (or equivalently orientation preserving, immersed) disks in $\mathbb{R}^2 = \mathbb{C}$ having boundary on $\pi_{xy}(K)$ and having a (convex) corner mapped to a positive quadrant at a and (convex) corners mapped to negative quadrants at b_1, \dots, b_n appearing in counter-clockwise order. Here, we use the Reeb signs

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FIGURE 1. The Reeb signs (left) and orientation signs (right) of quadrants at a Reeb chord. The shaded quadrants have orientation sign -1 .

for quadrants as in Figure 1 (left). We have

$$\partial a = \sum_{n \geq 0} \sum_{b_1, \dots, b_n} \sum_{[u] \in \mathcal{M}(a; b_1, \dots, b_n)} \iota(u) \cdot w(u)$$

where $w(u) = r_1 r_2 \cdots r_N$ is the product of basepoints and negative corners that appear along u ordered as they appear (counter-clockwise) along the boundary of u starting at the positive corner at a . In this product, a t_i generator corresponding to a basepoint $*_i$ appear with exponent $+1$ (resp. -1) when the orientations of ∂u and K agree (resp. disagree) at $*_i$. The coefficient $\iota(u) \in \{\pm 1\}$ is given by the product

$$\iota(u) = \iota' \iota_0 \iota_1 \cdots \iota_n.$$

Here, ι' is $+1$ (resp. -1) when the orientation of the initial arc of ∂u that is just counter-clockwise from the positive corner of u agrees (resp. disagrees) with the orientation of K , and ι_0, \dots, ι_n are the orientation signs of the quadrants covered by the various (positive and negative) corners of u . As depicted in Figure 1 (right), at each (xy) -crossing the two quadrants to the right (resp. to the left) of the understrand (with respect to its orientation) have -1 (resp. $+1$) orientation sign.

2.1.1. \mathbb{Z} -gradings on $\mathcal{A}(K)$ with multiple basepoints. As long as every component of K has at least one basepoint, it is possible to define a \mathbb{Z} -grading on $\mathcal{A}(K)$. (The case of a single basepoint per component is essentially as in [9].) First, choose degrees $|t_j| \in 2\mathbb{Z}$ for the generators associated to the basepoints, $*_1, \dots, *_\ell$ of K , subject only to the restriction that the sum of the degree of all basepoints on a given component $K_i \subset K$ is equal to $-2r(K_i)$. Next, choose a \mathbb{Z} -valued Maslov potential, μ , for $K \setminus \{*_1, \dots, *_\ell\}$, such that when $*_i$ is passed in the direction of the orientation of K_i the value of μ decreases by $|t_i|$. Since μ is \mathbb{Z} -valued, following the definition above provides integer degrees, $|a_i| \in \mathbb{Z}$, for the a_i .

Remark 2.1. The requirement that $|t_i|$ is even is not strictly necessary. However, if some of the t_i have odd degree (we will not actually use this case later in the article), we can no longer assume that the parity of μ is compatible with the orientation of K , i.e. even (resp. odd) at points where K is oriented in the increasing (resp. decreasing) x -direction. When this occurs, we modify the definition of ∂ by, for each a_i , multiplying all terms of ∂a_i by an extra sign ι'' that is $+1$ (resp. -1) if the parity of μ is (resp. is not) compatible with the orientation of K at the overstrand of a_i .

Proposition 2.2. (1) For any choice of μ , $(\mathcal{A}(K), \partial)$ is a \mathbb{Z} -graded DGA, i.e. ∂ has degree -1 , and satisfies $\partial^2 = 0$.
 (2) If (K, μ) and (K', μ') are Legendrian isotopic (as basepointed links, preserving Maslov potentials), then $(\mathcal{A}(K), \partial_1)$ and $(\mathcal{A}(K'), \partial_2)$ are stable tame isomorphic.¹
 (3) Let $(\mathcal{A}_1, \partial_1)$ and $(\mathcal{A}_2, \partial_2)$ be formed using two collections of basepoints, B_1 and B_2 , for a common Legendrian K . Suppose that B_1 and B_2 are identical except for a single component $K_i \subset K$ where B_1 only has one basepoint, $*$, and B_2 has a sequence of basepoints, $*_1, \dots, *_n$, all appearing in a small neighborhood of $*$ and ordered according to the orientation of K . Let $t \in \mathcal{A}_1$ and $t_1, \dots, t_n \in \mathcal{A}_2$ denote the generators corresponding to $*$ and $*_1, \dots, *_n$. Then, $(\mathcal{A}_2, \partial_2)$ is the free product with amalgamation

$$\mathcal{A}_2 = \mathcal{A} *_{t=t_1 \dots t_n} \mathbb{Z}\langle t_1^{\pm 1}, \dots, t_n^{\pm 1} \rangle$$

with ∂_2 induced by ∂_1 and the 0 differential on $\mathbb{Z}\langle t_1^{\pm 1}, \dots, t_n^{\pm 1} \rangle$.

¹See [9] for a complete definition of stable tame isomorphism which is a type of equivalence for DGAs equipped with explicit generating sets. For the fully non-commutative DGA, the definition of elementary isomorphism needs to be expanded to allow for isomorphisms that multiply a single a_i generator on the right or left by some $t_j^{\pm 1}$ to account for the effect of repositioning base points. As a result, when DGAs are considered up to tame isomorphism, the degree distribution of the set of a_i generators is only well defined mod the g.c.d. of all $|t_j|$.

Proof. This is closely related to Theorem 2.20 and 2.21 of [21] (which considers the $\mathbb{Z}/2$ -coefficient case with $\mathbb{Z}/2r(K)$ -grading).

Item (3) is immediately verified from definitions. [The algebra \mathcal{A}_2 is just \mathcal{A}_1 with $t^{\pm 1}$ removed from the generating set and replaced with $t_1^{\pm 1}, \dots, t_n^{\pm 1}$. The differentials are related as claimed since every time $*$ appears along the boundary of a disk used in computing ∂_1 , the entire sequence $*_1, \dots, *_n$ appears in its place when computing ∂_2 .]

For Items (1) and (2), the result is standard in the case where every component has a single basepoint, cf. [6, 9, 19]. Combining this result with item (3), it suffices to consider the case where $K = K'$ except for the basepoint sets, B_1 and B_2 , which are related by isotopy in K . Moreover, we can assume B_2 is obtained from B_1 by pushing a basepoint with generator t_j through a crossing a_l following the orientation of K . When t_j is pushed across the overstrand of a_l , one checks that the graded algebra isomorphism ϕ that fixes all generators other than a_l and maps $a_l \mapsto t_j^{-1}a_l$ is a chain map. (Here, if t_j has odd degree, the presence of the ι'' term that modifies the definition of ∂a_l becomes important when verifying that $\partial_2 \circ \phi(a_l) = (-1)^{|t_j|} t_j^{-1} \partial_2(a_l)$ agrees with $\phi \circ \partial_1(a_l) = \partial_1(a_l)$.) When t_j crosses past a_l along the understrand, ϕ is modified to have $a_l \mapsto a_l t_j$. \square

Remark 2.3. (1) Categorically, (3) is the statement that $(\mathcal{A}_2, \partial_2)$ is the pushout (in the category of DGAs over \mathbb{Z}) of $(\mathcal{A}_1, \partial_1)$ and $\mathbb{Z}\langle t_1^{\pm 1}, \dots, t_l^{\pm 1} \rangle$ with respect to the inclusion of the subalgebra $\mathbb{Z}\langle t^{\pm 1} \rangle \hookrightarrow \mathcal{A}_1$ and the homomorphism $\mathbb{Z}\langle t^{\pm 1} \rangle \rightarrow \mathbb{Z}\langle t_1^{\pm 1}, \dots, t_l^{\pm 1} \rangle$, $t \mapsto t_1 \cdots t_l$.

(2) When K has only one component with a single basepoint, $(\mathcal{A}(K), \partial)$ is independent of the choice of μ . When K has one component, but multiple basepoints, $(\mathcal{A}(K), \partial)$, is uniquely determined by the choice of degrees for the t_i . For multi-component links, different choices of μ can lead to different gradings on $(\mathcal{A}(K), \partial)$.

2.2. Ruling polynomials. We also briefly recall the ruling polynomial invariants of a Legendrian link K in $J^1\mathbb{R}$ or J^1S^1 equipped with a \mathbb{Z}/m -valued Maslov potential, μ , where $m \mid 2r(K)$. The m -graded ruling polynomial of K is

$$R_K^m(z) = \sum_{\rho} z^{j(\rho)},$$

where the sum is over all m -graded normal rulings of K and $j(\rho) = \#(\text{switches}) - \#(\text{right cusps})$. Here, an m -graded normal ruling of K is a combinatorial structure associated to the front projection of K . Normal rulings were introduced independently by Fuchs and Chekanov-Pushkar, cf. [10] and [5]; a detailed definition of normal rulings for links in $J^1\mathbb{R}$ and J^1S^1 may be found, for instance, in both [25] and [21]. The m -graded ruling polynomial satisfies the following skein relations:

(i)

$$\text{images/SR1.}\{\text{ps,eps,pdf}\} \text{ not found (or no BBox)} - \text{images/SR2.}\{\text{ps,eps,pdf}\} \text{ not found (or no BBox)} = z \left(\delta_1 \text{images/SR3.}\{\text{ps,eps,pdf}\} \text{ not found (or no BBox)} \right)$$

(ii)

$$\text{images/ZigZag1.}\{\text{ps,eps,pdf}\} \text{ not found (or no BBox)} = \text{images/ZigZag2.}\{\text{ps,eps,pdf}\} \text{ not found (or no BBox)} = 0$$

(iii)

$$\text{images/Unknot.}\{\text{ps,eps,pdf}\} \text{ not found (or no BBox)} \sqcup K = z^{-1}K$$

where in (i), δ_1 (resp. δ_2) is 1 if the the two strands that intersect in the first (resp. second) term on the left hand side have equal Maslov potentials, and is 0 otherwise; see [24, 25] for the cases $m = 1, 2$. For Legendrian links in $J^1\mathbb{R}$, the ruling polynomials are uniquely characterized by the skein relations (i)-(iii), together with the normalization $R_U^m(z) = z^{-1}$ on the standard ($tb = -1$) Legendrian unknot

$$U = \text{images/Unknot.}\{\text{ps,eps,pdf}\} \text{ not found (or no BBox)}.$$

The proof from Lemma 6.10 of [25] (which concerns the 2-graded case), applies with a trivial modification to show that using the relations (i)-(iii) any Legendrian link in J^1S^1 can be written as a

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FIGURE 2. The basic front, $A_m \subset J^1 S^1$, for $m \geq 1$ winds m times around S^1 with $m - 1$ crossings and no cusps.

$\mathbb{Z}[z^{\pm 1}]$ -linear combination of products of the basic fronts, A_m , as pictured in Figure 2. Here, the product of two Legendrians in $J^1 S^1$ is defined by stacking front diagrams vertically, and we allow the basic front factors to be equipped with \mathbb{Z}/m -valued Maslov potentials of arbitrary value.

Remark 2.4. (1) The skein relation (iii) follows from (i) and (ii) when $K \neq \emptyset$.
 (2) When K has only one component, $R_K^m(z)$ is independent of the choice of \mathbb{Z}/m -valued Maslov potential. This is not true for multi-component links.

3. REPRESENTATION NUMBERS

In this section, we define various Legendrian isotopy invariants from counts of representations of the Chekanov-Eliashberg DGA.

3.1. Representations of a DGA. Let (\mathcal{A}, ∂) be a \mathbb{Z} -graded DGA, $\mathcal{A} = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}_k$, and let (\mathcal{B}, δ) be a (unital) DGA graded by \mathbb{Z}/M , $\mathcal{B} = \bigoplus_{k \in \mathbb{Z}/M} \mathcal{B}_k$. With a divisor $m \mid M$ fixed, for any $k \in \mathbb{Z}$, we write

$$\mathcal{B}_k^m = \bigoplus_{\substack{l \in \mathbb{Z}/M \\ l = k \bmod m}} \mathcal{B}_l.$$

Definition 3.1. An m -graded representation from (\mathcal{A}, ∂) to (\mathcal{B}, δ) is a (unital) algebra homomorphism $f : (\mathcal{A}(K), \partial) \rightarrow (\mathcal{B}, \delta)$ that satisfies

$$f \circ \partial = \delta \circ f \quad \text{and} \quad f(\mathcal{A}_k) \subset \mathcal{B}_k^m, \quad \forall k \in \mathbb{Z}.$$

3.2. Defining m -graded representation numbers. Given a Legendrian link $K = \sqcup_{i=1}^c K_i \subset J^1 \mathbb{R}$, fix $m \geq 0$ with $m \mid 2r(K)$. If m is even, we impose the requirement that $r(K) = 0$ (for all components of K). Let μ be a Maslov potential for K that is \mathbb{Z} -valued if m is even and $\mathbb{Z}/2m$ -valued if m is odd. (For $\mathbb{Z}/2m$ -valued Maslov potentials, we maintain the requirement that μ is even (resp. odd) valued when K is oriented in the positive (resp. negative) x -direction.) We will define a class of Legendrian isotopy invariant m -graded representation numbers for (K, μ) that, in the case $c = 1$, are independent of μ .

Remark 3.2. When m is even, the assumption that $r(K) = 0$ is required for the proof of Legendrian invariance of m -graded representation numbers in Proposition 3.9. For 1-component Legendrian knots, in some cases of interest, no generality is lost by this assumption. Indeed, it will be shown in Corollary 6.2 that the existence of 2-graded representations of certain types implies that $r(K) = 0$.

We begin by making the following additional choices:

- (A1) Equip K with a collection of basepoints so that every component of K has at least 1 basepoint.
- (A2) As in Section 2.1.1, assign degrees, $|t_i|$, to generators associated to basepoints with the restriction that:

- (a) If m is even, then all t_i have $|t_i| = 0$.
- (b) If m is odd, then $2m \mid |t_i|$ for all basepoints.

Then, if m is odd, fix an additional \mathbb{Z} -valued Maslov potential, $\tilde{\mu}$, (discontinuous at basepoints with $|t_i| \geq 0$) that agrees with μ when reduced mod $2m$.

- (A3) For each component $K_i \subset K$, fix an initial basepoint, t_{e_i} , among the basepoints on K_i .

Let $(\mathcal{A}(K), \partial)$ be the resulting \mathbb{Z} -graded version of the Chekanov-Eliashberg DGA as in Section 2.1.1.

Next, given a \mathbb{Z}/M -graded DGA (\mathcal{B}, δ) with $m \mid M$, we choose a subset $T_i \subset (\mathcal{B}_0^m)^*$ (consisting of invertible elements in \mathcal{B}_0^m) for each component $K_i \subset K$, and we write $\mathbf{T} = \{T_i\}_{i=1}^c$. Denote by

$$\overline{\text{Rep}}_m(K, (\mathcal{B}, \delta), \mathbf{T})$$

the set of all m -graded representations from $(\mathcal{A}(K), \partial)$ to (\mathcal{B}, δ) with the property that

$$f(s_i) \in T_i, \quad \text{for all } 1 \leq i \leq c,$$

where s_i is the product of all invertible generators from basepoints of K_i ordered with t_{e_i} first and so that the factors appear in accordance with the orientation of K_i .

Definition 3.3. *The shifted Euler characteristic of K centered at $k \in \mathbb{Z}$ is*

$$\chi^k = \sum_{l \geq 0} (-1)^l r_{k+l} + \sum_{l < 0} (-1)^{l+1} r_{k+l},$$

where r_l is the number of Reeb chords of Λ of degree l . Alternatively, if we define

$$\eta^k : \mathbb{Z} \rightarrow \mathbb{Z}, \quad \eta^k(l) = \begin{cases} (-1)^{l-k}, & l \geq k, \\ (-1)^{l-k+1}, & l < k, \end{cases}$$

then $\chi^k = \sum_i \eta^k(|a_i|)$ where the sum is over all Reeb chords of K .

Consider the following conditions on (\mathcal{B}, δ) :

(B1) \mathcal{B} has a finite number of elements.

(B2) For all $k \in \mathbb{Z}$, $|\mathcal{B}_k^m| = |\mathcal{B}_{-k}^m|$. (Here, and throughout the article, $|X|$ indicates the cardinality of a set X .)

Definition 3.4. *Suppose that (\mathcal{B}, δ) satisfies (B1)-(B2). We then define m -graded representation numbers of K as*

$$(3.1) \quad \text{Rep}_m(K, (\mathcal{B}, \delta), \mathbf{T}) = \left(\lim_{N \rightarrow +\infty} \prod_{k \in \mathbb{Z}, |k| \leq N} |\mathcal{B}_k^m|^{-(\chi^k/2)} \right) \cdot |(\mathcal{B}_0^m)^* \cap \ker \delta|^{-\ell} \cdot |\overline{\text{Rep}}_m(K, (\mathcal{B}, \delta), \mathbf{T})|,$$

where ℓ is the total number of basepoints on K .

We will show in Proposition 3.10 that the $\text{Rep}_m(K, (\mathcal{B}, \delta), \mathbf{T})$ are Legendrian isotopy invariants of (K, μ) and are independent of the other choices involved in their definition. In the proof, we show that stabilizing a DGA \mathcal{A} in degree k changes $|\overline{\text{Rep}}_m(K, (\mathcal{B}, \delta), \mathbf{T})|$ by a factor of $|\mathcal{B}_k^m|$, thus leaving (3.1) unchanged. The following lemma shows the limit in (3.1) exists.

Lemma 3.5. *The sequence $(b_N)_{N=1}^\infty$ with $b_N = \prod_{k \in \mathbb{Z}, |k| \leq N} |\mathcal{B}_k^m|^{-(\chi^k/2)}$ is eventually constant. In particular, the limit exists in (3.1).*

Proof. Note that when $|l| < k$, we have $\eta^k(l) = (-1)^{l-k+1} = -(-1)^{l+k} = -\eta^{-k}(l)$. It follows that $\chi^{-k} = -\chi^k$ holds once $k > |\deg(a_i)|$ for all i . Using (B2), we see that once k is large enough all $|\mathcal{B}_k^m|^{-\chi^k/2}$ and $|\mathcal{B}_{-k}^m|^{-\chi^{-k}/2}$ terms cancel in the product. Thus, b_N is eventually constant. \square

Remark 3.6. (1) The assumption (B2) can be removed if $m = 0$, as the finiteness of \mathcal{B} implies $|\mathcal{B}_k^0| = |\mathcal{B}_k| = 1$ for $|k| \gg 0$.

(2) If $m > 0$ is odd, then (B2) can be removed if the limit in (3.1) is replaced with

$$\lim_{N \rightarrow \infty} \left(\prod_{k \in \mathbb{Z}, |k| \leq 2mN} |\mathcal{B}_k^m|^{-(\chi^k/2)} \right).$$

Definition 3.7. *It is also convenient to introduce reduced representation numbers defined by*

$$(3.2) \quad \widetilde{\text{Rep}}_m(K, (\mathcal{B}, \delta), \mathbf{T}) = |\mathcal{B}_0^m|^{-1/2} |(\mathcal{B}_0^m)^* \cap \ker \delta| \cdot \text{Rep}_m(K, (\mathcal{B}, \delta), \mathbf{T}).$$

The following example shows that, with this normalization of the representation numbers and some assumptions about (\mathcal{B}, δ) , the standard Legendrian unknot has reduced representation number 1.

Example 3.8. Let (\mathcal{B}, δ) be a DGA satisfying (B1) and (B2). Assume further that the subset $T_1 \subset (\mathcal{B}_0^m)^*$ contains all elements of the form $-1 + \delta x$ with $x \in \mathcal{B}_m^1$; in particular, we require that all such elements are invertible. Under these assumptions, we compute the representation number for the standard Legendrian unknot, U .

The DGA of U has a single Reeb chord generator, b , with $|b| = 1$, as well as an invertible generator $t^{\pm 1}$ for the base point on U . The differential is

$$\partial b = t + 1, \quad \partial t = 0.$$

An m -graded representation $f : (\mathcal{A}(U), \partial) \rightarrow (\mathcal{B}, \delta)$ is uniquely determined by $f(b) \in \mathcal{B}_1^m$ since the representation equation $f \circ \partial(b) = \delta \circ f(b)$ implies

$$(3.3) \quad f(t) = -1 + \delta \circ f(b).$$

Moreover, $f(t)$ as defined by (3.3) is invertible and belongs to T_1 , so $f(b) \in \mathcal{B}_1^m$ may be chosen arbitrarily. Thus, $\overline{\text{Rep}}_m(U, (\mathcal{B}, \delta), \mathbf{T})$ is in bijection with B_1^m . Using (B2) and that U has $\chi^k = -\chi^{-k}$ for $k \geq 2$, we compute

$$\begin{aligned} \text{Rep}_m(U, (\mathcal{B}, \delta), \mathbf{T}) &= (|\mathcal{B}_{-1}^m|^{-1/2} |\mathcal{B}_0^m|^{1/2} |\mathcal{B}_1^m|^{-1/2}) \cdot |(\mathcal{B}_0^m)^* \cap \ker \delta|^{-1} \cdot |\mathcal{B}_1^m| \\ &= |\mathcal{B}_0^m|^{1/2} \cdot |(\mathcal{B}_0^m)^* \cap \ker \delta|^{-1}. \end{aligned}$$

In addition, the reduced representation number satisfies $\widetilde{\text{Rep}}_m(U, (\mathcal{B}, \delta), \mathbf{T}) = 1$.

3.3. Legendrian invariance. In establishing invariance when m is odd, it will be useful to have that the limit used in (3.1) depends only on the mod $2m$ degree of Reeb chords. (This is false when m is even.)

To give a precise statement, when m is odd let

$$(3.4) \quad \sigma_m = \sum_{l=0}^{m-1} (-1)^l |\{a_i \mid \deg(a_i) = l \bmod m\}|$$

and for $1 \leq r < m$ set

$$\nu_m^r = \sum_{l=0}^{m-1} (-1)^l |\{a_i \mid \deg(a_i) = r + l \bmod 2m\}|.$$

Lemma 3.9. *When m is odd and (b_N) is as in Lemma 3.5,*

$$\lim_{N \rightarrow +\infty} b_N = |\mathcal{B}_0^m|^{-\sigma_m/2} \cdot \prod_{r=1}^{m-1} |\mathcal{B}_r^m|^{-\nu_m^r}$$

Proof. Write $b_{2mN} = \beta_N^0 \beta_N^1 \cdots \beta_N^{m-1}$ where

$$\beta_N^r = \prod_{\substack{|k| \leq 2mN, \\ k \equiv r \pmod{m}}} |\mathcal{B}_k^m|^{-\chi^k/2}$$

Then, by definition of B_k^m , $\beta_N^r = |B_r^m|^{-c_N^r/2}$ where

$$c_N^r = \sum_{\substack{|k| \leq 2mN, \\ k \equiv r \pmod{m}}} \chi^k,$$

and to establish the lemma we show that $\lim_{N \rightarrow +\infty} c_N^0 = \sigma_m$ and $\lim_{N \rightarrow +\infty} c_N^r = 2\nu_m^r$ for $1 \leq r \leq m-1$.

Let $\mathbb{Z}^{-\infty, \infty}$ denote the (free) \mathbb{Z} -module whose elements are bi-infinite sequences of integers, $(s_n)_{n \in \mathbb{Z}}$, with only finitely many non-zero terms, and let $e_i = (\delta_{n,i})_{n \in \mathbb{Z}}$ denote its standard basis. The definition of χ^k extends to provide a homomorphism

$$X^k : \mathbb{Z}^{-\infty, \infty} \rightarrow \mathbb{Z}, \quad X^k((s_n)) = \sum_{l \geq 0} (-1)^l s_{k+l} + \sum_{l < 0} (-1)^{l+1} s_{k+l}.$$

Notice that

$$X^k(e_{i-1} + e_i) = \begin{cases} 2, & i = k \\ 0, & i \neq k. \end{cases}$$

Moreover, the sequence $C_N^r = \sum_{|k| \leq 2mN, k \equiv r \pmod{m}} X^k$ is eventually constant when applied to any $(s_n) \in \mathbb{Z}^{-\infty, \infty}$. [This is because once k is far enough outside an interval containing the support of (s_n) , $X^k((s_n)) = -X^{k+m}((s_n))$.] Therefore, the point-wise limit $C^r = \lim_{N \rightarrow \infty} C_N^r$ exists and is a homomorphism $\mathbb{Z}^{-\infty, \infty} \rightarrow \mathbb{Z}$. Note that

$$(3.5) \quad \chi^k = X^k((r_n)) \quad \text{and} \quad \lim_{N \rightarrow \infty} c_N^r = C^r((r_n)),$$

where r_n is the number of degree n Reeb chords of K . Similarly, there are homomorphisms

$$S_m : \mathbb{Z}^{-\infty, \infty} \rightarrow \mathbb{Z}, \quad S_m((s_n)) = \sum_{l=0}^{m-1} (-1)^l \left(\sum_{i=l \bmod m} s_i \right)$$

$$T_m^r : \mathbb{Z}^{-\infty, \infty} \rightarrow \mathbb{Z}, \quad T_m^r((s_n)) = \sum_{l=0}^{m-1} (-1)^l \left(\sum_{i=r+l \bmod 2m} s_i \right)$$

such that

$$(3.6) \quad \sigma_m = S_m((r_n)) \quad \text{and} \quad \nu_m^r = T_m^r((r_n)).$$

In view of (3.5) and (3.6), the proof is completed by verifying that $S_m = C^0$ and $2T_m^r = C^r$, $1 \leq r \leq m-1$. This is done by easily computing values on the basis $\{e_0\} \cup \{e_{i-1} + e_i \mid i \in \mathbb{Z}\}$ as

$$C^0(e_0) = S_m(e_0) = 1, \quad C^0(e_{i-1} + e_i) = S_m(e_{i-1} + e_i) = \begin{cases} 2, & i = 0 \bmod m \\ 0, & \text{else,} \end{cases}$$

$$C^r(e_0) = 2T_m^r(e_0) = 0, \quad C^r(e_{i-1} + e_i) = 2T_m^r(e_{i-1} + e_i) = \begin{cases} 2, & i = r \bmod m \\ 0, & \text{else.} \end{cases}$$

□

Recall that K is a Legendrian link with $m \mid 2r(K)$, $m \geq 0$, and $r(K) = 0$ if m is even; the Maslov potential μ for K is \mathbb{Z} -valued (resp. $\mathbb{Z}/2m$ -valued) if m is even (resp. odd).

Proposition 3.10. *For any (\mathcal{B}, δ) and \mathbf{T} as above, the representation number $\text{Rep}_m(K, (\mathcal{B}, \delta), \mathbf{T})$ is a Legendrian isotopy invariant of (K, μ) . If K has only one component, then $\text{Rep}_m(K, (\mathcal{B}, \delta), \mathbf{T})$ is independent of μ .*

Proof. Suppose that (K, μ) is equipped with basepoints and, in the case m is odd, an additional \mathbb{Z} -valued Maslov potential as specified by the choices (A1)-(A3). When K and K' are related by a (basepoint and Maslov potential preserving) Legendrian isotopy, then, by Proposition 2.2 (2), their \mathbb{Z} -graded DGAs, $(\mathcal{A}(K), \partial)$ and $(\mathcal{A}(K'), \partial')$, are stable tame isomorphic. That is, after performing some number of algebraic stabilizations (where two new generators of degrees $|a| = |b| + 1 = k$ with differentials $\partial a = b$ and $\partial b = 0$ are added) on $\mathcal{A}(K)$ and $\mathcal{A}(K')$ we obtain DGAs $S\mathcal{A}(K)$ and $S\mathcal{A}(K')$ that are isomorphic via a tame isomorphism, ϕ . In particular, ϕ fixes the invertible generators t_i corresponding to basepoints, so that the pullback ϕ^* gives a bijection between the two relevant sets of representations of $S\mathcal{A}(K)$ and $S\mathcal{A}(K')$. When m is even, as all t_i have degree 0, the tameness of ϕ implies that the degree distribution of generators, and hence also all χ^k , agree for $S\mathcal{A}(K)$ and $S\mathcal{A}(K')$. It follows that representation numbers as in (3.1) computed from $S\mathcal{A}(K)$ and $S\mathcal{A}(K')$ are equal. When m is odd, all t_i have their degrees divisible by $2m$, therefore a tame isomorphism between $S\mathcal{A}(K)$ and $S\mathcal{A}(K')$ preserves the mod $2m$ degree distribution of generators. We can then apply Lemma 3.9 to conclude the representation numbers are equal.

Next, we check that $\mathcal{A}(K)$ and $S\mathcal{A}(K)$ produce the same representation numbers. Stabilizing a DGA \mathcal{A} in degree k changes $|\overline{\text{Rep}}_m(K, (\mathcal{B}, \delta), \mathbf{T})|$ by a factor of $|\mathcal{B}_k^m|$, as the extensions of a given m -graded representation of \mathcal{A} to the stabilization arise from choosing $f(a) \in \mathcal{B}_k^m$ arbitrarily and putting $f(b) = \delta f(a)$. Since stabilizing in degree k increases χ^k by 2 and leaves all χ^l with $l \neq k$ unchanged, the product in (3.1) remains invariant.

We next check independence of the choices (A1)-(A3). For (A2), note that no additional choice is made when m is even; when m is odd, Lemma 3.9 shows that the augmentation numbers only depend on the degree distribution of generators mod $2m$, and this is uniquely determined by the original $\mathbb{Z}/2m$ -valued Maslov potential, μ . For (A3), independence of the choice of initial basepoints, t_{e_i} , follows since the location of basepoints can be cyclically permuted by a Legendrian isotopy of K .

For (A1), we check that the representation numbers are independent of the number of basepoints used. The algebra (\mathcal{A}, ∂) with a single basepoint t on a component $K_i \subset K$ is related to an algebra (\mathcal{A}', ∂) with multiple (cyclically ordered) basepoints t_1, \dots, t_l on K_i as in Theorem 2.2 (3). As a result, m -graded representations $f : (\mathcal{A}', \partial) \rightarrow (\mathcal{B}, \delta)$ are in bijection with pairs (g, h) of m -graded representations where $g : (\mathcal{A}, \partial) \rightarrow (\mathcal{B}, \delta)$, $h : (\mathbb{Z}\langle t_1^{\pm 1}, \dots, t_l^{\pm 1} \rangle, 0) \rightarrow (\mathcal{B}, \delta)$, and $g(t) = h(t_1 \cdots t_l)$. All

of the t_i are cycles of degree 0 mod m , so for a given g , $g(t) \in (\mathcal{B}_0^m)^* \cap \ker \delta$ and there are always $|(\mathcal{B}_0^m)^* \cap \ker \delta|^{l-1}$ ways to choose h . [Put $h(t_1) = g(t)(h(t_2 \cdots t_l))^{-1}$, and choose $h(t_i)$ with $2 \leq i \leq l$ arbitrarily.] This increase in the number of representations is counteracted by the $|(\mathcal{B}_0^m)^* \cap \ker \delta|^{-\ell}$ factor in (3.1).

The final statement about independence of μ when K is connected follows since, in the connected case, any other Maslov potential for K is related to μ by the addition of an overall constant. Thus, the degree distribution of generators and the resulting grading on $\mathcal{A}(K)$ are independent of μ . \square

3.4. Augmentation numbers. Representations into a target of the form $(\mathcal{B}, \delta) = (\mathbb{F}, 0)$ with \mathbb{F} a field located in graded degree 0 are also called **augmentations** of (\mathcal{A}, ∂) ; the set of all m -graded augmentations to \mathbb{F} is denoted $\overline{\text{Aug}}_m(K, \mathbb{F})$. When $\mathbb{F} = \mathbb{F}_q$ is the finite field of order q and all $T_i = \mathbb{F}_q^*$ the representation numbers defined above are called **augmentation numbers**, denoted $\text{Aug}_m(K, q)$, and satisfy

$$\text{Aug}_m(K, q) = q^{-\sigma_m/2} (q-1)^{-\ell} \cdot |\overline{\text{Aug}}_m(K, \mathbb{F}_q)|$$

with ℓ the total number of basepoints on K and

$$(3.7) \quad \sigma_m = \lim_{N \rightarrow \infty} \sum_{\substack{|k| \leq N \\ k \equiv 0 \pmod{m}}} \chi^k.$$

Note that in terms of the degree distribution of Reeb chords, $(r_n)_{n \in \mathbb{Z}}$,

$$(3.8) \quad \text{for } m = 0, \quad \sigma_0 = \chi^0 = \sum_{l \geq 0} (-1)^l r_l + \sum_{l < 0} (-1)^{l+1} r_l,$$

$$(3.9) \quad \text{for even } m > 0, \quad \sigma_m = \sum_{k \in \mathbb{Z}} (2k+1) s_k, \quad \text{where } s_k = \sum_{l=0}^{m-1} (-1)^l r_{mk+l},$$

$$(3.10) \quad \text{for odd } m, \quad \sigma_m = \sum_{l=0}^{m-1} (-1)^l \left(\sum_{i \equiv l \pmod{m}} r_i \right).$$

The latter two formulas are verified as in the proof of Lemma 3.9; σ_m can be viewed as the unique homomorphism $\mathbb{Z}^{-\infty, \infty} \rightarrow \mathbb{Z}$ that satisfies $\sigma_m(e_0) = 1$ and, for all $i \in \mathbb{Z}$, $\sigma_m(e_{i-1} + e_i) = \begin{cases} 2, & i \equiv 0 \pmod{m}, \\ 0, & \text{else.} \end{cases}$

Remark 3.11. From the calculation of σ_m in (3.8)-(3.10), one sees that the normalization for $\text{Aug}_m(K, q)$ used above agrees with that used in [20] and, in the case $q = 2$ with $m = 0$ or m odd, in [23]. The normalization used in [14] differs slightly.

It was proven in [14] that the collection of m -graded augmentation numbers and the m -graded ruling polynomial are equivalent invariants.

Theorem 3.12 ([14, 20]). *Let $K \subset J^1\mathbb{R}$ be a Legendrian link equipped with a $\mathbb{Z}/2r(K)$ -valued Maslov potential, and let $m \geq 0$ have $m \mid 2r(K)$. If m is even, assume $r(K) = 0$. Then, for any prime power q ,*

$$\text{Aug}_m(K, q) = R_K^m (q^{1/2} - q^{-1/2}).$$

Proof. Remark 3.3 (ii) in [14] observes the formula in the case where $m = 0$ or m is odd as a variant of Theorem 1.1 of [14]. When m is even, the result is obtained in Proposition 16 of [20]. \square

Remark 3.13. In [18], a unital A_∞ -category, $\text{Aug}_+(K)$, is introduced having as objects the augmentations of K ; see [3] for a non-unital precursor. It is shown in [20] that $\text{Aug}_0(K, q)$ can alternatively, and perhaps more naturally, be interpreted as the homotopy cardinality of $\text{Aug}_+(K)$. More general representation categories have been defined in [4], but we do not pursue this direction any further in the present article.

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FIGURE 3. An xy -diagram (only topologically accurate) for a Legendrian $m(5_2)$ knot.

3.5. Representations on a DG-vector space. Let (V, d) be a \mathbb{Z} -graded vector space and let $d : V \rightarrow V$ be a (cohomologically graded) differential with degree $+1 \bmod m$, i.e. $d^2 = 0$ and $d(V_k^m) \subset V_{k+1}^m$. There is an induced differential on $\text{End}(V)$ given by the graded commutator with d ,

$$\delta : \text{End}(V) \rightarrow \text{End}(V), \quad \delta(T) = [d, T] = d \circ T - (-1)^{|T|} T \circ d,$$

where when m is odd we assume that V is defined over a field of characteristic 2. Moreover, $(-\text{End}(V), \delta)$ is a \mathbb{Z}/m -graded DGA (with homological grading) where $-\text{End}(V)$ denotes $\text{End}(V)$ equipped with the negative of its standard grading collapsed mod m ,

$$(-\text{End}(V))_k = \bigoplus_{l \in \mathbb{Z}} \text{Hom}(V_{l+k}, V_l), \quad -\text{End}(V) = \bigoplus_{k \in \mathbb{Z}/m} (-\text{End}(V))_k^m.$$

Assuming that V is finite dimensional over the finite field \mathbb{F}_q , considering the **graded dimension**

$$\mathbf{n} : \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}, \quad \mathbf{n}(j) = \dim(V_j)$$

leads to the formula

$$|(-\text{End}(V))_k| = q^{\sum_{l \in \mathbb{Z}} \mathbf{n}(l+k) \cdot \mathbf{n}(l)}.$$

In particular, $(-\text{End}(V), \delta)$ satisfies (B2), so representation numbers are defined when V is finite.

When $(\mathcal{B}, \delta) = (-\text{End}(V), \delta)$ we denote the sets of representations, $\overline{\text{Rep}}_m(K, (\mathcal{B}, \delta), \mathbf{T})$, by $\overline{\text{Rep}}_m(K, (V, d), \mathbf{T})$, and when V is finite we notate the associated representation numbers as

$$\text{Rep}_m(K, (V, d), \mathbf{T}).$$

Example 3.14. We compute here the 2-graded representation number, $\text{Rep}_2(K, (\mathbb{F}_q^2, 0), GL(2, \mathbb{F}_q))$, with \mathbb{F}_q^2 in grading 0, for the Legendrian $m(5_2)$ knot pictured in Figure 3. Since $\mathbf{T} = GL(2, \mathbb{F}_q)$, no restriction is placed on the image of t . We shorten notation to $\text{Rep}_2(K, \mathbb{F}_q^2) := \text{Rep}_2(K, (\mathbb{F}_q^2, 0), GL(2, \mathbb{F}_q))$, and note that in Section 7 this is called the *total 2-dimensional representation number*. See also Example 6.6 for an example where \mathbf{T} is a proper subset of $GL(n, \mathbb{F}_q)$.

Proposition 3.15. *There is a bijection*

$$\overline{\text{Rep}}_2(K, \mathbb{F}_q^n) \leftrightarrow \{(A, B) \in \text{Mat}(n, \mathbb{F}_q) \times \text{Mat}(n, \mathbb{F}_q) \mid E_{-1}(AB) = \{0\}\}$$

where $E_{-1}(AB)$ denotes the (-1) -eigenspace for AB .

Proof. The generators of $\mathcal{A}(K)$ as pictured in Figure 3 have degrees

$$|b| = -2, \quad |c_1| = |c_2| = |c_3| = 0, \quad |e_1| = |e_2| = |e_3| = |e_4| = 1, \quad |a| = 2, \quad |t| = 0,$$

and the only non-zero differentials are

$$\partial e_1 = t + (-c_3)(1 + ba), \quad \partial e_2 = 1 + (1 + ab)c_1, \quad \partial e_3 = 1 + c_1c_2, \quad \partial e_4 = 1 + c_2c_3.$$

We claim that the map $f \in \overline{\text{Rep}}_2(K, \mathbb{F}_q^2) \mapsto (A, B) = (f(a), f(b))$ gives the required bijection. Note that the equation $0 = f \circ \partial(e_2)$ shows that $(1 + f(a)f(b))f(c_1) = -1$, so that $1 + f(a)f(b)$ is invertible, i.e. -1 is not an eigenvalue of AB as required. The inverse map takes a pair (A, B) with $E_{-1}(AB) = \{0\}$ to the 2-graded representation defined by

$$\begin{aligned} f(a) &= A, \quad f(b) = B, \quad f(c_1) = -(1 + AB)^{-1}, \\ f(c_2) &= 1 + AB, \quad f(c_3) = -(1 + AB)^{-1}, \quad f(t) = -(1 + AB)^{-1}(1 + BA) \end{aligned}$$

where one should note $(1 + BA)$ is invertible (making the definition of $f(t)$ valid) since AB and BA have the same eigenvalues. [For $\lambda \in \mathbb{F}_q^*$, if $v \neq 0$ and $v \in E_\lambda(AB)$, then $B(v) \neq 0$ and $B(v) \in E_\lambda(BA)$.] \square

Proposition 3.16. *The total count of 2-graded representations on $(\mathbb{F}_q^2, 0)$ is*

$$|\overline{\text{Rep}}_2(K, \mathbb{F}_q^2)| = q^2 - q^3 + 2q^5 - q^6 - q^7 + q^8$$

Proof. Use the bijection from Proposition 3.15, and note that the number of pairs of 2×2 matrices (A, B) such that -1 is not an eigenvalue for AB is

$$|\overline{\text{Rep}}_2(K, \mathbb{F}_q^2)| = X_2 \cdot Y_2 + X_1 \cdot Y_1 + X_0 \cdot Y_0$$

where

$$X_k = |\{C \in \text{Mat}(2, \mathbb{F}_q) \mid \text{rank}(C) = k \text{ and } E_{-1}(C) = \{0\}\}|$$

and Y_k is the number of ways to factor a rank k matrix C into a product $C = AB$ with $A, B \in \text{Mat}(2, \mathbb{F}_q)$. For carrying out the counts X_k and Y_k , it is useful to fix, for each line $\ell \in \mathbb{P}^1 := \mathbb{P}(\mathbb{F}_q^2)$, a complementary vector $v_\ell \in \mathbb{F}_q^2$ such that $\ell \oplus \text{Span}_{\mathbb{F}_q}\{v_\ell\} = \mathbb{F}_q^2$. The counts are as follows.

- (1) $X_2 = (q^2 - 1)(q^2 - q) - [(q + 1)(q^2 - q - 1) + 1]$: Here, the different terms correspond to writing $X_2 = |\text{GL}(2, \mathbb{F}_q)| - (|W_1| + |W_2|)$ where $W_i = \{D \in \text{GL}(2, \mathbb{F}_q) \mid \dim E_{-1}(D) = i\}$. It is standard that $|\text{GL}(2, \mathbb{F}_q)| = (q^2 - 1)(q^2 - q)$ and $W_2 = \{-I\}$. Finally, notice there is a bijection

$$\{(\ell, w) \in \mathbb{P}^1 \times \mathbb{F}_q^2 \mid \ell \in \mathbb{P}^1, w \neq -v_\ell, \text{ and } w \notin \ell\} \leftrightarrow W_1$$

where (ℓ, w) is mapped to the matrix D with $E_{-1}(D) = \ell$ and $D(v_\ell) = w$. For each $\ell \in \mathbb{P}^1$ there are $(q^2 - q - 1)$ choices for w , so $|W_1| = |\mathbb{P}^1| \cdot (q^2 - q - 1) = (q + 1)(q^2 - q - 1)$ as required.

- (2) $X_1 = (q^2 - q - 1)(q + 1)$: We have $|X_1| = |R_1| - |S_1|$ where R_1 is the set of all rank 1 matrices and $S_1 \subset R_1$ are those that have -1 as an eigenvalue. Note that S_1 is in bijection with the set of ordered pairs (ℓ, w) with $w \in \ell$, so we can compute

$$|R_1| - |S_1| = (1 \cdot (q^2 - 1) + (q^2 - 1) \cdot q) - ((q + 1) \cdot q)$$

where the 1-st (resp. 2-nd) term for $|R_1|$ is the count of rank 1 matrices whose first column is 0 (resp. is non-zero).

- (3) $X_0 = 1$: This is obvious.
- (4) $Y_2 = (q^2 - 1)(q^2 - q)$: When $C \in \text{GL}(2, \mathbb{F}_q)$, and $C = AB$, it must be the case that $A, B \in \text{GL}(2, \mathbb{F}_q)$. As $B = A^{-1}C$ is uniquely determined by A which may be chosen arbitrarily, we have $Y_2 = |\text{GL}(2, \mathbb{F}_q)|$.
- (5) $Y_1 = 2(q^2 - 1)(q^2 - q) + (q^2 - 1)q$: When C has rank 1 and $C = AB$, there are 3 disjoint possibilities for (A, B) : $A \in \text{GL}(2)$ and $B \in R_1$; $A \in R_1$ and $B \in \text{GL}(2)$; or $A \in R_1$ and $B \in R_1$. In the first two cases, the rank 1 matrix is uniquely determined by C and the rank 2 matrix which may be chosen arbitrarily. These two cases account for the $2(q^2 - q)(q^2 - q) = 2|\text{GL}(2)|$ term. In the last case, when $C = AB$ with all matrices rank 1, note that $\ker(B) = \ker(C) = \ell$ for some $\ell \in \mathbb{P}^1$. A B with this property is then uniquely determined by Bv_ℓ which may be chosen arbitrarily in $\mathbb{F}_q^2 \setminus \{0\}$. For each of these $(q^2 - 1)$ choices of B , A must be chosen to satisfy $A(B(v_\ell)) = Cv_\ell$ and to have 1-dimensional kernel. Thus, we have $|\mathbb{P}^1 \setminus \{\text{Span}\{v_\ell\}\}| = q$ choices for A .
- (6) $Y_0 = q^4 + (q^2 - 1)(q + 1)q^2 + (q^2 - 1)(q^2 - q)$: The collection of ordered pairs (A, B) with $AB = 0$ is subdivided into disjoint subsets $T_0 \sqcup T_1 \sqcup T_2$ where the subscript denotes the rank of A . Note that $|T_0| = q^4$ since $A = 0$ and B may be arbitrary. In T_1 , given $A \in R_1$, B must be chosen with $\text{im } B \subset \ker A$ so there are q^2 choices for B (since each column of B may be an arbitrary vector in $\ker A$). Thus, $|T_1| = |R_1|q^2 = (q^2 - 1)(q + 1)q^2$. Finally, in T_2 there are $|\text{GL}(2)|$ choices for A , while B must be zero.

□

With the count of representations in hand, we now compute the 2-graded representation number $\text{Rep}_2(K, \mathbb{F}_q^2)$. As $(\mathcal{B}, \delta) = (-\text{End}(\mathbb{F}_q^2), 0)$ is concentrated in degree 0, we have $|\mathcal{B}_k^m| = \begin{cases} q^4, & k \equiv 0 \pmod m \\ 1, & \text{else.} \end{cases}$ The degree distribution for K is $r_{-2} = 1, r_0 = 3, r_1 = 4, r_2 = 1$ and all other $r_k = 0$. Therefore, the factors from (3.1) are

$$\lim_{N \rightarrow +\infty} \prod_{k \in \mathbb{Z}, |k| \leq N} |\mathcal{B}_k^2|^{-(\chi^k/2)} = (q^4)^{-(\chi^{-2} + \chi^0 + \chi^2)/2} = (q^4)^{-(-1+1+1)/2} = q^{-2}$$

and

$$|(\mathcal{B}_0^m)^* \cap \ker \delta|^{-1} = [(q^2 - 1)(q^2 - q)]^{-1}.$$

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FIGURE 4. The positive braid $\beta = \sigma_1\sigma_2\sigma_1\sigma_3$ pictured in the front projection (left) and in its resolved Lagrangian projection (right).

Thus, we have

$$(3.11) \quad \text{Rep}_2(K, \mathbb{F}_q^2) = q^{-2} \cdot [(q^2 - 1)(q^2 - q)]^{-1} \cdot (q^2 - q^3 + 2q^5 - q^6 - q^7 + q^8),$$

and the reduced representation number is

$$\widetilde{\text{Rep}}_2(K, \mathbb{F}_q^2) = q^{-4}(q^2 - q^3 + 2q^5 - q^6 - q^7 + q^8) = q^{-2} - q^{-1} + 2q^1 - q^2 - q^3 + q^4.$$

4. BRAIDS AND PATH MATRICES

A braid with only positive (in the sense of writhe) crossings may be viewed as a Legendrian $\beta \subset J^1S^1$. In [15], Kálmán associated, in the context of front projections, a matrix to a positive braid called its path matrix. These matrices play a key role in describing the DGA of a Legendrian satellite formed with positive braid as pattern. In this section, we consider path matrices in both the xz - and xy -projection setting, and prove in Proposition 4.6 an identity for the differential of an xy -path matrix that will be crucial in Section 6. We then introduce classes of braids that we call “path injective” and “path generated” for which the path matrix controls certain aspects of the DGA. We conclude the section by considering positive permutation braids which, when represented with reduced permutation braid words, are shown to satisfy both of the above conditions (Proposition 4.13) and to provide a decomposition of $GL(n, \mathbb{F})$ via their path matrices (Proposition 4.14). In fact, this coincides with the celebrated Bruhat decomposition of $GL(n, \mathbb{F})$.

4.0.1. *Notational convention.* We will use the following notation:

Convention 4.1. For an algebra, A , denote by $\text{Mat}(n, A)$ the algebra of $n \times n$ matrices with entries from A . For a linear map $\alpha : A_1 \rightarrow A_2$, we use the same notation $\alpha : \text{Mat}(n, A_1) \rightarrow \text{Mat}(n, A_2)$ for the linear map obtained from applying α entry-by-entry.

Note that if $\alpha : A_1 \rightarrow A_2$ is an algebra homomorphism or derivation, then so is $\alpha : \text{Mat}(n, A_1) \rightarrow \text{Mat}(n, A_2)$.

4.1. **Positive braids and path matrices.** A positive n -stranded braid with endpoints at $x = 0$ and $x = 1$ may be considered as a Legendrian link $\beta \subset J^1S^1$; the diagram of the braid is the front projection of the link. We number braid strands from top to bottom, so that positive braids may be written as products (composed left to right) of the braid group generators σ_i , $1 \leq i \leq n - 1$ where $\sigma_1 \in B_n$ and $\sigma_{n-1} \in B_n$ are crossings of the top two strands and bottom two strands respectively.

The resolution procedure from [22] provides an xy -diagram for β of a standard form. Topologically, this diagram is obtained from the front projection of β by viewing $S^1 = [0, 1]/\{0, 1\}$ and adding a full dip to the right of the crossings of β . See Figure 4. When considering the algebra $\mathcal{A}(\beta)$, we label the crossings of the xy -projection of β as

$$p_1, \dots, p_s; \quad x_{i,j}, y_{i,j}, \text{ for } 1 \leq i < j \leq n,$$

where the p_i are the crossings from the front projection of β labeled from left to right; $x_{i,j}$ (resp. $y_{i,j}$) is the crossing in the left (resp. right) half of the dip where the i -th strand of β crosses over the j -th strand of β .

For a positive braid $\beta \subset J^1S^1$ (with orientation), we next consider several versions of the *path matrices* introduced in [15]. Cut $S^1 \times \mathbb{R}$ along the vertical line where $x = 0$ and $x = 1$ to view the xz - and xy -projections of β as subsets of $[0, 1] \times \mathbb{R}$. Call a continuous path $\gamma : [0, 1] \rightarrow [0, 1] \times \mathbb{R}$ with $p_{[0,1]} \circ \gamma = \text{id}_{[0,1]}$ and with image in the xz -projection (resp. xy -projection) of β an **xz -section** (resp. **xy -section**) of β . We say that an xz - or xy -section, γ , has **downward** (resp. **upward**) **negative corners** if at all of the corners of γ (these can only occur at crossings of the projection) the region of $[0, 1] \times \mathbb{R}$ that lies above (resp. below) γ covers a single quadrant of the crossing with negative Reeb sign. To each section γ , we associate a sign $\iota(\gamma)$ and a word $w(\gamma)$ in the generators of $\mathcal{A}(\beta)$.

- For both xz - and xy -sections: $\iota(\gamma)$ is the product of orientation signs from the corners of γ (as in Figure 1).
- For xz -sections: When γ has downward (resp. upward) negative corners we orient γ in the increasing (resp. decreasing) x -direction. Then, $w(\gamma)$ is the product of corners that γ passes ordered according to the orientation of γ .
- For xy -sections: The word $w(\gamma)$ is as in the xz -case except that the invertible generators $t_i^{\pm 1}$ also appear whenever a basepoint is passed. (The exponent ± 1 is determined by whether the orientations of γ and β agree or not at t_i .)

Definition 4.2. The **left-to-right xz -path matrix**, $P_\beta^{xz} \in \text{Mat}(n, \mathcal{A}(\beta))$, has (i, j) -entry

$$\sum_{\gamma} \iota(\gamma) w(\gamma)$$

where the sum is over xz -sections of β with downward negative corners with left (resp. right) endpoint on the i -th (resp. j -th) strand of β . The **xy -path matrix**, $P_\beta^{xy} \in \text{Mat}(n, \mathcal{A}(\beta))$ is defined in the same manner using xy -sections.

Similarly, **right-to-left path matrices**, Q_β^{xz} and Q_β^{xy} , are defined to have (i, j) -entry obtained from summing over sections with upward corners that have their right (resp. left) endpoint on the i -th (resp. j -th) strand of β .

Example 4.3. If the crossings in the braid in Figure 4 are labeled p_1, \dots, p_4 from left to right, then the left-to-right xz -path matrix is

$$P_\beta^{xz} = \begin{pmatrix} (-1)^{\mu_3} p_2 + (-1)^{\mu_2 + \mu_3} p_1 p_3 & (-1)^{\mu_2} p_1 & (-1)^{\mu_4} p_4 & 1 \\ (-1)^{\mu_3} p_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Here, the sign $(-1)^{\mu_i}$ is $+1$ (resp. -1) if the i -th strand (as ordered at the left side of the braid) is oriented to the right (resp. left).

Typically, we will place a collection of basepoints $*_1, \dots, *_n$ with corresponding generators t_1, \dots, t_n on the strands of β at an x -value just to the left of the crossings of β ; here, $*_i$ belongs to the i -th strand of β . Fix a Maslov potential μ_β for β , and let (μ_1, \dots, μ_n) be the values of μ_β at the basepoints $*_1, \dots, *_n$. We define (invertible) diagonal matrices

$$(4.1) \quad \Sigma = \text{diag}((-1)^{\mu_1}, \dots, (-1)^{\mu_n}), \quad \Delta = \text{diag}(t_1^{(-1)^{\mu_1}}, \dots, t_n^{(-1)^{\mu_n}}).$$

Moreover, let X and Y denote *strictly upper triangular* matrices with entries $x_{i,j}$ and $y_{i,j}$ for $1 \leq i < j \leq n$.

Proposition 4.4. Suppose that $\beta \subset J^1 S^1$ has basepoints placed as above, and that the xy -diagram of β is related to the xz -diagram via the resolution procedure. Then,

$$P_\beta^{xy} = \Delta \cdot P_\beta^{xz} \cdot (I + X\Sigma).$$

Proof. Recalling that Maslov potentials are required to be even (resp. odd) along strands oriented to the right (resp. left), this follows examining the xy -projection. [When a path with downward negative corners passes through the dip, because of Reeb signs at most a single corner can occur, and this can only happen at an $x_{i,j}$ generator.] \square

Proposition 4.5. Let $\alpha, \beta \subset J^1 S^1$ be positive n -stranded braids with xy -diagrams and basepoints as above. The path matrices satisfy:

- (1) $P_{\alpha*\beta}^{xz} = P_\alpha^{xz} \cdot P_\beta^{xz}$,
- (2) $Q_\beta^{xz} = (P_\beta^{xz})^{-1}$, and $Q_\beta^{xy} = (P_\beta^{xy})^{-1}$.
- (3) The (i, j) -entry of P_β^{xy} has degree $\mu_i - \mu_j$ in $\mathcal{A}(\beta)$.

Proof. Over $\mathbb{Z}/2$, the xz -statements from (1) and (2) are found in Section 3 of [15], and are extended readily to the \mathbb{Z} -coefficient case. For the xy -statement, note that

$$Q_\beta^{xy} = W \cdot Q_\beta^{xz} \cdot \Delta^{-1} = W \cdot (P_\beta^{xz})^{-1} \cdot \Delta^{-1},$$

where W is the right-to-left path matrix associated to the dip. Right to left paths, γ , through the dip starting at the i -th strand and ending at the j -th strand can have many corners at the $x_{i,j}$ (but none at the $y_{i,j}$), and have words of the form $((-1)^{\mu_{j_1}+1}x_{i,j_1})((-1)^{\mu_{j_2}+1}x_{j_1,j_2})\cdots((-1)^{\mu_{j_m}+1}x_{j_{m-1},j})$ for some $i < j_1 < \cdots < j_m < j$ with $m \geq 0$. The sum of such words is precisely the (i,j) -entry of $(I + X\Sigma)^{-1} = I + (-X\Sigma) + (-X\Sigma)^2 + \cdots$, so $W = (I + X\Sigma)^{-1}$ and $Q_\beta^{xy} = (P_\beta^{xy})^{-1}$ follows from Proposition 4.4.

To prove (3), consider a path γ oriented left-to-right with downward corners, and suppose $w(\gamma) = c_1 \cdots c_m$. (The c_l are either xz -crossings of β or $x_{r,s}$ generators.) Note that the value of the Maslov potential μ along γ changes from p to $p - |c_l|$ when the corner at c_l is passed, so $\mu_j = \mu_i - \sum_l |c_l| = \mu_i - |w(\gamma)|$. The result follows. \square

Proposition 4.6. *For any positive braid β , in the Chekanov-Eliashberg DGA of β we have the identity*

$$(4.2) \quad \Sigma \cdot \partial(P_\beta^{xy}) = P_\beta^{xy}(Y\Sigma) - (Y\Sigma)P_\beta^{xy}.$$

Proof. For the purpose of induction, we will prove a more general statement. Consider a subset $I \times \mathbb{R} \subset S^1 \times \mathbb{R}$, with I either an open interval or S^1 , of some xy -diagram such that a positive braid β appears in $I \times \mathbb{R}$ bordered on the left and on the right by two *possibly distinct* dipoles. We allow the possibility that the diagram may have other crossings outside of $I \times \mathbb{R}$. Let X_0, Y_0 (resp. X_1, Y_1) denote strictly upper triangular matrices formed with the generators from the dip that occurs before (resp. after) β . Let Σ_0 and Σ_1 be diagonal matrices with respective entries $(-1)^{\mu_i^0}$ and $(-1)^{\mu_i^1}$ where μ_i^0 and μ_i^1 denote the value of the Maslov potential on the i -th strand near the left and right dip respectively. In this context, the path matrix of β , P_β^{xy} , has entries in the DGA of the larger diagram, and has its (i,j) -entry defined via considering paths that begin on the i -th strand just to the right of the X_0, Y_0 dip and proceed from left to right until they reach the j -th strand to the right of the X_1, Y_1 dip.

We will show by induction on the length of β that

$$(4.3) \quad \Sigma_0 \cdot \partial P_\beta^{xy} = P_\beta^{xy}(Y_1\Sigma_1) - (Y_0\Sigma_0)P_\beta^{xy}.$$

When the dip to the left and right of β are actually the same dip, this reduces to (4.2).

Base Case: $\beta = \emptyset$. We compute from the definition

$$\Sigma_0 \cdot \partial(I + X_1\Sigma_1) = (I + X_1\Sigma_1)(Y_1\Sigma_1) - \Delta^{-1}(Y_0\Sigma_0)\Delta(I + X_1\Sigma_1).$$

[If necessary, see [26] or [16] where the disks that contribute to the differential are pictured.] Thus, using Proposition 4.4 and the signed Leibniz rule we compute

$$\begin{aligned} \Sigma_0 \cdot \partial(P_\beta^{xy}) &= \Sigma_0 \cdot \Delta \partial(I + X_1\Sigma_1) = \Delta(\Sigma_0 \cdot \partial(I + X_1\Sigma_1)) \\ &= \Delta(I + X_1\Sigma_1)(Y_1\Sigma_1) - (Y_0\Sigma_0)\Delta(I + X_1\Sigma_1) = P_\beta^{xy}(Y_1\Sigma_1) - (Y_0\Sigma_0)P_\beta^{xy}. \end{aligned}$$

The inductive step, has two parts.

Step 1: We establish the result for $\beta = \sigma_k$, where $1 \leq k < n$. Let p be the generator associated with the crossing in σ_k of strands k and $k+1$. Let V_k be the $n \times n$ identity matrix with the 2×2 block $\begin{pmatrix} (-1)^{\mu_k^1} p & 1 \\ 1 & 0 \end{pmatrix}$ replacing the k -th and $(k+1)$ -th entries on the diagonal. Since $P_{\sigma_k}^{xz} = V_k$,

$$P_{\sigma_k}^{xy} = \Delta V_k(I + X_1\Sigma_1)$$

and one can check that

$$\Sigma_1 \partial X_1 \Sigma_1 = (I + X_1\Sigma_1)Y_1\Sigma_1 - V_k^{-1}\Delta^{-1}\widehat{Y}_0\Sigma_0\Delta V_k(I + X_1\Sigma_1),$$

where \widehat{Y}_0 is Y_0 with the $(k, k+1)$ -entry replaced by a zero. [See [26] or [16] for individual disks.] Noting that $\mu_{k+1}^1 + |p| \equiv \mu_k^1 \pmod{2}$ and using the signed Leibniz rule, we then compute

$$(4.4) \quad \begin{aligned} \Sigma_0 \partial P_{\sigma_k}^{xy} &= \Sigma_0 \Delta((\partial((-1)^{\mu_k^1} p))E_{k,k})(I + X_1\Sigma_1) \\ &\quad + \Sigma_0 \Delta V_k' \left(\Sigma_1(I + X_1\Sigma_1)Y_1\Sigma_1 - \Sigma_1 V_k^{-1}\Delta^{-1}\widehat{Y}_0\Sigma_0\Delta V_k(I + X_1\Sigma_1) \right), \end{aligned}$$

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FIGURE 5. The xy -diagrams for the algebras $(\mathcal{A}_1, \partial_1)$ and $(\mathcal{A}_2, \partial_2)$.

where V'_k is V_k with $\begin{pmatrix} (-1)^{\mu_k^1 p} & 1 \\ 1 & 0 \end{pmatrix}$ replaced by $\begin{pmatrix} (-1)^{\mu_{k+1}^1 p} & 1 \\ 1 & 0 \end{pmatrix}$ and $E_{i,j}$ is the $n \times n$ matrix with a single nonzero entry of 1 as the (i,j) -entry. Since

$$\partial p = (-1)^{\mu_k^0 + 1} t_k^{(-1)^{\mu_k^0 + 1}} y_{k,k+1}^0 t_{k+1}^{(-1)^{\mu_{k+1}^0}},$$

the first term in (4.4) is

$$\left(\begin{pmatrix} (-1)^{\mu_k^1 + 1} y_{k,k+1}^0 t_{k+1}^{(-1)^{\mu_{k+1}^0}} \end{pmatrix} E_{k,k} \right) (I + X_1 \Sigma_1) = - \left((y_{k,k+1}^0 E_{k,k+1}) \Sigma_0 \Delta V_k \right) (I + X_1 \Sigma_1),$$

as $\mu_k^1 = \mu_{k+1}^0$. One can check that $\Sigma_0 \Delta V'_k \Sigma_1 = \Delta V_k$ and so $\Sigma_0 \Delta V'_k \Sigma_1 V_k^{-1} \Delta^{-1} = I$. Therefore

$$\begin{aligned} \Sigma_0 \partial P_{\sigma_k}^{xy} &= - (y_{k,k+1}^0 E_{k,k+1}) \Sigma_0 \Delta V_k (I + X_1 \Sigma_1) + \Delta V_k (I + X_1 \Sigma_1) Y_1 \Sigma_1 \\ &\quad - \widehat{Y}_0 \Sigma_0 \Delta V_k (I + X_1 \Sigma_1) \\ &= \Delta V_k (I + X_1 \Sigma_1) Y_1 \Sigma_1 - Y_0 \Sigma_0 \Delta V_k (I + X_1 \Sigma_1) \\ &= P_{\sigma_k}^{xy} (Y_1 \Sigma_1) - (Y_0 \Sigma_0) P_{\sigma_k}^{xy} \end{aligned}$$

as desired.

Step 2: We show that if (4.3) holds for β_1 and β_2 , then it holds for $\beta_1 * \beta_2$.

Consider algebras $(\mathcal{A}_1, \partial_1)$ and $(\mathcal{A}_2, \partial_2)$ corresponding to the two xy -diagrams in Figure 5. For the algebra \mathcal{A}_1 the diagram has three sets of dips with corresponding matrices of generators X_k, Y_k and Maslov potential matrices Σ_k , for $0 \leq k \leq 2$; the braid β_k appears between the $k-1$ and k dips, for $1 \leq k \leq 2$. [The resolution procedure from [22] can always be modified to add extra sets of dips to the xy -projection between any crossings of the xz -projection; see [26].] Moreover, basepoints t_i^k , $1 \leq i \leq n$, are placed just to the left of each β_k , and are placed into matrices Δ_{k-1} with exponents (± 1) given by the corresponding entries of Σ_{k-1} . For \mathcal{A}_2 , the dip between β_1 and β_2 is removed, and there are no basepoints between β_1 and β_2 . The generators for \mathcal{A}_2 are the same as for \mathcal{A}_1 , except there are no Δ_1, X_1 and Y_1 generators.

In \mathcal{A}_1 , (i.e. in the context of the upper diagram from Figure 5), the respective path matrices for β_1 and β_2 are

$$(4.5) \quad P_{\beta_1}^{xy} = \Delta_0 P_{\beta_1}^{xz} (I + X_1 \Sigma_1), \quad P_{\beta_2}^{xy} = \Delta_1 P_{\beta_2}^{xz} (I + X_2 \Sigma_2).$$

In \mathcal{A}_2 , the path matrix for $\beta_1 * \beta_2$ is

$$(4.6) \quad P_{\beta_1 * \beta_2}^{xy} = \Delta_0 P_{\beta_1 * \beta_2}^{xz} (I + X_2 \Sigma_2) = \Delta_0 P_{\beta_1}^{xz} P_{\beta_2}^{xz} (I + X_2 \Sigma_2).$$

There is a DGA homomorphism $\Phi : (\mathcal{A}_1, \partial_1) \rightarrow (\mathcal{A}_2, \partial_2)$ that is the composition

$$\mathcal{A}_1 \rightarrow \mathcal{A}'_1 := \mathcal{A}_1|_{\Delta_1=I} / \langle X_1, \partial X_1 \rangle \xrightarrow{\cong} \mathcal{A}_2.$$

The first map specializes the t_i^1 to 1, and then projects to the quotient by the 2-sided ideal, $\langle X_1, \partial X_1 \rangle$, generated by the entries of X_1 and their differentials. [Note that $\langle X_1, \partial X_1 \rangle$ is a differential ideal.] The map from $\psi : \mathcal{A}_2 \rightarrow \mathcal{A}'_1$ that takes generators of \mathcal{A}_2 to the equivalence class in \mathcal{A}'_1 of the corresponding generator from \mathcal{A}_1 is an algebra isomorphism (because $\partial x_{i,j}^1 = y_{i,j}^1 + \dots$), and ψ^{-1} is the second map in the definition of Φ .

Claim: ψ is in fact a DGA isomorphism. The required formula $\partial'_1 \circ \psi = \psi \circ \partial_2$ with ∂'_1 the differential on \mathcal{A}'_1 induced by ∂_1 is essentially Sivek's van Kampen Theorem from [27]. We outline the argument in our setting:

The disks that contribute to ∂_1 and ∂_2 are identical except that (1) any disk for ∂_2 whose image intersects a vertical line ℓ separating β_1 and β_2 do not appear in ∂_1 , and (2) disks for ∂_1 with punctures at any of the $x_{i,j}^1$ or $y_{i,j}^1$ do not appear in ∂_2 . The disks of type (2) have one of the following forms:

(2a) Disks with a positive puncture at $x_{i,j}$. Here, note that we can write

$$(4.7) \quad \partial_1 x_{i,j}^1 = y_{i,j} + w_{i,j} + w_x$$

where $w_{i,j}$ (resp. w_x) is the contribution from disks without any (resp. with at least one) negative punctures at some other $x_{i,j}^1$. Note that the disks that produce $w_{i,j}$ are in bijection with “bordered disks” in the \mathcal{A}_2 diagram that in place of the positive puncture at $x_{i,j}$ have a right boundary segment mapped to the line segment $\ell_{i,j} \subset \ell$ that has endpoints on the i -th and j -th strands of the Legendrian.

(2b) Disks with at least one negative puncture at the $y_{i,j}$. These are in bijection with “bordered disks” in \mathcal{A}_2 that have left boundary segments along the $\ell_{i,j}$ in place of the punctures at the $y_{i,j}$.

In \mathcal{A}'_1 , we have $[x_{i,j}] = [\partial x_{i,j}] = 0$, and using (4.7) this gives that $[y_{i,j}] = [w_{i,j}]$. Thus, for a generator a of \mathcal{A}_2 , $\psi^{-1} \circ \partial'_1 \circ \psi(a)$ is computed from $\partial_1 a$ by replacing all occurrences of $x_{i,j}$ with 0 and $y_{i,j}$ with $w_{i,j}$. Since the xy -diagram used for \mathcal{A}_2 was formed using the resolution procedure and the dips prevent disks from wrapping around the S^1 factor, it can be shown, cf. [27], that any disk of Type (1) is obtained by starting with a bordered disk of Type (2b) and then gluing bordered disks of Type (2a) to it along the vertical segments $\ell_{i,j}$ corresponding to negative punctures at the $y_{i,j}$. It follows that the above procedure for computing $\psi^{-1} \circ \partial'_1 \circ \psi(a)$ produces precisely $\partial_2(a)$ as required.

With the claim established so that ψ , and hence Φ , is a DGA map, we return to Step 2. Comparing (4.5) and (4.6) with the definition of Φ leads to

$$(4.8) \quad \Phi(P_{\beta_1}^{xy} P_{\beta_2}^{xy}) = P_{\beta_1 * \beta_2}^{xy}.$$

In $(\mathcal{A}_1, \partial_1)$, we use the Leibniz rule (observing that the (i, j) -entry of the path matrix $P_{\beta_1}^{xy}$ has degree $\mu_i^0 - \mu_j^1$) and the inductive hypothesis to compute

$$\begin{aligned} \Sigma_0 \cdot \partial_1(P_{\beta_1}^{xy} P_{\beta_2}^{xy}) &= \Sigma_0 \left[(\partial_1 P_{\beta_1}^{xy}) P_{\beta_2}^{xy} + (\Sigma_0 P_{\beta_1}^{xy} \Sigma_1) (\partial_1 P_{\beta_2}^{xy}) \right] \\ &= (\Sigma_0 \cdot \partial_1 P_{\beta_1}^{xy}) P_{\beta_2}^{xy} + P_{\beta_1}^{xy} (\Sigma_1 \cdot \partial_1 P_{\beta_2}^{xy}) \\ &= \left[P_{\beta_1}^{xy} (Y_1 \Sigma_1) - (Y_0 \Sigma_0) P_{\beta_1}^{xy} \right] \cdot P_{\beta_2}^{xy} + P_{\beta_1}^{xy} \cdot \left[P_{\beta_2}^{xy} (Y_2 \Sigma_2) - (Y_1 \Sigma_1) P_{\beta_2}^{xy} \right] \\ &= P_{\beta_1}^{xy} P_{\beta_2}^{xy} (Y_2 \Sigma_2) - (Y_0 \Sigma_0) P_{\beta_1}^{xy} P_{\beta_2}^{xy}. \end{aligned}$$

Applying Φ to the above calculation and using the identity (4.8), gives the required equality

$$\Sigma_0 \cdot \partial_2 P_{\beta_1 * \beta_2}^{xy} = P_{\beta_1 * \beta_2}^{xy} (Y_2 \Sigma_2) - (Y_0 \Sigma_0) P_{\beta_1 * \beta_2}^{xy}.$$

□

4.2. Path matrices and augmentations. Consider a positive braid, $\beta \in J^1 S^1$, (with xy -diagram obtained from resolution and basepoints $*_1, \dots, *_n$ placed to the left of the crossings of β as in Section 4.1). Notice that the entries of the path matrix P_{β}^{xy} belong to the (unital) sub-algebra $\mathcal{B} \subset \mathcal{A}(\beta)$ generated by p_1, \dots, p_s ; $x_{i,j}$ for $1 \leq i < j \leq n$; and $t_1^{\pm 1}, \dots, t_n^{\pm 1}$, i.e. \mathcal{B} is the sub-algebra spanned by words not containing any of the $y_{i,j}$.

As a result, for any (unital, associative) ring R , a ring homomorphism

$$\alpha : \mathcal{B} \rightarrow R$$

produces an invertible $n \times n$ -matrix with entries in R by applying α entry-by-entry to P_{β}^{xy} .

Definition 4.7. Given a ring R , denote by $\text{Ring}(\mathcal{B}, R)$ the set of ring homomorphisms from \mathcal{B} to R . We say that β is **R -path injective** if the map

$$(4.9) \quad \text{Ring}(\mathcal{B}, R) \rightarrow \text{Mat}(n, R), \quad \alpha \mapsto \alpha(P_{\beta}^{xy})$$

is injective. We denote the image of this map as $B_{\beta} \subset \text{Mat}(n, R)$ and call it the **path subset** of β .

We will also use an m -graded version of the path subset. The Maslov potential on β provides a \mathbb{Z} -grading on $\mathcal{B} \subset \mathcal{A}(\beta)$, and we view R as sitting in grading degree 0. For fixed $m \geq 0$, we define the **m -graded path subset** of β , B_{β}^m , to be the image under the map (4.9) of the subset $\text{Ring}_m(\mathcal{B}, R) \subset \text{Ring}(\mathcal{B}, R)$ consisting of ring homomorphisms that preserve grading mod m .

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FIGURE 6. Legendrian isotopic braids. The right is a reduced braid word.

Definition 4.8. We say that β is **path generated**, if \mathcal{B} is generated as a ring by \mathbb{Z} , the $t_i^{\pm 1}$, and the entries of P_β^{xy} .

Remark 4.9. For Theorem 6.3 to hold, the path generated condition may be replaced with a seemingly weaker “ R -path generated” where we require that the free product $R * \mathcal{B}$ is generated by R , $t_i^{\pm 1}$, and P_β^{xy} . However, we currently do not know of examples of braids that are R -path generated but not \mathbb{Z} -path generated.

The following braids will give us examples of braids which are both R -path injective and path generated.

Definition 4.10. A **positive permutation braid** is a positive braid where every pair of strands crosses at most once.

Positive permutation braids are in one-to-one correspondence with elements of the symmetric group S_n (see [7]). In particular, if $\beta = \sigma_{i_1} \cdots \sigma_{i_r}$, where the σ_i are the standard generators of the braid group B_n , then β corresponds to the permutation with each σ_i replaced by the transposition of i and $i+1$. Equivalently, the permutation π associated to a braid β has $\pi(i) = j$ if the strand in position i at the *right* side of β is in position j when followed *left* to the left side of β . In [13], positive permutation braids play an important role in Garside’s solution of the word and conjugacy problems in the braid group B_n .

After a Legendrian isotopy, it is always possible to represent a positive braid by a braid word where the product $\sigma_i \sigma_{i+1} \sigma_i$ does not appear for any i by repeatedly replacing $\sigma_i \sigma_{i+1} \sigma_i$ with $\sigma_{i+1} \sigma_i \sigma_{i+1}$, see Figure 6. Such braid words are called **reduced braid words**. The braid in Figure 4 is an example of a positive permutation braid which is not reduced and corresponds to the permutation $(12)(23)(12)(34) = (134)$. The reduced braid word for the braid in Figure 4 is $\sigma_2 \sigma_1 \sigma_2 \sigma_3$. The basic fronts A_m , $m \geq 1$ (as in Figure 2) are examples of reduced positive permutation braids.

Restated in terms of our notation and with signs added, the following proposition from Kálmán gives us the form of the xz -path matrix for a reduced permutation braid word for a positive permutation braid. We use the notation $P_\pi = \sum_{i=1}^n E_{\pi(i),i}$ for the permutation matrix associated $\pi \in S_n$.

Proposition 4.11 ([15] Proposition 3.5). *Let β be a reduced positive permutation braid and let $\pi \in S_n$ be the associated permutation. The path matrix P_β^{xz} is obtained from the permutation matrix P_π as follows: Changes are only made to entries that are above the 1 in their column and to the left of the 1 in their row. At each such position, a single crossing label multiplied by $(-1)^{\mu_k}$, where μ_k is the value of the Maslov potential on the lower strand incident to the crossing, appears in P_β^{xz} .*

In other words, there is exactly one 1 in each row and column of P_β^{xz} and if an entry in P_β^{xz} is 1, then all entries either below or to the right of it are zero and all entries either above or to the left of it are either $\pm p_k$ for some k or zero. The positions that carry different entries in P_β^{xz} and P_π are in one-to-one correspondence with the crossings of β .

Example 4.12. Let β be the braid in Figure 4 and let $\beta' = \sigma_2 \sigma_1 \sigma_2 \sigma_3$ be the reduced braid word for β . If the crossings of β are labeled from left to right by p_1, \dots, p_4 and the crossings of β' by q_1, \dots, q_4 , then the path matrix of β , P_β^{xy} , is as in Example 4.3, while in the reduced case

$$P_{\beta'}^{xz} = \begin{pmatrix} (-1)^{\mu_3} q_2 & (-1)^{\mu_2} q_3 & (-1)^{\mu_4} q_4 & 1 \\ (-1)^{\mu_3} q_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Proposition 4.13. *If β is a reduced positive permutation braid and R is a unital ring, then β is R -path injective and path generated.*

Proof. Let $P_\beta^{xz} = (P_{i,j})_{1 \leq i,j \leq n}$. By Proposition 4.11, for each k there exist unique i, j such that $P_{i,j} = \pm p_k$. Relabel the p_k by their position in P_β^{xz} : let p_i^j be the p_k in the i -th row and j -th column of P_β^{xz} (for those positions (i, j) where the entry is $\pm p_k$ rather than 0 or 1). We then have that

$$P_{i,j} = \begin{cases} 1 & \text{if } \pi(j) = i, \\ 0 & \text{if } \pi^{-1}(i) > j, \\ 0 & \text{if } i > \pi(j), \\ \pm p_i^j & \text{else.} \end{cases}$$

This is because the entry $P_{i,j}$ appears above (resp. to the left of) the 1 in the j -th column (resp. i -th row) of P_π if and only if $i > \pi(j)$ (resp. $\pi^{-1}(i) > j$).

(path generated) Recalling that $P_\beta^{xy} = \Delta \cdot P_\beta^{xz} \cdot (I + X\Sigma)$, we note that the entries of P_β^{xy} are just entries of $P_\beta^{xz} \cdot (I + X\Sigma)$ multiplied on the left by some $t_i^{\pm 1}$. Therefore, to show that β is path generated it suffices to show that \mathcal{B} is generated as a ring by \mathbb{Z} , $t_i^{\pm 1}$, and the entries of $P_\beta^{xz} \cdot (I + X\Sigma)$.

Let $\mathcal{B}_k \subset \mathcal{B}$ denote the subring generated by \mathbb{Z} , the $t_i^{\pm 1}$, and entries from the first k columns of $P_\beta^{xz}(I + X\Sigma)$. We will prove by induction on k that $p_i^j, x_{i,j} \in \mathcal{B}_k$ whenever $j \leq k$. In particular, when $n = k$, we get that $\mathcal{B}_n = \mathcal{B}$ as required.

(Induction) Let $k \geq 1$, and suppose the statement is known for smaller values of k . First, we check that the $x_{i,k} \in \mathcal{B}_k$, for all $i < k$. To this end, recall that $P_{\pi(i),i} = 1$ for all i and $P_{\pi(i),j} = 0$ for all $i < j$. Thus, if $i < k$, then the $(\pi(i), k)$ -entry of $P_\beta^{xz} \cdot (I + X\Sigma)$ is

$$\begin{aligned} (P_\beta^{xz} \cdot (I + X\Sigma))_{(\pi(i), k)} &= \sum_{\ell=1}^{k-1} (-1)^{\mu_k} P_{\pi(i), \ell} x_{\ell, k} + P_{\pi(i), k} \\ &= \sum_{\ell=1}^{i-1} (-1)^{\mu_k} P_{\pi(i), \ell} x_{\ell, k} + x_{i, k}. \end{aligned}$$

The inductive hypothesis gives that the $P_{\pi(i), \ell}$ are in \mathcal{B}_k ; arguing inductively in i , we conclude that $x_{i, k} \in \mathcal{B}_k$ for all $i < k$.

For generators of the form p_i^k , there exists $\mu \in \mathbb{Z}$ such that $P_{i, k} = (-1)^\mu p_i^k$. We see that

$$\begin{aligned} (P_\beta^{xz} \cdot (I + X\Sigma))_{(i, k)} &= \sum_{\ell=1}^{k-1} (-1)^{\mu_k} P_{i, \ell} x_{\ell, k} + P_{i, k} \\ &= \sum_{\ell=1}^{k-1} (-1)^{\mu_k} P_{i, \ell} x_{\ell, k} + (-1)^\mu p_i^k. \end{aligned}$$

Since $P_{i, \ell}, x_{\ell, k} \in \mathcal{B}_k$ for all $\ell < k$, we get that $p_i^k \in \mathcal{B}_k$.

(R-path injective) Let $\alpha, \gamma \in \text{Ring}(\mathcal{B}, R)$ such that $\alpha(P_\beta^{xy}) = \gamma(P_\beta^{xy})$. To show that $\alpha(p_i) = \gamma(p_i)$, $\alpha(x_{i,j}) = \gamma(x_{i,j})$, and $\alpha(t_i) = \gamma(t_i)$ for all i and j , we will prove that $\alpha(x_{i,j}) = \gamma(x_{i,j})$ for all $i < j$, $\alpha(P_{i,j}) = \alpha\left(t_i^{(-1)^{\mu_i+1}}\right) \gamma\left(t_i^{(-1)^{\mu_i}} P_{i,j}\right)$ for all i and j , and $\alpha(t_{\pi(i)}) = \gamma(t_{\pi(i)})$ for all i .

Looking at the first column of P_β^{xy} tells us

$$\alpha\left(t_i^{(-1)^{\mu_i}} P_{i,1}\right) = \gamma\left(t_i^{(-1)^{\mu_i}} P_{i,1}\right)$$

and so

$$\alpha(P_{i,1}) = \alpha\left(t_i^{(-1)^{\mu_i+1}}\right) \gamma\left(t_i^{(-1)^{\mu_i}} P_{i,1}\right)$$

as desired. In particular, if $i = \pi(1)$, then we have

$$1 = \alpha(P_{\pi(1),1}) = \alpha\left(t_{\pi(1)}^{(-1)^{\mu_{\pi(1)}+1}}\right) \gamma\left(t_{\pi(1)}^{(-1)^{\mu_{\pi(1)}}} P_{\pi(1),1}\right) = \alpha\left(t_{\pi(1)}^{(-1)^{\mu_{\pi(1)}+1}}\right) \gamma\left(t_{\pi(1)}^{(-1)^{\mu_{\pi(1)}}}\right)$$

since $P_{\pi(1),1} = 1$. Thus $\alpha(t_{\pi(1)}) = \gamma(t_{\pi(1)})$.

Now induct on k , the column of P_β^{xy} . Assume that

- $\alpha(x_{i,j}) = \gamma(x_{i,j})$ for $i < j < k$,
- $\alpha(P_{i,j}) = \alpha\left(t_i^{(-1)^{\mu_i+1}}\right) \gamma\left(t_i^{(-1)^{\mu_i}} P_{i,j}\right)$ for $j < k$, and
- $\alpha(t_{\pi(j)}) = \gamma(t_{\pi(j)})$ for $j < k$.

We now establish these three identities in the case that $j = k$.

To obtain the first identity, we will prove by induction on j that $\alpha(x_{j,k}) = \gamma(x_{j,k})$ for all $j < k$. Looking at the k -th column of P_β^{xy} , we see that

$$\alpha\left(t_i^{(-1)^{\mu_i}} \left[\sum_{\ell=1}^{k-1} (-1)^{\mu_k} P_{i,\ell} x_{\ell,k} + P_{i,k} \right]\right) = \gamma\left(t_i^{(-1)^{\mu_i}} \left[\sum_{\ell=1}^{k-1} (-1)^{\mu_k} P_{i,\ell} x_{\ell,k} + P_{i,k} \right]\right)$$

for all i .

($j = 1$) By setting $i = \pi(j) = \pi(1)$, we see that

$$\begin{aligned} \alpha\left(t_{\pi(1)}^{(-1)^{\mu_{\pi(1)}}} \left[\sum_{\ell=1}^{k-1} (-1)^{\mu_k} P_{i,\ell} x_{\ell,k} + P_{i,k} \right]\right) &= \gamma\left(t_{\pi(1)}^{(-1)^{\mu_{\pi(1)}}} \left[(-1)^{\mu_k} \alpha(P_{\pi(1),1} x_{1,k}) \right]\right) \\ &= \gamma\left(t_{\pi(1)}^{(-1)^{\mu_{\pi(1)}}} (-1)^{\mu_k} \alpha(x_{1,k})\right), \end{aligned}$$

since $P_{\pi(1),\ell} = 0$ if $\ell > 1$. Similarly we see that

$$\gamma\left(t_{\pi(1)}^{(-1)^{\mu_{\pi(1)}}} \left[\sum_{\ell=1}^{k-1} (-1)^{\mu_k} P_{i,\ell} x_{\ell,k} + P_{i,k} \right]\right) = \gamma\left(t_{\pi(1)}^{(-1)^{\mu_{\pi(1)}}} (-1)^{\mu_k} x_{1,k}\right).$$

Thus $\alpha(x_{1,k}) = \gamma(x_{1,k})$.

(Induction) Now suppose for $\ell < j$, $\alpha(x_{\ell,k}) = \gamma(x_{\ell,k})$. Setting $i = \pi(j)$, we have

$$\begin{aligned} &\alpha\left(t_{\pi(j)}^{(-1)^{\mu_{\pi(j)}}} \left[\sum_{\ell=1}^{k-1} (-1)^{\mu_k} P_{\pi(j),\ell} x_{\ell,k} + P_{\pi(j),k} \right]\right) \\ &= \gamma\left(t_{\pi(j)}^{(-1)^{\mu_{\pi(j)}}} \left[\sum_{\ell=1}^{j-1} (-1)^{\mu_k} \alpha(P_{\pi(j),\ell} x_{\ell,k}) + (-1)^{\mu_k} \alpha(P_{\pi(j),j} x_{j,k}) \right]\right) \text{ since } P_{\pi(j),\ell} = 0 \text{ if } \ell > j \\ &= \gamma\left(t_{\pi(j)}^{(-1)^{\mu_{\pi(j)}}} \left[\sum_{\ell=1}^{j-1} (-1)^{\mu_k} \alpha\left(t_{\pi(j)}^{(-1)^{\mu_{\pi(j)}+1}}\right) \gamma\left(t_{\pi(j)}^{(-1)^{\mu_{\pi(j)}}} P_{\pi(j),\ell}\right) \gamma(x_{\ell,k}) + (-1)^{\mu_k} \alpha(x_{j,k}) \right]\right) \\ &= \gamma\left(t_{\pi(j)}^{(-1)^{\mu_{\pi(j)}}} \left[\sum_{\ell=1}^{j-1} \gamma(P_{\pi(j),\ell} x_{\ell,k}) + (-1)^{\mu_k} \alpha(x_{j,k}) \right]\right), \end{aligned}$$

since $\alpha(t_{\pi(j)}) = \gamma(t_{\pi(j)})$ and $\alpha(x_{\ell,k}) = \gamma(x_{\ell,k})$. Similarly,

$$\gamma\left(t_{\pi(j)}^{(-1)^{\mu_{\pi(j)}}} \left[\sum_{\ell=1}^{k-1} (-1)^{\mu_k} P_{i,\ell} x_{\ell,k} + P_{i,k} \right]\right) = \gamma\left(t_{\pi(j)}^{(-1)^{\mu_{\pi(j)}}} \left[\sum_{\ell=1}^{j-1} (-1)^{\mu_k} P_{\pi(j),\ell} x_{\ell,k} + (-1)^{\mu_k} x_{j,k} \right]\right).$$

Therefore $\alpha(x_{j,k}) = \gamma(x_{j,k})$ as desired.

We turn now towards establishing the 2nd and 3rd bullet points of the main induction. Note that the 2nd equation involving $P_{i,k}$ holds when $i = \pi(j)$ for some $j < k$ since in this case $P_{i,k} = 0$. When

$i = \pi(j)$ for some $k \leq j$, we have

$$\begin{aligned} \gamma \left(t_i^{(-1)^{\mu_i}} \left[\sum_{\ell=1}^{k-1} (-1)^{\mu_k} P_{i,\ell} x_{\ell,k} + P_{i,k} \right] \right) &= \alpha \left(t_i^{(-1)^{\mu_i}} \left[\sum_{\ell=1}^{k-1} (-1)^{\mu_k} P_{i,\ell} x_{\ell,k} + P_{i,k} \right] \right) \\ &= \alpha \left(t_i^{(-1)^{\mu_i}} \right) \left[\sum_{\ell=1}^{k-1} (-1)^{\mu_k} \alpha \left(t_i^{(-1)^{\mu_i+1}} \right) \gamma \left(t_i^{(-1)^{\mu_i}} P_{i,\ell} x_{\ell,k} \right) + \alpha(P_{i,k}) \right] \\ &= \sum_{\ell=1}^{k-1} (-1)^{\mu_k} \gamma \left(t_i^{(-1)^{\mu_i}} P_{i,\ell} x_{\ell,k} \right) + \alpha \left(t_i^{(-1)^{\mu_i}} P_{i,k} \right). \end{aligned}$$

So

$$\begin{aligned} \alpha(P_{i,k}) &= \alpha \left(t_i^{(-1)^{\mu_i+1}} \right) \left[\gamma \left(t_i^{(-1)^{\mu_i}} \left[\sum_{\ell=1}^{k-1} (-1)^{\mu_k} P_{i,\ell} x_{\ell,k} + P_{i,k} \right] \right) - \sum_{\ell=1}^{k-1} (-1)^{\mu_k} \gamma \left(t_i^{(-1)^{\mu_i}} P_{i,\ell} x_{\ell,k} \right) \right] \\ &= \alpha \left(t_i^{(-1)^{\mu_i+1}} \right) \gamma \left(t_i^{(-1)^{\mu_i}} P_{i,k} \right). \end{aligned}$$

If $i = \pi(k)$, then

$$1 = \alpha \left(t_{\pi(k)}^{(-1)^{\mu_{\pi(k)}+1}} \right) \gamma \left(t_{\pi(k)}^{(-1)^{\mu_{\pi(k)}}} \right)$$

so $\alpha(t_{\pi(k)}) = \gamma(t_{\pi(k)})$.

So we have shown that

- $\alpha(x_{i,j}) = \gamma(x_{i,j})$ for $i < j$,
- $\alpha(P_{i,j}) = \alpha(t_i^{(-1)^{\mu_i+1}}) \gamma(t_i^{(-1)^{\mu_i}} P_{i,j})$ for all i, j , and
- $\alpha(t_{\pi(i)}) = \gamma(t_{\pi(i)})$ for all i .

But $\pi \in S_n$, so $\alpha(t_{\pi(i)}) = \gamma(t_{\pi(i)})$ for all i if and only if $\alpha(t_i) = \gamma(t_i)$ for all i . Thus $\alpha(P_{i,j}) = \gamma(P_{i,j})$ for all i, j . So

- $\alpha(x_{i,j}) = \gamma(x_{i,j})$ for all $i < j$,
- $\alpha(P_{i,j}) = \gamma(P_{i,j})$ for all i, j , and
- $\alpha(t_i) = \gamma(t_i)$ for all i .

By Proposition 4.11, we know for each ℓ there exist i, j and $\mu \in \mathbb{Z}$ such that $P_{i,j} = (-1)^\mu p_\ell$, so $\alpha(p_\ell) = \gamma(p_\ell)$ for all ℓ and we are done. \square

4.3. A decomposition of $GL(n, \mathbb{F})$ from path matrices. We conclude this section by establishing the following decomposition of $GL(n, \mathbb{F})$ into path subsets (as in Definition 4.7) of positive permutation braids.

Proposition 4.14. *For each permutation in S_n , fix a positive permutation braid, β , with reduced braid word. For any field, \mathbb{F} , we have*

$$GL(n, \mathbb{F}) = \bigsqcup_{\beta \in S_n} B_\beta.$$

Proof. Step 1. $GL(n, \mathbb{F}) = \bigcup_{\beta \in S_n} B_\beta$.

The path subset B_β consists of all matrices of the form $DS_\pi U$ where D is invertible diagonal; U is upper triangular with 1's on the main diagonal; and (in view of Proposition 4.11) S_π is a matrix obtained from the permutation matrix for π , P_π , (where π is the permutation associated to β) by allowing arbitrary (i, j) -entries $s_{i,j} \in \mathbb{F}$ in positions that are above the 1 of P_π in column j and to the left of the 1 of P_π in row i . Therefore, it suffices to show that for any $A \in GL(n, \mathbb{F})$ there exists matrices D , U , and S_π as above such that

$$(4.10) \quad DAU = S_\pi$$

(since then $A = D^{-1}S_\pi U^{-1} \in B_\beta$).

Given A , we arrive at (4.10) via the following procedure:

- (1) In the 1-st column of A , locate the lowest nonzero entry, say it is in row i_1 . Choose the $(i_1, 1)$ -entry of D to scale this entry to 1.

- (2) Right multiply A by an upper-triangular matrix of the form $U_1 = I + u_{1,j}E_{1,j}$ where the $u_{1,j}$ are chosen to zero out all entries to the right of the 1 in position $(i_1, 1)$.
- (3) Move on to the next column and repeat this procedure.

Note that the sequence of rows i_1, \dots, i_n where the 1's in columns $1, \dots, n$ are arranged at Step (1) of the procedure must all be distinct. [When i_m is determined all the entries to the right of the 1's in positions $(i_1, 1)$, through $(i_{m-1}, m-1)$ have already become 0. However, there is still at least one non-zero entry in column m since $A \in GL(n, \mathbb{F})$.] Therefore, the resulting matrix, $DA(U_1 U_2 \cdots U_n)$, has the required form S_π .

Step 2. The B_β are disjoint.

If this were not the case, then for distinct permutations $\pi_1 \neq \pi_2$, we can arrange

$$S_{\pi_1} = DS_{\pi_2}U$$

for some D, U, S_{π_m} as above. Considering the left most column where the position of the lowest non-zero entry (necessarily a 1) of S_{π_1} and S_{π_2} differs leads to a contradiction. [WLOG we can assume the position of this 1 in S_{π_1} is lower than in S_{π_2} . Then, there is no way for the corresponding entry of $DS_{\pi_2}U$ to become non-zero.] □

Remark 4.15. In fact, the decomposition $GL(n, \mathbb{F}) = \bigsqcup_{\beta \in S_n} B_\beta$ is precisely the Bruhat Decomposition of $GL(n, \mathbb{F})$ into double cosets

$$GL(n, \mathbb{F}) = \bigsqcup_{\pi \in S_n} BP_\pi B$$

where B is the Borel subgroup of upper triangular invertible matrices. To see this, note that the matrices S_π have the form VP_π with V upper triangular, so that $B_\beta \subset BP_\pi B$ follows. The reverse inclusion then also holds since both collections of subsets partition $GL(n, \mathbb{F})$.

5. THE DGA OF A SATELLITE

In this section, we consider satellites $S(K, \beta)$ with pattern an n -stranded positive braid $\beta \in J^1 S^1$, and give a computation of the (fully non-commutative) DGA $(\mathcal{A}(S(K, \beta)), \partial)$.

5.1. Satellites via the xy -projection. Given a (connected) Legendrian knot $K \subset J^1 \mathbb{R}$ and a Legendrian link $L \subset J^1 S^1$, recall that the Legendrian satellite of K with pattern L is formed as follows: Using the Weinstein Tubular Neighborhood Theorem, we find a contactomorphism $\psi : N' \rightarrow N(K)$ from a neighborhood of the 0-section $N' \subset J^1 S^1$ to a neighborhood of K , $N(K) \subset J^1 \mathbb{R}$. The satellite, $S(K, L)$, is then obtained by scaling the y - and z -coordinates of L to shrink it into N' , and then applying ψ .

When $K \subset J^1 \mathbb{R}$ is equipped with a basepoint $*$ and $\beta \in J^1 S^1$ is a positive braid with n -strands, we form a basepointed xy -diagram for the satellite $S(K, \beta)$ using the following procedure. (See Figure 7 for an illustration.)

- (1) Begin with an xy -diagram for β as in Section 4.1. Basepoints, $*_1, \dots, *_n$, appear just to the left of the crossings of β , so that $*_i$ belongs to the i -th strand (as numbered with decreasing y -coordinate).
- (2) Next, this xy -diagram for β is placed in an annular thickening of the xy -projection of K , so that all crossings of $\pi_{xy}(\beta)$ occur in a small neighborhood of $*$. Here, we use the orientation of K and the blackboard framing to identify an immersed neighborhood of the xy -diagram of K with $S^1 \times (-\epsilon, \epsilon)$.

Note that as required in Section 3, every component of $S(K, \beta)$ receives at least one basepoint, although there may be components with multiple basepoints. For instance, if $\beta = \sigma_1 \sigma_2 \sigma_1 \sigma_3 \in B_4$, then $S(K, \beta)$ has 2 components with basepoints $*_1, *_3, *_4$ on one component and $*_2$ on the other, as seen in Figure 7.

Remark 5.1. To see that the above construction produces an xy -diagram for $S(K, \beta)$, note that the contactomorphism ψ may be chosen to preserve the Reeb vector field, and hence induces a well defined map of Lagrangian projections $\pi_{xy}(N') \rightarrow \pi_{xy}(N(K))$. Alternatively, an xy -diagram of the

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FIGURE 7. The top figure is an xy -diagram of $S(K, \beta)$ for K a right-handed trefoil and β the braid from Figure 4. In the top figure, t_i corresponds to the basepoint $*$ with label i . The bottom left figure gives the labels for the crossings in the lattice of crossings of $S(K, \beta)$ arising from the crossing a_k of K if the strands have the given labels (as in the three left-most lattices of the top figure). The bottom right figure gives the labels of the crossings in the x, y -lattice (the dip).

form described above can be obtained via the combinatorial description (via xz -projections) of $S(K, \beta)$ from [22].

5.2. Generators and grading. Outside of a neighborhood of $*$ where the crossings from β are located, the xy -diagram of $S(K, \beta)$ consists of n parallel copies of K . We number these copies of K from 1 to n so that the labeling increases as the coordinate transverse to K decreases; the i -th copy corresponds to the part of the xy -projection of β that stretches around the S^1 factor from the right side of the dip to the basepoint $*_i$.

Choose a labeling of the Reeb chords of K as a_1, \dots, a_r . Along with the invertible generator $t^{\pm 1}$ corresponding to the basepoint $*$, the a_k generate $\mathcal{A}(K)$.

The generators of $\mathcal{A}(S(K, \beta))$ are as follows:

- (1) For each $1 \leq k \leq r$, there is a collection of n^2 crossings of $S(K, L)$ arranged in a lattice at the location of a_k . We label them as

$$a_{i,j}^k, \quad 1 \leq i, j \leq n$$

so that the overstrand (resp. understrand) of $a_{i,j}^k$ belongs to the i -th (resp. j -th) copy of K .

- (2) The crossings of the xy -diagram of β also appear as crossings of $S(K, L)$. We label them as

$$p_1, \dots, p_s; \quad x_{i,j}, y_{i,j}, \quad \text{for } 1 \leq i < j \leq n$$

as in Section 4.1.

- (3) Label the invertible generators corresponding to the basepoints $*_i$ as

$$t_i, \quad 1 \leq i \leq n.$$

We choose the grading

$$|t_i| = -2r(K)$$

for all invertible generators, t_i . Then, a choice of \mathbb{Z} -valued (with the value decreasing by $-2r(K)$ at $*$, as in Section 2.1.1) Maslov potential, μ_K , on K and \mathbb{Z} -valued Maslov potential μ_β on β (continuous at base points) specifies a \mathbb{Z} -valued (with the value decreasing by $-2r(K)$ at all $*_1, \dots, *_{n-1}$) Maslov potential on $S(K, \beta)$ via $\mu = \mu_K + \mu_\beta$. Letting (μ_1, \dots, μ_n) be the values of μ_β at the base points $*_1, \dots, *_{n-1}$, the grading of generators of $S(K, \beta)$ satisfies

$$|a_{i,j}^k| = |a_k| + \mu_i - \mu_j; \quad |x_{i,j}| = \mu_i - \mu_j; \quad |y_{i,j}| = \mu_i - \mu_j - 1; \quad |t_i| = -2r(K)$$

and the grading of p_i in $\mathcal{A}(S(K, \beta))$ agrees with its grading in $\mathcal{A}(\beta)$.

Notice that this choice of basepoints and Maslov potential for $S(K, \beta)$ is consistent with (A1)-(A3) of Section 3.2, so that $\mathcal{A}(S(K, \beta))$ is suitable for computing m -graded augmentation numbers of $S(K, \beta)$ for any $m \mid 2r(K)$. (Note that the rotation number of a component $C \subset S(K, \beta)$ corresponding to a component of β with winding number n' around S^1 is $r(C) = n'r(K)$, so any divisor of $2r(K)$ also divides $2r(S(K, \beta))$. When computing augmentation numbers of $S(K, \beta)$ with m even, we will assume $r(K) = 0$, so that $r(S(K, \beta)) = 0$ also holds as required.)

5.3. The differential. Collect generators of $\mathcal{A}(S(K, \beta))$ into $n \times n$ matrices as follows. For $1 \leq k \leq r$, let

$$A_k = (a_{i,j}^k), \quad 1 \leq k \leq r.$$

As in Section 4, define *strictly upper triangular* matrices X and Y to have (i, j) -entries $x_{i,j}$ and $y_{i,j}$ when $1 \leq i < j \leq n$ and 0 when $i \geq j$, and (invertible) diagonal matrices Σ and Δ as in (4.1).

Define an algebra homomorphism

$$(5.1) \quad \Phi : \mathcal{A}(K) \rightarrow \text{Mat}(n, \mathcal{A}(S(K, \beta))), \quad a_k \mapsto A_k \Sigma, \quad t \mapsto P_\beta^{xy}.$$

(Note that P_β^{xy} is invertible; see Proposition 4.5.) As in Convention 4.1, the differential on $\mathcal{A}(S(K, \beta))$ applied entry-by-entry produces $\partial : \text{Mat}(n, \mathcal{A}(S(K, \beta))) \rightarrow \text{Mat}(n, \mathcal{A}(S(K, \beta)))$.

Proposition 5.2. *The differential in $\mathcal{A}(S(K, \beta))$ agrees with the differential for $\beta \in J^1 S^1$ on the sub-algebra $\mathcal{A}(\beta) \subset \mathcal{A}(S(K, \beta))$ and satisfies*

$$(5.2) \quad \Sigma \cdot \partial(A_k \Sigma) = \Phi \circ \partial(a_k) - (Y \Sigma) \cdot \Phi(a_k) + (-1)^{|a_k|} \Phi(a_k) \cdot (Y \Sigma),$$

$$(5.3) \quad \Sigma \cdot \partial(Y \Sigma) = -(Y \Sigma)^2.$$

In particular, the formula for ∂P_β^{xy} established in Proposition 4.6 also holds in $\mathcal{A}(S(K, \beta))$.

Proof. Following [21], we will divide the disks contributing terms to the differential for $\mathcal{A}(S(K, \beta))$ into two disjoint sets: **thin disks** and **thick disks**. Thin disks are those completely contained in the neighborhood $N(K)$ of the xy -diagram of K where the satellite construction is completed and all other disks are thick. Note that the disks used in computing the differential in $\mathcal{A}(\beta)$ correspond to thin disks in $S(K, \beta)$. After some consideration, we see that the only disks contributing to ∂p_k , $\partial x_{i,j}$, and $\partial y_{i,j}$ are exactly these thin disks.

It only remains to compute the differentials of $a_{i,j}^k$ and $y_{i,j}$ for all i, j, k . Since $\partial \Sigma = 0$, the signed Leibniz rule tells us $\partial(A_k \Sigma) = (\partial A_k) \cdot \Sigma$ and $\partial(Y \Sigma) = (\partial Y) \cdot \Sigma$, so it suffices to show the following:

$$\begin{aligned} \partial(A_k) &= \Sigma \Phi(\partial(a_k)) \Sigma - \Sigma Y \Sigma A_k + (-1)^{|a_k|} \Sigma A_k \Sigma Y, \\ \partial(Y) &= -(\Sigma Y)^2. \end{aligned}$$

Differential of A_k :

(Thick disks) Each thick disk, \tilde{u} , of $S(K, \beta)$ corresponds to a disk, u , of K as follows. Observe that (since neighborhoods of positive or negative punctures only cover a single quadrant at crossings) each time the boundary of a thick disk for $S(K, \beta)$ passes through the lattice of crossings $(a_{i,j}^k)$ corresponding to a crossing a_k of K , it can have at most one puncture. Then, \tilde{u} will have a negative (resp. positive) puncture at some $a_{i,j}^k$ if and only if u has a negative (resp. positive) puncture at a_k . In addition, when the boundary \tilde{u} enters into the region near the base point of K where the crossings from the xy -diagram of β appear it must follow a path as in the definition of the left-to-right (resp. right-to-left) path matrix P_β^{xy} (resp. Q_β^{xy}), assuming the orientation of the boundary of the thick disk agrees (resp. disagrees) with the orientation of K when it enters the region; see Definition 4.2. The corresponding disk u simply passes by the base point.

In particular, for all $1 \leq k \leq m$ and all $1 \leq i, j \leq n$, the thick disks which contribute to $\partial(a_{i,j}^k)$ are in many-to-one correspondence with disks which contribute to ∂a_k . The negative corners of a disk for $\partial(a_{i,j}^k)$ correspond to the negative corners of a disk for $\partial(a_k)$, unless the boundary of the disk passes through the basepoint. We will discuss this case later. We will drop the k 's from a_k and $a_{i,j}^k$ when it remains clear which generator we are discussing. Consider the disk contributing to ∂a with a positive corner at a and negative corners at b_1, \dots, b_ℓ . This disk corresponds to the term $\iota'_{\iota_0} b_{\iota_1} \iota_1 \cdots b_\ell \iota_\ell$ in ∂a . (Recall sign conventions from Section 2.) The disks contributing to $\partial(a_{i,j})$ corresponding to this disk have a positive (Reeb sign) corner at $a_{i,j}$ and negative corners at b_{i_{r-1}, i_r}^r for $1 \leq r \leq \ell$, where $i_0 = i$, $i_\ell = j$, and $1 \leq i_r \leq n$ for all $0 \leq r \leq \ell$. Recall that the orientation of strand r agrees with the orientation of K if and only if $\mu_r \equiv 0 \pmod{2}$. Moreover, the orientation sign for the corner at a (resp. b_r) will agree with the orientation sign for the corner at $a_{i,j}$ (resp. b_{i_{r-1}, i_r}^r) if and only if the orientation

of strand j (resp. i_r) agrees with the orientation of K . Therefore such a disk contributes the term

$$\begin{aligned} & (-1)^{\mu_0} \iota' (-1)^{\mu_{i_\ell} \iota_0} b_{i_0, i_1}^1 (-1)^{\mu_{i_1} \iota_1} \cdots b_{i_{\ell-1}, i_\ell}^\ell (-1)^{\mu_{i_\ell} \iota_\ell} \\ &= \iota' \iota_0 \iota_1 \cdots \iota_\ell [(-1)^{\mu_{i_0}}] \cdot [b_{i_0, i_1}^1 (-1)^{\mu_{i_1}}] \cdots [b_{i_{\ell-1}, i_\ell}^\ell (-1)^{\mu_{i_\ell}}] \cdot [(-1)^{\mu_{i_\ell}}]. \end{aligned}$$

Summing over all $1 \leq i_1, \dots, i_{\ell-1} \leq n$ gives the (i, j) -entry in the matrix

$$\iota' \iota_0 \iota_1 \cdots \iota_\ell \Sigma \cdot (B_1 \Sigma) \cdots (B_\ell \Sigma) \cdot \Sigma = \Sigma \cdot \Phi(\iota' \iota_0 \iota_1 \cdots \iota_\ell b_1 \cdots b_\ell) \cdot \Sigma.$$

Now consider the disks for ∂a_k where the boundary of the disk passes through the basepoint. Such a disk, u , contributes a term of the form $\iota(u) \cdot c_1 c_2 \cdots c_\ell$ where some c_i arise from negative corners at Reeb chords and some $c_i = t^{\pm 1}$ arise when the boundary of u passes the basepoint of K . (The $t_i^{\pm 1}$ terms do not contribute to the sign $\iota(u) \in \{\pm 1\}$.) The corresponding thick disks for $\partial(a_{i,j})$ again arise from sequences $i = i_0, i_1, \dots, i_\ell = j$, $1 \leq i_1, \dots, i_{\ell-1} \leq n$ together with, for each $c_r = t$ (resp. $c_r = t^{-1}$) term, a left-to-right (resp. right-to-left) section of the xy -diagram of β , beginning on strand i_{r-1} of β and ending on strand i_r of β , with corners as in the definition of the path matrix P_β^{xy} (resp. Q_β^{xy}); see Definition 4.2. (For terms with $c_r = b_k$, the corresponding thick disk has a negative corner at b_{i_{r-1}, i_r}^k as before). From the definition of the path matrices, since

$$\Phi(t) = P_\beta^{xy} \quad \text{and} \quad \Phi(t^{-1}) = (P_\beta^{xy})^{-1} = Q_\beta^{xy} \quad (\text{by Proposition 4.5 (2)}),$$

we see that once again the contribution to $\partial(a_{i,j})$ from all such thick disks is the (i, j) -entry of $\Sigma \cdot \Phi(\iota(u) c_1 \cdots c_\ell) \cdot \Sigma$.

This completes the entry-by-entry check that the contribution of thick disks to ∂A_k is the term $\Sigma \cdot \Phi(\partial(a_k)) \cdot \Sigma$.

(Thin disks) There are two types of thin disks contributing to $\partial(a_{i,j})$: triangles with negative corners at $y_{i,\ell}$ and $a_{\ell,j}$ for some $i < \ell$ and triangles with negative corners at $a_{i,\ell}$ and $y_{\ell,j}$ for some $\ell < j$. The former contributes the term

$$\iota' \iota_0 y_{i,\ell} \iota_1 a_{\ell,j} \iota_2 = (-1)^{\mu_i + 1} \iota_0 y_{i,\ell} (-1)^{\mu_\ell} a_{\ell,j} \iota_2 = -(-1)^{\mu_i} y_{i,\ell} (-1)^{\mu_\ell} a_{\ell,j}$$

to $\partial(a_{i,j})$ since $\iota_0 = \iota_2$ and thus $-\Sigma Y \Sigma A_k$ to ∂A_k . The latter contributes the term $\iota' \iota_0 a_{i,\ell} \iota_1 y_{\ell,j} \iota_2$. By considering the orientations of strand j and the two possible configurations of the disk (the quadrant covered by the disk near $a_{i,j}^k$ can either be to the right or the left of $a_{i,j}^k$), one can check that

$$\iota' \iota_2 = (-1)^{|a_{ij}^k|+1} = (-1)^{|a_k|+\mu_i+\mu_j+1}.$$

The signs of the quadrants covered by the disk near $a_{i,\ell}^k$ and $a_{i,j}^k$ only depends on whether strands j and ℓ are oriented the same direction or not, so $\iota_0 \iota_1 = (-1)^{\mu_j + \mu_\ell + 1}$. Therefore the disk contributes

$$\iota' \iota_0 a_{i,\ell} \iota_1 y_{\ell,j} \iota_2 = (-1)^{|a_k|+\mu_i+\mu_j+1} (-1)^{\mu_j + \mu_\ell + 1} a_{i,\ell} y_{\ell,j} = (-1)^{|a_k|} (-1)^{\mu_i} a_{i,\ell} (-1)^{\mu_\ell} y_{\ell,j}$$

to $\partial(a_{i,j})$ and thus $(-1)^{|a_k|} \Sigma A_k \cdot \Sigma Y$ to ∂A_k .

Differential of Y : As in [16, 26], the only disks contributing to $\partial y_{i,j}$ are disks with three corners, a positive corner at $y_{i,j}$ and negative corners at $y_{i,k}$ and $y_{k,j}$ for some $i < k < j$. Note that the disks are all contained in the dip and the signs for the disks are the same whether the disks are viewed in $J^1 S^1$ or $S(\Lambda, \beta)$. We will view the disks as being in $J^1 S^1$. We know strand i is oriented to the right if and only if $\mu_i \equiv 0 \pmod{2}$, so $\iota' = (-1)^{\mu_i + 1}$. The orientation signs at the corners depend only on the orientation of the under-strand at the crossing, so the disk contributes the term

$$(-1)^{\mu_i + 1} (-1)^{\mu_j + 1} y_{i,k} (-1)^{\mu_k} y_{k,j} (-1)^{\mu_j + 1} = -(-1)^{\mu_i} y_{i,k} (-1)^{\mu_k} y_{k,j}.$$

So

$$\partial y_{i,j} = - \sum_{i < k < j} (-1)^{\mu_i} y_{i,k} (-1)^{\mu_k} y_{k,j}.$$

Therefore $\partial Y = -\Sigma Y \Sigma Y$. □

Corollary 5.3. *Any generator $x = a_k$ or $x = t$ of $\mathcal{A}(K)$ satisfies*

$$\Sigma \cdot \partial \circ \Phi(x) = \Phi \circ \partial(x) - (Y \Sigma) \cdot \Phi(x) + (-1)^{|x|} \Phi(x) \cdot (Y \Sigma).$$

Proof. The case $x = a_k$ is in Proposition 5.2; the case $x = t$ is Proposition 4.6. \square

Remark 5.4. (i) A partial computation of $S(K, \beta)$ over $\mathbb{Z}/2$ is done in [21], Theorem 4.16 from the front diagram perspective.
(ii) The special case of the n -copy, i.e. when β is the identity braid, appears in [18], Proposition 3.25. That computation has some differences in signs when compared with our Proposition 5.2 as a result of different (but equivalent, see [19]) sign conventions for defining the differential over \mathbb{Z} .

6. SATELLITE RULING POLYNOMIALS FROM REPRESENTATION NUMBERS

We are now prepared to construct a correspondence from augmentations of $\mathcal{A}(S(K, \beta))$ to a set consisting of ordered pairs (d, f) where d is a differential on a fixed n -dimensional graded vector space, V_β , and f is a representation of $\mathcal{A}(K)$ on (V_β, d) . The image of t under f is restricted to lie in the path subset of β , $B_\beta \subset GL(n, \mathbb{F})$.

6.1. Augmentations of satellites as representations. Let the satellite link, $S(K, \beta) \subset J^1\mathbb{R}$, be constructed from the knot $K \subset J^1\mathbb{R}$ and n -stranded positive braid $\beta \subset J^1S^1$ as in Section 5.1. In particular, we have basepoints $* \in K$ and $*_1, \dots, *_n \in \beta$, and \mathbb{Z} -valued Maslov potentials μ_K and μ_β where μ_K is possibly discontinuous at $*$.

Given a field, \mathbb{F} , form a \mathbb{Z} -graded \mathbb{F} -vector space

$$V_\beta = \text{Span}_{\mathbb{F}}\{e_i \mid 1 \leq i \leq n\}, \quad |e_i| = \mu_i := \mu_\beta(*_i).$$

In the statement and proof of the following theorem, we implicitly use the ordered basis $\{e_1, \dots, e_n\}$ to identify $\text{End}(V_\beta)$ with the matrix ring $\text{Mat}(n, \mathbb{F})$. Using this identification, we write $B_\beta^m \subset GL(V_\beta)$, where B_β^m is the path subset of β (from Definition 4.7). Note that we call a linear map $A : V_\beta \rightarrow V_\beta$ **strictly upper triangular** when its matrix is, i.e. if $Ae_j = \sum_{i < j} a_{i,j}e_i$. Recall how δ is induced by d from Section 3.5.

Recall the notations $\overline{\text{Rep}}_m(K, (V, d), B)$ and $\overline{\text{Aug}}_m(K, \mathbb{F})$ from Section 3, and the algebra homomorphism $\Phi : \mathcal{A}(K) \rightarrow \text{Mat}(n, \mathcal{A}(S(K, \beta)))$ from (5.1).

Theorem 6.1. *Let $m \mid 2r(K)$, and let \mathbb{F} be a field. If m is odd, then assume $\text{char}(\mathbb{F}) = 2$.*

Given an m -graded augmentation $\epsilon : \mathcal{A}(S(K, \beta)) \rightarrow \mathbb{F}$, there is a strictly upper triangular differential

$$d : V_\beta \rightarrow V_\beta$$

with $\deg(d) = +1 \pmod m$ and an m -graded representation

$$f : (\mathcal{A}(K), \partial) \rightarrow (-\text{End}(V_\beta), \delta)$$

(with δ induced by d) defined by the matrices

$$(6.1) \quad d = \epsilon(Y\Sigma) \quad \text{and} \quad f = \epsilon \circ \Phi.$$

This correspondence $\epsilon \mapsto (d, f)$ defines a map

$$\overline{\text{Aug}}_m(S(K, \beta), \mathbb{F}) \rightarrow \bigsqcup_d \overline{\text{Rep}}_m(K, (V_\beta, d), B_\beta^m),$$

where the union is over all strictly upper triangular differentials d with $\deg(d) = +1 \pmod m$.

Moreover, the map is injective when β is \mathbb{F} -path injective, and is surjective when β is path generated.

Proof. First, observe that $\epsilon \mapsto (d, f)$ as in (6.1) defines a surjection, Ψ , from the set, $\text{Ring}_m(\mathcal{A}(S(K, \beta)), \mathbb{F})$, of ring homomorphisms that preserve grading mod m to the set of ordered pairs (d, f) with $d : V_\beta \rightarrow V_\beta$ a strictly upper-triangular linear map of degree $+1 \pmod m$ and $f : \mathcal{A}(K) \rightarrow -\text{End}(V)$ a degree $0 \pmod m$ ring homomorphism with $f(t) \in B_\beta$. [To verify the statement about grading, note that

$$de_j = \sum_{i < j} \epsilon(y_{i,j})(-1)^{\mu_j} e_i.$$

Since $|y_{i,j}| = \mu_i - \mu_j - 1$, when ϵ is m -graded, if e_i appears with nonzero coefficient in de_j we must have $\mu_i - \mu_j - 1 = 0 \pmod m$. This is equivalent to $|e_i| = |e_j| + 1 \pmod m$. Thus, d has degree $+1 \pmod m$ in $\text{End}(V_\beta)$. Similarly, since $|a_{i,j}^k| = \mu_i - \mu_j + |a_k|$, the map $(\epsilon \circ f)(a_k)$ has degree $-|a_k| \pmod m$ in $\text{End}(V)$ which translates to degree $|a_k| \pmod m$ in $-\text{End}(V)$. Finally, from Proposition 4.5 (3), the (i, j) -entry of the path matrix P_β^{xy} has degree $\mu_i - \mu_j$ in $\mathcal{A}(\beta)$, so $(\epsilon \circ f)(t) = \epsilon(P_\beta^{xy})$ has degree $0 \pmod m$.

m . The statement about surjectivity follows from the definition of B_β^m , and since any matrix entries of d or $f(a_k)$ that are allowed to be non-zero by the mod m grading condition can be made arbitrary by the choice of ϵ .]

Next, since the values of a ring map $\epsilon \in \text{Ring}_m(\mathcal{A}(S(K, \beta)), \mathbb{F})$ on the generators $a_{i,j}^k$ (resp. $y_{i,j}$) are always uniquely determined by the matrix entries of $f(a_k)$ (resp. d), we see that Ψ is also injective when β is \mathbb{F} -path injective.

Claim 1. If ϵ satisfies the augmentation equation, $\epsilon \circ \partial = 0$, then the pair (d, f) satisfies $d^2 = 0$ and $f \circ \partial = \delta \circ f$.

Using the formulas from Proposition 5.2 and Corollary 5.3, we observe the following three equivalences:

$$(6.2) \quad (\epsilon \circ \partial)(Y) = 0 \quad \Leftrightarrow \quad [\epsilon(Y\Sigma)]^2 = 0 \quad \Leftrightarrow \quad d^2 = 0.$$

$$(6.3) \quad \begin{aligned} (\epsilon \circ \partial)(A_k) = 0 &\quad \Leftrightarrow \quad \epsilon(\Sigma \cdot \partial(A_k \Sigma)) = 0 \\ &\Leftrightarrow \quad \epsilon \left(\Phi \circ \partial(a_k) - (Y\Sigma) \cdot \Phi(a_k) + (-1)^{|a_k|} \Phi(a_k) \cdot (Y\Sigma) \right) = 0 \\ &\Leftrightarrow \quad f \circ \partial(a_k) = \epsilon(Y\Sigma) \cdot f(a_k) - (-1)^{|a_k|} f(a_k) \cdot \epsilon(Y\Sigma) \\ &\Leftrightarrow \quad f \circ \partial(a_k) = \delta \circ f(a_k) \end{aligned}$$

A similar sequence of equivalences establishes

$$(6.4) \quad (\epsilon \circ \partial)(P_\beta^{xy}) = 0 \quad \Leftrightarrow \quad f \circ \partial(t) = \delta \circ f(t).$$

Thus, when $\epsilon \circ \partial = 0$ holds, we see that $d^2 = 0$ and the representation equation $f \circ \partial = \delta \circ f$ holds when applied to any generator of \mathcal{A} . This establishes Claim 1, which shows that the restriction of Ψ to $\overline{\text{Aug}}_m(S(K, \beta), \mathbb{F})$ indeed has its image in $\bigsqcup_d \text{Rep}_m(K, (V_\beta, d), B_\beta)$.

The remaining statement about surjectivity follows from combining the surjectivity of Ψ with the following:

Claim 2. When β is path generated, the converse to Claim 1 also holds.

Supposing $\Psi(\epsilon) = (d, f)$ with $d^2 = 0$ and $\delta \circ f = f \circ \partial$, the equivalences (6.2) and (6.3) show that $\epsilon \circ \partial(z) = 0$ for generators of the form $z = a_{i,j}^k$ and $z = y_{i,j}$; it is always the case that $\epsilon \circ \partial(t_i^{\pm 1}) = 0$. Finally, from the equivalence (6.4) we have $\epsilon \circ \partial(P_\beta^{xy}) = 0$; when β is path generated this implies (with notation as in 4.2) $\epsilon \circ \partial|_{\mathcal{B}} = 0$ so that $\epsilon \circ \partial(z) = 0$ holds for the remaining generators $z = x_{i,j}$ and $z = p_i$. \square

The following corollary generalizes a result of Sabloff from [26].

Corollary 6.2. Let $K \subset J^1\mathbb{R}$ be a (connected) Legendrian knot with base point $*$. If the DGA $\mathcal{A}(K, *)$ has a 2-graded representation on a DG-vector space $(V, 0)$ with V non-zero and concentrated in degree 0, then $r(K) = 0$.

Proof. Let $f \in \overline{\text{Rep}}_2(K, (V, 0), GL(V))$ where $\dim V = n$, and fix an isomorphism $V \cong \mathbb{F}^n$ leading to $GL(V) \cong GL(n, \mathbb{F})$. By Proposition 4.14, $f(t) \in B_\beta$ for some reduced positive permutation braid $\beta \in S_n$. Then, equipping β with the constant Maslov potential $\mu_\beta \equiv 0$, we have that $f \in \overline{\text{Rep}}_2(K, (V_\beta, 0), B_\beta^2)$. Since β is path generated and \mathbb{F} -path injective (by Proposition 4.13), Theorem 6.1 shows that $S(K, \beta)$ has a 2-graded augmentation to \mathbb{F} . Consequently, (see [16, Theorem 1.1]) $S(K, \beta)$ has a 2-graded normal ruling. Then, a result of Sabloff, ([26, Theorem 1.3] extended in a straight-forward manner to the multi-component case), shows that the sum of the rotation numbers of the components of $S(K, \beta)$ must be 0. This sum is $n \cdot r(K)$, so $r(K) = 0$ follows. \square

To state a precise formula for augmentation numbers of satellites in terms of representation numbers, we introduce some preliminary notation. Given an n -stranded positive braid $\beta \subset J^1 S^1$ equipped with a Maslov potential, μ_β , let $(q_l)_{l \in \mathbb{Z}}$ denote the degree distribution of the xz -crossings of β , i.e. q_l is the number of p_i generators with $|p_i| = l$. For $m \geq 0$, let $\lambda_m(\beta) = \sigma_m((q_l))$, where we view σ_m (defined in (3.7) or (3.8)-(3.10)), as a \mathbb{Z} -module homomorphism $\mathbb{Z}^{-\infty, \infty} \rightarrow \mathbb{Z}$. Note that when μ_β is constant, $\lambda_m(\beta)$ is just the length of β .

Theorem 6.3. *Suppose that $K \subset J^1\mathbb{R}$ is a (connected) Legendrian knot, and let $m \mid 2r(K)$ and \mathbb{F}_q a finite field. If m is odd, then assume $\text{char}(\mathbb{F}_q) = 2$; if m is even, then assume $r(K) = 0$.*

If $\beta \subset J^1S^1$ is path generated and \mathbb{F}_q -path injective, then we have

$$(6.5) \quad \text{Aug}_m(S(K, \beta), q) = q^{-\lambda_m(\beta)/2} (q^{1/2} - q^{-1/2})^{-n} \cdot \sum_d \widetilde{\text{Rep}}_m(K, (V_\beta, d), B_\beta^m)$$

where the sum is over all upper triangular differentials $d : V_\beta \rightarrow V_\beta$ with $\deg(d) = +1 \pmod m$.

Proof. From the definition, we have

$$(6.6) \quad \text{Aug}_m(S(K, \beta), q) = q^{-\sigma_m(S(K, \beta))/2} (q - 1)^{-n} |\overline{\text{Aug}}_m(S(K, \beta), \mathbb{F}_q)|.$$

Write

$$\sigma_m(S(K, \beta)) = X_a + X_{x,y} + X_p$$

where the 3 terms are the contribution to $\sigma_m(S(K, \beta))$ (as defined in equation (3.7)) from the degree distribution of Reeb chords of the form $a_{i,j}^k$; $x_{i,j}$ or $y_{i,j}$; and p_i respectively.

Lemma 6.4. *Let $\mathcal{B} = -\text{End}(V_\beta)$. The following identities hold:*

- (1) $q^{-X_a/2} = \lim_{N \rightarrow +\infty} \prod_{k \in \mathbb{Z}, |k| \leq N} |\mathcal{B}_k^m|^{-(\chi^k/2)}.$
- (2) $X_{x,y} = \sum_{k \in \mathbb{Z}/m} 2 \binom{\mathbf{n}(k)}{2}$ where $\mathbf{n} : \mathbb{Z}/m \rightarrow \mathbb{Z}_{\geq 0}$ is the \mathbb{Z}/m -graded dimension of V_β , i.e. $\mathbf{n}(k) = |\{e_i \mid |e_i| = k \pmod m\}|.$
- (3) $X_p = \lambda_m(\beta).$

Proof of Lemma 6.4. (1): Notice that as $|\mathcal{B}_k^m| = q^{\dim \mathcal{B}_k^m}$ we have

$$\prod_{k \in \mathbb{Z}, |k| \leq N} |\mathcal{B}_k^m|^{-(\chi^k/2)} = q^{-(\sum_{k \in \mathbb{Z}, |k| \leq N} (\dim \mathcal{B}_k^m) \chi^k)/2},$$

so it suffices to show that

$$(6.7) \quad X_a = \lim_{N \rightarrow +\infty} \sum_{k \in \mathbb{Z}, |k| \leq N} (\dim \mathcal{B}_k^m) \chi^k.$$

Fix $M \gg 0$ so that

$$(6.8) \quad \chi^{-k} = -\chi^k \quad \text{and} \quad \mathcal{B}_k = \mathcal{B}_{-k} = \{0\} \quad \text{whenever } |k| \geq M.$$

Then, using that $\dim \mathcal{B}_k^m = \dim \mathcal{B}_{-k}^m$ holds for all k , we have that

$$B := \lim_{N \rightarrow +\infty} \sum_{k \in \mathbb{Z}, |k| \leq N} (\dim \mathcal{B}_k^m) \chi^k = \sum_{|k| \leq M} (\dim \mathcal{B}_k^m) \chi^k.$$

Lemma 6.5. *For $b \in \mathbb{Z}$, let χ_a^b denote the contribution to $\chi^b(S(K, \beta))$ from generators of the form $a_{i,j}^k$. Then,*

$$\chi_a^b = \sum_{k \in \mathbb{Z}} (\dim \mathcal{B}_k) \chi^{b+k}(K)$$

where the sum is over the finitely many terms with $\mathcal{B}_k \neq \{0\}$.

Proof of Lemma 6.5. Let r_l (resp. s_l) denote the number of degree l Reeb chords of K (resp. of $S(K, \beta)$ of the form $a_{i,j}^k$), and set

$$m_k = |\{(i, j) \mid 1 \leq i, j \leq n, \mu_j - \mu_i = k\}|.$$

Note that $s_p = \sum_{k \in \mathbb{Z}} m_k r_{p+k}$, so we can compute

$$\begin{aligned}
 \chi_a^b &= \sum_{l \geq 0} (-1)^l s_{b+l} + \sum_{l < 0} (-1)^{l+1} s_{b+l} \\
 &= \sum_{l \geq 0} (-1)^l \left(\sum_{k \in \mathbb{Z}} m_k r_{b+l+k} \right) + \sum_{l < 0} (-1)^{l+1} \left(\sum_{k \in \mathbb{Z}} m_k r_{b+l+k} \right) \\
 &= \sum_{k \in \mathbb{Z}} m_k \cdot \left(\sum_{l \geq 0} (-1)^l r_{b+l+k} + \sum_{l < 0} (-1)^{l+1} r_{b+l+k} \right) \\
 &= \sum_{k \in \mathbb{Z}} (\dim \mathcal{B}_k) \cdot \chi^{b+k}(K).
 \end{aligned}$$

□

Returning now to the proof of (1) in Lemma 6.4, if $m = 0$ we have $X_a = \chi_a^0$ and $\mathcal{B}_k^m = \mathcal{B}_k$, so the required equality (6.7) is simply Lemma 6.5.

In the case where $m \neq 0$, we fix $N \gg M$ to be large enough so that the following hold:

- (a) $X_a = \chi_a^0 + \sum_{c=1}^N (\chi_a^{cm} + \chi_a^{-cm})$, (see equation (3.7)).
- (b) For all l with $|l| \leq M$, if $\mathcal{B}_k \neq 0$, then $k \in [l - Nm, l + Nm]$.

We show that $X_a - \chi_a^0 = B - \chi_a^0$ by calculating as follows:

$$\begin{aligned}
 X_a - \chi_a^0 &= \sum_{c=1}^N (\chi_a^{cm} + \chi_a^{-cm}) \\
 \text{(Lemma 6.5)} \quad &= \sum_{c=1}^N \left(\sum_{|k| \leq M} (\dim \mathcal{B}_k) \chi^{cm+k} + \sum_{|k| \leq M} (\dim \mathcal{B}_k) \chi^{-cm+k} \right) \\
 \text{(Replace } k \text{ with } -k \text{ in 2nd sum)} \quad &= \sum_{c=1}^N \left(\sum_{|k| \leq M} (\dim \mathcal{B}_k) \chi^{cm+k} + \sum_{|k| \leq M} (\dim \mathcal{B}_{-k}) \chi^{-(cm+k)} \right) \\
 \text{(Use (B2))} \quad &= \sum_{c=1}^N \left(\sum_{k \in \mathbb{Z}} \dim \mathcal{B}_k (\chi^{cm+k} + \chi^{-(cm+k)}) \right) \\
 \text{(Reindex } l = cm + k) \quad &= \sum_{c=1}^N \left(\sum_{l \in \mathbb{Z}} \dim \mathcal{B}_{l-cm} (\chi^l + \chi^{-l}) \right) \\
 \text{(Use (6.8))} \quad &= \sum_{c=1}^N \left(\sum_{|l| \leq M} \dim \mathcal{B}_{l-cm} (\chi^l + \chi^{-l}) \right) \\
 &= \sum_{|l| \leq M} (\chi^l + \chi^{-l}) \sum_{c=1}^N \dim \mathcal{B}_{l-cm} \\
 \text{(Replace } l \text{ with } -l \text{ in 2nd sum)} \quad &= \sum_{|l| \leq M} \chi^l \left(\sum_{c=1}^N \dim \mathcal{B}_{l-cm} \right) + \sum_{|l| \leq M} \chi^{-l} \left(\sum_{c=1}^N \dim \mathcal{B}_{-l-cm} \right) \\
 \text{(Use (B2))} \quad &= \sum_{|l| \leq M} \chi^l \left(\sum_{c=1}^N (\dim \mathcal{B}_{l-cm} + \dim \mathcal{B}_{l+cm}) \right) \\
 \text{(Use (b))} \quad &= \sum_{|l| \leq M} \chi^l \cdot (\dim \mathcal{B}_l^m - \dim \mathcal{B}_l) \\
 \text{(Lemma 6.5)} \quad &= B - \chi_a^0.
 \end{aligned}$$

(2): Next, since $|x_{i,j}| = |y_{i,j}| + 1$, these pairs of generators cancel in all χ^l with $l = 0 \bmod m$, except when

$$|x_{i,j}| = 0 \bmod m \quad \Leftrightarrow \quad i < j \text{ and } \mu(i) = \mu(j) \bmod m,$$

in which case there is a single such l where the contribution from $x_{i,j}$ and $y_{i,j}$ to χ^l is 2 rather than 0. As the \mathbb{Z}/m -graded dimension of V_β satisfies $\mathbf{n}(k) = |\{i \mid \mu(i) = k \bmod m\}|$, it follows that

$$X_{x,y} = \sum_{k \in \mathbb{Z}/m} 2 \binom{\mathbf{n}(k)}{2}.$$

(3): Note that $X_p = \lambda_m(\beta)$ by definition. □

With $q^{-\sigma_m(S(K,\beta))/2}$ evaluated using Lemma 6.4 and making use of the bijection from Theorem 6.1, equation (6.6) leads to

$$\begin{aligned} \text{Aug}_m(S(K, \beta), q) &= q^{-\lambda_m(\beta)/2} q^{-\sum_{k \in \mathbb{Z}/m} (\mathbf{n}(k)^2 - \mathbf{n}(k))/2} (q-1)^{-n} \\ &\quad \times \sum_d \left(\lim_{N \rightarrow +\infty} \prod_{k \in \mathbb{Z}, |k| \leq N} |\mathcal{B}_k^m|^{-(\chi^k/2)} \right) \cdot |\overline{\text{Rep}}_m(K, (V_\beta, d), B_\beta)| \\ &= q^{-\lambda_m(\beta)/2} q^{-\sum_{k \in \mathbb{Z}/m} (\mathbf{n}(k)^2 - \mathbf{n}(k))/2} (q-1)^{-n} \sum_d |\mathcal{B}_0^m|^{1/2} \cdot \widetilde{\text{Rep}}_m(K, (V_\beta, d), B_\beta), \end{aligned}$$

since, by Definition 3.4 and 3.7,

$$\begin{aligned} \widetilde{\text{Rep}}_m(K, (\mathcal{B}, \delta), \mathbb{T}) &= |\mathcal{B}_0^m|^{-1/2} |(\mathcal{B}_0^m)^* \cap \ker \delta| \\ &\quad \times \left(\lim_{N \rightarrow +\infty} \prod_{k \in \mathbb{Z}, |k| \leq N} |\mathcal{B}_k^m|^{-(\chi^k/2)} \right) \cdot |(\mathcal{B}_0^m)^* \cap \ker \delta|^{-\ell} \cdot |\overline{\text{Rep}}_m(K, (\mathcal{B}, \delta), \mathbf{T})|, \end{aligned}$$

where $\ell = 1$ since we are considering the original knot K . Recall from Section 3.5 that $|\mathcal{B}_0^m|^{1/2} = q^{\sum_{k \in \mathbb{Z}/m} \mathbf{n}(k)^2/2}$, so this becomes

$$\begin{aligned} &q^{-\lambda_m(\beta)/2} q^{\sum_{k \in \mathbb{Z}} \mathbf{n}(k)/2} (q-1)^{-n} \sum_d \widetilde{\text{Rep}}_m(K, (V_\beta, d), B_\beta) \\ &= q^{-\lambda_m(\beta)/2} (q^{1/2} - q^{-1/2})^{-n} \sum_d \widetilde{\text{Rep}}_m(K, (V_\beta, d), B_\beta). \end{aligned}$$

□

Example 6.6. We will consider the satellite of the right-handed trefoil with the basic front A_2 (as in Figure 2) and show by direct computation that the equation in Theorem 6.3 holds for $q = 2$ when $m = 0$.

Let K be the Legendrian right-handed trefoil with crossings labeled as in Figure 8. We have $|a_1| = |a_2| = |a_3| = 0$, $|a_4| = |a_5| = 1$, and $|t| = 0$ with differential

$$\begin{aligned} \partial a_4 &= t^{-1} + a_1 + a_3 + a_1 a_2 a_3 \\ \partial a_5 &= 1 - a_1 - a_3 - a_3 a_2 a_1 \\ \partial a_1 &= \partial a_2 = \partial a_3 = 0 \end{aligned}$$

Let $\beta = A_2$ with both strands oriented to the right. Label the crossing in β by p and for ease of notation set $x = x_{12}$ and $y = y_{12}$. If we set $\mu_1 = 0 = \mu_2$ (and thus $\Sigma = I$), then $|p| = 0$. The differential satisfies $\partial p = -t_1^{-1} y t_2$ and the path matrix is

$$P_\beta^{xy} = \Delta \begin{pmatrix} p & px + 1 \\ 1 & x \end{pmatrix},$$

where Δ is the matrix with t_1 and t_2 along the diagonal.

images/trefoil.{ps,eps,pdf} not found (or no BBox)

FIGURE 8. A right-handed Legendrian trefoil.

Since $\mu_1 = \mu_2$, in $S(K, \beta)$, we have $|a_{ij}^1| = |a_{ij}^2| = |a_{ij}^3| = 0$, $|a_{ij}^4| = |a_{ij}^5| = 1$, $|p| = |x| = 0$, $|y| = -1$, and $|t_1| = |t_2| = 0$. The definition of the differential and Proposition 5.2 give us that the differential satisfies

$$\begin{aligned}\partial p &= -t_1^{-1} y t_2 \\ \partial X &= (I + X)Y - \Delta^{-1} Y \Delta (I + X) \\ \partial Y &= 0 \\ \partial A_k &= -Y A_k + A_k Y, \quad k = 1, 2, 3 \\ \partial A_4 &= (P_\beta^{xy})^{-1} + A_1 + A_3 + A_1 A_2 A_3 - Y A_4 - A_4 Y \\ \partial A_5 &= I - A_1 - A_3 - A_3 A_2 A_1 - Y A_5 - A_5 Y\end{aligned}$$

Between the definition of the differential and the gradings of the generators, all 0-graded augmentations $\epsilon : \mathcal{A}(S(K, \beta), \partial) \rightarrow \mathbb{F}_q$ with $q = 2$ satisfy

$$\begin{aligned}\epsilon(t_1) &= \epsilon(t_2) = 1 \\ \epsilon(Y) &= \epsilon(A_4) = \epsilon(A_5) = 0 \\ 0 &= \epsilon((P_\beta^{xy})^{-1} + A_1 + A_3 + A_1 A_2 A_3) \\ 0 &= \epsilon(I - A_1 - A_3 - A_3 A_2 A_1)\end{aligned}$$

With Mathematica one can check that

$$|\overline{\text{Aug}}_0(S(K, \beta), \mathbb{F}_2)| = 146.$$

Recall from (3.8) that $\sigma_0 = \chi^0$, where χ^0 is the shifted Euler characteristic for $S(K, \beta)$. As the shifted Euler characteristic of $S(K, \beta)$ is $\chi^0 = (-1)^{-1+1}1 + (-1)^0 14 + (-1)^1 8 = 7$, by definition, the augmentation number for $q = 2$ of $S(K, \beta)$ is

$$\text{Aug}_0(S(K, \beta), 2) = 2^{-\sigma_0/2} (2 - 1)^{-\ell} |\overline{\text{Aug}}_0(S(K, \beta), \mathbb{F}_2)| = \frac{73\sqrt{2}}{8}.$$

To compute the right hand side of the equation in Theorem 6.3, we first compute $|\overline{\text{Rep}}_0(K, (V_\beta, d), B_\beta^0)|$ for all upper triangular differentials $d : V_\beta \rightarrow V_\beta$ with $\deg(d) = +1$. Recall that $V_\beta = \text{Span}(e_1, e_2)$ where $|e_i| = \mu_i = 0$. Thus, the only such differential is $d = 0$.

First, find the 0-graded path subset

$$B_\beta^0 = \left\{ \alpha(P_\beta^{xy}) \mid 0\text{-graded } \alpha \in \text{Ring}(\mathcal{B}, \mathbb{F}_2) \right\}.$$

This amounts to finding for which matrices $A \in GL(2, \mathbb{F}_2)$ we can solve

$$\alpha \begin{pmatrix} p & px + 1 \\ 1 & x \end{pmatrix} = A$$

for $\alpha(p)$ and $\alpha(x)$ (since $\alpha(t_1) = \alpha(t_2) = 1$). Thus

$$B_\beta^0 = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\} \subset GL(2, \mathbb{F}_2).$$

Since $d = 0$ on V_β induces the differential $\delta = 0$ on $-\text{End}(V_\beta)$ and by definition

$$\overline{\text{Rep}}_0(K, (V_\beta, 0), B_\beta^0) = \{ 0\text{-graded } f : (\mathcal{A}(K), \partial) \rightarrow (-\text{End}(V_\beta), \delta) \text{ s.t. } f \circ \partial = \delta \circ f, f(t) \in B_\beta^0 \},$$

$|\overline{\text{Rep}}_0(K, (V_\beta, 0), B_\beta^0)|$ is the number of 0-graded $f : (\mathcal{A}(K), \partial) \rightarrow (-\text{End}(V_\beta), \delta)$ such that

$$\begin{aligned}0 &= \delta \circ f(a_4) = f \circ \partial(a_4) = f(t^{-1} + a_1 + a_3 + a_1 a_2 a_3) \\ 0 &= \delta \circ f(a_5) = f \circ \partial(a_5) = f(1 - a_1 - a_3 - a_3 a_2 a_1) \\ f(t) &\in (B_\beta^0)^*\end{aligned}$$

Using Mathematica, one can check that there are 40 such f if $f(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, 33 if $f(t) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, 40 if $f(t) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and 33 if $f(t) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Thus $|\overline{\text{Rep}}_0(K, (V_\beta, d), B_\beta^0)| = 146$. Notice that this matches Theorem 6.1 which tells us $|\overline{\text{Aug}}_0(S(K, \beta), \mathbb{F}_2)| = |\overline{\text{Rep}}_0(K, (V_\beta, d), B_\beta^0)|$.

Now we have the 0-graded representation number of K for $q = 2$ is

$$\text{Rep}_0(K, (V_\beta, 0), B_\beta^0)$$

$$\begin{aligned} &= \left(\lim_{N \rightarrow +\infty} \prod_{k \in \mathbb{Z}, |k| \leq N} |(-\text{End}(V_\beta))_k^0|^{-\chi^k/2} \right) \cdot |(-\text{End}(V_\beta))_0^* \cap \ker \delta|^{-\ell} \cdot |\overline{\text{Rep}}_0(K, (V_\beta, 0), B_\beta^*)| \\ &= |(-\text{End}(V_\beta))_0|^{-\chi^0/2} \cdot |(-\text{End}(V_\beta))_0^*|^{-1} \cdot 146 \text{ since } \delta = 0 \text{ and } (-\text{End}(V_\beta))_k^0 = (-\text{End}(V_\beta))_k \neq 0 \text{ iff } k = 0 \\ &= (2^4)^{-\frac{1}{2}} \cdot 6^{-1} \cdot 146 \text{ since } (-\text{End}(V_\beta))_0 = \text{End}(V_\beta) \text{ and } \chi^0 = 1 \\ &= \frac{73}{12} \end{aligned}$$

and thus the 0-graded reduced representation number for $q = 2$ is

$$\begin{aligned} \widetilde{\text{Rep}}_0(K, (V_\beta, 0), B_\beta^0) &= |(-\text{End}(V_\beta))_0|^{-\frac{1}{2}} |(-\text{End}(V_\beta))_0^* \cap \ker \delta| \cdot \text{Rep}_0(K, (V_\beta, 0), B_\beta^0) \\ &= (2^4)^{-\frac{1}{2}} |(-\text{End}(V_\beta))_0^*| \cdot \frac{73}{12} = \frac{1}{4} \cdot 6 \cdot \frac{73}{12} = \frac{73}{8}. \end{aligned}$$

Finally, we see that the right hand side of the equation in Theorem 6.3 is

$$2^{-\lambda_0(\beta)/2} \left(\sqrt{2} - \frac{1}{\sqrt{2}} \right)^{-2} \cdot \sum_d \widetilde{\text{Rep}}_0(K, (V_\beta, d), B_\beta^0) = 2^{-\frac{1}{2}} \cdot 2 \cdot \frac{73}{8} = \frac{73\sqrt{2}}{8}$$

as desired, where the sum is over all upper triangular differentials $d : V_\beta \rightarrow V_\beta$ with $\deg(d) = +1$.

6.2. Satellite ruling polynomials via representation numbers. For any fixed Legendrian link, $L \subset J^1 S^1$, equipped with a \mathbb{Z}/m -valued Maslov potential, the m -graded ruling polynomial, $K \mapsto R_{S(K,L)}^m(z)$, provides a Legendrian isotopy invariant of (connected) Legendrian knots $K \subset J^1 \mathbb{R}$ with $m \mid 2r(K)$. Note that a choice of \mathbb{Z}/m -valued Maslov potential on K should be made to equip $S(K, L)$ with a \mathbb{Z}/m -valued Maslov potential, but $R_{S(K,L)}^m(z)$ is independent of this choice. We refer to these invariants as **satellite ruling polynomials**. Note that in the case that $L = \beta$ is a positive permutation braid (and $r(K) = 0$ if m is even), the combination of Theorems 3.12 and 6.3 gives the formula

$$R_{S(K,\beta)}^m(z) \big|_{z=q^{1/2}-q^{-1/2}} = q^{-\lambda_m(\beta)/2} (q^{1/2} - q^{-1/2})^{-n} \sum_d \widetilde{\text{Rep}}_m(K, (V_\beta, d), B_\beta^m)$$

as stated in Theorem A from the Introduction. (Sum over strictly upper triangular differentials, d , of degree $+1 \pmod{m}$.)

Example 6.7. In Example 6.6, we saw that if $q = 2$, K is the right-handed trefoil in Figure 8, and $\beta = A_2$, then

$$q^{-\lambda_m(\beta)/2} (q^{1/2} - q^{-1/2})^{-n} \sum_d \widetilde{\text{Rep}}_m(K, (V_\beta, d), B_\beta^m) = \frac{73\sqrt{2}}{8}.$$

Using a *Mathematica* program written by J. Sabloff, we see that the 0-graded ruling polynomial for $S(K, \beta)$ is $R_{S(K,\beta)}^0(z) = 3z^{-1} + 9z + 6z^3 + z^5$ and so

$$R_{S(K,\beta)}^0(z) \big|_{z=q^{1/2}-q^{-1/2}} = \frac{73\sqrt{2}}{8},$$

as stated in Theorem A.

The following corollary shows that the Chekanov-Eliashberg algebra of the original knot K already contains all information about satellite ruling polynomials of K .

Corollary 6.8. Fix $m \geq 0$, and consider the class of Legendrian knots with $r(K) = 0$ if m is even, and $m \mid 2r(K)$ if m is odd. For any fixed $L \subset J^1 S^1$ equipped with a \mathbb{Z}/m -valued Maslov potential, the satellite ruling polynomial invariant

$$K \mapsto R_{S(K,L)}^m(z)$$

is determined by L and the Chekanov-Eliashberg algebra of K .

Proof. The satellite ruling polynomials $R_{S(K,L)}^m(z)$ satisfy the ruling polynomial skein relations with respect to the pattern L . As discussed in Section 2, after applying the skein relations L can be written as a $\mathbb{Z}[z^{\pm 1}]$ -linear combination of products of the basic fronts A_m (equipped with arbitrary \mathbb{Z}/m -valued Maslov potentials). Any such product is a positive permutation braid with reduced braid word, and so we can write

$$R_{S(K,L)}^m(z) = \sum_{\beta} c_{\beta} R_{S(K,\beta)}^m(z)$$

with $c_{\beta} \in \mathbb{Z}[z^{\pm 1}]$ depending only on L , where the sum is over all positive permutation braid words β . That $R_{S(K,L)}^m(z)$ is determined by $\mathcal{A}(K)$ at $z = q^{1/2} - q^{-1/2}$ for infinitely many values of q follows from an application of Theorem A. Moreover, this uniquely determines $R_{S(K,L)}^m$ since it is a Laurent polynomial in z . (Note that a choice of \mathbb{Z} -valued Maslov potential on β lifting the \mathbb{Z}/m -valued potential on β is required for the representation numbers appearing on the right hand side of (6.5) to be defined, but this choice can be made independent from K —e.g. by lifting to the range $[0, m)$.) \square

7. REPRESENTATION NUMBERS FROM COLORED RULING POLYNOMIALS

The formula from Theorem 6.3 shows how augmentation numbers (and hence also ruling polynomials) of satellites of K can be written in terms of higher dimensional representation numbers of K . In this section, we provide a converse-type formula by realizing a total n -dimensional representation number, where no restriction is made on the image of t , in terms of satellite ruling polynomials. In fact, the particular combination of satellite ruling polynomials used is itself a natural object from the point of view of quantum topology, as it is the Legendrian analog of the colored HOMFLY-PT polynomial.

Definition 7.1. Let $m \geq 0$ have $m \mid 2r(K)$ and $m \neq 1$. Define the **n -colored m -graded ruling polynomial** of K to be

$$R_{n,K}^m(q) = \frac{1}{\alpha_n} \sum_{\beta \in S_n} q^{l(\beta)/2} R_{S(K,\beta)}^m(z)$$

where each positive permutation braid β has Maslov potential 0; $l(\beta)$ is the length of β ; $z = q^{1/2} - q^{-1/2}$; and

$$\alpha_n = (q^{1/2})^{n(n-1)/2} [n][n-1] \cdots [1]$$

with $[r] = \frac{q^{r/2} - q^{-r/2}}{q^{1/2} - q^{-1/2}}$.

Remark 7.2. This is in strict analogy with the definition of the colored HOMFLY-PT polynomial. See Section 7.1, below.

Definition 7.3. We refer to the m -graded representation number with $V = \mathbb{F}_q^n$ concentrated in degree 0 (necessarily with $d = 0$) and $T_1 = GL(V)$ as the **total n -dimensional representation number** of K , and write

$$Rep_m(K, \mathbb{F}_q^n) := Rep_m(K, (V, 0), GL(n, \mathbb{F}_q)).$$

Theorem 7.4. Let $K \subset J^1\mathbb{R}$ be Legendrian, and let $m \geq 0$ have $m \mid 2r(K)$. In addition assume $m \neq 1$, and that $r(K) = 0$ if m is even. Then,

$$Rep_m(K, \mathbb{F}_q^n) = R_{n,K}^m(q).$$

Proof. By Legendrian invariance of R^m , we may assume the β used in computing $R_{n,K}^m$ have reduced braid words.

Note that since V_{β} is concentrated in degree 0 and $m \neq 1$, the only $d : V_{\beta} \rightarrow V_{\beta}$ of degree $+1 \bmod m$ is $d = 0$. Thus, using Theorem A, we compute

$$\begin{aligned} R_{n,K}^m(q) &= \frac{1}{\alpha_n} \sum_{\beta \in S_n} q^{l(\beta)/2} \cdot R_{S(K,\beta)}^m(z) \\ &= \frac{1}{\alpha_n} \sum_{\beta \in S_n} q^{l(\beta)/2} q^{-\lambda_m(\beta)/2} z^{-n} \cdot \widetilde{Rep}_m(K, (\mathbb{F}_q^n, 0), B_{\beta}^m). \end{aligned}$$

Now, since all p_i and $x_{i,j}$ generators have degree 0, $\lambda_m(\beta) = l(\beta)$ and $B_\beta^m = B_\beta$, so the above becomes

$$\begin{aligned}
 & \frac{1}{\alpha_n} \sum_{\beta \in S_n} z^{-n} \cdot \widetilde{\text{Rep}}_m(K, (\mathbb{F}_q^n, 0), B_\beta). \\
 &= \frac{1}{\alpha_n z^n} \cdot \widetilde{\text{Rep}}_m(K, (\mathbb{F}_q^n, 0), GL(\mathbb{F}_q^n)) \\
 (7.1) \quad &= \frac{1}{(q^{1/2})^{(n^2-n)/2} \prod_{m=1}^n q^{-m/2}(q^m - 1)} \cdot |\mathcal{B}_0|^{-1/2} |GL(\mathbb{F}_q^n)| \cdot \text{Rep}_m(K, \mathbb{F}_q^n)
 \end{aligned}$$

where at the first equality we used that $GL(\mathbb{F}_q^n) = \bigsqcup_{\beta \in S_n} B_\beta$ by Proposition 4.14, and at the last we used the definition of the reduced representation numbers. Using that $|\mathcal{B}_0| = q^{n^2}$ and

$$|GL(\mathbb{F}_q^n)| = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1}) = q^{(n^2-n)/2} \prod_{m=1}^n (q^m - 1),$$

(7.1) simplifies to

$$\frac{q^{-n^2/2} q^{(n^2-n)/2}}{(q^{1/2})^{(n^2-n)/2} (q^{1/2})^{(-n^2-n)/2}} \cdot \text{Rep}_m(K, \mathbb{F}_q^n) = \text{Rep}_m(K, \mathbb{F}_q^n).$$

□

Remark 7.5. We did not define colored ruling polynomials in the case $m = 1$, since the above proof breaks down. (When applying Theorem 6.3, the many differentials with $d \neq 0$ need to be included.) We leave the correct definition of $R_{n,K}^1$, i.e. one where Theorem 7.4 will hold, as an open problem.

7.1. Relation with colored HOMFLY-PT polynomials. The **framed HOMFLY-PT polynomial** is an invariant of framed links in \mathbb{R}^3 that assigns to $K \subset \mathbb{R}^3$ a Laurent polynomial $P_K \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ which is characterized by the skein relations (pictured with blackboard framing)

(7.2)

$$\begin{array}{c}
 \text{images/HSR1.}\{\text{ps,eps,pdf}\} \text{ not found (or no BBox)} \quad \text{---} \quad \text{images/HSR2.}\{\text{ps,eps,pdf}\} \text{ not found (or no BBox)} \quad \text{---} \quad \text{images/HSR3.}\{\text{ps,eps,pdf}\} \text{ not found (or no BBox)} \\
 (7.3)
 \end{array}$$

$$\begin{array}{c}
 \text{images/HSR4.}\{\text{ps,eps,pdf}\} \text{ not found (or no BBox)} \quad \text{---} \quad \text{images/HSR6.}\{\text{ps,eps,pdf}\} \text{ not found (or no BBox)} \quad \text{---} \quad \text{images/HSR5.}\{\text{ps,eps,pdf}\} \text{ not found (or no BBox)} \\
 (7.4)
 \end{array}$$

together with a choice of normalization for the unknot (framed with self linking number 0) which we take to be $P_{\text{images/HSR7.}\{\text{ps,eps,pdf}\} \text{ not found (or no BBox)}} = (a - a^{-1})/z$. The **unframed HOMFLY-PT polynomial**, $\widehat{P}_K \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$, is then defined by

$$\widehat{P}_K = a^{-w(K)} P_K$$

where $w(K)$ is the writhe, i.e. self linking number, of K . Note that \widehat{P}_K is independent of the framing of K .

Definition 7.6. Given a framed knot $K \subset \mathbb{R}^3$ and a positive n -stranded braid β with the blackboard framing, a diagram for the (framed) satellite link $S(K, \beta) \subset \mathbb{R}^3$ arises from placing the projection annulus for β into an annular neighborhood of a diagram for K with blackboard framing. The **framed n -colored HOMFLY-PT polynomial** of K is defined by

$$P_{n,K}(a, q) = \frac{1}{\alpha_n} \sum_{\beta \in S_n} q^{l(\beta)/2} P_{S(K, \beta)}(a, z) \Big|_{z=q^{1/2}-q^{-1/2}}.$$

An **unframed n -colored HOMFLY-PT polynomial** can be defined by the normalization

$$\widehat{P}_{n,K}(a, q) = \left(a^n q^{n(n-1)/2} \right)^{-w(K)} P_{n,K}(a, q).$$

Remark 7.7. (1) That $\widehat{P}_{n,K}(a, q)$ is independent of the choice of framing on K follows from Theorem 5.5 of [2] (with $a = v^{-1}$, $q^{1/2} = s$, and $x = 1$).

- (2) In [2, 17], more general colored HOMFLY-PT polynomials $P_{\lambda,K}$ are defined where λ is a partition. The n -colored HOMFLY-PT polynomial is the case where $\lambda = n$ is a partition with only one part.
- (3) Notice that $\widehat{P}_{n,K}(a, q)$ belongs to the sub-ring of $\mathbb{Q}(a, q^{1/2})$ that is the localization of $\mathbb{Z}[a^{\pm 1}, q^{\pm 1/2}]$ obtained from inverting $q^{r/2} - q^{-r/2}$, $r \geq 1$. In particular, $\widehat{P}_{n,K}$ is a Laurent polynomial in a .

We now recall two results relating Legendrian knots with the HOMFLY-PT polynomial.

Theorem 7.8 ([12]). *For any Legendrian link $K \subset J^1\mathbb{R}$,*

$$tb(K) + |r_{tot}(K)| \leq -\deg_a \widehat{P}_K,$$

where $r_{tot}(K)$ is the sum of the rotation numbers of the components of K .

Recall that any Legendrian knot K has a framing specified by the contact planes, and the self-linking number of this framing is $tb(K)$. Therefore, an equivalent statement of Theorem 7.8 is that

$$\deg_a P_K \leq -|r_{tot}(K)|.$$

In particular, the framed HOMFLY-PT polynomial of a Legendrian knot does not contain any positive powers of a , so it is possible to specialize $a^{-1} = 0$.

Theorem 7.9 ([24]). *For any Legendrian link $K \subset J^1\mathbb{R}$, the 2-graded ruling polynomial is the specialization*

$$R_K^2(z) = P_K(a, z)|_{a^{-1}=0}.$$

We now generalize these results to apply to the n -colored HOMFLY-PT and ruling polynomials.

Theorem 7.10. *Let $K \subset J^1\mathbb{R}$ be a (connected) Legendrian knot. For all $n \geq 1$,*

$$(7.4) \quad tb(K) + |r(K)| \leq \frac{1}{n} \deg_a \widehat{P}_{n,K}, \text{ and}$$

$$(7.5) \quad \deg_a P_{n,K} \leq -n \cdot |r(K)|.$$

Moreover, the specialization of $P_{n,K}$ at $a^{-1} = 0$ is the 2-graded colored ruling polynomial, i.e.

$$(7.6) \quad R_{n,K}^2(q) = P_{n,K}(a, q)|_{a^{-1}=0}.$$

If, in addition $r(K) = 0$, then

$$(7.7) \quad \text{Rep}_2(K, \mathbb{F}_q^n) = P_{n,K}(a, q)|_{a^{-1}=0}.$$

Proof. For a Legendrian $K \subset J^1\mathbb{R}$, the satellites $S(K, \beta)$ appearing in the definition of $P_{n,K}$ are themselves Legendrian links with $r_{tot}(K) = n \cdot r(K)$, so Theorem 7.8 gives the inequality

$$\deg_a P_{n,K} = \deg_a \left(\frac{1}{\alpha_n} \sum_{\beta \in S_n} q^{l(\beta)/2} P_{S(K, \beta)} \right) \leq -n \cdot |r(K)|.$$

Inequality (7.4), then follows from (7.5) and the definition of $\widehat{P}_{n,K}$ since

$$\deg_a \widehat{P}_{n,K} = \deg_a P_{n,K} - n \cdot tb(K) \leq -n \cdot (|r(K)| + tb(K)).$$

For (7.6), we just apply Theorem 7.9 to compute

$$\begin{aligned} R_{n,K}^2(q) &= \frac{1}{\alpha_n} \sum_{\beta \in S_n} q^{l(\beta)/2} R_{S(K, \beta)}^2(q^{1/2} - q^{-1/2}) \\ &= \frac{1}{\alpha_n} \sum_{\beta \in S_n} q^{l(\beta)/2} P_{S(K, \beta)}(a, q^{1/2} - q^{-1/2})|_{a^{-1}=0} = P_{n,K}|_{a^{-1}=0}. \end{aligned}$$

Finally, (7.7) is (7.6) combined with Theorem 7.4. \square

Corollary 7.11. (1) *The 2-graded total n -dimensional representation number $\text{Rep}_2(K, \mathbb{F}_q^n)$ depends only on the underlying framed knot type of K .*

- (2) *The Chekanov-Eliashberg algebra of K has a 2-graded representation on $(\mathbb{F}_q^n, 0)$ for some q if and only if $tb(K) = \frac{1}{n} \deg_a \widehat{P}_{n,K}$. In particular, if such a representation exists, then K must have maximal Thurston-Bennequin number within its smooth isotopy class.*

Remark 7.12. The final statement of part (2) of the Corollary was proven for the special case $q = 2$ in [21, Theorem 4.9].

Example 7.13. For the Legendrian $m(5_2)$ knot, K , from Figure 3, we used the HOMFLY-PT polynomial implementation from *Sage* to compute that the 2-colored HOMFLY-PT polynomial is

$$\begin{aligned} P_{2,K}(a, q) = P_{2,O}(a, q) \cdot a^{-10} q^{-5} \cdot & \left(a^8 (q^8 - q^7 - q^6 + 2q^5 - q^3 + q^2) \right. \\ & + a^6 (q^8 + q^7 - 2q^6 + 3q^4 - q^3 - q^2 + q) - a^4 (2q^5 - 2q^3 + q^2 + q - 1) \\ & \left. - a^2 (q^4 - q^2 + q + 1) + q \right). \end{aligned}$$

Here, $P_{2,O}(a, q) = \frac{(a - a^{-1})(aq^{1/2} - a^{-1}q^{-1/2})}{(q^{1/2} - q^{-1/2})(q - q^{-1})}$ where O is the 0-framed unknot, so we have

$$\begin{aligned} P_{2,K}|_{a^{-1}=0} &= \frac{q^{1/2}}{(q^{1/2} - q^{-1/2})(q - q^{-1})} \cdot q^{-5} (q^8 - q^7 - q^6 + 2q^5 - q^3 + q^2) \\ &= q^{-2} [(q^2 - 1)(q^2 - q)]^{-1} (q^8 - q^7 - q^6 + 2q^5 - q^3 + q^2). \end{aligned}$$

Note that this agrees precisely with the direct computation of $\text{Rep}_2(K, \mathbb{F}_q^2)$ from (3.11).

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