# A Copula-based Multivariate Degradation Analysis for Reliability Prediction

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*SUMMARY & CONCLUSIONS*

Degradation test often involves multivariate Performance Characteristics (PCs) to be analyzed to make reliability prediction. As a result, the complex dependency structure among PCs needs to be addressed. In this paper, we develop a flexible copula-based multivariate model for analyzing high-dimensional degradation process. A two-stage method for parameters estimation is developed as an efficient statistical inference scheme. Finally, a real LED dataset is analyzed by the proposed approach.

## 1 INTRODUCTION

One of the main objectives of a reliability engineer is to analyze and predict a product's lifetime. However, since modern products can last for a very long time, it is difficult to assess the reliability of these products via the traditional life tests or accelerated life tests. Instead, degradation tests or accelerated degradation tests, which utilize product performance measurements, can provide a useful and efficient way. In the last decade, degradation data analysis has become more important in reliability assessment than ever before [1].

In literature, there are two major frameworks for modeling degradation - general path model and stochastic process model. By utilizing regression approach, the general path model is to fit the degradation path with appropriate parameters and random effects to account for unit-to-unit variability [2]. Recent developments on general path model include [3, 4, 5]. Alternatively, the stochastic process model assumes the data are generated from a stochastic process, such as Wiener process [6], Gamma process [7], and Inverse Gaussian process [8, 9]. In most previous studies, researchers considered only one product Performance Characteristic (PC); however, in reality, more than one failure mechanisms may contribute to product failure [10]. As a result, multiple PCs are required to be monitored in practice. Moreover, if there exist interactions between these mechanisms, the product's reliability will differ a lot from the case of considering single PC separately, because the product's overall performance is affected by multiple failure modes simultaneously. The past related work of multivariate degradation analysis either assume these multiple PCs as independent factors or they are dependent with a known multivariate joint distribution. However, the independent PC assumption may not match the engineering reality very well, and assigning a multivariate joint distribution to PCs may not be a suitable solution too [11], as it is difficult to find an appropriate joint distribution in most cases. Thus, a more flexible, yet reasonable, multivariate model is desired.

The goal of this paper is to establish a copula-based multivariate degradation model. With the use of copula function, we are able to separate the correlations between two PCs from their marginal distributions. To infer unknown parameters, we develop a two-stage estimation method. In the first stage, we estimate the parameters of possible marginal models for each PC. Akaike information criterion (AIC) is employed to compare the goodness-of-fit of candidate models. In the second stage, the association parameter of copula function is estimated. The main advantage of this two-stage method is its easy-to-implementation and computational efficiency. Based on the estimated model, the product reliability can be predicted. To illustrate the proposed approach, a numerical example about Light-Emitting Diode (LED) degradation data is presented.

The rest of the paper is organized as follows. In Section 2, three common univariate stochastic process models are introduced. Then, Section 3 elaborates the multivariate modeling with copula function. Section 4 provides the marginal reliability as well as joint reliability function under the framework of copulas. In Section 5, the method of two-stage parameters estimation is described. The numerical example is given in Section 6 followed by summary in Section 7.

## 2 UNIVARIATE MODELING

A degradation process is a result of material deterioration with inherent randomness. Therefore, it is natural to model a degradation process as a stochastic process [12]. As mentioned above, there are three typical stochastic process models being assumed in the literature and they are Wiener process, Gamma process and Inverse Gaussian (IG) process. But before discussing their mechanisms, the concept of a more general stochastic process- Lévy process needs to be introduced.

### 2.1 Lévy Process

In probability theory, a Lévy process represents the motion of a point whose successive displacements are random and independent, and statistically identical over different time intervals of the same length [13]. Actually, it can be viewed as the continuous-time analog of a random walk. If defined using mathematical language, it is indicated as below.

According to [14], a stochastic process $X = \{X_t : t \geq 0\}$ is said to be a Lévy process if it satisfies the following properties:

- $X_0 = 0$ almost surely.
- Independence of increments: For any $0 \leq t_1 < t_2 < \dots < t_\infty < \infty$, $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent.
- Stationary increments: For any $s < t$, $X_t - X_s$ is equal in distribution to $X_{t-s}$.
- Continuity in probability: For any $s > 0$ and $t \geq 0$, it holds that $\lim_{s\to 0} P(|X_{t+s} - X_t|) = 0$.

The above important properties of stationary independent increments imply that the increments of a Lévy process are independent and identically distributed (i.i.d.) whenever the time intervals of the increments are in equal length and any pairwise time intervals do not overlap.

In addition, Sato [15] shows that every infinitely divisible distribution corresponds in a natural way to a Lévy process. Besides, Steutel and Kent [16] provides a list of infinitely divisible distributions including Normal distribution, Gamma distribution, and IG distribution. Consequently, as subgroups of Lévy process, the three typical univariate degradation models-Wiener process, Gamma process, and IG process preserve all the aforementioned properties.

### *2.2 Wiener Process*

Wiener process assumes the increment $X_{s+t} - X_s$ is normally distribution with mean 0 and variance $\sigma^2 t$. If $\sigma^2 = 1$, it is called standard Wiener process or standard Brownian motion. By considering a gradual drift of the mean value of degradation, a general Wiener process model is described as

$$W(t) = \mu t + \sigma B(t) \tag{1}$$

where $\mu$ is drift parameter, $\sigma$ is diffusion parameter, and $B(t)$ is a standard Wiener process.

Considering the case under degradation testing, the measurement, $Y_i(t_j)$ of $i^{th}$unit at the corresponding time $t_j$ are obtained. According to the property of independence of increments, $\Delta Y_i(t_j)$ is subject to normal distribution with mean, $\mu\Delta\Lambda(t_j)$ and variance, $\sigma^2\Delta\Lambda(t_j)$, where $\Delta Y_i(t_j) = Y_i(t_j) - Y_i(t_{j-1})$, $t_0 = 0$, and $\Delta\Lambda(t_j) = \Lambda(t_j,\beta) - \Lambda(t_{j-1},\beta) = t_j^\beta - t_{j-1}^\beta$ for $i = 1, 2, \dots, N$, $j = 1, 2, \dots, M$. Here, $\Lambda(t_j,\beta) = t_j^\beta$ is used to transform time scale to make PCs be linear with time [17]. Thus, the individual increment $\Delta Y_i(t_j) \sim N\left(\mu\Delta\Lambda(t_j,\beta), \sigma^2\Delta\Lambda(t_j,\beta)\right)$, with probability density function (pdf) and cumulative density function (cdf) given by

$$f_W\left(\Delta Y_i(t_j)\right) = \frac{1}{\sqrt{2\pi\Delta\Lambda(t_j,\beta)\sigma^2}} \exp\left\{-\frac{\left(\Delta Y(t_j) - \mu\Delta\Lambda(t_j,\beta)\right)^2}{2\sigma^2\Delta\Lambda(t_j,\beta)}\right\}, \tag{2}$$

$$F_W\left(\Delta Y_i(t_j)\right) = \Phi\left[\frac{\Delta W_i(t_j) - \mu\Delta\Lambda(t_j,\beta)}{\sigma\sqrt{\Delta\Lambda(t_j,\beta)}}\right]. \tag{3}$$

### *2.3 Gamma Process*

Like Wiener process stating that the increments follow Normal distribution, Gamma process is built based on Gamma distribution. Basically, it is a stochastic process with independent, non-negative increments having a Gamma distribution with an identical scale parameter [18].

The pdf of Gamma distribution is shown as below:

$$f_X(x) = \frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-\frac{x}{\theta}}, \tag{4}$$

where $k > 0$ is shape parameter and $\theta > 0$ is scale parameter.

Suppose a degradation process is governed by $Gamma\ (k\Lambda(t,\beta), \theta)$, the increment $\Delta Y_i(t_j)$ has the following pdf and cdf

$$f_G\left(\Delta Y_i(t_j)\right) = \frac{\Delta Y_i(t_j)^{k\Delta\Lambda(t_j,\beta)-1}}{\Gamma\left(k\Delta\Lambda(t_j,\beta)\right)\theta^{k\Delta\Lambda(t_j,\beta)}} \exp\left\{-\frac{\Delta Y_i(t_j)}{\theta}\right\}, \tag{5}$$

$$F_G\left(\Delta Y_i(t_j)\right) = \frac{1}{\Gamma\left(k\Delta\Lambda(t_j,\beta)\right)} \gamma\left(k\Delta\Lambda(t_j,\beta), \frac{\Delta Y_i(t_j)}{\theta}\right), \tag{6}$$

where $\gamma$ is the lower incomplete gamma function.

### *2.4 IG Process*

Even though the existing Wiener process and Gamma process we discussed in the above can be applied to degradation analysis, there are still many scenarios in which the two processes do not fit the data well [9]. In this situation, IG process may be a good alternative.

The pdf of IG distribution is indicated as

$$f_X(x) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left\{\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right\}, \tag{7}$$

where $\mu$ is mean and $\lambda$ is shape parameter. Thus, it is possible to obtain independent random variables, $\Delta Y_i(t_j)$, that are subject to $IG\left(\mu\Delta\Lambda(t_j,\beta), \lambda\Delta\Lambda(t_j,\beta)^2\right)$

$$f_{IG}\left(\Delta Y_i(t_j)\right) = \left(\frac{\lambda\Delta\Lambda(t_j,\beta)^2}{2\pi\Delta Y_i(t_j)^3}\right)^{\frac{1}{2}} \exp\left\{\frac{-\lambda\left(\Delta Y_i(t_j) - \mu\Delta\Lambda(t_j,\beta)\right)^2}{2\mu^2\Delta Y_i(t_j)}\right\}, \tag{8}$$

$$F_{IG}\left(\Delta Y_i(t_j)\right) = \Phi\left[\sqrt{\frac{\lambda}{\Delta Y_i(t_j)}}\left(\frac{\Delta Y_i(t_j)}{\mu} - \Delta\Lambda(t_j,\beta)\right)\right] + \exp\left\{\frac{2\lambda\Delta\Lambda(t_j,\beta)}{\mu}\right\} \times \Phi\left[-\sqrt{\frac{\lambda}{\Delta Y_i(t_j)}}\left(\frac{\Delta Y_i(t_j)}{\mu} + \Delta\Lambda(t_j,\beta)\right)\right]. \tag{9}$$

## *3 MULTIVARIATE MODELING WITH COPULA FUNCTION*

While performing degradation analysis, monitoring more than one PC is necessary in some cases. An example is a lighting system consisting of many LED lamps for different purposes of lighting [19]. The design and the characteristic of the LED system may generate two or more PCs, such as light intensity and chromatic change, etc. that reflect products performance. Sometimes, there are interactions among these PCs due to common or similar failure mechanisms. In such situations, a bivariate or multivariate degradation model is needed for accurately estimating the reliability of products [20]. Copula function is a powerful tool for modeling the dependency of multivariate [21].

A copula is a function that connects the joint distribution function with individual marginal distribution functions. However, it simplifies this process by separating the learning of marginal distributions from the learning of dependence structure [22]. The definition of copula function is given by [21]

A $p$-dimentional copula function $C$, of which domain is $\boldsymbol{u} \in [0,1]^p$, has the following properties:

- Zero-grounded: $C(u_1, u_2, \dots, u_p) = 0$ if at least one coordinate of $\boldsymbol{u}$ is 0.
- Uniform margins: if all coordinates of $\boldsymbol{u}$ are 1 except $u_k$, then
$$C(1, \dots, 1, u_k, 1, \dots, 1) = u_k.$$
- $p$-increasing: for each hyperrectangle $B = \prod_{k=1}^{p} [x_k, y_k]^p$, the $C$-volume is non-negative:
$$\int_B dC(u) = \sum_{z \in \times_{k=1}^{p}\{x_k, y_k\}} (-1)^{p(z)} C(z) \geq 0,$$
where the $p(\boldsymbol{z}) = \#\{i: z_i = x_i\}$.

If we replace $(u_1, u_2, \dots, u_p)$ with $\left(F_1(x_1), F_2(x_2), \dots, F_p(x_p)\right)$, where $F_k(x_k)$ is the cdf of random variable $X_k$, then an important theorem, Sklar's theorem [21] is obtained as below.

Sklar's Theorem: Let $X = (X_1, X_2, \dots, X_p)$ be a random vector with marginal distributions $F_1(x_1), F_2(x_2), \dots, F_N(x_p)$, and let $H$ be their joint cumulative distribution function. Then, there exists a copula function $C$ such that

$$H(x_1, x_2, \dots, x_p) = C\left(F_1(x_1), F_2(x_2), \dots, F_p(x_p)\right). \quad (10)$$

This theorem states that there exists a copula function $C$, which uniquely defines joint cdf $H$. Furthermore, the pdf of joint distribution can be derived as

$$h(x_1, x_2, \dots, x_p) = c\left(F_1(x_1), F_2(x_2), \dots, F_p(x_p)\right) \prod_{k=1}^{p} f_k(x_k), \quad (11)$$

where $f_k(x_k)$ is the marginal pdf of $X_k$ and $c\left(F_1(x_1), F_2(x_2), \dots, F_p(x_p)\right)$ is copula density function, which can be achieved by taking partial derivative of copula function.

Due to different construction routes, there are three commonly used classes of copulas- Elliptical copulas, Archimedean copulas, and extreme-value copulas. Among them, Archimedean copulas have a wide range of applications because they can be constructed easily and can be extended from 2-dimension to $p$-dimension when some conditions are satisfied [1]. Thus, in this paper, three functions that belong to the Archimedean family, 2-dimension Gumbel copula, Clayton copula, and Frank copula, are introduced. Inside these functions, there is an association parameter $\delta$, which is used to measure the dependency between two variables. Note that the relationship between Kendall's correlation $\tau$ and the association parameter $\delta$ is also given.

- Gumbel copula
$$C(u_1, u_2) = \exp\left\{-\left[(-\log u_1)^{\frac{1}{\delta}} + (-\log u_2)^{\frac{1}{\delta}}\right]^{\delta}\right\}, \quad (12)$$
where $\delta \in (0,1]$ and $\tau = 1 - \delta$.
- Clayton copula
$$C(u_1, u_2) = \max\left(\left(u_1^{-\delta} + u_2^{-\delta} - 1\right)^{-\frac{1}{\delta}}, 0\right), \quad (13)$$
where $\delta \in [-1, \infty)/\{0\}$ and $\tau = \frac{\delta}{2+\delta}$.
- Frank copula
$$C(u_1, u_2) = -\frac{1}{\delta}\log\left\{1 + \frac{[\exp(-\delta u_1) - 1][\exp(-\delta u_2) - 1]}{\exp(-\delta) - 1}\right\}, \quad (14)$$
where $\delta \in (-\infty, 0) \cup (0, \infty)$ and $\tau = 1 + 4\frac{D_1(\delta) - 1}{\delta}$ with $D_1(\delta) = \frac{1}{\delta}\int_0^{\delta} \frac{t}{e^t - 1} dt$ being a Debye function.

Apparently, with the introduction of copulas into multivariate degradation modeling, two major benefits are achieved immediately.

- Marginal models and dependency structure can be separated. This feature eases the process of parameters estimation, which will be discussed in section 5.
- There is also no restriction on marginal models. They can be any distribution that comes from continuous univariate models.

## 4 RELIABILITY FUNCTION

Suppose the trend of a product PC being monitored in degradation test is decreasing over time, a "soft failure" happens when the PC measurement reaches a critical point. Then, the product reliability is defined as the probability of the product's performance level decrease is less than the threshold $\omega$ at a given time. One can monitor the degradation process of PCs so as to infer the failure time- $T$ of the product. On the other hand, the reliability at given time- $t$ can be estimated as well. Thus, the lifetime is defined as $T_\omega = \inf\{t: \Delta Y(t) < \omega\}$, where $\Delta Y(t) = -(Y(t) - Y_0)$ and $Y_0$ is the initial performance value.

### 4.1 Marginal Reliability

For an individual PC, each degradation process is demonstrated by the path of a single PC. Thus, the marginal reliability function can be acquired easily from the cdf of univariate models except the scenario of Wiener process because of its non-monotonicity. However, Folks and Chhikara [23] proves that the first passage time (i.e. $T_\omega$) follows an inverse Gaussian distribution, $IG(\frac{\omega}{\mu}, \frac{\omega^2}{\sigma^2})$, under Wiener process.

- Wiener process
$$R(t) = P(T_\omega > t) = 1 - P(T_\omega \leq t)$$
$$= 1 - \Phi\left[\frac{\mu t^\beta - \omega}{\sigma\sqrt{t^\beta}}\right] - \exp\left(\frac{2\mu\omega}{\sigma^2}\right)\Phi\left(-\frac{\mu t^\beta + \omega}{\sigma\sqrt{t^\beta}}\right). \quad (15)$$

- Gamma process

$$R(t) = P(T_\omega > t) = P(\Delta Y(t) < \omega) = \frac{1}{\Gamma(kt^\beta)}\gamma\left(kt^\beta, \frac{\omega}{\theta}\right). \tag{16}$$

- IG process

$$R(t) = P(T_\omega > t) = P(\Delta Y(t) < \omega)$$
$$= \Phi\left[\sqrt{\frac{\lambda}{\omega}}\left(\frac{\omega}{\mu} - t^\beta\right)\right] + \exp\left(\frac{2\lambda t^\beta}{\mu}\right)\Phi\left(-\sqrt{\frac{\lambda}{\omega}}\left(\frac{\omega}{\mu} + t^\beta\right)\right). \tag{17}$$

### *4.2 Joint Reliability*

When two or more PCs are correlated with each other, the joint reliability needs to be considered. It is assumed that the product fails if any one PC reaches its corresponding threshold-$\omega_k$, where $k$ is the PC index. Denote the failure time of the $k$th PC by $T_k$, then the product lifetime is $T = \min(T_1, T_2, \dots, T_p)$, where $p$ is the total number of PCs being monitored. So the joint reliability can be expressed as

$$R(t) = P(T > t) = P(T_1 > t, T_2 > t, \dots, T_p > t)$$
$$= P(\Delta Y_1(t) < \omega_1, \Delta Y_2(t) < \omega_1, \dots, \Delta Y_p(t) < \omega_p). \tag{18}$$

If all PCs are assumed to be independent, Equation (18) becomes

$$R(t) = R_1(t) \times R_2(t) \times \cdots \times R_p(t). \tag{19}$$

However, if there exist correlations among these PCs, Equation (18) is essentially a copula function.

$$R(t) = C(R_1(t), R_2(t), \cdots, R_p(t); \delta). \tag{20}$$

Here, the marginal reliability in Equation (20) can be any one from Equations (15), (16) and (17). Thus, again, one can easily see that how big advantage it brings by introducing copulas into multivariate degradation modeling.

## *5 METHOD OF PARAMETERS ESTIMATION*

Consider a copula-based multivariate distribution, the density based on Equation (11) is given by

$$h(x_1, x_2, \dots, x_p;\ \Theta_1, \Theta_2, \dots, \Theta_p, \delta)$$
$$= c(F_1(x_1; \Theta_1), F_2(x_2; \Theta_2), \dots, F_p(x_p; \Theta_p); \delta)\prod_{k=1}^{p} f_k(x_k; \Theta_k) \tag{21}$$

where $\Theta_k$ is the parameters set for each marginal distribution.

Thus, the log-likelihood function in a multivariate degradation scenario with $N$ units, $M$ measuring time points and $p$ PCs is

$$\log L(\Theta_1, \Theta_2, \dots, \Theta_p, \delta)$$
$$= \sum_{i=1}^{N}\sum_{j=1}^{M}\log c(F_1(x_{ij1}; \Theta_1), F_2(x_{ij2}; \Theta_2), \dots, F_p(x_{ijp}; \Theta_p); \delta)$$
$$+ \sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{k=1}^{p} f_k(x_{ijk}; \Theta_k), \tag{22}$$

where $x_{ijk}$ is the corresponding dataset-$\left(\Delta Y_{ik}(t_j), \Delta\Lambda(t_j)\right)$.

Obviously, to carry out Maximum Likelihood Estimation (MLE), one needs to feed Equation (22) to an optimization routine. This may be a difficult task. However, the separation of margins and copula density suggests that we may firstly estimate marginal parameters and then infer the copula association parameter, leading to a two-stage method. In first stage, each PC is treated separately. All the univariate models discussed in Section 2 are considered as potential candidate models. The parameters embedded in each model for each PC are estimated using MLE. Then, AIC is deployed to compare the goodness-of-fit,

$$\mathrm{AIC} = 2k - 2\log\hat{L},$$

where $k$ is the number of parameters and $\hat{L}$ is the maximized value of likelihood function. In second stage, the cdf calculated from the best fitted marginal models are plugged into the three Archimedean copula functions mentioned in Section 3 to infer the association parameter. In short, it can be represented as

- Stage 1

$$\widehat{\Theta_k} = \underset{\Theta_k}{\operatorname{argmax}} \sum_{i=1}^{N}\sum_{j=1}^{M}\log f_k(x_{ij1}; \Theta_1)\ \forall k. \tag{23}$$

- Stage 2

$$\hat{\delta} = \underset{\delta}{\operatorname{argmax}} \sum_{i=1}^{N}\sum_{j=1}^{M}\log c(F_1(x_{ij1}; \widehat{\Theta_1}), F_2(x_{ij2}; \widehat{\Theta_2}),$$
$$\dots, F_p(x_{ijp}; \widehat{\Theta_p}); \delta). \tag{24}$$

By utilizing this method, each maximization task has a very relatively small number of parameters, greatly reducing the computational difficulty [24]. It is also asymptotically efficient [25].

## *6 NUMERICAL EXAMPLE*

To illustrate how to apply the copula-based multivariate model into degradation data analysis, we make use of an actual LED lamps dataset from Chaluvadi's PhD thesis [26]. This dataset presents a degradation testing result of LED lamps, of which lighting intensity is measured every 50 hours under a stress level of 40mA current. It has been widely analyzed by many researchers. For example, Ye et al. [27] and Tang et al. [28] did univariate modeling based on Wiener process, while Hao et al. [29] constructed bivariate model using Frank copula with Gamma process as marginal. Later in this section, we will compare our approach with theirs.

For demonstrating bivariate modeling, similarly to Hao et al. [29], we split the LED dataset into two streams as if the first half represents PC1 and the left indicates PC2, which are shown in Table 1. LED is considered to be failed if any PC value is under 30. In addition, the degradation path of each PC for every unit is demonstrated in Figure 1. Note, it is necessary to apply time scale transformation due to the nonlinear trend of the path.

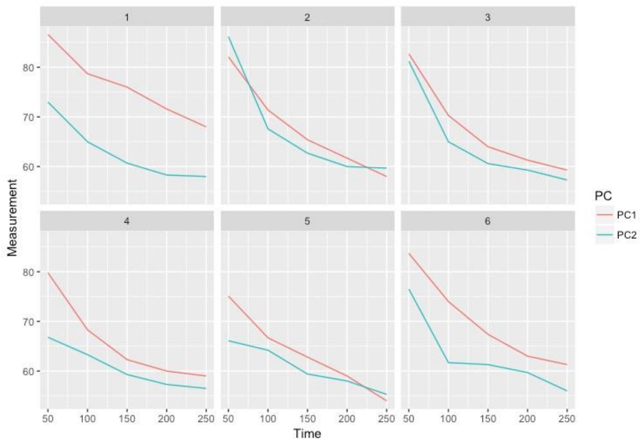


*Figure 1.Degradation Path of Each PC*

*Table 1 - LED Degradation Test Data*

| Unit | Inspection time (hour) | | | | | |
|---|---|---|---|---|---|---|
| | 0 | 50 | 100 | 150 | 200 | 250 |
| PC1 | | | | | | |
| 1 | 100 | 86.6 | 78.7 | 76.0 | 71.6 | 68.0 |
| 2 | 100 | 82.1 | 71.4 | 65.4 | 61.7 | 58.0 |
| 3 | 100 | 82.7 | 70.3 | 64.0 | 61.3 | 59.3 |
| 4 | 100 | 79.8 | 68.3 | 62.3 | 60.0 | 59.0 |
| 5 | 100 | 75.1 | 66.7 | 62.8 | 59.0 | 54.0 |
| 6 | 100 | 83.7 | 74.0 | 67.4 | 63.0 | 61.3 |
| PC2 | | | | | | |
| 1 | 100 | 73.0 | 65.0 | 60.7 | 58.3 | 58.0 |
| 2 | 100 | 86.2 | 67.6 | 62.7 | 60.0 | 59.7 |
| 3 | 100 | 81.2 | 65.0 | 60.6 | 59.3 | 57.3 |
| 4 | 100 | 66.8 | 63.3 | 59.3 | 57.3 | 56.5 |
| 5 | 100 | 66.1 | 64.2 | 59.4 | 58.0 | 55.3 |
| 6 | 100 | 76.5 | 61.7 | 61.3 | 59.7 | 56.0 |

### 6.1 *Correlation Analysis*

First, we check the Pearson correlation of negative increments between PCs for every unit. The results in Table 2 indicate the two PCs are highly correlated. Thus, building a bivariate degradation model is necessary.

*Table 2 - Correlation between PCs among Each Unit*

| Method | Correlation between PC1 and PC2 | | | | | |
|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 |
| Pearson | 0.95 | 0.787 | 0.979 | 0.888 | 0.967 | 0.899 |

### 6.2 *Selection of Marginal Model*

Then, the first stage of parameters estimation for each PC is conducted on every univariate model from section 2 with results shown in Table 3. It is found that Gamma process is appropriate to model both PCs due to lowest AIC. However, as stated earlier, Ye et al. [27] and Tang et al. [28] directly chose Wiener process as priori without checking other possible candidate models. This has impact on subsequent reliability assessment.

*Table 3 - Parameters Estimation of Marginal Model*

| PC | Wiener process | | | | |
|---|---|---|---|---|---|
| | $\mu$ | $\sigma$ | $\beta$ | AIC | Ranking |
| PC1 | 3.2205 | 1.5567 | 0.4566 | 139.3159 | 2 |
| PC2 | 7.8777 | 4.7098 | 0.3068 | 175.0414 | 3 |
| | Gamma process | | | | |
| | $k$ | $\theta$ | $\beta$ | AIC | Ranking |
| PC1 | 3.8473 | 0.8358 | 0.4569 | 137.7911 | 1 |
| PC2 | 2.7694 | 2.7204 | 0.3149 | 157.4284 | 1 |
| | IG process | | | | |
| | $\mu$ | $\lambda$ | $\beta$ | AIC | Ranking |
| PC1 | 3.3693 | 11.1856 | 0.4485 | 139.5361 | 3 |
| PC2 | 8.8137 | 17.4782 | 0.2862 | 158.5518 | 2 |

### 6.3 *Selecting the Copula Function*

In the next step, the second stage of parameters estimation is performed via utilizing "copula" package in R. After inserting the estimated parameters of marginal, the association parameter of every possible copula function can be inferred. Table 4 indicates that Gumbel copula is appropriate to describe the bivariate distribution. But Hao et al. [29] directly deployed Frank copula, which will also affect reliability prediction.

*Table 4 - Parameters Estimation of Copula Function*

| Copula | $\delta$ | $\tau$ | AIC | Ranking |
|---|---|---|---|---|
| Gumbel | 1.358 | 0.2638 | -4.847779 | 1 |
| Clayton | 0.0081 | 0.0040 | 1.999144 | 3 |
| Frank | 1.925 | 0.2064 | 0.8024627 | 2 |

### 6.4 *Reliability Assessment*

After marginal and joint models are decided, the reliability can be calculated according to Equation (15)- (20).

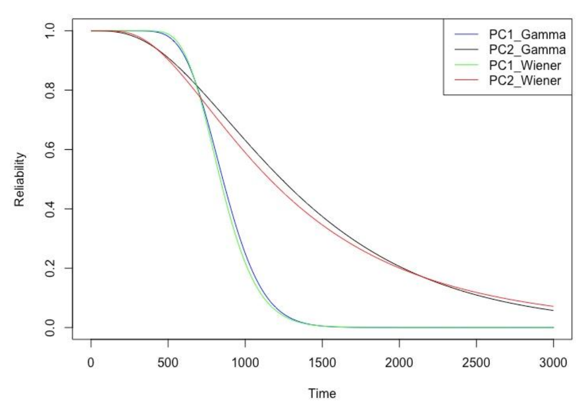


*Figure 2. Comparison of Marginal Reliability Curves*

Figure 2 presents the marginal reliability plot of both PCs for Gamma process and Wiener process. In our calculation, we compare the goodness-of-fit of three potential marginal models and conclude that Gamma process is a proper model. However, if Wiener process is arbitrarily pre-determined as priori without checking the assumption, the reliability curve of PC2 deviates from its case of Gamma process obviously.

Then, a comparison study of joint reliability based on independent, dependent with Frank copula and dependent with

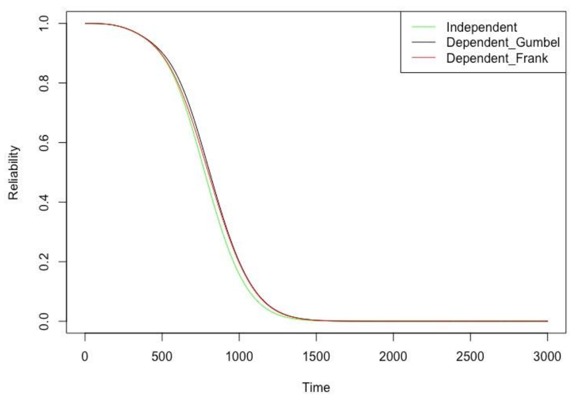


*Figure 3. Comparison of Joint Reliability Curves*

Gumbel copula cases are demonstrated on Figure 3. Apparently, the reliability of the first two scenarios are underestimated comparing to that of Gumbel copula due to either not considering dependence or underestimating the significance of dependence.

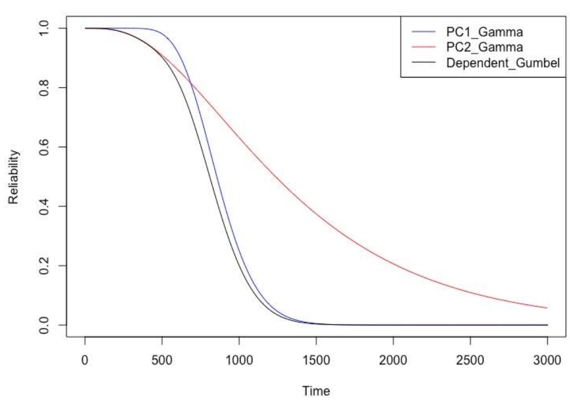


*Figure 4. Reliability Curves with Gamma Process as Marginal and Gumbel Copula as Joint*

Lastly, Figure 4 shows the marginal reliability line based on the Gamma process for each PC and the joint reliability line based on the Gumbel copula function.

## 7 SUMMARY

In this paper, we proposed a multivariate degradation modeling approach for reliability prediction. This approach provides a structured framework for both univariate and multivariate modeling along with their reliability function estimation. Our model is developed based on the concept of copula function; thus it is able to separate marginal modeling and correlation estimation and results in a two-stage model building process. In the case study, the marginal Gamma process and the joint Gumbel copula are obtained. Comparison study with other researchers' results are also presented. In conclusion, our approach is more flexible to various data structures and it leads to an efficient implementation of multivariate degradation analysis.

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