Analyzing ALT Data with Time-varying Stress Profile

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Key Words: accelerated life test, stress profile, proportional hazard, Poisson regression, reliability prediction

SUMMARY & CONCLUSIONS

We are interested in analyzing the accelerated life testing data obtained under time-varying stresses. This is a generalization of step-stress accelerated life test, where the stress levels are kept constant at each step. Our study shows that the time-dependent proportional hazard model, commonly appeared in the survival data analysis literature, is not applicable for ALTs, because this approach does not take accounts of the cumulative damage that the stress profile exerts on test units. Instead, we assume that products possess constant failure rates over small time intervals, and the change of failure rate by the stress variable still have the proportional hazard property. With these assumptions, it is possible to formulate the data according to a generalized linear model and statistical inferences on model parameters can be carried out. We demonstrate our models and inference procedures by using both synthetic and real datasets.

1 INTRODUCTION

Analyzing the effect of time-varying stress on product reliability prediction was initially investigated in Nelson (1980) for an electrical insulation test under low temperature. The test protocol can be described as follows: One test unit or a group of test units are first tested under the lowest stress level. If a test unit fails, it would be removed from the test; for the remaining test units, after a certain test period, the stress level will be elevated to a higher level in order to hasten their failure process further. This type of test is labeled as step-stress accelerated life test (SSALT). One of the purposes of employing SSALT is to produce more failures within a limited testing period. In addition, the failure time data obtained under different stress levels can be used to infer life-stress models, thus eliminating the need of setting up another reliability test on a different batch of test units at a different stress level. Therefore, in theory, this test will save precious test specimens as well as the effort of test setup.

Stepping up stress level is the most common approach of SSALT; however, for various reasons, we also see some step-down test protocols in literature. Furthermore, some researchers considered the stress ramp-up and ramp-down periods during stress level transitions. Van Dorp and Mazzuchi (2004) described some time-varying stress ALT procedures, including progressive, regressive, and profile step-stress ALTs. In particular, the progressive SSALT, where the stress intensity is changed in an ascending order, is the most common testing procedure used in practice. We can generalize these test protocols to be the test with time-varying stress profiles. This stress profile could be a monotonic increasing or decreasing step or continuous function, or even a non-monotonic function. Nevertheless, from the failure or survival time observations of all test units, we can study the relationship between product lifetime and stress variables, and further predict any lifetime characteristic of interest under the normal use stress condition.

In this paper we investigate the applicability of the well-known proportional hazard (PH) model on profiled-stress ALTs. It is noticed that the extension of PH model to time-dependent covariate has been widely discussed in the survival data analysis literature, especially for medical applications; however, it has not been accepted for ALT data analysis. It is our interest to investigate the reason and to propose an alternative approach, while still maintain the PH property for failure rate to a certain extent.

The paper is organized into four sections hereafter: Section 2 gives a brief literature review on SSALT and the cumulative damage assumption. In Section 3, we discuss the PH property and the implication of preserving the PH property for profiled-stress ALTs. In Section 4, both synthetic and real data are used to demonstrate the results of different modeling statistical inference approaches. Lastly, Section 5 concludes the paper and points out the limitations of our proposed approach and some topics for future investigation.

2 LITERATURE REVIEW

For an ALT, it is reasonable to assume that a stress variable acting on a product will have a lasting effect on the product's lifetime; that is, the stress effect is not transient, but accumulated over time to damage the product's integrity, thus its lifetime. Nelson (1980) argued that the failure probabilities at a stress transition time point should keep the same value. This lead to a compression of lifetime at higher stress levels, and the product lifetime distribution is composite distribution due to the cumulative exposure (CE) time of the product to time-varying stresses. Bhattacharyya and Soejoeti (1989) and Khamis and Higgins (1998) used a tampered failure rate model where the logarithm of the product failure rate is a linear function of the stress level. This model can also be explained by the well-known proportional hazard (PH) concept. It has been shown that when the failure time is exponentially distributed the CE SSALT model coincides with the PH model (Lee and Pan (2010), Sha and Pan, (2014)).

The first treatment of survival data analysis using a Generalized Linear Model (GLM) was given by Aitkin and Clayton (1980) and Whitehead (1980). This approach was also summarized in McCullagh and Nelder (1983). With the GLM formulation, the model parameter estimation can be carried out through the Iteratively Weighted Least Squares (IWLS) method. Barbosa and Louzada-Neto (1994) and Barbosa et al. (1996) applied this GLM technique on a constant-stress ALT with either a Weibull failure time distribution or piecewise exponential distribution. Wang and Kececioglu (2000) applied the IRWLS algorithm to estimate model parameters in Weibull ALT models. They concluded that this method is effective and numerically stable.

3 *PH PROPERTY AND TIME-DEPENDENT STRESS*

To deal with time-varying covariates, it seems to be natural to directly apply the PH model with time-dependent covariates, as used in many medical applications (see, e.g., Therneau et al. (2021)). However, this model is not suitable for modeling the effect of external life accelerating stresses when the cumulative damage of stress profile on product life needs to be taken into account. In the ALT literature, the cumulative damage model is widely accepted because it is consistent with the general understanding of physics of failures of materials. As shown in this section, we may still introduce the PH property into failure rates at different stress levels, but the cumulative hazard function will not demonstrate the separation of baseline hazard (time-dependent) and effect of stress (time-independent) terms, thus this model is not the PH model with time-dependent covariates.

3.1 *PH Regression*

In the PH model, we assume the hazard function satisfies

$$\lambda(t;x) = \lambda_0(t)\exp(\beta x),$$

where $\lambda_0(t)$ is the baseline hazard function and $\exp(\beta x)$ is the effect of stress variable x exerted on the failure rate. Notice that if the stress variable is a constant, then only the baseline hazard function is a function of time.

The PH model specifies the ratio of two hazard functions with different covariate values is time independent, i.e.,

$$\frac{\lambda(t;x_1)}{\lambda(t;x_2)} = \exp(\beta(x_1 - x_2)).$$

Due to this property, to estimate the coefficient β, the PH regression utilized the partial likelihood function so that the baseline hazard function needs not to be explicitly specified. That is the reason that PH models are regarded as semi-parametric models.

From the above hazard function, the cumulative hazard and reliability functions are derived to be, respectively,

$$\Lambda(t;x) = \int_0^t \lambda_0(u)\exp(\beta x)du = \Lambda_0(t)\exp(\beta x),$$

and

$$R(t;x) = \exp\big(-\Lambda(t)\big) = [R_0(t)]^{\exp(\beta x)},$$

where $\Lambda_0(t) = \int_0^t \lambda_0(u)du$, is the baseline cumulative hazard function and $R_0(t) = \exp(-\Lambda_0(t))$, the baseline reliability function. Again, these functions are time dependent, but the cumulative hazard function is factored out two terms – baseline cumulative hazard function, which is time dependent, and the effect of covariate, which is time independent.

3.2 *Incorporating Stress Profile*

Now, extending the PH regression to modeling ALTs with time-varying stress variables, we have

$$\lambda(t;x(t)) = \lambda_0(t)\exp(\beta x(t)). \qquad (2)$$

Then the hazard ratio becomes

$$\frac{\lambda(t;x_1(t))}{\lambda(t;x_2(t))} = \exp(\beta(x_1(t) - x_2(t))),$$

which is not time independent in general.

In the survival data analysis literature, however, the PH model has been applied on the cases with time-dependent covariates. This is argued through the counting process of failure events. Let $Y_i(t)$ be the indicator that subject i is at risk (survived up to) at time t and $N_i(t)$ be the cumulative number of events for the subject up to time t. The empirical cumulative hazard estimator, given a particular covariate profile, is given by

$$\widehat{\Lambda}\big(t;x(t)\big) = \sum_{i=1}^{n}\int_0^t \frac{\exp\left(\hat{\beta}x(u)\right)dN_i(u)}{\sum_j Y_j(u)\exp\left(\hat{\beta}x_j(u)\right)}$$

where the summation over index j in the denominator refers to the at-risk set at any point of event. In practice (software implementation), the hazard function is only evaluated at times of events. This is like discretizing the covariate profile to a step function, assuming the covariate is constant before the next event. This approach does not incorporate the concept of cumulative damage, thus not suitable for ALT analysis.

We propose an alternative approach that combines the PH property and the concept of cumulative damage, although this approach will bring more restrictions on the lifetime model than the PH regression approach. These assumptions are:

1. Over a short time interval, product failure rate is a constant;
2. The PH property is preserved for these constant failure rates with their stress levels being the average stress level over the time interval.

In fact, the above assumptions have defined a stepwise exponential lifetime distribution, which is a strong assumption to applications.

Figure 1 shows a general stress profile. It is segmented to several time intervals and their corresponding constant failure rates are shown below. Here, $(\tau_1, \tau_2, \dots)$ are segmentation points and $(\lambda_1, \lambda_2, \dots)$ are the corresponding failure rates over these intervals. Assuming the PH property, we have

$$\frac{\lambda_i}{\lambda_j} = \exp(\beta(x_i - x_j))$$

where x_i and x_j are the average stress level at intervals i and j, respectively.

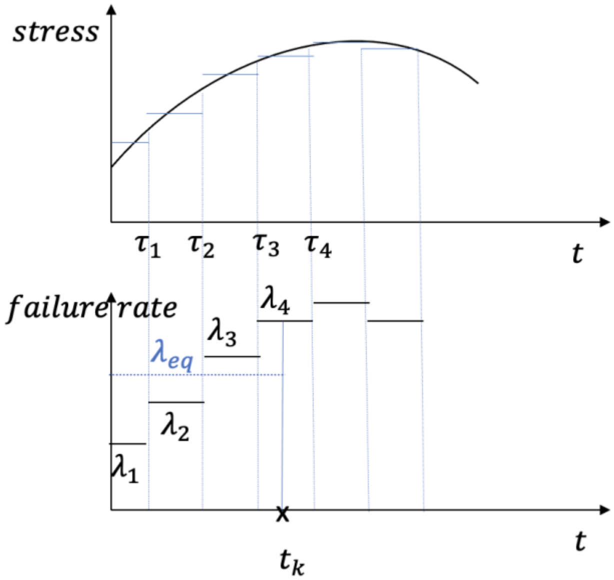

Figure 1. A stress profile and its corresponding stepwise failure rate function.

As illustrated in Figure 1, suppose there is a test unit, unit k, failed at time t_k and this failure time falls into the time interval (τ_3, τ_4). Then, the cumulative hazard experienced by this unit is found to be

$$\Lambda(t_k) = \lambda_1\tau_1 + \lambda_2(\tau_2 - \tau_1) + \lambda_3(\tau_3 - \tau_2) + \lambda_4(t_k - \tau_3)$$
$$= \lambda_4 t_k + (\lambda_3 - \lambda_4)\tau_3 + (\lambda_2 - \lambda_3)\tau_2 + (\lambda_1 - \lambda_2)\tau_1$$

Note that this cumulative hazard is equivalent to testing the unit under a pseudo-constant stress level, $\lambda_{eq}(t_k)$, for a time period of t_k. This leads to

$$\lambda_{eq} = \lambda_4 + (\lambda_3 - \lambda_4)\frac{\tau_3}{t_k} + (\lambda_2 - \lambda_3)\frac{\tau_2}{t_k} + (\lambda_1 - \lambda_2)\frac{\tau_1}{t_k}$$

Also note that even though $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ possess the PH property, λ_{eq} does not, thus we cannot apply the PH regression analysis directly with λ_{eq}.

We examine the likelihood function for this test unit. The reliability function of this test unit is given by

The reliability function of this test unit is given by

$$R(t_k) = \exp\big(-\Lambda(t_k)\big) = \exp\,(-\lambda_{eq} t_k)$$

And the failure density function is
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$$f(t_k) = \lambda_{eq}\exp\big(-\lambda_{eq} t_k\big)$$

It becomes

$$f(t_k) = \exp(-\lambda_1\tau_1)\exp\big(-\lambda_2(\tau_2 - \tau_1)\big)\exp\big(-\lambda_3(\tau_3 - \tau_2)\big)\lambda_4\exp\big(-\lambda_4(t_k - \tau_3)\big)$$
$$= R_1(\tau_1)R_2(\tau_2 - \tau_1)R_3(\tau_3 - \tau_2)f_4(t_k - \tau_3)$$

where

$$R_i(t) = \exp\,(-\lambda_i t)$$

and

$$f_i(t) = \lambda_i \exp\,(-\lambda_i t)$$

The above derivation shows that this test unit can be treated as four independent test units and each of them is tested on a distinct stress level (thus a distinct constant failure rate) over a short time interval. We name these independent test units to be equivalent test units. The first three equivalent units survive their tests, while the last one fails.

To generalize the above discussion, a test with a stepwise failure rate can be decomposed to multiple tests with individual failure rates and the failure event could only happen at the last time interval. See Figure 2.

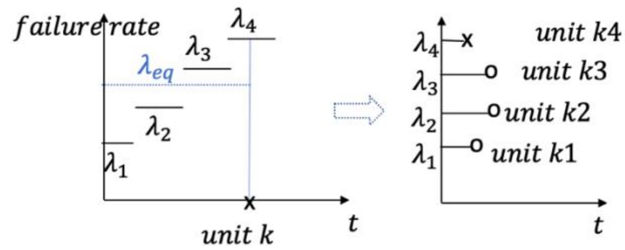

Figure 2. Decompose one test to multiple tests.

Following the approach described above, for a general stress profile applied on test units during an ALT experiment, we may discretize the profile and then apply the cumulative damage model with the PH property to formulate the likelihood function.

3.3 *Poisson Regression*

The PH property defines the relationship between failure rate and stress level to be a log-linear function such as

$$\log\lambda_i = \alpha + \beta x_i$$

Here we define the baseline hazard to be as $\lambda_0 = \exp\,(\alpha)$. In the following discussion, we use subscript k to index for all equivalent test units, instead of original test units. Then, the total likelihood function is given by

$$L = \prod_k f_k^{c_k}(t_k) R_k^{1-c_k}(t_k)$$

where c_k is an indicator variable for right censoring. So, with exponential distributions at each stress levels, we have

$$L = \prod_k \lambda_k^{c_k}\exp(\lambda_k t_k) = \prod_k \mu_k^{c_k}\exp(-\mu_k)\, t_k^{-c_k}$$

where μ_k is defined as $\mu_k = \lambda_k t_k$.

We notice that this likelihood function consists of two parts, i.e., $\mu_k^{c_k}\exp\,(-\mu_k)$ and $t_k^{-c_k}$. The first part is a Poisson probability mass function for variable c_k and mean parameter μ_k, where

$$\log\mu_k = \alpha + \beta x_k + \log t_k$$

The second part is a function of failure time only, not related to the distribution mean, μ_k, or regression coefficients, α and β. Therefore, to maximize the likelihood function to find regression coefficient estimates, we only need to maximize the first part, which is the same as treating c_k as a Poisson variable. This Poisson regression has a log link function with an offset term.

4 NUMERICAL STUDIES

4.1 *Synthetic Dataset*

Consider a ramp stress linearly increasing from 0 stress level to 10 stress level over a time period of 10 time units. Assume there are 10 test units being tested under this stress profile and it is observed one failure at t=6, 2 failures at t=8 and t=9, 3 failures at t=10, and the failure times of remaining test units are right censored at the end of the test. Figure 3 illustrates the stress profile and the failure/survival events observed under this profile. Using this synthetic dataset, we want to investigate the following:

1. Will the time-dependent Cox PH model work for this case?
2. How to implement the cumulative damage model with the PH property for this case?
3. Will the fineness of discretizing time interval have a significant impact on model parameter estimation?

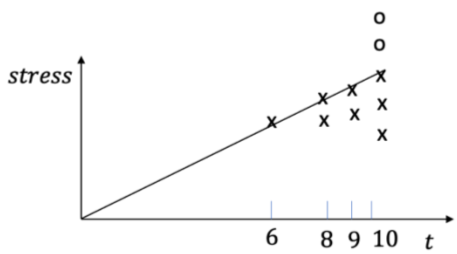

Figure 3. Ramp stress and ten test units' failure and survival time

Analysis 1: Following the conventional survival data analysis, we approximate the stress profile by discretizing it at the times of failures, while stress levels are set at the levels when failures occur.

id	start	end	stress	failure
1	0	6	6	1
2	0	6	6	0
2	6	8	8	1
3	0	6	8	0
3	6	8	8	1
4	0	6	6	0
4	6	8	8	0
…	…	…	…	…
10	9	10	10	0

The R code for the PH regression is given below:

fit1 <- coxph(Surv(start, end, failure) ~ log(stress), data=analysis1)

Note that we did a log transformation of stress variable. This is because we regard the stress as a pressure type of stress and its physical acceleration model follows an inverse power function. After transformation, the stress is not longer linearly increasing over time.

```
                 coef  exp(coef)   se(coef) z Pr(>|z|)
log(stress) -7.853e-01  4.560e-01  1.071e+08 0        1
```

The result from this analysis is disappointing. As one can see the coefficient estimate is negative, which means that this model predicts a lower failure rate at a higher stress level. This is in contradiction to our general understanding of ALT. The error is due to the lack of consideration of cumulative damage in the ordinary PH regression model. We also notice that the coefficient estimate is very unreliable, as its standard error is huge. Therefore, we do not recommend the PH regression for profiled-stress ALT analysis.

Analysis 2: Introduce a constant baseline hazard; that is, assume exponential lifetime distributions for the failure times of test units. Then, we can use the Poisson regression approach to estimate reliability function. The dataset is presented below. The stress profile is discretized by time intervals with one time unit. We treat the "failure" variable as the response variable. As shown in the previous section, the likelihood function of this dataset is the likelihood of Poisson variables with an offset term, $\ln(\tau_{i+1} - \tau_i)$. We set all time intervals to be one time unit.

id	eq. unit	time	stress	failure
1	1	1	0.5	0
1	2	1	1.5	0
1	3	1	2.5	0
1	4	1	3.5	0
1	5	1	4.5	0
1	6	1	5.5	1
2	7	1	0.5	0
…	…	…	…	…
10	90	1	9.5	0

R code: fit2< - glm(failure ~ log(stress) + offset(log(time)), data=analysis2, family = poisson(link = "log"))

```
Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)  -12.021      4.152  -2.895  0.00379 **
log(stress)    5.095      1.974   2.581  0.00984 **
```

The above results are more consistent with what are expected, as the stress coefficient estimate is positive and it is statistically significant.

Analysis 3: We widen the time interval to 2 time units to discretize the stress profile.

id	eq. unit	time	stress	failure
1	1	2	1	0
1	2	2	3	0
1	3	2	5	1
1	4	2	1	0

1	5	2	3	0
1	6	2	5	0
2	7	2	7	1
…	…	…	…	…
10	46	2	9	0

```
Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept)  -10.174      3.562  -2.856  0.00429 **
log(stress)    4.209      1.720   2.447  0.01441 *
```

It turns out that this stress coefficient estimate is quite different from that of Analysis 2, and the statistical significance of this estimate is reduced too. This indicates that a wider time interval will decrease the effect of cumulative damage of stress on test unit lifetime, although the wider time interval will reduce computational efforts. Therefore, we need to be careful on how to segment a stress profile. It is suspected that the optimal segmenting scheme is related to the rate of change of stress profile, but this conjecture has yet to be validated.

4.2 *Real Dataset*

Nelson (1980) provides a dataset obtained from an SSALT of power cable insulation at cryogenic temperature and the stress variable is voltage. Ten steps of stress are applied on the test units during the testing and, based on the setting of holding times at the stress levels, there are four testing stress profiles. For the purpose of illustration, here we only use two stress profiles in this paper. Due to the partial data, the estimation results presented in this paper cannot be compared to the original study.

There are 6 test units and each of them are tested for 10 minutes at stress 5, 10, 15, 20kv without failures. Then, they are tested at remaining stress level, 26, 28.5, 31, 33.4, 36 and 38.5kv. Three test units are tested at these levels with a holding time of 60 minutes, and the other three with a holding time of 15 minutes. The following table present the data. The stress level has been adjusted for cable thickness.

id	level	time	stress	failure
1	1	10	5.133	0
1	2	10	5.826	0
1	3	10	6.231	0
1	4	10	6.519	0
1	5	60	6.782	0
1	6	60	6.783	0
1	7	60	6.957	0
1	8	60	7.032	0
1	9	60	7.107	0
1	10	30	7.174	0
2	1	10	5.133	0
2	2	10	5.826	0
2	3	10	6.231	0
2	4	10	6.519	0
2	5	60	6.782	0
2	6	60	6.873	0
2	7	60	6.957	0
2	8	60	7.032	0
2	9	60	7.107	0
2	10	5	7.174	0
3	1	10	5.185	0
3	2	10	5.878	0
3	3	10	6.284	0
3	4	10	6.571	0
3	5	60	6.834	0
3	6	60	6.926	0
3	7	60	7.010	0
3	8	60	7.084	0
3	9	60	7.159	0
3	10	5	7.226	1
4	1	10	5.221	0
4	2	10	5.915	0
4	3	10	6.320	0
4	4	10	6.608	0
4	5	15	6.870	0
4	6	15	6.962	0
4	7	15	7.046	0
4	8	15	7.121	0
4	9	2	7.95	1
5	1	10	5.221	0
5	2	10	5.915	0
5	3	10	6.320	0
5	4	10	6.608	0
5	5	15	6.870	0
5	6	15	6.962	0
5	7	15	7.046	0
5	8	15	7.121	0
5	9	13	7.195	1
6	1	10	5.221	0
6	2	10	5.915	0
6	3	10	6.320	0
6	4	10	6.608	0
6	5	15	6.870	0
6	6	15	6.962	0
6	7	15	7.046	0
6	8	15	7.121	0
6	9	13	7.195	1

The analysis show that the stress variable does have significant effect on elevating the failure rate of cable, thus shortening its lifetime.

```
Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept)  -50.404      8.510  -5.923 3.16e-09 ***
stress         6.302      1.149   5.483 4.18e-08 ***
```

With this parametric model, we can predict the reliability at various stress levels. The first three low stress levels (5, 10, 15kv) generate no failure observations in the ALT experiment. By plugging in their values into the loglinear function of failure rate, the following reliability functions are obtained for the three lowest stress levels, respectively.

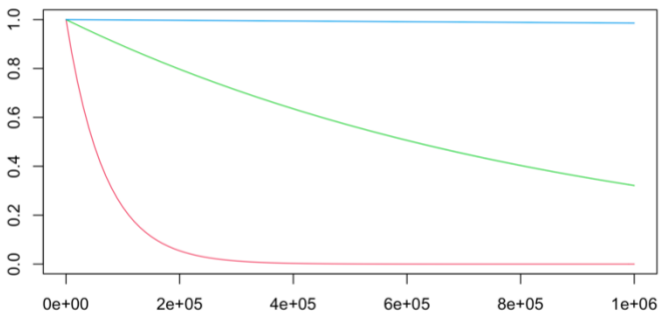

Figure 4. Reliability prediction at the three lowest stress levels

5 CONCLUSIONS

In this study, we investigate the PH regression model for the profiled-stress ALT analysis. It is found that due to the cumulative damage concept the PH property cannot be maintained in the overall product failure time likelihood function; however, under the stricter assumption of stepwise exponential distribution, the PH property can be preserved for the failure rates of constant stress levels over a short time interval. This in turn facilitates the data analysis, as we can estimate the stress effect via a Poisson regression approach. As aforementioned, the optimal stress profile segmentation scheme needs to be further researched. Furthermore, extending this approach to Weibull distribution seems to be possible, worth of further investigation.

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BIOGRAPHIES

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