COMPARING STUDENT PROOFS TO EXPLORE A STRUCTURAL PROPERTY IN ABSTRACT ALGEBRA

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Abstract: Connecting and comparing across student strategies has been shown to be productive for students in the elementary and secondary classrooms. We have recently been working on a project converting such practices from K-12 level to the undergraduate classroom. In this paper, we share a particular instantiation of this practice in an abstract algebra setting. Students compare across two common proof approaches to showing that the Abelian property is preserved by isomorphism. We share a complete sample lesson where students make sense of the theorem, these proofs, then leverage the difference between them in order to modify both proofs and mathematical statements. We conclude with the students' reflections on the activities, and share our learnings from adapting best practices from K-12 to this new setting.

Keywords: best practices in instruction, abstract algebra, proof presentations

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BIOGRAPHICAL SKETCHES

K. Melhuish is PI on an NSF grant looking at orchestrating discussions in the Abstract Algebra classroom. She is also PI on an NSF grant focused on operationalizing and measuring best teaching practices at the K-8 level. She is interested in the intersection of advanced mathematics and best practices in K-8 instruction as a means to support instructors engaging students in authentic mathematical activity while still meeting coverage goals.

K. Lew is co-PI on an an NSF grant looking at orchestrating discussions in the Abstract Algebra classroom and co-PI on an NSF grant looking at students' comprehension of Real Analysis proofs. Her interests focus on the learning and teaching of proof, specifically in how to help students navigate the difficulties associated with proofs and proving.

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Abstract: Connecting and comparing across student strategies has been shown to be productive for students in elementary and secondary classrooms. We have recently been working on a project converting such practices from the K-12 level to the undergraduate classroom. In this paper, we share a particular instantiation of this practice in an abstract algebra setting. Students compare across two common proof approaches to showing that the Abelian property is preserved by isomorphism. We share a complete sample lesson where students make sense of the theorem and the two proof approaches, then leverage the differences between them in order to modify both proofs and mathematical statements. We conclude with the students' reflections on the activities, and share our learnings from adapting best practices from K-12 to this new setting.

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1 INTRODUCTION

Recently, we have been teaching abstract algebra using an inquiry-oriented approach [5]. One of the struggles in implementing a more student-centered curriculum has been orchestrating discussions around proof in ways that are productive for students while still centering their ideas. In order to address this struggle, we have been experimenting with incorporating best practices for orchestrating discussion from the elementary and secondary level (e.g., [8]).

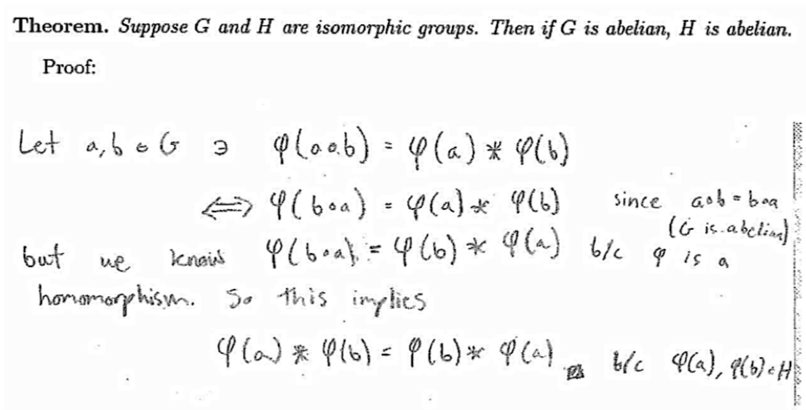
The model of five practices for discussion facilitation is a compilation of an expansive research base and was created to aid teachers who are new to the more student-centered and inquiry-oriented approaches to teaching [8]. The five practices are: (a.) anticipating student responses to mathematical tasks, (b.) monitoring students' responses, (c.) selecting particular students to present their responses, (d.) purposefully sequencing the student responses that will be displayed, and (e.) helping the class make mathematical connections between different responses. Furthermore, such practices can be enriched by the intentional use of public records ([7]) and integration of visual representations ([3]).We leveraged these K-12 best practices and related literature to design tasks to help college instructors orchestrate discussions around proving.

In this article, we share examples from implementing one focal task, related to a common theorem in an introductory abstract algebra class, that has been developed with a focus on connecting and comparing across student strategies (e.g., [4]). By having students move beyond just sharing their strategies, to connecting and comparing, they are positioned to engage with authentic mathematical activities such as analyzing and validating proofs, modifying statements, and using diagrams and examples to explore statements and strategies. In the next sections, we will share the task, examples from our most recent implementation of the task, feedback from students, and conclude with a reflection on components of instructor facilitation.

1.1 The Task

Theorem 1 *Suppose G and H are isomorphic groups. Then if G is Abelian, H is Abelian.*

The focal task was developed around the proof of the standard structural property in the theorem above. As a reminder, a group is a nonempty set with a binary operation (often notated as $\circ$ or $*$) such that the group contains an identity, inverses, is closed under the operation, and the operation is associative. An abelian group is a group whose operation is commutative. A group homomorphism is a function between



Theorem. *Suppose G and H are isomorphic groups. Then if G is abelian, H is abelian.*

Proof:

Let $a, b \in G \ni \varphi(a \circ b) = \varphi(a) * \varphi(b)$

$\iff \varphi(b \circ a) = \varphi(a) * \varphi(b)$ since $a \circ b = b \circ a$ (G is abelian)

but we know $\varphi(b \circ a) = \varphi(b) * \varphi(a)$ b/c φ is a homomorphism. So this implies

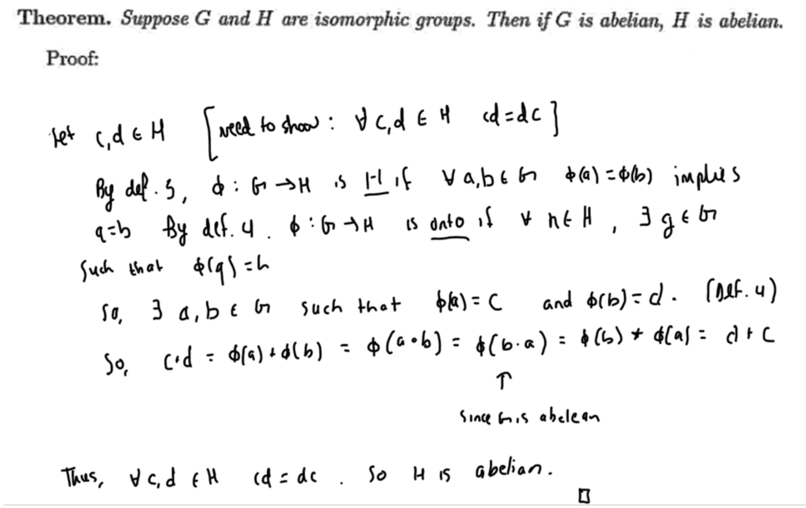
$\varphi(a) * \varphi(b) = \varphi(b) * \varphi(a)$ ■ b/c $\varphi(a), \varphi(b) \in H$

Figure 1. *G*-First Approach.

two groups such that the group operation is preserved. Two groups are isomorphic if there exists a homomorphism between the groups that is one-to-one and onto.

Our motivations to develop this particular task are twofold: (1) proving this type of statement is ubiquitous to introductory abstract algebra curricula yet prior research has shown that students often are unable to successfully construct a proof of the claim and (2) there are multiple approaches to setting up this proof, one often more productive than the other [1], which provides the opportunity for students to compare and contrast proof approaches.

In fact, in some of our prior research, we have found that students often produce the proof found in Figure 1. The reader likely notices that in the G-first approach, the arbitrary elements are selected from G rather than H even though the goal is to make an argument about H. This is unsurprising as students frequently begin with assumptions, apply some known information, and arrive at conclusions. However, this proof ultimately makes an argument about the image of the elements commuting without leveraging surjectivity to argue that all elements in H are necessarily images of elements in G. The thoughtful reader may speculate that this is just an omission of a detail; however, we



Theorem. *Suppose G and H are isomorphic groups. Then if G is abelian, H is abelian.*

Proof:

Let $c, d \in H$ [need to show: $\forall c, d \in H$ $cd = dc$]

By def. 5, $\phi : G \to H$ is 1-1 if $\forall a, b \in G$ $\phi(a) = \phi(b)$ implies $a = b$. By def. 4, $\phi : G \to H$ is onto if $\forall h \in H$, $\exists g \in G$ such that $\phi(g) = h$

So, $\exists a, b \in G$ such that $\phi(a) = c$ and $\phi(b) = d$. (Def. 4)

So, $c \cdot d = \phi(a) \cdot \phi(b) = \phi(a \cdot b) = \phi(b \cdot a) = \phi(b) \cdot \phi(a) = d \cdot c$

↑ since G is abelean

Thus, $\forall c, d \in H$ $cd = dc$. So H is abelian. □

Figure 2. *H*-First Approach.

have established that students produced identical proofs for the false statement without the necessary surjective requirement [1]. This reflects that this error was more substantial and that students could benefit from the error being explicitly addressed in instruction.

Figure 2 presents a contrasting approach (which we call the H-first approach) where the student began with arbitrary elements in H, used surjectivity, and arrived at the conclusion that these arbitrary elements commute.

We note that the bones of the argument are similar in both approaches. In fact, we have found that the majority of students believe both proofs are valid. This task design entails having both arguments available providing grounds for making the comparisons, noticing what is the same and different, and ultimately making sense of the consequences of these differences.

1.2 The Setting

The task was implemented in a lab setting with four undergraduate mathematics majors who had previously taken a course in elementary group theory. Our goal in implementing the task in the lab setting was to pilot and subsequently refine the task in preparation for use in a classroom setting. The iteration of this task discussed in the following sections was the second group of students with which this task was implemented.

1.3 Outline of Lesson Components

While the implementation data presented in this paper was conducted in a lab setting, we have since piloted this task in a classroom setting, and share the approximate timings in our 80-minute classroom session.

1. Students familiarize themselves with the theorem (∼15 min)
   (a) Refresh on terms
   (b) Determine givens and conclusions
   (c) Anticipate (or share) a proof approach
2. Students produce/make sense of the proof approaches (∼25 min)
   (a) Students/group of students presents a G-first approach
   (b) Listening students share what makes sense and what they have a questions about
   (c) Students/group of students presents an H-first approach
   (d) Listening students share what makes sense and what they have a questions about
3. Students compare proof approaches (∼25 min)
4. Students analyze and modify proofs and statements to arrive at valid statement and proof pairings (∼15 min)

Regarding (1c), for the lab setting, students anticipated a proof approach by thinking about how they would prove the theorem. We then provided student generated proofs for them to consider in part (2). For an in class implementation, an instructor may have students prepare proofs of the theorem as part of a homework activity prior to class and then have students share their approaches in small groups or pairs.

This task was designed to be added to existing abstract algebra course curricula, whether that curriculum is traditionally lecture-based or inquiry-oriented. Although this structure and the task were developed in the setting of abstract algebra, we note that the overall structure of the task may be suitably adapted for use with other theorems/statements in various proof-based undergraduate mathematical content domains, especially those that offer multiple proving approaches.

2 ILLUSTRATION OF IMPLEMENTING THE TASK

2.1 Familiarizing with Theorem and Terms

The first part of the task served to provide students access to the theorem and anticipate student approaches. Students were given *private reasoning time* to think about the terminology in the proof, and to *sketch out how you might go about proving it.* At this point, we explicitly stated we were not wanting them to actually complete the proof. After they had some time to digest the theorem, we began with the prompt: *So what are the types of things that we think about when we're going to prove something?*

The students responded with the givens and what we want to prove. We created a public record on the board notating what the students suggested fall into each of these categories. (See Figure 3.)

We also used this time to unpack various vocabulary asking students to explicate what the words meant in their "given" and "to prove" statements (e.g., *What does it mean for G and H to be isomorphic?*). We similarly kept a record of each of the definitions of Abelian and isomorphic (including one-to-one and onto) on the board. Generally, students were able to remember the definitions. However, later in the task we return to meaning making around vocabulary connected to functions (see Appendix A).

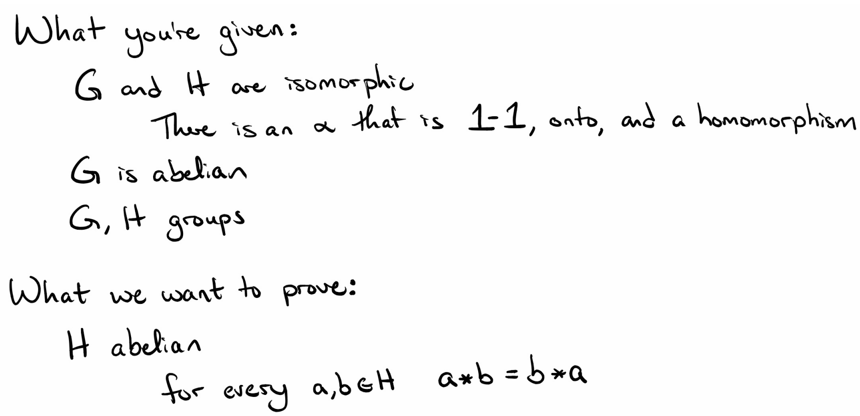


Figure 3. Public Record on the Whiteboard of Student Identified "Givens" and "Want to prove". (Recreated for clarity.)

2.2 Presenting Approaches to the Proof

Because this was done in a lab setting, we provided the two pairs of students with the two proof approaches found in Figures 1 and 2. Based on prior research, we knew these were the two most common approaches. When implementing in a full class setting, an instructor may want to intentionally look for students taking these two approaches to share their work. The students were prompted:

> Presenting Group: *Be prepared to explain this proof approach to your classmates. This explanation should include a function diagram that connects to the proof approaches.*
>
> Listening Group: *What is one thing about this proof approach that makes sense to you? What is something that you have a question about?*

2.2.1 Presentation of the G-First Approach

When implementing this task, we select the group presenting the G-first approach first as this is the most common approach students take. The partners first had time to make sense of their approach, then went to the board to explain the general structure of the proof. This included a focus on the use of the homomorphism property and Abelian property to warrant the claims. See Figure 4.

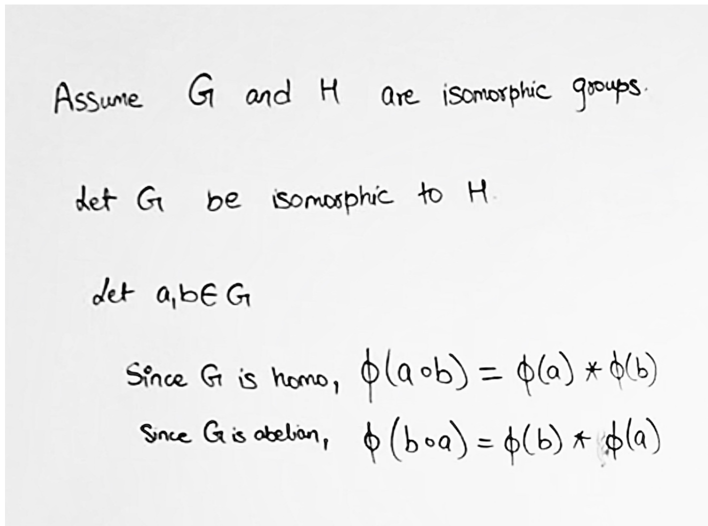


Figure 4. Students' Presentation of the Argument in G-First Proof

The students then created a function diagram explaining:

> So, this is group G, and we have these two elements in G. And then this maps ... to the group H, which will contain, thank you, $\phi(a) * \phi(b)$, which is equal to $\phi(b) * \phi(a)$. And those two elements will map to these two elements in H. (See Figure 5.)

The listening pair of students explained that the approach made sense and revoiced how the homomorphism was leveraged in the argument. One student asked about the role of one-to-one in the argument. A presenting student explained that "we were given that they were isomorphic" with their partner adding that "our proof was using the homomorphism." We took this opportunity to ask, "the one-to-one and onto piece wasn't part of the approach that you were looking at?" with the students agreeing, "[T]he only thing that we had to use was homomorphism and abelian."

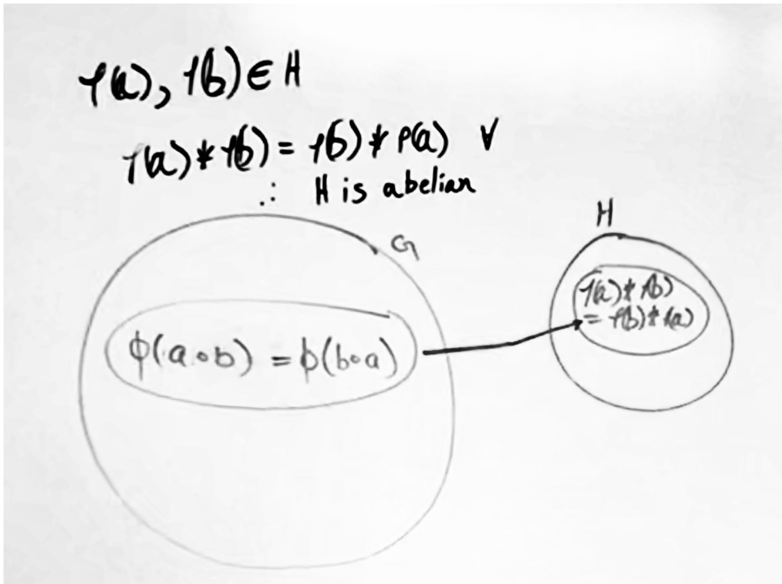


Figure 5. Students' Function Diagram Representation.

We also took this opportunity to prompt the students to clarify their function diagram by asking where "a and b live?" The presenting group argued for keeping ϕ in the domain group. We gave all the students time to think about modifying the function diagram with their partners. After some negotiation, they arrived at the two function diagrams in Figure 6.

2.2.2 Presentation of the H-First Approach

The students presented the second argument in two parts. First they explained the definitions of one-to-one and onto using a function diagram. They continued to outline the proof (see Figure 7.) They then leveraged a second function diagram to explain where the various elements were: "So, you have ϕ defined by G mapping from the dot to the star. So, c star d equals $\phi(a)$ star $\phi(b)$." (see Figure 8.)

The focus of questions for this proof was about notation and operators. One student remarked:

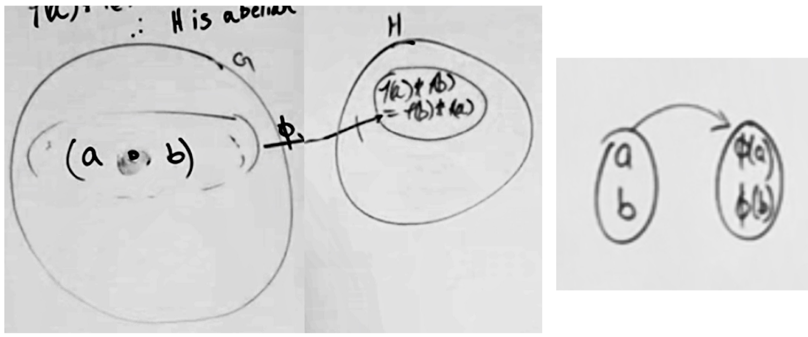


Figure 6. Students' Modified Function Diagram Representation.

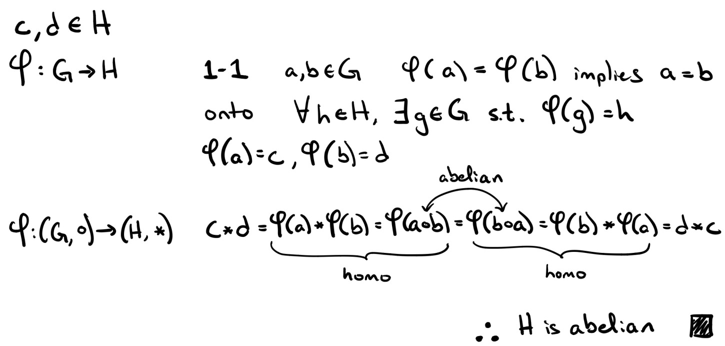


Figure 7. Students' Presentation of the Argument in the H-first Proof. (Recreated for clarity.)

> When we were talking about the operations, I thought it was really well-done, writing down the operations, what you were using, although this equation at the bottom ... you all did switch back and forth of operation, or dot, or star, your saying of it.

This again led to a conversation about operation and the domain in which particular elements lied. First, one of the presenters went through each expression to explain when the operation was from G versus when it was from H. The student who voiced the concern suggested it might have "been helpful to label" when elements were in G versus when they

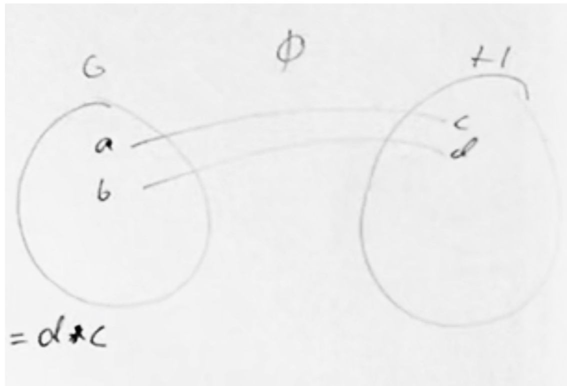


Figure 8. Students' Function Diagram Representation.

were in H. A lot of cross-talk erupted as the students continued to grapple with when the operation was from which group, until reaching a consensus after a clarification that the elements a and b are in G, but then a whole expression was in H. In our experience, this conversation was important as students often struggle to make sense of what exactly the objects are in proofs.

2.3 Comparing Across Approaches

The second part of the task was the focal piece: comparing and connecting across the two proof approaches to provide a tool for analysis. The students were prompted to spend time with their partners *thinking about what's the same and what's different about these approaches.*

After the students had an opportunity to talk with their partners, they were asked to share out to the others to motivate discussion with the whole group. First, the students pointed out the common warrants across the proofs:

Student A: They both use homomorphism and abelian properties.
[Instructor revoices and scribes on board]
Student B: To prove the main portion of the proofs.

Next, the students picked up on the different ways that the elements were labelled as they were introduced in the proofs ("They both had a unique way of naming elements in H after they're mapped ... like y'all's came up with a whole new name, while ours we kept in terms of mapping."). This observation can give students an opportunity to think about and make sense of the impact of the decision to start in G or start in H. After a student mentioned going back-and-forth in the diagrams, we used this moment to leverage the diagram to further articulate this difference.

> Instructor: Can I ask in this side of the diagram, because we didn't talk too much about your diagram, did we start with elements in G or start with elements in H?
> Student B: We started in H...
> Student C: Yeah, they started ... they let c, d be in H, and then they said, "There exists a, and b in G."

The students easily agreed there was a difference, but at this point, as one student stated "I don't think that matters." The students also noticed some other differences including that the H-first approach brought up one-to-one and onto. (See Figure 9.)

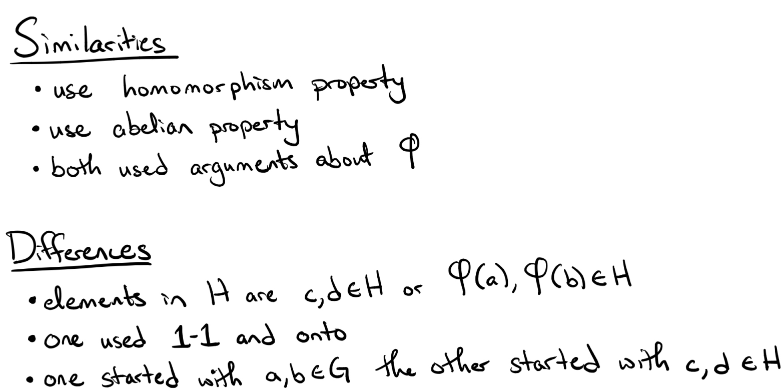


Figure 9. Whiteboard Record of Similarities and Differences. (Recreated for clarity.)

2.4 Analysis and Modification

2.4.1 Using the Proofs to Modify the Statement

After comparing the proofs, we asked students to think about modifying the statement. To motivate modifications, we asked, *So, the big question is, did we actually need all of the assumptions in this statement?*, further prompting the students in their small group to *[C]ome up with a list of which ones are actually needed to prove this statement.* After some debate, the students decided in their small group that the homomorphism property was the needed part of isomorphism.

> Student B: You would need everything for isomorphic, because you need to know that it is isomorphic.
> Student D: I mean, couldn't we prove it with homomorphism?
> Student B: If you say G and H are homo and if you-
> Student A: But if they used-
> Student C: So then you wouldn't-
> Student A: ... one-on-one and onto over there-
> Student B: But your [crosstalk]-
> Student D: Our proof worked.
> Student B: All you need to know is that G and H are homomorphic.

After their discussion started to die down, we asked *So, if you wanted to rewrite the statement, so it only has the assumptions we need it to have, what would be a different version of that statement?* The students said to keep abelian and to change "isomorphic" to "homomorphic." After a brief discussion of terminology, we arrived at the version of the statement in Figure 10.

2.4.2 Using Examples to Explore Statement Modification

As anticipated, the students produced a reasonable but false conjectured statement. We prompted them to test out their modified statement by testing examples to see if they could find a counterexample. The students began by asking what is lost (e.g, "Since we lost isomorphism, do we lose

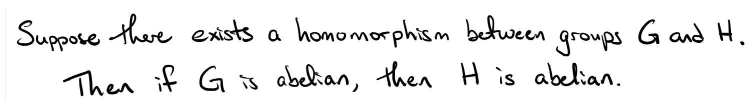


Figure 10. Whiteboard Record of Modified Statement. (Recreated for clarity.)

one-to-one, onto, well-defined?"). We clarified that a homomorphism is still a well-defined function.

We have found that during our implementations of this task, students struggled with where to begin to test examples and potentially find a counterexample. To scaffold their attempts, we provided targeted questions: *What would a counterexample to this statement even look like? ... what would be true about G, what would be true about H, and what would be true about ϕ in this counterexample?* From here the students were able to identify they wanted a group G that was Abelian, ϕ to be a homomorphism and H to be non-Abelian. The students then worked with their partners and began suggesting potential domain and co-domain groups. (See Figure 11).

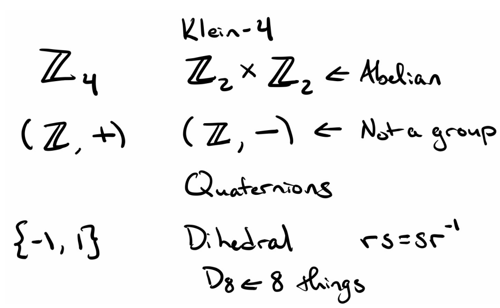


Figure 11. Whiteboard Record of Suggested Counterexamples. (Recreated for clarity.)

As a group, we decided to explore the example of $\{-1, 1\}$ under multiplication for G, and $\mathbb{D}_8$ in the role of H. The next challenge was to create homomorphisms between these two groups. We used a function diagram to support and notate their suggestions (see Figure 12). The diagram served a crucial role as students tend to need additional support to create the homomorphism. This is unsurprising as even at this level students often desire explicit-symbolic rules for functions [2]. We took this moment to emphasize finding "the easiest" map. We also reminded students that we know homomorphisms preserve identity. The first suggested map was to map 1 to the identity in the dihedral group, and -1 to another element (s, representing a "flip").

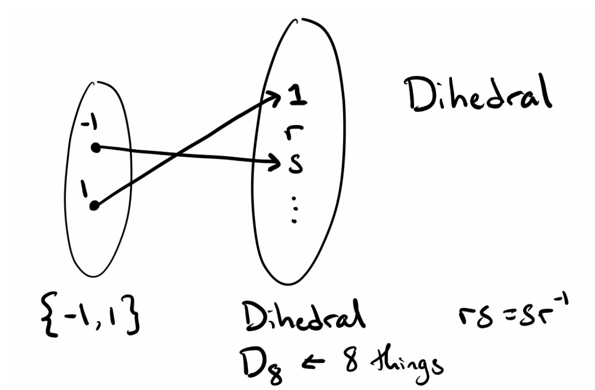


Figure 12. Sketch of Counterexample. (Recreated for clarity.)

The counterexample made it clear it is insufficient that ϕ is merely a homomorphism. We asked the students *What other things do we need to be able to make this argument?* They recognized that "onto" was needed, but after a great deal of debate remained unsure whether "one-to-one" was important with several students feeling "You need both of them."

2.4.3 Using Proofs to Explore Statement Modification

At this point, we redirected them to using the proof attempts as a tool for analysis by prompting students to identify where the one-to-one and onto assumptions would be needed, by asking *Where in the argument is onto and one-to-one used?* The students easily recognized the role of the onto assumption pointing out the line "So, $\exists a, b$ such that $\phi(a) = c$ and $\phi(b) = d$" where onto is "utilize[d] ..to create our images."

However, the students struggled to identify a portion of the proof that used one-to-one ultimately making statements in their group like the following:

> I think that's the biggest thing, it's not really used in the proof itself, but the argument part of it is just stated.

We used this as an opportunity to prompt students to segment the proof in order to identify which parts were setting up the assumptions and which was part of the actual proof argument. (See Figure 13.)

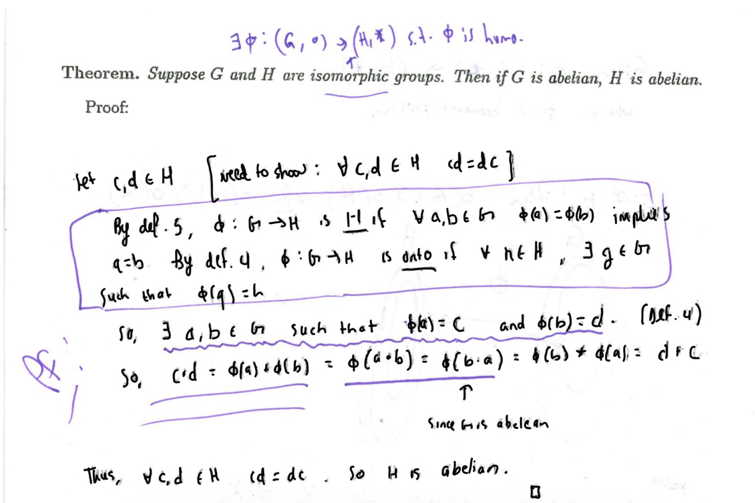


Figure 13. One Student's Segmented Proof.

After continued discussion, the students determined that "onto" was used in the argument, but one-to-one was not. At this point, we endorsed the fact that one-to-one was not needed to extinguish lingering doubt. The students then updated their modified statement to include an "onto homomorphism."

2.4.4 Patching the Proof

After determining that the onto assumption was needed and the one-to-one assumption was not, the last part of our task was to return to the proof where the use of onto was not explicit and figure out if there was a way to *fix* the proof so that the "onto" warrant is leveraged. The partners started talking back-and-forth in analyzing the G-first proof:

Student C: So, they let $a, b \in G$.
Student B: They just let the [inaudible]?
Student C: So, how do you know that $\phi(a)$ is mapping? ...
Student B: To $\phi(a)$?
Student C: To $\phi(a)$? Because it's onto, but they didn't say that.

A student from the other partner team similarly asked, "So, then, just a final statement saying ... $\phi(a)$... where the pre-image of $\phi(a)$ maps to a? Because it's onto? Because then that's the only thing we need is saying that $\phi(a)$ is actually- was mapped from something" identifying the crux of the issue.

We had conjectured that the students would alter the last line of the proof to include surjectivity, but during the discussion the students decided that would be "convoluted." Rather, they ultimately suggested to modify the G-first approach to begin with c and d in H as in the second approach.

We then asked, *Thinking back to what our statement is, that we're trying to prove, why is [it] helpful to introduce a "c", something from H, before introducing something from G?* Student C explained in response, "Because then we can show for certain that two elements, arbitrary two elements, are abelian [commute]." To verify that all the students saw this realization, we asked them to sketch a function diagram to highlight where the concern is about starting with arbitrary elements of G. The students offered the diagram in Figure 14. The diagram does indeed highlight that without the onto assumption, simply considering the images of elements of G will not necessarily include all possible elements of H. (We note that while talking through the diagram, Student D recognized that the element $\phi(b)$ in the group H is not the value of the function ϕ at b.)

We used this realization and recognition to emphasize the importance of using the conclusion of the statement to structure a proof, similar to

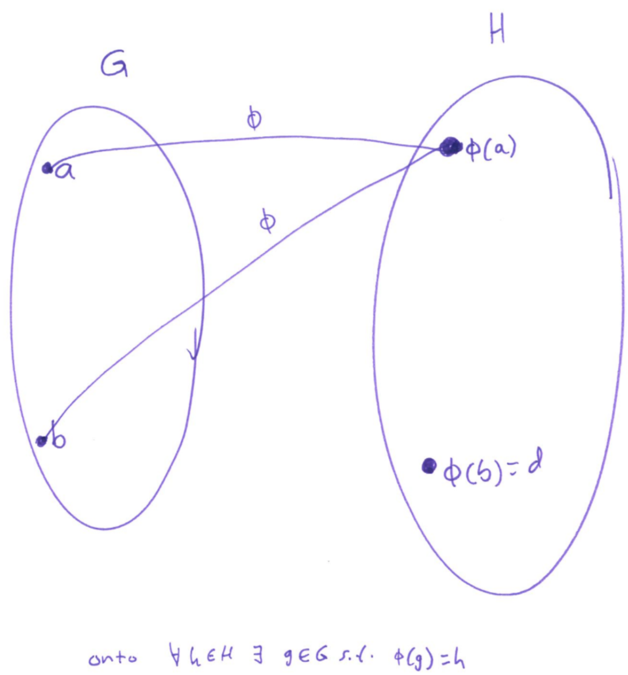


Figure 14. One Student's Function Diagram Demonstrating the Necessity of Surjectivity.

the concept of proof framework by [6]. This was a major breakthrough point and illustrated that the students had arrived at the importance of starting with arbitrary elements from H, a structuring choice that we found uncommon amongst abstract algebra students in our prior research [1].

3 Student Impressions

After completing the task, we prompted the students to reflect on the different activities that they engaged with and how they related to them and their own thinking about proof. We gave them the following prompt: *Did any of these activities that we did make you think about how you, yourself, work with proofs?*

The students responded positively explaining a number of aspects they appreciated. One student noted that they had not previously focused on what needs to actually be proven, explaining:

> I always follow definitions, so if it was last semester when I was proving this was abelian, I would've proved it was one-to-one, I would've proved it was onto, I would've proved it was homomorphism, then would've gone to abelian. But then now, you can skip some stuff.

She noted that proving this way would be twice as fast.

Another student focused on statement modification and thinking about how this type of activity was similar to their research experiences in mathematics:

> That's what I like about research, is trying to remove strengths. Is this stronger? Can I like in topology thats all we talked about was counterexamples and ... do I really need to use all I'm given?

Another student commented on the metacognition involved:

> I don't think about it near as openly. I'm like, "Let me just prove this real quick." I don't think about as far as "why does it do that?" I don't ask myself, "Why does it work? What could we tune, what could make it..."

In general, these students focused on aspects of writing proofs that we were aiming to highlight: the importance of analyzing the statement to be proven, using proofs to modify statements, and exploring the importance of making sense of the "why" behind the proof. These student impressions highlight the potential value of this teaching activity. By engaging the students with the material, encouraging them to discuss the relevant terms, compare proof approaches, and analyze/modify the proofs; we see students engaging in more authentic mathematical activities than in a traditional lecture-based classroom. Not only do we see evidence of students engaging in these activities, but also evidence that they are aware of and valued these activities.

4 Facilitation Reflection

As facilitators of this activity, we found a couple aspects of best practices particularly fruitful. First, the role of public records was huge in the implementation of this task. The students frequently referenced back to our four main records: givens and what we want to prove, the similarities and differences, the outlines of the proofs, and the function diagrams. By leaving these available to students, they could continue to reason from them.

We then found the choice to select and sequence student ideas powerful. This is an approach to orchestrating classes quite prevalent in the K-12 literature (e.g., [8]). We purposefully had students explore two approaches to the proof with a lot of similarities (same set of warrants), but fundamental differences (starting in G versus H, explicitly using the onto assumption). Students could focus on comparing across strategies in ways that make the difference more apparent. This led to productive discussions around the necessity of the one-to-one and onto assumptions and the difference in proof and statement alignment across the two student approaches. As such, we see this task as supporting the K-12 literature results: it's productive for students to compare and contrast strategies.

Finally we note the crucial role of visual representations (function diagrams) and example generation. It was through these visuals that students developed examples and counterexamples, and made connections from the context of the proof to their understanding of functions. It also provided a common ground that the instructor and students could discuss key elements of the proof such as the role of the onto assumption and how elements were selected.

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APPENDIX: A NOTE ON FUNCTIONS

In all implementations of this task, at some point the definition of function served a crucial role. This occurred in two ways: attending to everywhere-defined and attending to well-defined. Implementing a task like this provides grounds for a just-in-time need to talk about what properties functions have. (Function properties came up at different times with different students.) Here is some sample dialogue from the implementation in this paper:

On everywhere-defined. The notion that a function is defined everywhere on its domain came up early in the implementation. As the first group presented the G-first approach, it became clear they were unsure what guarantees that the elements a and b can be mapped to their image in H. We removed the isomorphism requirement and just asked if ϕ is a function (not necessarily injective or surjective) whether we can map a to $\phi(a)$ and b to $\phi(b)$. The students doubted this was the case. One student voiced:

> My only concern is because it matters what kind of function we're dealing with in this instance. For instance, $\frac{1}{x}$ ... if say, G worked for integers, and the function is $\frac{1}{x}$ well zero doesn't exist, so we can't always say for certain, "a and b will just map to whatever that is."

This provided us the opportunity to discuss a function being defined everywhere on its domain.

On well-defined. Later in the activity, when the students were attempting to identify where one-to-one was leveraged in the proof, the notion of well-defined was explored. In particular, the students often conflated the properties of being one-to-one and well-defined. The students conjectured that $ab = ba$ implying $\phi(ab) = \phi(ba)$ was "the only place where one-to-one is."

We took this time to address that this property was quite similar to one-to-one, but in a way it was "backwards." Another student then suggested this was the "well-defined" property of a function. This led to a brief digression into what well-defined means and how this property is different than one-to-one.