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THE P_2^1 MARGOLIS HOMOLOGY OF CONNECTIVE TOPOLOGICAL MODULAR FORMS

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Abstract

The element P_2^1 of the mod 2 Steenrod algebra $\mathcal{A}$ has the property $(\mathsf{P}_2^1)^2 = 0$. This property allows one to view P_2^1 as a differential on $H_*(X, \mathbb{F}_2)$ for any spectrum X. Homology with respect to this differential, $\mathcal{M}(X, \mathsf{P}_2^1)$, is called the P_2^1 Margolis homology of X. In this paper we give a complete calculation of the P_2^1 Margolis homology of the 2-local spectrum of topological modular forms *tmf* and identify its $\mathbb{F}_2$ basis via an iterated algorithm. We apply the same techniques to calculate P_2^1 Margolis homology for any smash power of *tmf*.

Convention. Throughout this paper we work in the stable homotopy category of spectra localized at the prime 2.

1. Introduction

The connective E_∞ ring *spectrum of topological modular forms tmf* has played a vital role in computational aspects of chromatic homotopy theory over the last two decades [**Goe10**], [**DFHH14**]. It is essential for detecting information about the chromatic height 2, and it has the rare quality of having rich Hurewicz image. There is a $K(2)$-local equivalence [**HM14**]

$$L_{K(2)} tmf \simeq E_2^{hG_{48}},$$

where E_2 is the second Morava E-theory at $p = 2$ and G_{48} is the maximal finite subgroup of the Morava stabilizer group $\mathbb{G}_2$. The spectrum $E_2^{hG_{48}}$ can be used to build the $K(2)$-local sphere spectrum (see [**BG18**]). The homotopy groups of *tmf* approximate both the stable homotopy groups of spheres and the ring of integral modular forms. In many senses, *tmf* is the chromatic height 2 analogue of connective real K-theory *ko*. Further, the homotopy groups of *tmf* are completely known [**Bau08**].

Let us now recall the definition of the element $\mathsf{P}_2^1 \in \mathcal{A}$. Milnor described the mod 2 dual Steenrod algebra $\mathcal{A}_*$ as the graded polynomial algebra [**Mil58**, App. 1]

$$\mathcal{A}_* \cong \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \dots],$$

where $|\xi_i| = 2^i - 1$. The Steenrod algebra $\mathcal{A}$ has an $\mathbb{F}_2$-basis dual to the monomial

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basis of $\mathcal{A}_*$. The elements of the $\mathbb{F}_2$-basis of $\mathcal{A}$ which are dual to $\xi_t^{2^s}$ are denoted by P_t^s, and the elements P_t^0 are denoted by Q_{t-1}. When $s<t$, the elements P_t^s are exterior power generators, i.e. $(\mathsf{P}_t^s)^2=0$. Thus, any left $\mathcal{A}$-module K can be regarded as a complex with differential given by the left multiplication by P_t^s (for $s<t$). This leads to the following definition.

Definition 1.1 ([**Mar83**])**.** Let K be any left $\mathcal{A}$-module and $0 \leqslant s < t$. Let

$$ {}^L\mathcal{P}_t^s \colon K \longrightarrow K $$

denote the left action by P_t^s. The *left* P_t^s *Margolis homology group* of K, $\mathcal{M}^L(K,\mathsf{P}_t^s)$, is defined as

$$ \mathcal{M}^L(K,\mathsf{P}_t^s) := \frac{\operatorname{Ker} {}^L\mathcal{P}_t^s \colon K \to K}{\operatorname{Im} {}^L\mathcal{P}_t^s \colon K \to K}. $$

For a right $\mathcal{A}$-module K, one can similarly define the *right* P_t^s *Margolis homology group* of K as

$$ \mathcal{M}^R(K,\mathsf{P}_t^s) := \frac{\operatorname{Ker} {}^R\mathcal{P}_t^s \colon K \to K}{\operatorname{Im} {}^R\mathcal{P}_t^s \colon K \to K}, $$

where ${}^R\mathcal{P}_t^s$ is the right action by P_t^s on K.

Notation 1.2. For a spectrum X, $\mathcal{M}(X,\mathsf{P}_t^s)$ will denote $\mathcal{M}^L(H^*(X),\mathsf{P}_t^s)$ or equivalently $\mathcal{M}^R(H_*(X),\mathsf{P}_t^s)$.

Computations of Margolis homology underly many essential computations in homotopy theory. For example, Adams work on $BP\langle 1\rangle$ cooperations [**Ada74**] relies on the computations of $\mathcal{M}(BP\langle 1\rangle,\mathsf{Q}_i)$ for $i=0,1$. Calculations like $\mathcal{M}(bo,\mathsf{Q}_i)$ for $i=0,1$ are essential ingredients in the work of Mahowald on *bo*-resolutions [**Mah81**]. More recently, Culver described $BP\langle 2\rangle$ resolutions [**Cul19**] by understanding $\mathcal{M}(BP\langle 2\rangle,\mathsf{Q}_i)$ for $i=0,1,2$. Computation of $\mathcal{M}(tmf^{\wedge n},\mathsf{Q}_2)$ is an essential ingredient in [**BBB$^+$**].

The element Q_i is primitive for all $i\in\mathbb{N}$. In other words, the comultiplication map Δ on $\mathcal{A}$ sends Q_i to

$$ \Delta(\mathsf{Q}_i) = \mathsf{Q}_i \otimes 1 + 1 \otimes \mathsf{Q}_i\,. \tag{1} $$

Consequently, Q_i acts on $H_*(X)$ as a derivation, namely it follows the Leibniz rule

$$ \mathsf{Q}_i(xy) = \mathsf{Q}_i(x)\cdot y + x\cdot \mathsf{Q}_i(y), $$

whenever X is a ring spectrum. The Leibniz rule implies the Künneth isomorphism [**Mar83**, Proposition 17, p. 343]

$$ \mathcal{M}(X\otimes Y,\mathsf{Q}_i) \cong \mathcal{M}(X,\mathsf{Q}_i)\otimes \mathcal{M}(Y,\mathsf{Q}_i) $$

and hence, $\mathcal{M}(X,\mathsf{Q}_i)$ is an $\mathbb{F}_2$ algebra whenever X is a ring spectrum. As a result, computation of Q_i Margolis homology and its description is often fairly straightforward.

On the other hand, for $s>0$, P_t^s is not a primitive element of $\mathcal{A}$. In particular,

$$ \Delta(\mathsf{P}_2^1) = \mathsf{P}_2^1 \otimes 1 + \mathsf{Q}_1\otimes \mathsf{Q}_1 + 1\otimes \mathsf{P}_2^1 $$

and its action on $H_*(X)$ for a ring spectrum X, does not follow the Leibniz rule.

Instead, we have

$$\mathsf{P}_2^1(xy) = \mathsf{P}_2^1(x)y + \mathsf{Q}_1(x)\,\mathsf{Q}_1(y) + x\,\mathsf{P}_2^1(y). \tag{2}$$

As a result, the product of two P_2^1 cycles may not necessarily be a P_2^1 cycle, hence $\mathcal{M}(X, \mathsf{P}_2^1)$ may not admit any multiplicative structure even if X is a ring spectrum. This is the main reason why the P_2^1 Margolis homology calculations are significantly more complicated.

Let us now consider the spectrum *tmf*. It is well-known ([**HM14**], [**Mat16**]) that

$$H_*(tmf; \mathbb{F}_2) \cong \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \zeta_5, \dots] \subset \mathcal{A}_*$$

is a subalgebra of $\mathcal{A}_*$. Here the elements ζ_i are the images of ξ_i under the antipode of the Hopf algebra $\mathcal{A}_*$ (see Section 2). The right action of Q_i is given by the formula (see [**Cul19**, §2] for details)

$$\mathsf{Q}_i(\zeta_n) = \zeta_{n-i-1}^{2^{i+1}}.$$

Then, since the Q_i are derivations, it can be easily seen that

$$\mathcal{M}(tmf, \mathsf{Q}_0) = \mathbb{F}_2[\zeta_1^8, \zeta_2^4], \qquad \mathcal{M}(tmf, \mathsf{Q}_1) = \frac{\mathbb{F}_2[\zeta_1^8, \zeta_3^2, \zeta_4^2, \dots]}{\langle \zeta_3^4, \zeta_4^4, \dots \rangle},$$

$$\text{and} \quad \mathcal{M}(tmf, \mathsf{Q}_2) = \frac{\mathbb{F}_2[\zeta_2^4, \zeta_3^2, \zeta_4^2, \dots]}{\langle \zeta_2^8, \zeta_3^8, \zeta_4^8, \dots \rangle}.$$

In this paper, we give a complete calculation of $\mathcal{M}(tmf^{\wedge r}, \mathsf{P}_2^1)$ for arbitrary $r \geqslant 1$. In fact, the calculation for $r > 1$ follows from the case $r = 1$, because after forgetting the internal grading one can construct a non-canonical isomorphism (see Section 4)

$$\mathcal{M}(tmf^{\wedge r}, \mathsf{P}_2^1) \cong \mathcal{M}(tmf, \mathsf{P}_2^1).$$

For the case $r = 1$, we give an iterated algorithm (see Definition 3.14) that constructs an $\mathbb{F}_2$-basis of $\mathcal{M}(tmf, \mathsf{P}_2^1)$. We give a complete description of $\mathcal{M}(tmf, \mathsf{P}_2^1)$ in Theorem 3.16 which is the main result of this paper. Although $\mathcal{M}(tmf, \mathsf{P}_2^1)$ is not an algebra, we notice that $\mathcal{M}(tmf, \mathsf{P}_2^1)$ is a module over an infinitely generated exterior algebra $\mathcal{S}$ (see Lemma 3.1 for a description of $\mathcal{S}$). Theorem 3.16 also describes $\mathcal{M}(tmf, \mathsf{P}_2^1)$ as an $\mathcal{S}$-module.

The key tool we use is the *length spectral sequence* (9), which we define in Section 2. The length spectral sequence admits a d_0 differential and a d_2 differential and collapses at the E_3 page. The Leibniz rule does hold for the d_0, but not for d_2. In order to work around this issue, we notice that the E_2 page admits an action of $\mathcal{S}$ (i.e. d_2 are $\mathcal{S}$ linear) and we use it to simplify the computation of $E_\infty = E_3$.

We also notice that almost identical calculations lead to a complete description of $\mathcal{M}(((B\mathbb{Z}/2)^{\times k})_+, \mathsf{P}_2^1)$. The methods developed in this paper can be considered as a blueprint for computations of P_t^1 Margolis homology of a variety of other $\mathcal{A}$-modules.

Our calculations of $\mathcal{M}(tmf^{\wedge r}, \mathsf{P}_2^1)$ have many applications, as the spectrum *tmf* has a wide range of applications, particularly in chromatic homotopy theory. First note that the cohomology of *tmf*, as a module over the Steenrod algebra $\mathcal{A}$, is isomorphic to (see [**HM14**], [**Mat16**])

$$H^*(tmf; \mathbb{F}_2) \cong \mathcal{A}/\!\!/\mathcal{A}(2), \tag{3}$$

where $\mathcal{A}(2)$ is the subalgebra of $\mathcal{A}$ generated by Sq^1, Sq^2 and Sq^4. This, and a change of rings isomorphism, imply that the E_2 page of the Adams spectral sequence converging to tmf_*X (for a spectrum X) is

$$E_2^{s,t} := Ext^{s,t}_{\mathcal{A}(2)}(H^*(X), \mathbb{F}_2). \tag{4}$$

One can detect infinite families in the E_2 page via the map

$$q\colon Ext^{s,t}_{\mathcal{A}(2)}(H^*(X), \mathbb{F}_2) \longrightarrow Ext^{s,t}_{\Lambda(\mathsf{P}^1_2)}(H^*(X), \mathbb{F}_2).$$

The codomain of q can be understood by calculating $\mathcal{M}(X, \mathsf{P}^1_2)$. Note that

$$Ext^{s,t}_{\Lambda(\mathsf{P}^1_2)}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[h_{2,1}],$$

where $|h_{2,1}| = (1, 6)$ and

$$\mathbb{F}_2[h_{2,1}] \otimes \mathcal{M}(X, \mathsf{P}^1_2) \subset Ext^{s,t}_{\Lambda(\mathsf{P}^1_2)}(H^*(X), \mathbb{F}_2)$$

accounts for all the elements with positive s filtration. This shows that the knowledge of $\mathcal{M}(X, \mathsf{P}^1_2)$ is crucial in detecting patterns in the E_2-page of (4).

Motivation I — Towards homotopy groups of $K(2)$-local sphere

Computation of the homotopy groups of $L_{K(n)}S^0$ — the sphere spectrum localized with respect to Morava K-theories $K(n)$ at various primes p and heights n — is the central question of chromatic homotopy theory. It is sometimes easier to compute $\pi_*L_{K(n)}X$ for finite complexes other than the sphere, although very little data like this is known at $n = p = 2$ anyway. Recently, Bhattacharya and Egger introduced a family of finite spectra Z [**BE20a**], and $\pi_*L_{K(2)}Z$ has been computed [**BBB$^+$, BE20b**], the first example of a finite complex at $p = 2$ whose $K(2)$-local homotopy groups are completely determined. The finite complex Z can be constructed from the sphere spectrum, by a succession of cofiber sequences of self-maps (see [**BE20a**]), the last one of which is

$$\Sigma^5 A_1 \wedge C\nu \xrightarrow{w} A_1 \wedge C\nu \longrightarrow Z.$$

In a quest to leverage the knowledge of $\pi_*L_{K(2)}Z$ to $\pi_*L_{K(2)}S^0$, one must first attempt to compute the $K(2)$-local homotopy groups of $A_1 \wedge C\nu$. Very briefly, our strategy is to use the v_2-local *tmf*-based Adams spectral sequence

$$E_1^{r,t} = v_2^{-1}\pi_t(tmf \wedge \overline{tmf}^{\wedge r} \wedge A_1 \wedge C\nu) \Longrightarrow \pi_{t-r}(L_{K(2)}A_1 \wedge C\nu)$$

and compare it with that of Z. One can identify the E_1-page of the above spectral sequence using the classical Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}}(H^*(tmf \wedge \overline{tmf}^{\wedge r} \wedge A_1 \wedge C\nu), \mathbb{F}_2) \Rightarrow \pi_{t-s}(tmf \wedge \overline{tmf}^{\wedge r} \wedge A_1 \wedge C\nu). \tag{5}$$

Because of (3) and the fact that $H^*(A_1 \wedge C\nu) \cong \mathcal{A}(2) /\!\!/ \Lambda(\mathsf{Q}_2, \mathsf{P}^1_2)$, and the change of rings isomorphism, the E_2-page of the spectral sequence (5) has the form

$$\mathrm{Ext}^{s,t}_{\Lambda(\mathsf{Q}_2, \mathsf{P}^1_2)}(H^*(\overline{tmf}^{\wedge r}), \mathbb{F}_2).$$

Hence, computation of $\mathcal{M}(tmf^{\wedge r}, \mathsf{P}^1_2)$ is essential for understanding the E_2-page of (5).

Motivation II — *tmf* resolution of the sphere spectrum

The connective spectrum bo is not a flat ring spectrum, hence the E_2 page of the bo-based Adams spectral sequence does not have a straightforward expression like the classical Adams spectral sequence. However, Lellmann and Mahowald [**LM87**] were able to calculate the d_1 differentials (also see [**BBB+20**]) and gave a description of the "v_1-periodic part" of the E_2-page. They identified the free Eilenberg–MacLane summand of $bo^{\wedge r}$. To identify this free summand one needs to identify the $\mathcal{A}(1)$ free summand of

$$H^*(bo^{\wedge r}) \cong \mathcal{A}/\!\!/\mathcal{A}(1)^{\otimes r}.$$

This can be done by calculating $\mathcal{M}(bo^{\wedge r}, \mathsf{Q}_0)$ and $\mathcal{M}(bo^{\wedge r}, \mathsf{Q}_1)$ and using the following theorem due to Margolis.

Theorem 1.3 ([**Mar83**, Chapter 19, Theorem 6])**.** *An $\mathcal{A}(n)$-module K is free if and only if $\mathcal{M}(K, \mathsf{P}^s_t) = 0$ whenever $s + t \leqslant n + 1$ with $s < t$.*

To emulate the strategy of Lellmann and Mahowald to understand the *tmf*-based Adams spectral sequence for S^0 one needs to first identify the $\mathcal{A}(2)$-free part of

$$H^*(tmf^{\wedge r}) \cong (\mathcal{A}/\!\!/\mathcal{A}(2))^{\otimes r}.$$

Potentially, this can be identified using the knowledge of $\mathcal{M}(tmf^{\wedge r}, \mathsf{Q}_i)$ for $i = 0, 1, 2$ and $\mathcal{M}(tmf^{\wedge r}, \mathsf{P}^1_2)$, along with Theorem 1.3.

Motivation III — Infinite loop space of *tmf*

There are $\mathcal{A}$-modules $J(k)$, called Brown–Gitler modules [**BG73**], which assemble into a doubly graded $\mathcal{A}$-algebra, denoted here by $J(*)^*$. Moreover, there is an $\mathcal{A}$-module isomorphism $J(*)^* \cong \mathbb{F}_2[x_1, x_2, \dots]$ where $x_i \in J(2^i)^1$ and the left $\mathcal{A}$ action on $J(*)^*$ is [**Sch94**]

$$\mathsf{Sq}(x_i) = x_i + x_{i-1}^2.$$

In fact, $J(k)^*$ can be thought of as inheriting this action by virtue of being a subobject of $\mathcal{A}$. Because of this, minor modifications to methods of this paper apply to the calculation of $\mathcal{M}(J(k), \mathsf{P}^1_2)$. By [**KM13**] there is a spectral sequence, obtained by studying Goodwillie towers, relating the knowledge of $H_*(tmf; \mathbb{F}_2)$ to that of $H_*(\Omega^\infty tmf; \mathbb{F}_2)$ (also see [**HM16**] which provides a spectral sequence relating the cohomology of *tmf* to the cohomology of its infinite loop-space $H^*(\Omega^\infty tmf; \mathbb{F}_2)$). Roughly speaking, this relies on computing certain derived functors, usually labeled Ω^∞_s, in the category of unstable modules over $\mathcal{A}$. It turns out that there is an isomorphism (see [**Goe86**] or [**HK00**])

$$\Omega^\infty_s \Sigma^{-t}(\mathcal{A}/\!\!/\mathcal{A}(2))_* \cong \mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(\mathbb{F}_2, J(*)),$$

so that these computations require an understanding of the $J(k)$ as modules over $\mathcal{A}(2)$, the hardest part of which is understanding how P^1_2 acts.

Organization of the paper

In Section 2, we recall some facts about the Steenrod algebra and its dual. We introduce the spectral sequence (9), which computes the P^1_2 Margolis homology of *tmf*, and discuss the d_0 differentials in it.

In Section 3, we compute the $E_3 = E_\infty$ page of the spectral sequence (9). We do that by introducing building blocks M_J and computing $\mathcal{M}(M_J, \mathsf{P}_2^1)$. Then we establish the relationship between $\mathcal{M}(tmf, \mathsf{P}_2^1)$ and $\mathcal{M}(M_J, \mathsf{P}_2^1)$ in Theorem 3.16.

In Section 4, we show how to apply the same methods to calculate P_2^1 Margolis homology for $tmf^{\wedge r}$ and $((B\mathbb{Z}/2)^{\times k})_+$. Theorem 3.16 essentially gives the complete answer in these cases.

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2. Action of P_2^1 and the length spectral sequence

The dual Steenrod algebra $\mathcal{A}_* = \pi_*(H\mathbb{F}_2 \wedge H\mathbb{F}_2)$ has the structure of a graded commutative algebra which Milnor [**Mil58**] showed to be a polynomial algebra

$$\mathcal{A}_* \cong \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \dots],$$

where $|\xi_i| = 2^i - 1$. Milnor defined $\mathsf{Sq}(r_1, r_2, \dots) \in \mathcal{A}$ as the dual of $\xi_1^{r_1}\xi_2^{r_2}\dots$ and showed that they form an $\mathbb{F}_2$ basis of the Steenrod algebra $\mathcal{A}$, known as the Milnor basis. The P_t^s elements are defined as

$$\mathsf{P}_t^s = \mathsf{Sq}(r_1, \dots), \text{ where } r_i = \begin{cases} 0, & i \neq t, \\ 2^s, & i = t. \end{cases}$$

The action of an element $a \in \mathcal{A}$ on an $\mathcal{A}$-algebra follows the product rule given by the Cartan formula, i.e.

$$a(x \cdot y) = \Sigma_i a_i'(x) \cdot a_i''(y),$$

where $\Delta(a) = \Sigma_i a_i' \otimes a_i''$ is the comultiplication in the Hopf algebra $\mathcal{A}$.

Remark 2.1. We would like to note that standard commonly used notation for the generators of the dual Steenrod algebra at $p = 2$ differs from the notation in the original paper [**Mil58**], and we are grateful to John Rognes for explaining this to us. In [**Mil58**, Appendix 1], Milnor denotes the polynomial generators of the dual Steenod algebra at $p = 2$ by ζ_i, so that $\mathcal{A}_* \cong \mathbb{F}_2[\zeta_1, \zeta_2, \dots]$ and defines $\mathsf{Sq}(r_1, r_2, \dots)$ as dual to the element $\zeta_1^{r_1}\zeta_2^{r_2}\cdots$. It has since become standard in the literature [**MT68, Ada74, Mar83**] to use a different notation and to denote the polynomial generators which were denoted by ζ_i in [**Mil58**, Appendix 1] by ξ_i, in order to match the notation for the odd primary Steenrod algebra. Hence in current standard notation $\mathsf{Sq}(r_1, r_2, \dots)$ is dual to $\xi_1^{r_1}\xi_2^{r_2}\cdots$. The symbol ζ_i is now usually used to denote the image of ξ_i under the antipode of the Hopf algebra $\chi\colon \mathcal{A}_* \to \mathcal{A}_*$, induced by the 'flip map' on

$H\mathbb{F}_2 \wedge H\mathbb{F}_2$. The elements $\zeta_i = \chi(\xi_i)$ can be computed recursively using the formula $\sum_{i+j=k} \xi_i^{2^j}\chi(\xi_j) = 0$, together with the assumption that $\xi_0 = 1$ and $\xi_i = 0$ when $i < 0$.

The homology of *tmf* is the subalgebra of $\mathcal{A}_*$ ([**HM14**], [**Mat16**, Theorem 5.13])

$$\mathfrak{T} := H_*(tmf; \mathbb{F}_2) \cong (\mathcal{A}/\!\!/\mathcal{A}(2))_* = \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \zeta_5, \dots].$$

Thus the action of $\mathcal{A}$ on $\mathfrak{T}$ is simply the restriction of the action of $\mathcal{A}$ on $\mathcal{A}_*$.

The right action of $\mathcal{A}$ on $\mathcal{A}_*$ is determined by the action of the total squaring operation $\mathsf{Sq} = 1 + \sum_{i>0} \mathsf{Sq}^i$ [**Pea14**, Lemma 3.6]

$$(\zeta_i)\,\mathsf{Sq} = \zeta_i + \zeta_{i-1}^2 + \zeta_{i-2}^4 + \cdots + \zeta_1^{2^{i-1}} + 1 \tag{6}$$

which is a ring homomorphism.

Remark 2.2 (Action of the total squaring operation). There are multiple ways to define the action of $\mathcal{A}$ on $\mathcal{A}_*$. While we will be using the action defined by (6), we would like to collect other commonly used actions here. By [**Mah81**], the right and left actions of Sq on ξ_i are given by the formulas

$$\mathsf{Sq}(\xi_i) = \xi_i + \xi_{i-1}^2, \qquad\qquad (\xi_i)\,\mathsf{Sq} = \xi_i + \xi_{i-1},$$

while the left action on ζ_i is

$$\mathsf{Sq}(\zeta_i) = \zeta_i + \zeta_{i-1} + \cdots + \zeta_1 + 1.$$

From these formulas we can derive

$$\mathsf{Q}_{i-1}(\xi_n) = \xi_{n-i}^{2^i}, \qquad\qquad (\zeta_n)\,\mathsf{Q}_{i-1} = \zeta_{n-i}^{2^i};$$

the second equation can also be found in [**Cul19**].

Important Notation 2.3. Since we only work with the right action of Sq in this paper, we will write $a(x)$ to denote the *right* action of $a \in \mathcal{A}$ on $x \in H_*(tmf)$ for the rest of the paper. Thus, from now on

$$\boxed{a(x) := (x)a.}$$

We now focus on the action of $\mathsf{P}_2^1 = \mathsf{Sq}(0,2) = \mathsf{Sq}^2\,\mathsf{Sq}^4 + \mathsf{Sq}^4\,\mathsf{Sq}^2$ on $\mathfrak{T}$. From (6), one can easily see that Sq^{2i} acts trivially on ζ_n, when $i > 0$ and $n \neq 1$. It follows immediately that

$$\mathsf{P}_2^1(\zeta_i) = 0.$$

Beware! This *does not* mean that $\mathsf{P}_2^1(\zeta_i\zeta_j) = 0$, as the Leibniz rule does not hold. Since $\Delta(\mathsf{P}_2^1) = \mathsf{P}_2^1 \otimes 1 + \mathsf{Q}_1 \otimes \mathsf{Q}_1 + 1 \otimes \mathsf{P}_2^1$, we obtain the product formula

$$\mathsf{P}_2^1(xy) = \mathsf{P}_2^1(x)y + \mathsf{Q}_1(x)\,\mathsf{Q}_1(y) + x\,\mathsf{P}_2^1(y). \tag{7}$$

Using $\mathsf{Q}_1(\zeta_i) = \zeta_{i-2}^4$, we get

$$\mathsf{P}_2^1(\zeta_i\zeta_j) = \zeta_{i-2}^4\zeta_{j-2}^4, \qquad\qquad \mathsf{P}_2^1(\zeta_i^2) = \zeta_{i-2}^8. \tag{8}$$

Formulas become more complicated for triple products, e.g.

$$\mathsf{P}_2^1(\zeta_i\zeta_j\zeta_k) = \zeta_{i-2}^4\zeta_{j-2}^4\zeta_k + \zeta_{i-2}^4\zeta_j\zeta_{k-2}^4 + \zeta_i\zeta_{j-2}^4\zeta_{k-2}^4,$$

and in general we have the following result.

Lemma 2.4. *The action of* P_2^1 *on* $\mathfrak{T}$ *is given by the formula*

$$\begin{aligned}\mathsf{P}_2^1(\zeta_{i_1}\dots\zeta_{i_n}) &= \sum_{1\leqslant j<k\leqslant n} \frac{\zeta_{i_1}\dots\zeta_{i_n}}{\zeta_{i_j}\zeta_{i_k}}\,\mathsf{Q}_1(\zeta_{i_j})\,\mathsf{Q}_1(\zeta_{i_k}) \\ &= \sum_{1\leqslant j<k\leqslant n} \zeta_{i_1}\dots\zeta_{i_{j-1}}\zeta_{i_j-2}^4\zeta_{i_{j+1}}\dots\zeta_{i_{k-1}}\zeta_{i_k-2}^4\zeta_{i_{k+1}}\dots\zeta_{i_n},\end{aligned}$$

where indices are allowed to repeat.

Proof. Follows from an inductive argument on n, using (7) and the facts that $\mathsf{P}_2^1(\zeta_i) = 0$ and $\mathsf{Q}_1(\zeta_i) = \zeta_{i-2}^4$. □

The technique developed in this paper begins with the following observation. Consider the subalgebra

$$\mathcal{E} := \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4^2, \zeta_5^2, \dots] \subset \mathfrak{T} = \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \zeta_5, \dots]$$

which we will call the *even* subalgebra of $\mathfrak{T}$, as every element in $\mathcal{E}$ has even grading. Since $|\,\mathsf{Q}_1\,| = 3$ and every element in $\mathcal{E}$ has even grading, Q_1 must act trivially on $\mathcal{E}$. Thus, P_2^1 restricted to $\mathcal{E}^{\otimes r}$ follows the Leibniz rule, therefore $(\mathcal{E}^{\otimes r}, \mathsf{P}_2^1)$ is a differential graded algebra, and hence, $\mathcal{M}(\mathcal{E}^{\otimes r}, \mathsf{P}_2^1)$ is an algebra. Using (8) and the Künneth isomorphism, we can easily deduce the following result.

Lemma 2.5. *The* P_2^1 *Margolis homology of* $\mathcal{E}$ *is given by*

$$\mathcal{M}(\mathcal{E}, \mathsf{P}_2^1) \cong \Lambda(\zeta_2^4, \zeta_3^4, \zeta_4^4, \dots).$$

Moreover

$$\mathcal{M}(\mathcal{E}^{\otimes r}, \mathsf{P}_2^1) \cong \mathcal{M}(\mathcal{E}, \mathsf{P}_2^1)^{\otimes r} \cong (\Lambda(\zeta_2^4, \zeta_3^4, \zeta_4^4, \dots))^{\otimes r}.$$

Notation 2.6. For a set A, we let $\mathbb{F}_2\langle A\rangle$ denote the $\mathbb{F}_2$-vector space which has the generating set A.

Now consider the quotient $\mathcal{K} := \mathfrak{T}/\!/\mathcal{E} \cong \mathbb{F}_2 \otimes_{\mathcal{E}} \mathfrak{T}$. We have an isomorphism $\mathcal{K} \cong \Lambda(\zeta_4, \zeta_5, \dots)$, and the induced action of Q_1 and P_2^1 on $\mathcal{K}$ is trivial. The algebra $\mathcal{K}$ admits a natural increasing filtration

$$G^p(\mathcal{K}) := \mathbb{F}_2\langle \zeta_{i_1}\dots\zeta_{i_k} | k \leqslant p\rangle,$$

induced by the length of the monomials. We call it the *length filtration*.

This length filtration on $\mathcal{K}$ induces an increasing filtration $\{G^p(\mathfrak{T})\}_{p\geqslant 0}$ on $\mathfrak{T}$, where $G^p(\mathfrak{T})$ is the pullback of $G^p(\mathcal{K})$ (in vector spaces) along the quotient map $\mathfrak{T} \twoheadrightarrow \mathcal{K}$

$$\begin{array}{ccc} G^p(\mathfrak{T}) & \longrightarrow & \mathfrak{T} \\ \downarrow & & \downarrow\!\!\downarrow \\ G^p(\mathcal{K}) & \longrightarrow & \mathcal{K}. \end{array}$$

Definition 2.7. Let I be a finite tuple of natural numbers, and for $I = \{i_1, \dots, i_n\}$ let ζ^I denote the monomial $\zeta_1^{i_1}\dots\zeta_n^{i_n}$. Then the *length* L of ζ^I is defined by

$$L(\zeta^I) = \sum_{j=1}^{|I|} (i_j \bmod 2).$$

In other words, $L(\zeta^I)$ counts the number of odd exponents in ζ^I. Then $G^p(\mathfrak{T})$ is the span of monomials ζ^I of length less than or equal to p

$$G^p(\mathfrak{T}) \cong \mathbb{F}_2\langle \zeta^I | L(\zeta^I) \leqslant p\rangle.$$

The length function L measures "how far" a given monomial in $\mathfrak{T}$ is from the even subalgebra $\mathcal{E}$. Since there is an $\mathbb{F}_2$-vector space isomorphism

$$\mathfrak{T} \cong \mathcal{E} \otimes \mathfrak{T}/\!/\mathcal{E} = \mathcal{E} \otimes \mathcal{K}$$

any monomial $m \in \mathfrak{T}$ can be uniquely written as $e \cdot k$ where $e \in \mathcal{E}$ and $k \in \mathcal{K}$.

Example 2.8. If $m = \zeta_3^4\zeta_5^5\zeta_8^3$, then there is an unique expression $m = e \cdot k$, where $e = \zeta_3^4\zeta_5^4\zeta_8^2 \in \mathcal{E}$ and $k = \zeta_5\zeta_8 \in \mathcal{K}$.

The following lemma shows that the action of Q_1 and P_2^1 preserves the length filtration.

Lemma 2.9. *Let $m \in \mathfrak{T}$ be any monomial.*

(i) If $m \in \mathcal{E}$, then $\mathsf{Q}_1(m) = 0$ and $\mathsf{P}_2^1(m) \in \mathcal{E}$.

(ii) If $m \notin \mathcal{E}$, then $\mathsf{Q}_1(m)$ is a sum of monomials of length exactly $L(m) - 1$ and

$$\mathsf{P}_2^1(m) = m_L + m_{L-2},$$

where m_L is a sum of monomials of length exactly $L(m)$ and m_{L-2} is a sum of monomials of length exactly $L(m) - 2$.

Proof. When $m \in \mathcal{E}$, $\mathsf{Q}_1(m) = 0$ by the Leibniz rule. Using Lemma 2.4 we have $\mathsf{P}_2^1(m) \in \mathcal{E}$ and $L(\mathsf{P}_2^1(m)) = L(m) = 0$.

Now assume $m \notin \mathcal{E}$, which means $m = e \cdot k$ for some $e \in \mathcal{E}$ and some $1 \neq k \in \mathcal{K}$. Note that k is of the form $\zeta_{i_1} \dots \zeta_{i_n}$, where indices do not repeat.

The action of Q_1 is given by the formula

$$\mathsf{Q}_1(\zeta_{i_1} \dots \zeta_{i_n}) = \sum_{k=1}^{n} \zeta_{i_1} \dots \zeta_{i_{k-1}} \zeta_{i_k - 2}^4 \zeta_{i_{k+1}} \dots \zeta_{i_n},$$

where we allow repetition of indices. Since Q_1 acts trivially on $\mathcal{E}$, it follows that

$$\mathsf{Q}_1(e \cdot k) = e \cdot \mathsf{Q}_1(k).$$

From the formula above we see that $\mathsf{Q}_1(k) \neq 0$ and $L(\mathsf{Q}_1(k)) = L(k) - 1$. Hence,

$$L(\mathsf{Q}_1(m)) = L(e \cdot \mathsf{Q}_1(k)) = L(\mathsf{Q}_1(k)) = L(k) - 1 = L(e \cdot k) - 1 = L(m) - 1.$$

Next, note that

$$\mathsf{P}_2^1(m) = \mathsf{P}_2^1(e) \cdot k + \mathsf{Q}_1(e) \cdot \mathsf{Q}_1(k) + e \cdot \mathsf{P}_2^1(k) = \mathsf{P}_2^1(e) \cdot k + e \cdot \mathsf{P}_2^1(k).$$

From Lemma 2.4, we see that $L(\mathsf{P}_2^1(k)) = L(\mathsf{P}_2^1(k)) - 2$ assuming $\mathsf{P}_2^1(k) \neq 0$. Now set $m_L = \mathsf{P}_2^1(e) \cdot k$ and $m_{L-2} = e \cdot \mathsf{P}_2^1(k)$ ☐

Lemma 2.10. *The Hopf algebra $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$ is commutative and cocommutative.*

Proof. Commutativity follows from the fact that P_2^1 and Q_1 commute, see [**AM71**], Lemma 1.3(2) (in the notation of [**AM71**], $\mathsf{P}_2^1 = P_2(2)$ and $\mathsf{Q}_1 = P_2(1)$). Cocommutativity follows from the fact that the diagram

$$\begin{array}{ccc} \Lambda(\mathsf{Q}_1, \mathsf{P}_2^1) & \xrightarrow{\Delta} & \Lambda(\mathsf{Q}_1, \mathsf{P}_2^1) \otimes \Lambda(\mathsf{Q}_1, \mathsf{P}_2^1) \\ & \searrow^{\Delta} & \downarrow \text{flip} \\ & & \Lambda(\mathsf{Q}_1, \mathsf{P}_2^1) \otimes \Lambda(\mathsf{Q}_1, \mathsf{P}_2^1) \end{array}$$

commutes, because of (1) and (2). □

If M is a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module then let $\mathcal{C}_M^\bullet$ denote the periodic chain complex

$$\dots \xrightarrow{\mathsf{P}_2^1} M \xrightarrow{\mathsf{P}_2^1} M \xrightarrow{\mathsf{P}_2^1} \dots .$$

Its homology groups are isomorphic in each degree, i.e.

$$H_i(\mathcal{C}_M^\bullet) \cong H_j(\mathcal{C}_M^\bullet)$$

for all $i, j \in \mathbb{Z}$. We use $\mathcal{M}(M, \mathsf{P}_2^1)$ to denote this common homology group. When $M = \mathfrak{T}$, the filtration $G^\bullet(\mathfrak{T})$ induces a filtration on $C_{\mathfrak{T}}^\bullet$. By Lemma 2.9, P_2^1 respects the length filtration. This means we have a short exact sequence of chain complexes

$$0 \longrightarrow \bigoplus_{p\in\mathbb{Z}} G^{p-1}(C_{\mathfrak{T}}^\bullet) \longrightarrow \bigoplus_{p\in\mathbb{Z}} G^{p}(C_{\mathfrak{T}}^\bullet) \longrightarrow \bigoplus_{p\in\mathbb{Z}} \frac{G^{p}(C_{\mathfrak{T}}^\bullet)}{G^{p-1}(C_{\mathfrak{T}}^\bullet)} \longrightarrow 0.$$

Upon taking the homology, this short exact sequence of chain complexes produces an exact couple, resulting in a spectral sequence

$$E_1^{p,q} := H^q\left(\frac{G^p(C_{\mathfrak{T}}^\bullet)}{G^{p-1}(C_{\mathfrak{T}}^\bullet)}\right) \Rightarrow H^q(C_{\mathfrak{T}}^\bullet).$$

We rewrite this spectral sequence as

$$E_1^p := \mathcal{M}\left(\frac{G^p(\mathfrak{T})}{G^{p-1}(\mathfrak{T})}, \mathsf{P}_2^1\right) \Rightarrow \mathcal{M}(tmf, \mathsf{P}_2^1), \tag{9}$$

and we call it the *length spectral sequence*.

The E_1 page of (9) is easy to calculate. Note that the length filtration $G^\bullet(\mathfrak{T})$ is multiplicative, i.e.

$$G^p(\mathfrak{T}) \cdot G^{p'}(\mathfrak{T}) \subset G^{p+p'}(\mathfrak{T}),$$

hence the associated graded

$$\bigoplus_{p\geqslant 0} \frac{G^p(\mathfrak{T})}{G^{p-1}(\mathfrak{T})} \cong \mathcal{E} \otimes \mathcal{K}$$

is an $\mathbb{F}_2$-algebra. The action of $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$ on $\mathcal{E} \otimes \mathcal{K}$ is defined using the Cartan formula as in the definition below.

Definition 2.11 ([**Mar83**], p. 186)**.** Let Γ be any Hopf algebra. For two Γ-modules M and N, the underlying $\mathbb{F}_2$ vector space of $M \otimes N$ is simply $M \otimes_{\mathbb{F}_2} N$, and Γ acts via the diagonal map, i.e.

$$a(m \otimes n) = \sum_i a_i(m) \otimes a_i'(n),$$

where $a \in \Gamma$ and $\Delta(a) = \sum_i a_i \otimes a_i'$, where Δ is the coproduct of the Hopf algebra.

Now we describe the action of P_2^1 on a monomial $m \in \bigoplus_{p \geqslant -1} \frac{G^p(\mathfrak{T})}{G^{p-1}(\mathfrak{T})}$. Write $m = e \otimes k$ for some $e \in \mathcal{E}$ and $k \in \mathcal{K}$. By Definition 2.11

$$\mathsf{P}_2^1(m) = \mathsf{P}_2^1(e \otimes k) = \mathsf{P}_2^1(e) \otimes k.$$

Since P_2^1 restricted to $\mathcal{E}$ follows the Leibniz rule, the E_1 page of (9) is also an $\mathbb{F}_2$-algebra and isomorphic to

$$E_1^* \cong \mathcal{M}(\mathcal{E} \otimes \mathcal{K}, \mathsf{P}_2^1) \cong \mathcal{M}(\mathcal{E}, \mathsf{P}_2^1) \otimes \mathcal{K} \cong \Lambda(\zeta_2^4, \zeta_3^4, \dots) \otimes \Lambda(\zeta_4, \zeta_5, \dots).$$

In order to avoid confusion regarding the multiplicative structure of E_1^*, it is convenient to rename the generators.

Notation 2.12. We set $x_i := \zeta_{i+3}$ and $t_i := \zeta_{i+1}^4$. Further, for finite subsets $I = \{i_1, \dots, i_n\} \subset \mathbb{N}$ and $J = \{j_1, \dots, j_m\} \subset \mathbb{N}$, we let t_I and x_I denote the monomials $t_{i_1} \dots t_{i_n}$ and $x_{j_1} \dots x_{j_m}$ respectively. We use $t_I x_J$ to denote the tensor product $t_I \otimes x_J$.

Lemma 2.9 and Lemma 2.10 imply that we have a commutative diagram of chain complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_p G^p(C^\bullet_{\mathfrak{T}}) & \longrightarrow & \bigoplus_p G^{p+1}(C^\bullet_{\mathfrak{T}}) & \longrightarrow & \bigoplus_p \frac{G^{p+1}(C^\bullet_{\mathfrak{T}})}{G^p(C^\bullet_{\mathfrak{T}})} & \longrightarrow & 0 \\
\| & & \downarrow{\scriptstyle \mathsf{Q}_1(\ )} & & \downarrow{\scriptstyle \mathsf{Q}_1(\ )} & & \downarrow{\scriptstyle \mathsf{Q}_1(\ )} & & \| \\
0 & \longrightarrow & \bigoplus_p G^{p-1}(C^\bullet_{\mathfrak{T}}) & \longrightarrow & \bigoplus_p G^{p}(C^\bullet_{\mathfrak{T}}) & \longrightarrow & \bigoplus_p \frac{G^{p}(C^\bullet_{\mathfrak{T}})}{G^{p-1}(C^\bullet_{\mathfrak{T}})} & \longrightarrow & 0.
\end{array}$$

Consequently there is an action of Q_1 on each page of (9), which shifts the length filtration by -1. In particular, we note $\mathsf{Q}_1(x_i) = t_i$ and in general

$$\mathsf{Q}_1(t_I x_J) = \sum_{j \in J} t_j t_I x_{J - \{j\}}. \tag{10}$$

Let $m \in \mathfrak{T}$ be any monomial, m_L and m_{L-2} be as in Lemma 2.9, and let $[m]$ denote the equivalence class in the E_1 page of (9) represented by m. Lemma 2.9 implies that the d_1 differential of (9) is trivial,

$$d_2([m]) = [m_{L-2}]$$

for the class of the monomial $m \in \mathfrak{T}$ in the E_1 page, and the spectral sequence (9) collapses at the E_3 page. If we write $m \in \mathfrak{T}$ as $m = e \cdot k$, where $e \in \mathcal{E}$ and $k \in \mathcal{K}$, then

$$d_2([m]) = [e \cdot \mathsf{P}_2^1(k)] = [e] \cdot [\mathsf{P}_2^1(k)].$$

This means that the d_2 differential of (9) is $\mathcal{M}(\mathcal{E}, \mathsf{P}_2^1)$-linear. It follows from the formula of Lemma 2.4 that

$$d_2(t_I x_J) = \sum_{K \in J[2]} t_K t_I x_{J-K}, \tag{11}$$

where $J[2]$ is the set of subsets of J which contain two elements.

The formula for the d_2 differentials is intimately related to the action of Q_1 on the E_2 page of (9). The $\Lambda(\mathsf{Q}_1)$-module structure on $E_2^\bullet$ (see (10)) can be extended to the $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module structure using the algebra structure of $E_2^\bullet$ and the product formula (7), together with

$$\mathsf{P}_2^1(x_i) = \mathsf{P}_2^1(t_i) = 0.$$

The action of P_2^1 that results from this procedure is

$$\mathsf{P}_2^1(t_I x_J) = \sum_{K \in J[2]} t_K t_I x_{J-K} \tag{12}$$

on the monomial basis, which can be extended to all of $E_2^\bullet$ using $\mathbb{F}_2$-linearity. Notice that the action we obtain through this process coincides with the formula for the d_2 differentials (11).

3. The reduced length

For convenience, we denote the E_2-page of (9) by

$$\mathcal{R} = \Lambda(t_i : i \geqslant 1) \otimes \Lambda(x_i : i \geqslant 1),$$

which is an $\mathbb{F}_2$-algebra, as well as a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module, where the actions of Q_1 and P_2^1 are given by (10) and (12) respectively. In this section we analyze the $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module structure of $\mathcal{R}$, which leads us to a description of

$$E_\infty^\bullet \cong \ldots \cong E_3^\bullet \cong H(E_2^\bullet, d_2) \cong \mathcal{M}(\mathcal{R}, \mathsf{P}_2^1).$$

The main idea here is to notice (this will be shown in Lemma 3.3) that the action of P_2^1 is linear with respect to the subalgebra

$$\mathcal{S} := \Lambda(t_i x_i | i \in \mathbb{N}_+) \subset \mathcal{R},$$

which implies that $\mathcal{M}(\mathcal{R}, \mathsf{P}_2^1)$ admits an $\mathcal{S}$-module structure.

Lemma 3.1. *The subalgebra $\mathcal{S} \subset \mathcal{R}$ is a trivial $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-submodule which splits off as a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module.*

Proof. For any element $t_I x_I \in \mathcal{S}$, it is clear from (10) and (11) that

$$\mathsf{Q}_1(t_I x_I) = 0 = \mathsf{P}_2^1(t_I x_I).$$

Thus $\mathcal{S}$ is a trivial submodule.

Now observe from (10) and (11) that none of the monomials $t_I x_I \in \mathcal{S}$ is a summand of $\mathsf{Q}_1(t_{I'} x_{J'})$ or $\mathsf{P}_2^1(t_{I'} x_{J'})$ for any choice of I' and J'. Hence, $\mathcal{S}$ is a split summand. □

Corollary 3.2. *Every element of $\mathcal{S}$ is a nonzero cycle in the $\mathcal{M}(\mathcal{R}, \mathsf{P}_2^1)$.*

Lemma 3.3. *The action of P_2^1 on $\mathcal{R}$ is $\mathcal{S}$-linear.*

Proof. It is enough to show that

$$\mathsf{P}_2^1(t_i x_i \cdot t_I x_J) = (t_i x_i) \cdot \mathsf{P}_2^1(t_I x_J). \tag{13}$$

If $i \in I$, then $t_i t_I = 0$. Hence both the LHS and the RHS of (13) are zero.

If $i \in J$, then $x_i x_J = 0$, hence LHS of (13) is zero. On the other hand,

$$\begin{aligned} RHS &= t_i x_i \cdot \sum_{K \in J[2]} t_K t_I x_{J-K} \\ &= \sum_{i \in K \in J[2]} t_i t_K t_I x_i x_{J-K} + t_i \cdot \left(\sum_{i \notin K \in J[2]} t_i t_I x_i x_{J-K} \right) = 0, \end{aligned}$$

as $t_i t_K = 0$ when $i \in K$ and $x_i x_{J-K} = 0$ when $i \notin K$.

Now consider the case when $i \notin I \cup J$. Let $I' = I \cup \{i\}$ and $J' = J \cup \{i\}$. Then,

$$\begin{aligned} \mathsf{P}_2^1(t_i x_i \cdot t_I x_J) = \mathsf{P}_2^1(t_{I'} x_{J'}) &= \sum_{K \in J'[2]} t_K t_{I'} x_{J'-K} \\ &= \sum_{i \in K \in J'[2]} t_K t_{I'} x_{J'-K} + \sum_{i \notin K \in J'[2]} t_K t_{I'} x_{J'-K} \\ &= \sum_{i \notin K \in J'[2]} t_K t_{I'} x_{J'-K} \\ &= t_i x_i \cdot \sum_{K \in J[2]} t_K t_I x_{J-K} \\ &= t_i x_i \cdot \mathsf{P}_2^1(t_I x_J). \end{aligned}$$

□

Remark 3.4. While the E_2 page of (9) admits an $\mathbb{F}_2$-algebra structure, the E_3 page does not admit any multiplicative structure. This is because the d_2 differentials do not follow the Leibniz rule and the product of d_2 cycles may not be a cycle. For example, x_i for all $i \in \mathbb{N}_+$, is a d_2-cycle, whereas $x_i x_j$ for $i \neq j$ supports a differential $d_2(x_i x_j) = t_i t_j$ by (11). Even if α, β and $\alpha \cdot \beta$ are P_2^1 cycles it is unclear that the pairing $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$ is well-defined in the E_3 page.

Corollary 3.5. *$\mathcal{M}(\mathcal{R}, \mathsf{P}_2^1)$ is a module over the ring $\mathcal{S}$.*

Proof. By Lemma 3.3, there exists a pairing $\mu \colon \mathcal{S} \otimes \mathcal{R} \to \mathcal{R}$ such that the diagram

$$\begin{array}{ccc} \mathcal{S} \otimes \mathcal{R} & \xrightarrow{\mu} & \mathcal{R} \\ \downarrow{\scriptstyle 1 \otimes \mathsf{P}_2^1} & & \downarrow{\scriptstyle \mathsf{P}_2^1} \\ \mathcal{S} \otimes \mathcal{R} & \xrightarrow[\mu]{} & \mathcal{R} \end{array}$$

commutes. It follows that $\mathcal{M}(\mathcal{R}, \mathsf{P}_2^1)$ is an $\mathcal{S}$ module. □

As a result, we only need to understand the action of P_2^1 on the generators of $\mathcal{R}$ when viewed as an $\mathcal{S}$-module. In order to approach this problem we introduce the notion of *reduced length*.

Definition 3.6. For any monomial $t_I x_J \in \mathcal{R}$ the *reduced length* ℓ is

$$\ell(t_I x_J) = |J - I| = |J \cap I^c| = |J| - |J \cap I|,$$

where I^c denotes the complement of I.

Note that the length of $t_I x_J \in \mathcal{R}$ is given by the formula $L(t_I x_J) = |J|$; in other words, it is counting the number of factors of x_J. Whereas, $\ell(t_I x_J)$ counts only those factors x_j in x_J for which t_j is not a factor of t_I. For example,

$$\ell(x_1) = \ell(t_1 x_1 x_2) = \ell(t_1 t_2 x_1 x_2 x_3) = \ell(t_1 t_2 t_3 x_4) = 1,$$
$$\ell(x_1 x_2) = \ell(t_1 x_1 x_2 x_3) = \ell(t_1 t_2 t_3 t_4 x_5 x_6) = 2.$$

Remark 3.7. The reduced length function ℓ measures "how far" a given monomial in $\mathcal{R}$ is from the subalgebra $\mathcal{S}$.

For each $i \in \mathbb{N}_+$, let $M_i := \Lambda(\mathsf{Q}_1)\{x_i\} \subset \mathcal{R}$ denote the $\Lambda(\mathsf{Q}_1, \mathsf{P}^1_2)$-submodule isomorphic to $\Lambda(\mathsf{Q}_1)$ and generated by x_i. For an indexing set $K \underset{\text{finite}}{\subset} \mathbb{N}_+$, let

$$M_K := \bigotimes_{j \in K} M_j \subset \mathcal{R}$$

with the convention that $M_\emptyset := \mathbb{F}_2$. If the indexing set is $[n] = \{1, \ldots, n\} \subset \mathbb{N}_+$, then we write $M_{[n]}$ to denote $M_{\{1,\ldots,n\}}$.

In Figure 1, Figure 2 and Figure 3 we present M_i, $M_{\{1,2\}}$ and $M_{\{1,2,3\}}$ respectively as a $\Lambda(\mathsf{Q}_1, \mathsf{P}^1_2)$-module. In these figures the dotted curved lines depict the action of Q_1 and dashed straight lines depict the action of P^1_2.

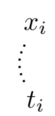

Figure 1: M_i as a module over $\Lambda(\mathsf{Q}_1, \mathsf{P}^1_2)$

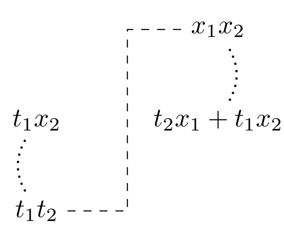

Figure 2: $M_{[2]}$ as a module over $\Lambda(\mathsf{Q}_1, \mathsf{P}^1_2)$, where $[2] = \{1, 2\}$

Note that the set $\mathcal{W} := \{t_I x_J \in \mathcal{R} | I \cap J = \emptyset\}$ forms a generating set for $\mathcal{R}$ as an $\mathcal{S}$-module as any monomial $t_I x_J \in \mathcal{R}$ can be uniquely written as a product of an element of $\mathcal{W}$ and a monomial in $\mathcal{S}$:

$$t_I x_J = t_{I \cap J} x_{I \cap J} \cdot t_{I - (I \cap J)} x_{J - (I \cap J)}.$$

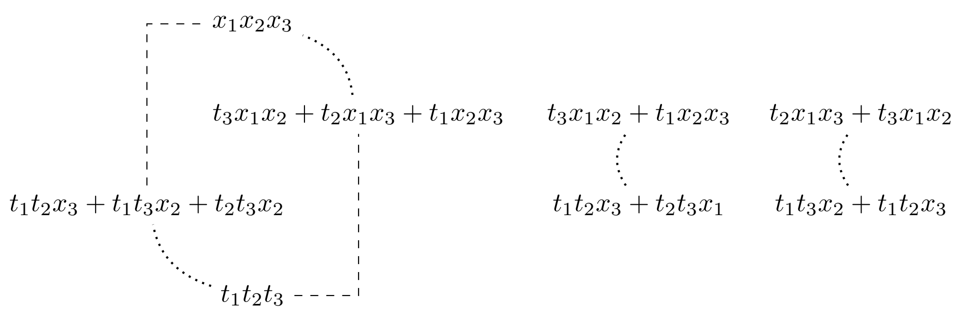

Figure 3: $M_{[3]}$ as a module over $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$, where $[3] = \{1, 2, 3\}$

For any finite subset $K \subset \mathbb{N}_+$,

$$\mathcal{W}_K := \{t_I x_J | I \cup J = K, I \cap J = \emptyset\} \subset \mathcal{W}$$

forms an $\mathbb{F}_2$-basis for M_K, i.e. $\mathbb{F}_2\langle \mathcal{W}_K \rangle = M_K$. Since

$$\mathcal{W} = \bigsqcup_{K \underset{\text{finite}}{\subset} \mathbb{N}_+} \mathcal{W}_K$$

and $\mathbb{F}_2\langle \mathcal{W}_K \rangle = M_K$ is closed under the action of Q_1 and P_2^1 (these actions preserve the total indexing set K, by (10) and (11)), we learn that

$$\mathcal{R}/\!/\mathcal{S} \cong \mathbb{F}_2 \otimes_{\mathcal{S}} \mathcal{R} \cong \mathcal{R} \otimes_{\mathcal{S}} \mathbb{F}_2 \cong \bigoplus_K M_K$$

is an isomorphism of $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-modules. Consequently, we have the following lemma.

Lemma 3.8. *Let $\mathcal{S}_K \subset \mathcal{S}$ denote the subalgebra $\Lambda(t_I x_I | I \subset \mathbb{N}_+ - K)$. There is a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module isomorphism*

$$\bigoplus_{K \underset{\text{finite}}{\subset} \mathbb{N}_+} \mathcal{S}_K \otimes M_K \cong \mathcal{R}.$$

Proof. Consider the $\mathbb{F}_2$-vector space isomorphism

$$\iota \colon \mathcal{R} \longrightarrow \bigoplus_{K \underset{\text{finite}}{\subset} \mathbb{N}_+} \mathcal{S}_K \otimes M_K$$

which sends

$$t_I x_J \mapsto t_{I\cap J} x_{I\cap J} \otimes t_{I-(I\cap J)} x_{J-(I\cap J)} \in \mathcal{S}_K \otimes M_K,$$

where $K = I \cup J - I \cap J$. The map ι^{-1} sends

$$t_B x_B \otimes t_{K-A} x_A \mapsto t_{B\cup(K-A)} \cdot x_{B\cup A},$$

where $A \subset K$. This map is also a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module isomorphism as $\mathcal{S}_K \subset \mathcal{S}$ is a trivial $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module by Lemma 3.1. □

Hence we can reduce the problem of computing $\mathcal{M}(\mathcal{R}, \mathsf{P}_2^1)$ to computing $\mathcal{M}(M_K, \mathsf{P}_2^1)$ for various finite subsets K of $\mathbb{N}_+$. Thus, we first need to understand the structure of M_K as a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module.

Remark 3.9. Let $[n]$ denote the indexing set $\{1, \dots, n\} \subset \mathbb{N}_+$. If $|K| = n$, then there exists the unique order preserving bijection

$$\iota \colon [n] \longrightarrow K$$

and it induces an isomorphism $\iota \colon M_{[n]} \xrightarrow{\cong} M_K$. Thus it is enough to understand $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module structure of $M_{[n]}$ for all $n \in \mathbb{N}_+$.

As depicted in Figure 3, $M_{[3]}$ splits as a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module

$$M_{[3]} \cong \Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)\{x_1x_2x_3\} \oplus \Lambda(\mathsf{Q}_1)\{t_3x_1x_2 + t_1x_2x_3\} \oplus \Lambda(\mathsf{Q}_1)\{t_2x_1x_3 + t_3x_1x_2\} \tag{14}$$

as a sum of a free $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module and two copies of $\Lambda(\mathsf{Q}_1)$.

Remark 3.10. The splitting of (14) is a consequence of Lemma 2.10. Since $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$ is cocommutative, for any $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module M and $\sigma \in \mathbb{F}_2[\Sigma_n]$, the induced map

$$\sigma \colon M^{\otimes n} \longrightarrow M^{\otimes n}$$

is a map of $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-modules. Note that in the group ring $\mathbb{F}_2[\Sigma_3]$, the identity element can be written as a sum of idempotent elements

$$\mathbb{1} = e + f_1 + f_2.$$

For example, one can choose $e = \mathbb{1} + (1\ 2\ 3) + (1\ 2\ 3)$, $f_1 = \mathbb{1} + (1\ 2) + (1\ 3) + (1\ 3\ 2)$ and $f_2 = \mathbb{1} + (1\ 2) + (1\ 3) + (1\ 2\ 3)$. Then we have

$$M^{\otimes 3} \cong e(M^{\otimes 3}) \oplus f_1(M^{\otimes 3}) \oplus f_2(M^{\otimes 3}).$$

When $M \cong \Lambda(\mathsf{Q}_1)$, we get the decomposition of (14).

The splitting of (14), along with the following fact about finite dimensional Hopf algebras, is the key to understanding the structure of M_K.

Theorem 3.11 ([**NZ89**])**.** *If $\mathcal{H}$ is a finite dimensional connected Hopf algebra over a field $\mathbb{F}$, then for any $\mathcal{H}$-module M, $\mathcal{H} \otimes M$ is a free $\mathcal{H}$-module.*

Let us denote by A the $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module isomorphic to $\Lambda(\mathsf{Q}_1)$ and let $B := A \otimes A$. Then using (14) and Theorem 3.11, we notice that

$$M_{[3]} \cong B \otimes A \cong \{\text{Free}\} \oplus A^{\oplus 2}, \qquad M_{[4]} \cong \{\text{Free}\} \oplus B^{\oplus 2}, \qquad M_{[5]} \cong \{\text{Free}\} \oplus A^{\oplus 4},$$

where $\{\text{Free}\}$ denotes a free $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$ module. This iterative process can be continued as described in Lemma 3.12 below. We use $A\{y\}$, resp. $B\{y\}$, to specify that y generates A, resp. B, as a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$ module. For example, $M_i \cong A\{x_i\}$.

Lemma 3.12. *There exist elements $h_{2r+1,i} \in M_{[2r+1]}$ with $\ell(h_{2r+1,i}) = r + 1$ such that, as a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module,*

$$M_{[2r+1]} \cong \{\mathit{Free}\} \oplus \left(\bigoplus_{i=1}^{2^r} A\{h_{2r+1,i}\} \right).$$

There exist elements $h_{2r,i} \in M_{[2r]}$ with $\ell(h_{2r+1,i}) = r + 1$ such that, as a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module

$$M_{[2r]} \cong \{\mathit{Free}\} \oplus \bigoplus_{i=1}^{2^{r-1}} B\{h_{2r,i}\}.$$

Proof. Our proof is by induction on r. From Figure 1, Figure 2 and Figure 3, the claim is true for $k = 1, 2, 3$. Note that

$$h_{1,1} = x_1, \qquad h_{2,1} = x_1x_2, \qquad h_{3,1} = (t_3x_1 + x_3t_1)x_2, \qquad h_{3,2} = (t_2x_3 + t_3x_2)x_1.$$

Now assume that the result is true for $2r - 1$, i.e.

$$M_{[2r-1]} \cong \{\text{Free}\} \oplus \bigoplus_{1\leqslant i\leqslant 2^{r-1}} A\{h_{2r-1,i}\},$$

where $\ell(h_{2r-1,i}) = r$ and $\{\text{Free}\}$ is a free $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module. It follows that

$$M_{[2r]} \cong M_{[2r-1]} \otimes M_{2r} \cong (\{\text{Free}\} \otimes A\{x_{2r}\}) \oplus \bigoplus_{1\leqslant i\leqslant 2^{r-1}} B\{h_{2r-1,i} \cdot x_{2r}\}.$$

By Theorem 3.11, the first summand is, again, a free module. Set

$$h_{2r,i} = h_{2r-1,i} \cdot x_{2r}$$

and notice $\ell(h_{2r-1,i}x_{2r}) = \ell(h_{2r-1,i}) + \ell(x_{2r}) = r + 1$.

To complete the inductive argument, observe

$$\begin{aligned} M_{[2r+1]} &\cong M_{[2r-1]} \otimes B\{x_{2r}x_{2r+1}\} \\ &\cong \left(\{\text{Free}\} \oplus \bigoplus_{1\leqslant i\leqslant 2^{r-1}} A\{h_{2r-1,i}\}\right) \otimes B\{x_{2r}x_{2r+1}\} \\ &\cong \{\text{Free}\} \oplus \bigoplus_{1\leqslant i\leqslant 2^{r-1}} (A\{h_{2r+1,2i-1}\} \oplus A\{h_{2r+1,2i}\}), \end{aligned}$$

where one can define the generators $h_{2r-1,j}$ from Figure 3 by replacing x_1, x_2, x_3 with $h_{2r-1,i}$, x_{2r} and x_{2r+1} respectively. More specifically, one can define

$$h_{2r+1,2i-1} = \mathsf{Q}_1(h_{2r-1,i} \cdot x_{2r+1}) \cdot x_{2r}, \qquad\qquad h_{2r+1,2i} = h_{2r-1,i} \cdot \mathsf{Q}_1(x_{2r}x_{2r+1}).$$

It is easy to check that $\ell(h_{2r+1,j}) = r + 1$. □

Following the proof of Lemma 3.12, we can provide an explicit basis of $\mathcal{M}(M_K, \mathsf{P}_2^1)$. By Remark 3.9 it suffices to provide a basis for $\mathcal{M}(M_{[n]}, \mathsf{P}_2^1)$ for all $n \geqslant 1$. We do so inductively (see Definition 3.14), however we must treat the odd and the even case separately, essentially because of Lemma 3.12. Since A is a trivial $\Lambda(\mathsf{P}_2^1)$-module, $\mathcal{M}(A, \mathsf{P}_2^1) \cong A$, and we get

$$\mathcal{M}(M_{[2r+1]}, \mathsf{P}_2^1) \cong \mathcal{M}\left(\bigoplus_{i=1}^{2^r} A\{h_{2r+1,i}\}, \mathsf{P}_2^1\right) \cong \bigoplus_{i=1}^{2^r} A\{h_{2r+1,i}\}.$$

Thus the collection

$$\{h_{2r+1,i} : 1 \leqslant i \leqslant 2^r\} \cup \{\mathsf{Q}_1(h_{2r+1,i}) : 1 \leqslant i \leqslant 2^r\}$$

is an $\mathbb{F}_2$-basis of $\mathcal{M}(M_{[2r+1]}, \mathsf{P}_2^1)$. When n is even, say $n = 2r$, then

$$\mathcal{M}(M_{[2r]}, \mathsf{P}_2^1) \cong \bigoplus_{i=1}^{2^{r-1}} \mathcal{M}(B\{h_{2r,i}\}, \mathsf{P}_2^1).$$

Now note that, if $B\{x \otimes y\} = A\{x\} \otimes A\{y\}$ (where x and y are generators), then

$$\{\mathsf{Q}_1(x) \otimes y, x \otimes \mathsf{Q}_1(y)\}$$

is an $\mathbb{F}_2$-basis of $\mathcal{M}(B\{x \otimes y\}, \mathsf{P}_2^1)$. Using the fact that

$$h_{2r,i} = h_{2r-1,i} \cdot x_{2r},$$

we get Corollary 3.13 and Definition 3.14 thereafter.

Corollary 3.13. *Let* $\mathcal{M}(M_K, \mathsf{P}_2^1)_l = \{x \in \mathcal{M}(M_K, \mathsf{P}_2^1) | \ell(x) = l\}$.
If $|K| = 2r+1$, *then*

$$\dim \mathcal{M}(M_K, \mathsf{P}_2^1)_l = \begin{cases} 2^r, & \text{if } l = r, r+1, \\ 0, & \text{otherwise.} \end{cases}$$

If $|K| = 2r$, *then*

$$\dim \mathcal{M}(M_K, \mathsf{P}_2^1)_l = \begin{cases} 2^r, & \text{if } l = r, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Lemma 3.12 implies

$$M_{[2r+1]} \cong \{\text{Free}\} \oplus \bigoplus_{1 \leqslant i \leqslant 2^r} A\{h_{2r+1,i}\},$$

where $\ell(h_{2r+1,i}) = r+1$. By Lemma 2.9 we have $\ell(\mathsf{Q}_1(h_{2r+1,i})) = r$. Thus $\{h_{2r+1,i}\}$ is the basis for $\mathcal{M}(M_{[2r+1]}, \mathsf{P}_2^1)_{r+1}$ and $\{\mathsf{Q}_1(h_{2r+1,i})\}$ is the basis for $\mathcal{M}(M_{[2r+1]}, \mathsf{P}_2^1)_r$. Applying Remark 3.9 we deduce the statement about dimension for any M_K with $|K| = 2r+1$.

For the even case we have from Lemma 3.12

$$M_{[2r]} \cong \{\text{Free}\} \oplus \bigoplus_{1 \leqslant i \leqslant 2^{r-1}} B\{h_{2r,i}\},$$

where $\ell(h_{2r,i}) = r+1$. Then for each i, $\mathcal{M}(B\{h_{2r,i}\}, \mathsf{P}_2^1) = \mathcal{M}(B\{h_{2r,i}\}, \mathsf{P}_2^1)_r$ is an $\mathbb{F}_2$ vector space of dimension 2 generated by $\{h_{2r-1,i} \cdot t_{2r}, \mathsf{Q}_1(h_{2r-1,i}) \cdot x_{2r}\}$. ☐

Definition 3.14. We define the basis $\mathcal{B}_{[n],l}$ of $\mathcal{M}(M_{[n]}, \mathsf{P}_2^1)_l$ for $0 \leqslant l \leqslant n$ inductively starting with $\mathcal{B}_{[1],0} = \{t_1\}$ and $\mathcal{B}_{[1],1} = \{x_1\}$. Suppose we have defined

$$\mathcal{B}_{[2r-1],l} := \begin{cases} \{h_{2r-1,1}, \dots, h_{2r-1,2^{r-1}}\} & \text{if } l = r, \\ \{\mathsf{Q}_1(h_{2r-1,1}), \dots, \mathsf{Q}_1(h_{2r-1,2^{r-1}})\} & \text{if } l = r-1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then define:

$$\mathcal{B}_{[2r],r} := \{h_{2r-1,1} \cdot t_{2r}, \dots, h_{2r-1,2^{r-2}} \cdot t_{2r}\} \cup \{\mathsf{Q}_1(h_{2r-1,1}) \cdot x_{2r}, \dots, \mathsf{Q}_1(h_{2r-1,2^{r-2}}) \cdot x_{2r}\}$$

and set $\mathcal{B}_{[2r],l} := \emptyset$ if $l \neq r$.

Now define $h_{2r+1,2i-1} = \mathsf{Q}_1(h_{2r-1,i}) \cdot (x_{2r+1} \cdot x_{2r})$ and $h_{2r+1,2i} = h_{2r-1,i} \cdot \mathsf{Q}_1(x_{2r} x_{2r+1})$ and set

$$\mathcal{B}_{[2r+1],l} := \begin{cases} \{h_{2r+1,1}, \dots, h_{2r+1,2^{r-2}}\} & \text{if } l = r+1, \\ \{\mathsf{Q}_1(h_{2r+1,1}), \dots, \mathsf{Q}_1(h_{2r+1,2^{r-2}})\} & \text{if } l = r, \\ \emptyset & \text{otherwise.} \end{cases}$$

We let $\mathcal{B}_{[n]}$ denote the union $\bigcup_l \mathcal{B}_{[n],l}$. Let $\mathcal{B}_K$ denote the image of the $\mathcal{B}_{[n]}$ under the isomorphism $\iota \colon M_{[n]} \to M_K$ of Remark 3.9.

Example 3.15 (Examples of $\mathcal{B}_K$). We explicitly identify $\mathcal{B}_{[n]}$ using Definition 3.14 for $n \leqslant 4$, and for $n = 1, 2, 3$ we can compare to Figures 1, 2 and 3, to see that $\mathcal{B}_{[n]}$ is indeed the basis for $\mathcal{M}(M_{[n]}, \mathsf{P}_2^1)$.

- $\mathcal{B}_{[1]} = \{t_1, x_1\}$,
- $\mathcal{B}_{[2]} = \{t_1x_2, t_2x_1\}$,
- $\mathcal{B}_{[3]} = \{t_1t_2x_3 + t_1t_3x_2, t_1t_2x_3 + t_2t_3x_1\} \cup \{t_3x_1x_2 + t_2x_1x_3, t_3x_1x_2 + t_1x_2x_3\}$,
- $\mathcal{B}_{[4]} = \{t_1t_2x_3x_4 + t_1t_3x_2x_4,\ t_1t_2x_3x_4 + t_2t_3x_1x_4,\ t_3t_4x_1x_2 + t_2t_4x_1x_3,\ t_3t_4x_1x_2 + t_1t_4x_2x_3\}$.

Note that $P_K := \mathbb{F}_2\langle \mathcal{B}_K \rangle \subset M_K$ is a split summand. This is because the inclusion map $P_K \hookrightarrow M_K$ induces $\mathcal{M}(-, \mathsf{P}_2^1)$-isomorphism, or equivalently, the quotient M_K/P_K is a free $\Lambda(\mathsf{P}_2^1)$-module.

Theorem 3.16. *Let K be a finite subset of $\mathbb{N}_+$. Let*

$$\mathcal{SB}_K := \{t_I x_I \cdot b \,|\, I \cap K = \emptyset \text{ and } b \in \mathcal{B}_K\} \subset \mathcal{R}.$$

Then

$$\mathcal{B} := \coprod_{K \underset{\text{finite}}{\subset} \mathbb{N}_+} \mathcal{SB}_K$$

forms a basis of the $\mathbb{F}_2$-vector space $\mathcal{M}(tmf, \mathsf{P}_2^1)$ and

$$\mathcal{M}(tmf, \mathsf{P}_2^1) \cong \bigoplus_{K \underset{\text{finite}}{\subset} \mathbb{N}_+} \mathbb{F}_2\langle \mathcal{SB}_K \rangle \cong \bigoplus_{K \underset{\text{finite}}{\subset} \mathbb{N}_+} \mathcal{S}_K \otimes \mathcal{M}(M_K, \mathsf{P}_2^1)$$

is an isomorphism of $\mathbb{F}_2$-vector spaces.

Proof. By Lemma 3.8, we have a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$ module isomorphism

$$\mathcal{R} \cong \bigoplus_{K \underset{\text{finite}}{\subset} \mathbb{N}_+} \mathcal{S}_K \otimes M_K.$$

Therefore, the linearity of the action of P_2^1 (see Corollary 3.5) with respect to elements in $\mathcal{S}$ gives us

$$\begin{aligned}\mathcal{M}(tmf, \mathsf{P}_2^1) &\cong \mathcal{M}(\mathcal{R}, \mathsf{P}_2^1) \cong \mathcal{M}\left(\textstyle\bigoplus_{K \underset{\text{finite}}{\subset} \mathbb{N}_+} \mathcal{S}_K \otimes M_K, \mathsf{P}_2^1\right) \\ &\cong \textstyle\bigoplus_{K \underset{\text{finite}}{\subset} \mathbb{N}_+} \mathcal{S}_K \otimes \mathcal{M}(M_K, \mathsf{P}_2^1) \cong \bigoplus_{K \underset{\text{finite}}{\subset} \mathbb{N}_+} \mathcal{S}_K \otimes P_K \\ &\cong \textstyle\bigoplus_{K \underset{\text{finite}}{\subset} \mathbb{N}_+} \mathbb{F}_2\langle \mathcal{SB}_K \rangle. \end{aligned}$$

□

Remark 3.17. Let e denote the exchange map $e\colon \mathcal{R} \to \mathcal{R}$ which sends

$$e\colon t_I x_J \mapsto t_J x_I.$$

It seems to be the case that $[m] \in \mathcal{M}(tmf, \mathsf{P}_2^1)$ if and only if $[e(m)] \in \mathcal{M}(tmf, \mathsf{P}_2^1)$. The source of such symmetry is unclear to the authors, although it might be related to Spanier–Whitehead duality.

Finally, we would like to say a word about the module structure of $\mathcal{M}(tmf, \mathsf{P}_2^1)$ over $\mathcal{S}$. Note that the collection of elements

$$\mathcal{B}_{\mathcal{S}} := \{t_I x_I | I \underset{\text{finite}}{\subset} \mathbb{N}_+\}$$

forms an $\mathbb{F}_2$-basis of $\mathcal{S}$. The $\mathcal{S}$-module structure on $\mathcal{M}(tmf, \mathsf{P}_2^1)$ is extended from a pairing at the level of bases

$$\mathcal{B}_{\mathcal{S}} \otimes \mathcal{S}\mathcal{B}_K \xrightarrow{\mu} \mathcal{S}\mathcal{B}_K$$
$$s \otimes (s' \cdot b) \mapsto \begin{cases} (s \cdot s') \cdot b, & \text{if } I \cap K = \emptyset, \\ 0, & \text{if } I \cap K \neq \emptyset. \end{cases}$$

Remark 3.18. Recall that $H_*(tmf)$ was described in terms of ζ_i. We can convert an element of the Margolis homology expressed in terms of t_i and x_i back to an expression involving ζ_i using the identifications of Notation 2.12. For example,

$$t_4 t_9 x_2 x_6 + t_2 t_9 x_4 x_6$$

can be identified with the class represented by element $\zeta_5^5 \zeta_{10}^4 \zeta_9 + \zeta_3^4 \zeta_{10}^4 \zeta_7 \zeta_9 \in \mathfrak{T}$.

4. P_2^1 Margolis homology of $tmf^{\wedge r}$ and $((B\mathbb{Z}/2)^{\times k})_+$

4.1. P_2^1 Margolis homology of $tmf^{\wedge r}$

Note that

$$H_*(tmf^{\wedge r}) \cong H_*(tmf)^{\otimes r} \cong \mathfrak{T}^{\otimes r}.$$

We first extend the notion of length to $\mathfrak{T}^{\otimes r}$. For a monomial $\zeta^{I_1}| \dots |\zeta^{I_r}$ for $\zeta^{I_i} \in \mathfrak{T}^{\otimes r}$, which is a tensor product of monomials in $\mathfrak{T}$, we define

$$L(\zeta^{I_1}| \dots |\zeta^{I_r}) = L(\zeta^{I_1}) + \dots + L(\zeta^{I_r}).$$

We define the even subalgebra $\mathbb{E}_r$ of $\mathfrak{T}^{\otimes r}$ as the span of those monomials in $\mathfrak{T}^{\otimes r}$ whose lengths are zero. Observe that,

$$\mathbb{E}_r \cong \mathcal{E}^{\otimes r}.$$

The notion of length leads to an increasing filtration on $\mathfrak{T}^{\otimes r}$, called the length filtration, by setting

$$G^p(\mathfrak{T}^{\otimes r}) = \{(\zeta^{I_1}| \dots |\zeta^{I_r}) \,|\, L(\zeta^{I_1}| \dots |\zeta^{I_r}) \leqslant p\}.$$

Let $\mathbb{K}_r = \mathcal{K}^{\otimes r}$, where $\mathcal{K}$ is as defined in Section 2. Just like in the case $r = 1$, we get a length spectral sequence and its E_1 page is

$$E_1^{\bullet} \cong \mathcal{M}(\mathbb{E}_r, \mathsf{P}_2^1) \otimes \mathbb{K}_r \Rightarrow \mathcal{M}(tmf^{\wedge r}, \mathsf{P}_2^1). \tag{15}$$

Since the action of P_2^1 follows the Leibniz rule when restricted to $\mathcal{E}$, we get

$$\mathcal{M}(\mathbb{E}_r, \mathsf{P}_2^1) \cong \mathcal{M}(\mathcal{E}, \mathsf{P}_2^1)^{\otimes r}.$$

Notation 4.1. For shorthand, we denote

$$x_{i,j} = (\underbrace{1|\dots|1}_{j-1}|\zeta_{i+3}|\underbrace{1|\dots|1}_{r-j}), \qquad t_{i,j} = (\underbrace{1|\dots|1}_{j-1}|\zeta_{i+1}^4|\underbrace{|\dots|1}_{r-j}).$$

With this notation we have

$$\mathsf{Q}_1(x_{i,j}) = t_{i,j}.$$

Using Notation 4.1, we see that the E_1 page of the length spectral sequence (15), as an algebra, is isomorphic to

$$\mathcal{R}_r := \Lambda(t_{i,j} : i \in \mathbb{N} - \{0\}, 1 \leqslant j \leqslant r) \otimes \Lambda(x_{i,j} : i \in \mathbb{N} - \{0\}, 1 \leqslant j \leqslant r).$$

It is easy to see that the map induced by the reindexing map

$$\iota\colon (i,j) \mapsto r(i-1) + j,$$

produces a (non-canonical) isomorphism of algebras between $\mathcal{R}_r$ (the E_2 page of (15)) and $\mathcal{R}$ (the E_2 page of (9)), after forgetting the internal grading. This is also an isomorphism of $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-modules. Thus we have an isomorphism

$$\iota_*\colon \mathcal{M}(\mathit{tmf}, \mathsf{P}_2^1) \xrightarrow{\cong} \mathcal{M}(\mathit{tmf}^{\wedge r}, \mathsf{P}_2^1)$$

induced by the ι. Therefore, Theorem 3.16 essentially gives a complete calculation of $\mathcal{M}(\mathit{tmf}^{\wedge r}, \mathsf{P}_2^1)$.

Example 4.2. For example, let us assume $r = 3$. Then the element $t_2t_4x_6x_9 + t_2t_6x_4x_9 \in \mathcal{M}(\mathit{tmf}, \mathsf{P}_2^1)$ (see Example 3.15) corresponds to the element

$$t_{1,2}t_{2,1}x_{2,3}x_{3,3} + t_{1,2}t_{2,3}x_{2,1}x_{3,3} \in \mathcal{M}(\mathit{tmf}^{\wedge 3}, \mathsf{P}_2^1)$$

under the bijection obtained from the above reindexing. When expressed in terms of ζ_is (see Notation 4.1), the same element can be expressed as

$$\zeta_3^4|\zeta_2^4|\zeta_5\zeta_6|1 + \zeta_5|\zeta_2^4|\zeta_3^4\zeta_6|1.$$

Remark 4.3 (P_2^1 Margolis homology of Brown–Gitler spectra)*.* It is well-known that

$$H_*(\mathit{tmf}) \cong \bigoplus_{i \geqslant 0} H_*(\Sigma^{8i} bo_i),$$

where bo_i are certain Brown–Gitler spectra associated with bo. In [**Mah81**] Mahowald defined a multiplicative weight function, which is given by $w(\zeta_i) = 2^{i-1}$. $H_*(\Sigma^{8i} bo_i)$ is the summand of $H_*(\mathit{tmf})$ which consists of elements of Mahowald weight exactly equal to $8i$. We assign Mahowald weight of $t_{i,j}$ and $x_{i,j}$ as

$$w(t_{i,j}) = w(x_{i,j}) = 2^{i+1}.$$

It follows that the Margolis homology $\mathcal{M}(bo_{q_1} \wedge \cdots \wedge bo_{q_r}, \mathsf{P}_2^1)$ is a summand of $\mathcal{M}(\mathit{tmf}^{\wedge r}, \mathsf{P}_2^1)$. It consists of all polynomials of $\mathcal{M}(\mathit{tmf}^{\wedge r}, \mathsf{P}_2^1)$ expressed in terms of $x_{i,j}$ and $t_{i,j}$ such that $w(x_{i,j}) = w(t_{i,j}) = 4q_j$.

Remark 4.4. While it is true that $\mathcal{R}_r \cong \mathcal{R}^{\otimes r}$, as an $\mathbb{F}_2$-algebra as well as a $\Lambda(\mathsf{Q}_1, \mathsf{P}_2^1)$-module, it is not useful for the purposes of calculating $\mathcal{M}(\mathcal{R}_r, \mathsf{P}_2^1)$. This is because

P^1_2 does not obey the Leibniz rule and

$$\mathcal{M}(\mathcal{R}_r, \mathsf{P}^1_2) \ncong \mathcal{M}(\mathcal{R}, \mathsf{P}^1_2)^{\otimes r}.$$

However we overcome this difficulty by producing a $\Lambda(\mathsf{Q}_1, \mathsf{P}^1_2)$-module isomorphism ι_* at the expense of forgetting the internal grading.

4.2. P^1_2 Margolis homology of $((B\mathbb{Z}/2)^{\times k})_+$

The space $B\mathbb{Z}/2$ is also known as $\mathbb{RP}^\infty$, the real infinite-dimensional projective space. It is well-known that

$$H^*((B\mathbb{Z}/2)_+, \mathbb{F}_2) \cong \mathbb{F}_2[x]$$

and therefore

$$H^*(((B\mathbb{Z}/2)^{\times k})_+, \mathbb{F}_2) \cong \mathbb{F}_2[x_1, \dots x_k].$$

It can be seen that $\mathsf{P}^1_2(x_i) = 0$ and $\mathsf{Q}_1(x_i) = x_i^4$. We again define the length function on the monomials in the usual way

$$L(x_1^{i_1} \dots x_k^{i_k}) = (i_1 \bmod 2) + \cdots + (i_k \bmod 2).$$

The even complex $\mathcal{E}$, which is the span of elements of length zero, is isomorphic to

$$\mathcal{E} = \mathbb{F}_2[x_1^2, \dots, x_k^2].$$

It can be seen that $\mathsf{P}^1_2(x_i^2) = x_i^8$. Now observe that Q_1 acts trivially on $\mathcal{E}$, hence P^1_2 acts as a derivation and, therefore,

$$\mathcal{M}(\mathcal{E}, \mathsf{P}^1_2) \cong \Lambda(x_1^4, \dots, x_k^4).$$

Now the length function gives us an increasing length filtration

$$G^p(\mathbb{F}_2[x_1, \dots, x_k]) = \mathbb{F}_2\langle x_1^{i_1} \dots x_k^{i_k} : L(x_1^{i_1} \dots x_k^{i_k}) \leqslant p\rangle.$$

This results in a length spectral sequence which only has d_0 and d_2 differentials. If we denote x_i^4 by t_i for convenience, we can see that the action of Q_1 on the E_1-page of the length spectral sequence

$$E_1^\bullet = \Lambda(t_1, \dots, t_k) \otimes \Lambda(x_1, \dots, x_k) \Rightarrow \mathcal{M}(((B\mathbb{Z}/2)^{\times k})_+, \mathsf{P}^1_2)$$

is determined by the formula $\mathsf{Q}_1(x_i) = t_i$ and the Leibniz rule. Since the d_2-differentials are determined by the Q_1-action on the E_1-page, we conclude that the length spectral sequence above is a sub spectral sequence of (9), in fact, isomorphic to it when $k = \infty$. Thus, when k is finite, we can recover a complete description of $\mathcal{M}(((B\mathbb{Z}/2)^{\times k})_+, \mathsf{P}^1_2)$ from Theorem 3.16. More precisely, we obtain

$$\mathcal{M}(((B\mathbb{Z}/2)^{\times k})_+, \mathsf{P}^1_2) \cong \bigoplus_{K \subset [k]} S_K \otimes \mathcal{M}(M_K, \mathsf{P}^1_2),$$

where $S_K = \Lambda(t_i x_i \,|\, i \in [k] - K)$ and $\mathcal{M}(((B\mathbb{Z}/2)^{\times k})_+, \mathsf{P}^1_2)$ is a module over $\mathcal{S}_{[k]}$.

Example 4.5. $\mathcal{M}(\mathbb{RP}^\infty_+, \mathsf{P}^1_2) \cong \mathbb{F}_2\langle x_1, t_1, t_1 x_1\rangle$, where the internal degrees of x_1 and t_1

are 1 and 4 respectively and $\mathcal{S}_{[1]} = \Lambda(t_1x_1)$. Similarly,

$$\mathcal{M}((\mathbb{RP}^\infty \times \mathbb{RP}^\infty)_+, \mathsf{P}_2^1) \cong \mathbb{F}_2\langle x_1, x_2, t_1, t_2, t_1x_1, t_2x_2, \\ t_1x_2, t_2x_1, t_1x_1x_2, t_2x_2x_1, t_1t_2x_2, t_1t_2x_1\rangle,$$

where the internal degrees of x_i and t_i are 1 and 4 respectively. Here $\mathcal{S}_{[2]} = \Lambda(t_1x_1, t_2x_2)$. If we denote

$$H^*((\mathbb{RP}^\infty \times \mathbb{RP}^\infty)_+) \cong \mathbb{F}_2[y, z],$$

where $|y| = |z| = 1$, then one may choose $x_1 = [x]$, $x_2 = [y]$, $t_1 = [x^4]$ and $t_2 = [y^4]$.

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