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Spectral analysis of Euler–Bernoulli beam model with distributed damping and fully non-conservative boundary feedback matrix

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Abstract. The distribution of natural frequencies of the Euler–Bernoulli beam resting on elastic foundation and subject to an axial force in the presence of several damping mechanisms is investigated. The damping mechanisms are: (i) an external or viscous damping with damping coefficient $(-a_0(x))$, (ii) a damping proportional to the bending rate with the damping coefficient $a_1(x)$. The beam is clamped at the left end and equipped with a four-parameter $(\alpha,\ \beta,\ \kappa_1,\ \kappa_2)$ linear boundary feedback law at the right end. The 2×2 boundary feedback matrix relates the control input (a vector of velocity and its spacial derivative at the right end) to the output (a vector of shear and moment at the right end). The initial boundary value problem describing the dynamics of the beam has been reduced to the first order in time evolution equation in the state Hilbert space of the system. The dynamics generator has a purely discrete spectrum (the vibrational modes). Explicit asymptotic formula for the eigenvalues as the number of an eigenvalue tends to infinity have been obtained. It is shown that the boundary control parameters and the distributed damping play different roles in the asymptotical formulas for the eigenvalues of the dynamics generator. Namely, the damping coefficient a_1 and the boundary controls κ_1 and κ_2 enter the leading asymptotical term explicitly, while damping coefficient a_0 appears in the lower order terms.

Keywords: Non-selfadjoint operator, dynamics generator, vibrational modes, distributed damping, boundary control parameters, spectral asymptotics

1. Introduction

The present paper is concerned with the asymptotic properties of the eigenmodes of the Euler–Bernoulli beam model subject to two distributed damping mechanisms and a four-parameter family of non-conservative boundary conditions. At the left end the beam is clamped, while at the right end it is subject to linear feedback type conditions with a feedback matrix depending on four control parameters: α, β, κ_1, and κ_2. In addition, there are the following damping mechanisms: (i) an external viscous damping with damping coefficient $(-a_0(x))$ and (ii) a damping which is proportional to the bending rate with damping coefficient $a_1(x)$. In some particular cases, e.g. when $|\alpha| + |\beta| \geqslant 0$ and $\kappa_1 = \kappa_2 = 0$, the system is dissipative, i.e. the energy of the system is a decreasing function of time. However, when $\kappa_1 + \kappa_2 \neq 0$, the system is neither dissipative, nor conservative. In our approach, the initial boundary-value problem describing the beam dynamics is reduced to the evolution equation in the Hilbert state space, $\mathcal{H}$, equipped with the energy norm. This evolution equation is completely determined by its dynamics generator, $i\mathcal{L}$, which is an unbounded non-skew-selfadjoint matrix differential operator in $\mathcal{H}$. The eigenmodes (the vibrational modes) and mode shapes of the system are defined as the eigenvalues and the generalized eigenvectors of the operator $i\mathcal{L}$. It is technically convenient to study spectral and asymptotic properties of the operator $\mathcal{L}$ (rather than $i\mathcal{L}$). Clearly, the dynamics generator $i\mathcal{L}$ and the operator $\mathcal{L}$ have the same generalized eigenvectors. In what follows, for the set of the vibrational modes we use the notation $\{\nu_n\}_{n\in\mathbb{Z}'}$, where $\mathbb{Z}' = \mathbb{Z} \setminus \{0\}$ and for the set of the eigenvalues of the operator $\mathcal{L}$ we use the notation $\{\lambda_n\}_{n\in\mathbb{Z}'}$, which yields $\nu_n = i\lambda_n$.

The main object of interest in the present paper is the asymptotic distribution of vibrational modes for the problem having a combination of distributed damping and boundary controls. To the best of our knowledge such an analysis has never been done before. The main result – Theorem 7.2 – provides asymptotic approximation for the vibrational modes when the number of the mode tends to infinity. The leading asymptotical terms contain both the distributed damping coefficient and boundary control parameters.

At this moment we would like to emphasize that there exists an extensive literature devoted to different aspects of the Euler–Bernoulli model dynamics, such as asymptotic, spectral and stability analysis of the model (both linear and nonlinear versions) and control of the corresponding distributed parameter system. We mention below several recent works where the model has been used in contemporary research directions. One of them is concerned with developing unmanned aerial vehicles (UAVs) in aeronautics. For intelligence missions, surveillance and environmental research, highly flexible unmanned airframes, that have been designed recently, allow for high-altitude and long duration flights (Patil and Hodges [24]; Patil et al. [25]). In particular, a light and flexible long-span object in flight (high aspect-ratio 'flying wing' configuration) can be modelled as an elastic beam with both ends free. A boundary feedback stabilization of such beam-like structures could be of great interest both in control theory and in engineering practice. In works (Patil and Hodges [24]; Patil et al. [25]) a computer theoretical methodology for a highly flexible wing has been presented. The authors use geometrically exact beam theory for elastic deformations coupled with aerodynamic theory of large motion airfoils.

Paulsen [27] has discussed another important area of applications of the modelling of large space structures, e.g. a large communication satellite or a space platform. In such a structure, different types of damping devices are installed at the joints of the beam elements to suppress vibrations. Without these dampers, small vibrations would persist and even slowly build up. In Paulsen [27] the model of serially connected non-collinear Euler–Bernoulli beams with dissipative joints has been considered and numerical simulation results presented. Obviously, rigourous analytical results on the multi-beam model would be desirable.

A class of Euler–Bernoulli beam models with boundary and structural damping has been discussed in (Russell [28]; Chen and Russell [5]; Chen et al. [2,3]). We also mention works (Chen et al. [4]; Liu and Liu [18,19]) dealing with the models with viscous and Kelvin–Voigt damping. For a cantilever model, the lack of exponential stabilization under velocity feedback has been proven in Littman and Markus [17]; the energy multiplier method has been used in Conrad and Morgül [6] to prove the exponential stabilization under the linear boundary feedback control $\alpha h_t(1,t) + h_{xxxt}(1,t) = 0$. In Gottlieb [15] the author has shown the existence of different classes of non-homogeneous Euler–Bernoulli beam models with continuous density and flexural rigidity functions and different end conditions, that are analytically solvable and 'isospectral' to a homogeneous beam model of clamped–clamped end conditions.

From the numerous works on the inverse problem for the Euler–Bernoulli model, we refer to the paper of Gladwell [11]. It has been long known that two scaling factors and three spectra, corresponding to three different end conditions, are required to determine the cross-sectional area $A(x)$ and the cross-sectional moment of inertia $I(x)$. However, the necessary and sufficient conditions on the spectral data that yield 'positive' functions $A(x)$ and $I(x)$ have not been known. Such conditions have been derived in Gladwell [11].

In Wang and Chen [34] the authors study a slender beam with spatially non-homogeneous viscous damping and structural damping. For constant damping coefficients, it is well known that the structural damping induces a strong attenuation rate that is frequency proportional, while the viscous damping induces a constant attenuation rate for all frequencies. In Wang and Chen [34] the author have shown that for the case of variable damping coefficients, the asymptotic patterns of the spectra remain the same, i.e. the viscous damping causes an asymptotically constant shift in the attenuation rates; hence, it is overwhelmed by the structural damping effect.

We also mention the work of Paulsen [26], where the asymptotic distribution of the eigenfrequencies of in-plane vibrations of an Euler–Bernoulli beam curved as an arc of a circle has been computed. This result could be instrumental in the analysis of Euler–Bernoulli beam system describing UAVs (Patil and Hodges [24].)

Finally, we mention some recent papers where the models' boundary conditions are just particular cases of conditions (3.5) considered in the present paper. Fernandes da Silva et al. [10] carry out the dynamic analysis of a beam with the ends elastically restrained against rotation and translation or with ends connected to concentration masses or rotational inertia. The authors split the boundary conditions into two groups that they call classical conditions (describing e.g. beams with free ends, clamped ends or supported ends) and non-classical conditions (describing beams with the ends connected to masses, springs, rotational inertia and/or dampers). Numerical results on the models are presented and discussed in the paper. Among other models, Fernandes da Silva et al. [10] consider two models with dampers at the right end. The case of linear damping is modelled by boundary conditions corresponding to the combination $k_1 = k_2 = \beta = 0$ and $\alpha > 0$ from (3.5) of the present paper, while the case of torsional damping is modelled by boundary conditions corresponding to the case of $k_1 = k_2 = \alpha = 0$ and $\beta > 0$ from (3.5).

In Gorrec et al. [14] the Euler–Bernoulli beam model connected to nonlinear mass–spring systems is studied. The model is motivated by the control of compliant micro-mechanical systems (microgrippers) that are used for the manipulation of biological samples. These systems are represented by undamped Euler–Bernoulli model connected to mass–spring damper systems. The boundary conditions used in Gorrec et al. [14] correspond to the case given by (3.5) with the following values of the control parameters: $k_1 = k_2 = 0$, $\alpha = 1$, $\beta = -1$.

Hermansen and Thomsen [16] have suggested a practically efficient methodology for using measured vibrations to estimate linear boundary stiffness and damping of beams, while simultaneously estimating axial tension. Estimation is performed by fitting model boundary parameters to measured model vibration data. The authors consider the Euler–Bernoulli beam model with linear and rotational springs and dampers at its boundaries. The transverse and longitudinal boundary conditions consist of transverse and rotational springs (K_2 and K_4) and dampers (C_2 and C_4). In these boundary conditions non-dimensional quantities are used, i.e. $k_4 = K_4 l/EI$, $c_4 = C_4 l\omega_0/EI$, where l is the length of the beam and ω_0 is a characteristic angular frequency. Boundary condition (6) in the paper of Hermansen and Thomsen [16] is $u''(1,\tau) = -k_4 u'(1,\tau) - c_4 \dot{u}'(1,\tau)$, where $u(x,\tau)$ is the non-dimensional vertical displacement of the beam at location x and time τ, and the overdot stands for the time derivative. Direct comparison of condition (6) in Hermansen and Thomsen [16] with the first boundary condition in (3.6) of the present paper shows the two conditions coincide if we identify the parameters k_4 and c_4 of (6) with $k_1/EI(L)$ and $\beta/EI(L)$ of (3.6), respectively.

The present paper is a continuation of the study initiated in our works (Shubov and Shubov [32]; Shubov and Kindrat [30] and [31]). In the paper of Shubov and Shubov [32], we have considered the model whose boundary feedback matrix contained only two non-trivial parameters, k_1 and k_2 (with $\alpha = \beta = 0$). One of the main results of Shubov and Shubov [32] is related to the stability of the model, i.e. it is shown that even though the model is not dissipative, for the case when one of the control parameters is positive and the other is sufficiently small, the set of the eigenmodes is located in the left half-plane of the complex plane. We have derived 'the main identity' (that might be of interest in its own right) which establishes a relation between the eigenmodes and mode shapes of the non-conservative model corresponding to the case $(k_1, k_2) \neq (0,0)$ and the eigenmodes and mode shapes of the clamped-free conservative model corresponding to the case $(k_1, k_2) = (0,0)$. We suggest a hypothesis that a similar stability result can be proven for the multiparameter case $(|\alpha| + |\beta| + |k_1| + |k_2| > 0)$ as well. In our second paper (Shubov and Kindrat [30]), we have considered the case of a four-parameter feedback control matrix and have shown a number of results on the general spectral properties of the non-selfadjoint operator $\mathcal{L}$. In particular, it is shown that for any combination of the boundary parameters the corresponding operator, $\mathcal{L}$, is a finite-rank perturbation of one and the same selfadjoint operator, $\mathcal{L}_0$, where $i\mathcal{L}_0$ is the dynamics generator for a cantilever beam model. It is also shown that the non-selfadjoint operator, $\mathcal{L}$, shares a number of spectral properties specific to its selfadjoint counterparts. (i) Namely, we have introduced four selfadjoint operators (corresponding to the clamped-free, clamped-hinged, clamped-sliding and clamped-clamped beam models) and derived specific inequalities that describe the boundary behaviour of the eigenfunctions of these operators. We have obtained the generalization of the aforementioned results for the non-selfadjoint operator $\mathcal{L}$. (ii) We have shown that each selfadjoint problem has a simple spectrum, and a similar result holds for the non-selfadjoint operator $\mathcal{L}$, i.e. the geometric multiplicity of any eigenvalue of $\mathcal{L}$ is one, while the algebraic multiplicity of each eigenvalues is finite but not necessarily one. Thus, for each eigenvalue there could exist a finite chain of associate functions. (iii) Finally, it is shown in Shubov and Kindrat [30], that if exactly one control parameter is not equal to zero, the operator $\mathcal{L}$ does not have real eigenvalues. On the other hand, when there are two or more non-zero boundary parameters, there could be real eigenvalues depending on which parameters are non-zero. In the third work of the series (Shubov and Kindrat [31]) we derive explicit formulae describing the asymptotic distribution of the eigenvalues of the operator $\mathcal{L}$, as the number of an eigenvalue tends to infinity. As expected, the asymptotic results strongly depend on the boundary control parameters. Our goal is to obtain such asymptotic formulae that contain all four parameters. To this end, in some cases it is not enough to derive the main leading asymptotic terms, but also the next order terms to identify the role of all control parameters. We derive asymptotic approximations for the eigenvalues of the operator $\mathcal{L}$ (and hence for the vibrational modes) when all boundary control parameters are non-negative (practically, the most important case). It is shown that the asymptotic approximation for the eigenvalues of $\mathcal{L}$ strongly depends on whether parameter $\beta > 0$ or $\beta = 0$. It turns out that in this case when $\beta = 0$ the spectral asymptotics are controlled by the parameter $K = (1 - \kappa_1\kappa_2)/(\kappa_1 + \kappa_2)$. To briefly represent the main results, we use the following terminology.

We call a vibrational mode, ν_n, stable if the energy norm of the vector $\exp\{i\mathcal{L}t\}\Psi_n$, with Ψ_n being the corresponding mode shape is a decreasing function of time, i.e. we have

$$\left\|\exp\{i\mathcal{L}t\}\Psi_n\right\|_{\mathcal{H}} = \left|e^{\nu_n t}\right| \|\Psi_n\|_{\mathcal{H}} = e^{(\Re\nu_n)t}\|\Psi_n\|_{\mathcal{H}} \to 0 \quad \text{as } t \to \infty.$$

$\Re\nu_n$ is called the energy decay rate (Curtain and Zwart [7]). If $\Re\nu_n = 0$, the corresponding vibrational mode is called marginally stable. If $\Re\nu_n > 0$, the corresponding vibrational mode is called an unstable mode and $\Re\nu_n$ is the energy gain rate.

It is shown that for $K > 0$, $K \neq 1$, there exists an infinite countable set of stable vibrational modes and, at most, a finite set of unstable modes. The rate of energy decay of a stable mode depends on the number of a mode, i.e. if ν_n is the n^{th} vibrational mode, then for the energy decay rate, the following relation holds: $|\Re\nu_n/n| = O(1)$ as $|n| \to \infty$. If $K < 0$, $K \neq -1$, there exists an infinite set of unstable vibrational modes and at most a finite number of stable modes; the rate of energy gain is proportional to $|n|$. For the cases when $|K| = 1$, the stability of vibrational modes and energy decay rates depend on the parameter α. The spectral asymptotics for the case $\beta > 0$ behave as follows: distant vibrational modes have about the same rate of energy decay if $k_1k_2 < 1 + \alpha\beta$ and about the same rate of energy gain (but different if $k_1k_2 > 1 + \alpha\beta$). In addition, if $k_1k_2 = 1$ and $\alpha = 0$, the spectrum of the operator $\mathcal{L}$ is asymptotically close to the real axis which can be an indication that the corresponding non-selfadjoint operator $\mathcal{L}$ is 'close' to its selfadjoint counterpart.

2. Spectral properties of the dynamics generator

We consider the Euler–Bernoulli beam model, subject to the general four-parameter family of non-conservative linear boundary conditions and distributed damping acting along the beam. We assume that the beam rests on an elastic foundation, whose modules of elasticity is $\gamma(x)$ and is subject to an axial force $S(x)$. There are also the following damping mechanisms: (i) an external or viscous damping with the damping coefficient $(-a_0(x))$ and (ii) a damping which is proportional to the bending rate with the damping coefficient $a_1(x)$. The transverse displacement of the beam, $u(x,t)$, at position x and time t is governed by the following damped wave equation:

$$\begin{aligned}&\rho(x)u_{tt}(x,t) + \big(EI(x)u_{xx}(x,t)\big)_{xx} - \big(S(x)u_x(x,t)\big)_x + \gamma(x)u(x,t)\\ &\quad + a_0(x)u_t(x,t) - \big(a_1(x)u_{tx}(x,t)\big)_x = 0,\end{aligned} \tag{2.1}$$

where $\rho(x)$ is the density of the beam and $EI(x)$ is the modulus of elasticity.

Heuristic considerations. We begin with the derivation of the energy expression, which will be instrumental for obtaining the state space metric form. Without loss of generality, we assume that the beam is of a unit length. The beam is clamped at the left end, i.e. the solution of Eq. (2.1) satisfies the following left–end conditions:

$$u(0,t) = 0, \qquad u_x(0,t) = 0. \tag{2.2}$$

To derive the right-end conditions, we need some preliminary steps. Let us multiply Eq. (2.1) by $u_t(x,t)$ and integrate with respect to x. We obtain

$$\begin{aligned}&\int_0^1 \rho(x)u_{tt}(x,t)u_t(x,t)\,dx+\int_0^1\big(EI(x)u_{xx}(x,t)\big)_{xx}u_t(x,t)\,dx\\&\quad-\int_0^1\big(S(x)u_x(x,t)\big)_x u_t(x,t)\,dx+\int_0^1\gamma(x)u(x,t)u_t(x,t)\,dx\\&\quad+\int_0^1 a_0(x)u_t^2(x,t)\,dx-\int_0^1\big(a_1(x)u_{tx}(x,t)\big)_x u_t\,dx\equiv\sum_{i=1}^6 I_i(t)=0.\end{aligned}\tag{2.3}$$

Taking into account conditions (2.2), we get the following representations for the integrals $I_i(t)$, $i=1,\ldots,6$:

$$\begin{aligned}&(i)\quad I_1(t)=\frac{1}{2}\frac{d}{dt}\int_0^1\rho(x)u_t^2(x,t)\,dx,\\&(ii)\quad I_2(t)=\big(EI(x)u_{xx}(u,t)\big)_x\Big|_{x=1}u_t(1,t)-EI(1)u_{xx}(1,t)u_{xt}(1,t)\\&\qquad\qquad+\frac{1}{2}\frac{d}{dt}\int_0^1 EI(x)u_{xx}^2(x,t)\,dx,\\&(iii)\quad I_3(t)=-S(1)u_x(1,t)u_t(1,t)+\frac{1}{2}\frac{d}{dt}\int_0^1 S(x)u_x^2(x,t)\,dx,\\&(iv)\quad I_4(t)=\frac{1}{2}\frac{d}{dt}\int_0^1\gamma(x)u^2(x,t)\,dx,\qquad(v)\quad I_5(t)=\int_0^1 a_0(x)u_t^2(x,t)\,dx,\\&(vi)\quad I_6(t)=-\int_0^1\big(a_1(x)u_{tx}(x,t)\big)_x u_t(x,t)\,dx\\&\qquad\qquad=-a_1(1)u_{tx}(1,t)u_t(1,t)+\int_0^1 a_1(x)u_{tx}^2\,dx.\end{aligned}\tag{2.4}$$

Summing up expressions (i)–(vi) of (2.4), we obtain the following equation:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1\big[\rho(x)u_t^2(x,t)+EI(x)u_{xx}^2(x,t)+S(x)u_x^2(x,t)+\gamma(x)u^2(x,t)\big]\,dx\\&\quad+\int_0^1 a_0(x)u_t^2(x,t)\,dx+\int_0^1 a_1(x)u_{xt}^2(x,t)\,dx+\Big[\big(EI(x)u_{xx}(x,t)\big)_x\Big|_{x=1}\\&\quad-S(1)u_x(1,t)\Big]u_t(1,t)-\big[EI(1)u_{xx}(1,t)+a_1(1)u_t(1,t)\big]u_{xt}(1,t)=0.\end{aligned}\tag{2.5}$$

Definition 2.1. Let the energy functional, $\mathcal{E}(t)$, be defined by

$$\mathcal{E}(t) = \frac{1}{2}\int_0^1 \left[\rho(x)u_t^2(x,t) + EI(x)u_{xx}^2(x,t) + S(x)u_x^2(x,t) + \gamma(x)u^2(x,t)\right] dx. \tag{2.6}$$

Based on this definition, we rewrite Eq. (2.5) in the form

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) = &-\int_0^1 a_0(x)u_t^2(x,t)\,dx - \int_0^1 a_1(x)u_{xt}^2(x,t)\,dx \\ &- \left[\left(EI(x)u_{xx}(x,t)\right)_x\Big|_{x=1} - S(1)u_x(1,t)\right]u_t(1,t) \\ &+ \left[EI(1)u_{xx}(1,t) + a_1(1)u_t(1,t)\right]u_{xt}(1,t).\end{aligned} \tag{2.7}$$

Now we are in a positions to introduce the right–end boundary conditions for the model.

If the following conditions are satisfied at the right end $x = 1$:

$$\left(EI(x)u_{xx}\right)_x\Big|_{x=1} - S(1)u_x(1,t) = 0, \qquad EI(1)u_{xx}(1,t) + a_1(1)u_t(1,t) = 0. \tag{2.8}$$

then we call them the *generalized free–end conditions*.

If the following conditions are satisfied at the right end $x = 1$:

$$\left(EI(x)u_{xx}(x,t)\right)_x\Big|_{x=1} - S(1)u_x(1,t) = 0, \quad u_{xt}(1,t) = 0, \tag{2.9}$$

then we call them the *generalized sliding conditions*.

If the following conditions are satisfied at the right end $x = 1$:

$$EI(1)u_{xx}(1,t) + a_1(1)u_t(1,t) = 0, \quad u_t(1,t) = 0, \tag{2.10}$$

then we call them the *generalized pinned conditions*.

If at $x = 1$ the following conditions are satisfied at the right end:

$$u_{xt}(1,t) = 0, \qquad u_t(1,t) = 0, \tag{2.11}$$

then we call them the *clamped condition*.

In all four cases, i.e. for a clamped-free model, a clamped-sliding model, a clamped-pinned model, and clamped-clamped model, the corresponding system is *dissipative*, i.e. its energy dissipates in time. The aforementioned examples are well-known in the applied sciences; however, there could be other types of the boundary conditions generating non-conservative systems (Benaroya [1]; Chen et al. [4]; Gladwell [11,12]; Littman and Markus [17]; Russell [29]; Wang and Chen [34]).

In the present paper we consider the initial boundary–value problem defined by Eq. (2.1), the left–end conditions (2.2), one of the right–end conditions from the set (2.8)–(2.11), and a standard set of the initial conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x). \tag{2.12}$$

Operator reformulation of the problem. Let us rewrite the initial boundary value problem as the first order in time evolution equation in the state space of the system (the energy space). We assume that $EI(x)$, $\rho(x)$, $S(x)$, $\gamma(x)$, $a_0(x)$, and $a_1(x)$ are strictly positive functions from $C^2[0,1]$.

Let $\mathcal{H}$ be the Hilbert space of two–component functions obtained as a closure of smooth functions $\Phi(x) = [\varphi_0(x), \varphi_1(x)]^T$ such that $\varphi_0(0) = \varphi_0'(0) = 0$ in the following norm:

$$\|\Phi\|^2_{\mathcal{H}} = \frac{1}{2}\int_0^1 \left[EI(x)\left|\varphi_0''(x)\right|^2 + S(x)\left|\varphi_0'(x)\right|^2 + \gamma(x)\left|\varphi_0(x)\right|^2 + \rho(x)\left|\varphi_1(x)\right|^2\right] dx. \tag{2.13}$$

The energy space $\mathcal{H}$ is topologically equivalent to the space $\widetilde{H}^2_0(0,1) \times L^2(0,1)$, where $\widetilde{H}^2_0 = \{\varphi \in H^2(0,1) : \varphi(0) = \varphi'(0) = 0\}$. Let us introduce the following matrix expression:

$$\mathcal{L} = -i \begin{bmatrix} 0 & 1 \\ \mathfrak{M}_1 & \mathfrak{M}_2 \end{bmatrix}, \tag{2.14}$$

with $\mathfrak{M}_1$ and $\mathfrak{M}_2$ being the differential operations defined by

$$(\mathfrak{M}_1\varphi)(x) = -\frac{1}{\rho(x)}\frac{d^2}{dx^2}\left(EI(x)\frac{d^2\varphi(x)}{dx^2}\right) + \frac{1}{\rho(x)}\frac{d}{dx}\left(S(x)\frac{d\varphi(x)}{dx}\right) + \gamma(x)\varphi(x), \tag{2.15}$$

$$(\mathfrak{M}_2\varphi)(x) = -\frac{a_0(x)}{\rho(x)}\varphi(x) + \frac{1}{\rho(x)}\frac{d}{dx}\left(a_1(x)\frac{d\varphi(x)}{dx}\right). \tag{2.16}$$

Let $\mathcal{L}$ be defined on the domain

$$\begin{aligned}\mathcal{D}(\mathcal{L}) = \Big\{&\Phi \in \mathcal{H} : \Phi = (\varphi_0, \varphi_1)^T, \varphi_0 \in H^4(0,1), \varphi_1 \in H^2(0,1);\\ &\varphi_1(0) = \varphi_1'(0) = 0; \text{“right-end conditions”}\Big\}.\end{aligned} \tag{2.17}$$

Depending on the choice of the right–end conditions, we obtain different matrix differential operators in the space $\mathcal{H}$ (Mennicken and Möller [22]).

(i) If for the right–end conditions we take conditions (2.8), then the differential operator defined by (2.14)–(2.17), and (2.8) corresponds to the clamped–free model. We denote this operator by $\mathcal{L}_{cf}$. One can verify by direct calculations that the initial boundary value problem defined by Eq. (2.1), conditions (2.2), (2.8), and (2.12) can be represented as the evolution problem in $\mathcal{H}$:

$$\Psi_t(x,t) = i(\mathcal{L}_{cf}\Psi)(x,t), \qquad \Psi(x,0) = \left[\psi_0(x), \psi_1(x)\right]^T, \quad 0 \leqslant x \leqslant 1, t \geqslant 0. \tag{2.18}$$

(ii) If for the right–end conditions, we take conditions (2.9), then the differential operator defined by (2.14)–(2.17), and (2.9) corresponds to the clamped–sliding model. We denote this operator by $\mathcal{L}_{cs}$. One can verify that the initial boundary value problem defined by Eq. (2.1), conditions (2.2), (2.9), and (2.12) can be written as the evolution problem in $\mathcal{H}$:

$$\Psi_t(x,t) = i(\mathcal{L}_{cs}\Psi)(x,t), \qquad \Psi(x,0) = \left[\psi_0(x), \psi_1(x)\right]^T, \quad 0 \leqslant x \leqslant 1, t \geqslant 0. \tag{2.19}$$

(iii) If for the right-end condition we take conditions (2.10), then the operator defined by (2.14)–(2.17), and (2.10) corresponds to the clamped–pinned model. Denoting this operator by $\mathcal{L}_{cp}$, we can verify that the initial boundary value problem defined by Eq. (2.1), conditions (2.2), (2.10), and (2.12) can be written as an evolution problem in $\mathcal{H}$:

$$\Psi_t(x,t) = i(\mathcal{L}_{cp}\Psi)(x,t), \qquad \Psi(x,0) = \left[\psi_0(x), \psi_1(x)\right]^T, \quad 0 \leqslant x \leqslant 1, t \geqslant 0. \tag{2.20}$$

(*iv*) To deal with the clamped–clamped model, one has to consider a different state space $\widetilde{\mathcal{H}}$, which is obtained as a closure of smooth two–component functions $\Phi(x) = [\varphi_0(x), \varphi_1(x)]^T$ such that $\varphi_0(0) = \varphi_0'(0) = 0$ and $\varphi_0(1) = \varphi_0'(1) = 0$ in the norm (2.13). Thus $\widetilde{\mathcal{H}}$ is a proper subspace of $\mathcal{H}$, i.e., $\dim \mathcal{H}(\operatorname{mod} \widetilde{\mathcal{H}}) = 1$. If for the right–end conditions we take (2.11), then the operator defined in $\widetilde{\mathcal{H}}$ by (2.14)–(2.17), and (2.11) corresponds to the clamped–clamped model, $\mathcal{L}_{cc}$. The appropriate evolution problem in $\widetilde{\mathcal{H}}$ can be written in the form:

$$\Psi_t(x,t) = i(\mathcal{L}_{cc}\Psi)(x,t), \qquad \Psi(x,0) = \left[\psi_0(x), \psi_1(x)\right]^T, \quad 0 \leqslant x \leqslant 1, t \geqslant 0. \tag{2.21}$$

In what follows we focus on asymptotic and spectral analysis of the operator $\mathcal{L}_{cf}$. Using similar techniques asymptotic and spectral results can be obtained for each operator from the remaining set, i.e. for $\mathcal{L}_{cs}$, $\mathcal{L}_{cp}$, and $\mathcal{L}_{cc}$.

We recall that the operator $\mathcal{L}_{cf}$ is defined by matrix differential expressions (2.14)–(2.16) on the domain

$$\begin{aligned} \mathcal{D}(\mathcal{L}_{cf}) = \Big\{ & \Phi \in \mathcal{H} : \Phi = (\varphi_0, \varphi_1)^T, \varphi_0 \in H^4(0,1), \varphi_1 \in H^2(0,1); \\ & \varphi_1(0) = \varphi_1'(0) = 0; EI(1)\varphi_0''(1) + a_1(1)\varphi_1(1) = 0, \\ & \left(EI(x)\varphi_0''(x)\right)'\Big|_{x=1} - S(1)\varphi_0'(1) = 0 \Big\}. \end{aligned} \tag{2.22}$$

Our first result on the operator $\mathcal{L}_{cf}$ is the following statement.

Theorem 2.2. *$\mathcal{L}_{cf}$ is a dissipative operator in $\mathcal{H}$.*

Proof. A linear operator is said to be dissipative if its quadratic form has non-negative imagery part (see, e.g., Gohberg and Krein [13] and Szökefalvi-Nagy and Foias [33]). Thus, it suffices to show that $\Im(\mathcal{L}_{cf}\Phi, \Phi)_{\mathcal{H}} \geqslant 0$. We have for $\Phi \in \mathcal{D}(\mathcal{L}_{cf})$

$$i(\mathcal{L}_{cf}\Phi, \Phi)_{\mathcal{H}} = \left(\begin{pmatrix} \varphi_1(x) \\ \psi(x) \end{pmatrix}, \begin{pmatrix} \varphi_0(x) \\ \varphi_1(x) \end{pmatrix} \right)_{\mathcal{H}}, \tag{2.23}$$

where

$$\begin{aligned} \psi(x) = {} & -\frac{1}{\rho(x)}\left(EI(x)\varphi_0''\right)'' + \frac{1}{\rho(x)}\left(S(x)\varphi_0'(x)\right)' \\ & - \frac{\gamma(x)}{\rho(x)}\varphi_0(x) - \frac{a_0(x)}{\rho(x)}\varphi_1(x) + \frac{1}{\rho(x)}\left(a_1(x)\varphi_1(x)\right)'. \end{aligned} \tag{2.24}$$

Evaluating the inner product of (2.23) denoted by $\mathbf{J}$, we have

$$\begin{aligned}\mathbf{J} = \frac{1}{2}\int_0^1 \bigl\{ & EI(x)\varphi_1''(x)\overline{\varphi_0''(x)} + S(x)\varphi_1'(x)\overline{\varphi_0'(x)} + \gamma(x)\varphi_1(x)\overline{\varphi_0(x)} \\ & - \bigl(EI(x)\varphi_0''(x)\bigr)''\overline{\varphi_1(x)} + \bigl(S(x)\varphi_0'(x)\bigr)'\overline{\varphi_1(x)} - \gamma(x)\varphi_0(x)\overline{\varphi_1(x)} \\ & - a_0(x)\varphi_1(x)\overline{\varphi_1(x)} + \bigl(a_1(x)\varphi_1'(x)\bigr)'\overline{\varphi_1(x)}\bigr\}\,dx. \end{aligned} \tag{2.25}$$

Integrating by parts the integral (2.25) and representing $\mathbf{J}$ as the sum: $\mathbf{J} = \sum_{j=1}^{4} I_j + \widetilde{\mathbf{J}}$, we obtain the following representations for the integrals I_j, $j = 1, \dots, 4$:

$$\begin{aligned} I_1 &\equiv \int_0^1 EI(x)\bigl[\varphi_1''(x)\overline{\varphi_0''(x)} - \varphi_0''(x)\overline{\varphi_1''(x)}\bigr]\,dx = 2i\Im \int_0^1 EI(x)\varphi_1''(x)\overline{\varphi_0''(x)}\,dx, \\ I_2 &\equiv \int_0^1 S(x)\bigl[\varphi_1'(x)\overline{\varphi_0'(x)} - \varphi_0'(x)\overline{\varphi_1'(x)}\bigr]\,dx = 2i\Im \int_0^1 S(x)\varphi_1'(x)\overline{\varphi_0'(x)}\,dx, \\ I_3 &\equiv \int_0^1 \gamma(x)\bigl[\varphi_1(x)\overline{\varphi_0(x)} - \varphi_0(x)\overline{\varphi_1(x)}\bigr]\,dx = 2i\Im \int_0^1 \gamma(x)\varphi_1(x)\overline{\varphi_0(x)}\,dx, \\ I_4 &\equiv \int_0^1 \bigl[a_0(x)\bigl|\varphi_1(x)\bigr|^2 + a_1(x)\bigl|\varphi_1'(x)\bigr|^2\bigr]\,dx. \end{aligned} \tag{2.26}$$

Out of integral term, $\widetilde{\mathbf{J}}$, is given by:

$$\begin{aligned} \widetilde{\mathbf{J}} &= -\bigl(EI(x)\varphi_0''(x)\bigr)'\overline{\varphi_1(x)}\Big|_{x=1} + EI(1)\varphi_0''(1)\overline{\varphi_1'(1)} + S(1)\varphi_0'(1)\overline{\varphi_1(1)} + a_1(1)\varphi_1'(1)\overline{\varphi_1(1)} \\ &= -\Bigl[\bigl(EI(x)\varphi_0''(x)\bigr)'\Big|_{x=1} - S(1)\varphi_0'(1)\Bigr]\varphi_1(1) + \bigl[EI(1)\varphi_0''(1) + a_1(1)\varphi_1(1)\bigr]\overline{\varphi_1'(1)}. \end{aligned} \tag{2.27}$$

Taking into account the domain of $\mathcal{L}_{cf}$ (see (2.22)), we obtain that the expression $\widetilde{\mathbf{J}}$ of (2.27) is zero. Thus, substituting formulae (2.26) into (2.23), we obtain

$$\Im(\mathcal{L}_{cf}\Phi, \Phi)_{\mathcal{H}} = \int_0^1 \bigl[a_0(x)\bigl|\varphi_1(x)\bigr|^2 + a_1(x)\bigl|\varphi_1'(x)\bigr|^2\bigr]\,dx > 0.$$

The theorem is shown. $\square$

Remark 2.3. Using similar techniques, one can show that the operators $\mathcal{L}_{cp}$ and $\mathcal{L}_{cs}$ are dissipative in the space $\mathcal{H}$ and $\mathcal{L}_{cc}$ is dissipative in the space $\widetilde{\mathcal{H}}$.

3. Generalization of the model to the case of four-parameter boundary conditions

In this section, we introduce an additional mechanism for damping through parameter-dependent boundary conditions. To this end, we consider the generalized moment $M(x,t)$ and the generalized shear $Q(x,t)$ (Benaroya [1]; Gladwell [12]):

$$\begin{aligned} &M(x,t) = EI(x)u_{xx}(x,t) + a_1(x)u_t(x,t), \\ &Q(x,t) = \big(EI(x)u_{xx}(x,t)\big)_x - S(x)u_{xt}(x,t). \end{aligned} \tag{3.1}$$

Let the input, $U(t)$, and the output, $Y(t)$, be given as $\mathbb{R}^2$ – vectors

$$U(t) = \big[-Q(1,t), M(1,t)\big]^T \quad \text{and} \quad Y(t) = \big[u_t(1,t), u_{xt}(1,t)\big]^T, \tag{3.2}$$

where the superscript "T" stands for transposition. The feedback control law can be given as follows:

$$U(1) = \mathbb{K}Y(t), \quad \text{and} \quad \mathbb{K} = \begin{bmatrix} -\alpha & -\kappa_2 \\ -\kappa_1 & -\beta \end{bmatrix}, \tag{3.3}$$

with α, β, κ_1, and κ_2 being the control parameters. The feedback (3.3) can be written in the form

$$\begin{aligned} &\big(EI(x)u_{xx}(x,t)\big)_x\Big|_{x=1} - S(1)u_{xt}(1,t) = \alpha u_t(1,t) + \kappa_2 u_{xt}(1,t), \\ &EI(1)u_{xx}(1,t) + a_1(1)u_t(1,t) = -\kappa_1 u_t(1,t) - \beta u_{xt}(1,t). \end{aligned} \tag{3.4}$$

Having in mind conditions (3.4), we introduce a new matrix differential operator in $\mathcal{H}$ denoted by $\mathcal{L}_{cf}(\alpha,\beta,\kappa_1,\kappa_2)$, which will be our main object of interest. This operator is defined by the matrix expression given by (2.14)–(2.16) on the domain

$$\begin{aligned} \mathcal{D}\big(\mathcal{L}_{cf}(\alpha,\beta,\kappa_1,\kappa_2)\big) = \Big\{ &\Phi = (\varphi_0,\varphi_1)^T, \varphi_0 \in H^4(0,1), \varphi_1 \in H^2(0,1); \varphi_1(0) = \varphi_1'(0) = 0; \\ &EI(1)\varphi_0''(1) + a_1(1)\varphi_1(1) = -\kappa_1\varphi_1(1) - \beta\varphi_1'(1), \\ &\big(EI(x)\varphi_0''(x)\big)'\Big|_{x=1} - S(1)\varphi_1'(1) = \alpha\varphi_1(1) + \kappa_2\varphi_1'(1)\Big\}. \end{aligned} \tag{3.5}$$

Our first result is this section is the following statement.

Lemma 3.1. *For the case of non–negative parameters α, β, κ_1, and κ_2, the operator $\mathcal{L}_{cf}(\alpha,\beta,\kappa_1,\kappa_2)$ is dissipative in $\mathcal{H}$ when $\kappa_1 + \kappa_2 = 0$ and it is not dissipative when $\kappa_1 + \kappa_2 > 0$.*

Proof. To prove the result, it suffices to evaluate $\Im(\mathcal{L}_{cf}(\alpha,\beta,\kappa_1,\kappa_2)\Phi,\Phi)_{\mathcal{H}}$ for $\Phi \in \mathcal{D}(\mathcal{L}_{cf}(\alpha,\beta,\kappa_1,\kappa_2))$. Evaluating the inner product, we have to repeat all the steps that have been carried out for the operator $\mathcal{L}_{cf}$ in Theorem 2.2. As the result, we obtain

$$i\big(\mathcal{L}_{cf}(\alpha,\beta,\kappa_1,\kappa_2)\Phi,\Phi\big)_{\mathcal{H}} = \left(\begin{pmatrix}\varphi_1(x) \\ \psi(x)\end{pmatrix}, \begin{pmatrix}\varphi_0(x) \\ \varphi_1(x)\end{pmatrix}\right)_{\mathcal{H}}, \tag{3.6}$$

where ψ is given in (2.24). Evaluating the inner product of (3.6) denoted by $\mathbf{J}_1$ we obtain that $\mathbf{J}_1$ can be given by formula similar to (2.25) evaluated on the functions from the domain of $\mathcal{L}_{cf}(\alpha,\beta,\kappa_1,\kappa_2)$.

Integrating by parts we represent $\mathbf{J}_1$ as a sum: $\mathbf{J}_1 = \sum_{j=1}^{4} \widetilde{I}_j + \widetilde{\mathbf{J}}_1$, where the integrals $\widetilde{I}_j$, $j = 1, 2, 3, 4$, are given by the formulae similar to (2.26). Taking into account the domain (3.5), we obtain the following representation for out of integral term $\widetilde{\mathbf{J}}_1$:

$$\begin{aligned}\widetilde{\mathbf{J}}_1 &= \big[-\alpha\varphi_1(1) - \kappa_1\varphi_1'(1)\big]\overline{\varphi_1(1)} + \big[-\kappa_1\varphi_1(1) - \beta\varphi_1'(1)\big]\overline{\varphi_1'(1)} \\ &= -\alpha\big|\varphi_1(1)\big|^2 - \beta\big|\varphi_1'(1)\big|^2 - \kappa_2\varphi_1'(1)\overline{\varphi_1(1)} - \kappa_1\varphi_1(1)\overline{\varphi_1'(1)}. \end{aligned} \tag{3.7}$$

Substituting this representation for $\widetilde{\mathbf{J}}_1$ into (3.6) we obtain

$$\Im\big(\mathcal{L}_{cf}(\alpha, \beta, \kappa_1, \kappa_2)\Phi, \Phi\big)_{\mathcal{H}} = \alpha\big|\varphi_1(1)\big|^2 + \beta\big|\varphi_1'(1)\big|^2 + \Re\big[\kappa_1\varphi_1(1)\overline{\varphi_1'(1)} + \kappa_2\varphi_1'(1)\overline{\varphi_1(1)}\big]. \tag{3.8}$$

If $\kappa_1 + \kappa_2 = 0$, then $\Re[\kappa_1(\varphi_1(1)\overline{\varphi_1'(1)} - \kappa_2\varphi_1'(1)\overline{\varphi_1(1)})] = 0$ and the dissipativity of the operator $\mathcal{L}_{cf}(\alpha, \beta, \kappa_1, \kappa_2)$ follows from (3.8).

The lemma is shown. $\square$

To simplify the notations, without loss of generality, we will write $\mathcal{L}$ in place of $\mathcal{L}_{cf}(\alpha, \beta, \kappa_1, \kappa_2)$. To investigate the spectral properties of the operator $\mathcal{L}$, it is convenient to represent it as the sum, $\mathcal{L} = \mathcal{L}^0 + \mathfrak{M}$. The operator $\mathcal{L}^0$ is given by (2.14)–(2.16) in which $S(x) = \gamma(x) = 0$ and defined on the same domain as $\mathcal{L}$:

$$\mathcal{L}^0 = -i\begin{pmatrix} 0 & 1 \\ -\frac{1}{\rho(x)}\frac{d^2}{dx^2}(EI(x)\frac{d^2}{dx^2}\cdot) & -\frac{a_0(x)}{\rho(x)} + \frac{1}{\rho(x)}\frac{d}{dx}(a_1(x)\frac{d}{dx}\cdot) \end{pmatrix}, \tag{3.9}$$

with

$$\mathcal{D}\big(\mathcal{L}^0\big) = \mathcal{D}(\mathcal{L}). \tag{3.10}$$

The operator $\mathfrak{M}$ is given by the formula

$$\mathfrak{M} = -i\begin{pmatrix} 0 & 0 \\ \frac{1}{\rho(x)}\frac{d}{dx}(S(x)\frac{d}{dx}\cdot) - \frac{\gamma(x)}{\rho(x)} & 0 \end{pmatrix}. \tag{3.11}$$

It can be checked that $\mathfrak{M}$ is a bounded operator in $\mathcal{H}$. The following statement holds for $\mathcal{L}^0$.

Theorem 3.2. *1) $\mathcal{L}^0$ is an unbounded non–selfadjoint operator in $\mathcal{H}$ with compact resolvent, whose spectrum consists of a countable set of normal eigenvalues (i.e., each eigenvalue is an isolated point of the spectrum, whose algebraic multiplicity is finite (Locker [20]; Marcus [21]; Gohberg and Krein [13])). The set of the eigenvalues accumulates only at infinity.*

2) For any combination of the parameters, such that $\alpha + \beta + \kappa_1 + \kappa_2 > 0$, the operator $\mathcal{L}^0$ is a finite–rank perturbation of the operator $\mathcal{L}_0$ corresponding to the cantilever case when $\alpha = \beta = \kappa_1 = \kappa_2 = 0$. The fact that $\mathcal{L}_0$ is a perturbation of $\mathcal{L}^0$ should be understood in the following sense. The operators $(\mathcal{L}^0)^{-1}$ and $\mathcal{L}_0^{-1}$ exist and are related by the rule

$$\big(\mathcal{L}^0\big)^{-1} = \mathcal{L}_0^{-1} + \mathcal{T}, \tag{3.12}$$

where $\mathcal{T}$ is a finite–rank operator. The following formulae are valid for any $G = (g_0, g_1)^T \in \mathcal{H}$:

$$\big((\mathcal{L}^0)^{-1}G\big)(x) = \big[(\mathfrak{R}_0 g_0)(x) + \big(\mathfrak{R}_1 g_0'\big)(x) + (\mathfrak{R}_2 g_1)(x), i g_0(x)\big]^T, \tag{3.13}$$

where $\mathfrak{R}_0$, $\mathfrak{R}_1$, and $\mathfrak{R}_2$ are Volterra integral operators defined on a differentiable function ψ by the following formulae:

$$(i) \quad (\mathfrak{R}_0\psi)(x) = -i\int_0^x d\tau \int_0^\tau \frac{d\eta}{EI(\eta)} \int_\eta^1 d\xi \int_\xi^1 a_0(w)\psi(w)\,dw,$$

$$(ii) \quad (\mathfrak{R}_1\psi)(x) = -i\int_0^x d\tau \int_0^\tau \frac{d\eta}{EI(\eta)} \int_\eta^1 a_1(w)\psi'(w)\,dw,$$

$$(iii) \quad (\mathfrak{R}_2\psi)(x) = -i\int_0^x d\tau \int_0^\tau \frac{d\eta}{EI(\eta)} \int_\eta^1 d\xi \int_\xi^1 \rho(w)\psi(w)\,dw. \tag{3.14}$$

The operator $\mathcal{T}$ is defined by the formula

$$(\mathcal{T}G)(x) = \big[\widehat{g}(x), 0\big]^T, \qquad G(x) = \big[g_0(x), g_1(x)\big]^T, \tag{3.15}$$

where

$$\widehat{g}(x) = \big[\big(a_1(1) + \kappa_1\big)g_0(1) + \beta g_0'(1)\big]g(x) + \big[\alpha g_0(1) + \big(S(1) + \kappa_2 + a_1(1)\big)g_0'(1)\big]h(x), \tag{3.16}$$

with $g(x)$ and $h(x)$ being given by

$$g(x) = -i\int_0^x d\eta \int_0^\eta \frac{dw}{EI(w)}, \qquad h(x) = -i\int_0^x d\eta \int_0^\eta \frac{1-w}{EI(w)}\,dw. \tag{3.17}$$

The operator $\mathcal{L}_0$ is self–adjoint in a Hilbert space $\mathcal{H}_0$, whose norm can be obtained from the norm of $\mathcal{H}$ by setting $S = \gamma = 0$.

3) The decomposition similar to (3.12) is valid for the adjoint operator, i.e.,

$$\big((\mathcal{L}^0)^*\big)^{-1} = \big(\mathcal{L}_0^*\big)^{-1} + \mathcal{T}^*, \tag{3.18}$$

where $\mathcal{T}^*$ is given by the formulae similar to (3.15) and (3.16) in which α, β, κ_1, and κ_2 have been replaced with $(-\alpha)$, $(-\beta)$, $(-\kappa_1)$, and $(-\kappa_2)$ respectively.

Proof. To prove the decomposition (3.12), let us show that the equation $\mathcal{L}^0\Phi = F$ has a unique solution $\Phi \in \mathcal{D}(\mathcal{L})$ for any $F \in \mathcal{H}$. Rewriting this equation component–wise, we obtain the following system:

$$\varphi_1(x) = i f_0(x), \qquad \big(EI(x)\varphi_0''(x)\big)'' + a_0(x)\varphi_1(x) - \big(a_1(x)\varphi_1'(x)\big)' = -i\rho(x) f_1(x). \tag{3.19}$$

Eliminating φ_1 from system (3.19) we obtain the equation for φ_0:

$$\big(EI(x)\varphi_0''(x)\big)'' = -i\rho(x) f_1(x) - i a_0(x) f_0(x) + i\big(a_1(x) f_0'(x)\big)'. \tag{3.20}$$

Integrating Eq. (3.20) from x to 1 and taking into account the α – boundary condition from (3.4), we rewrite this equation as

$$\begin{aligned}\big(EI(x)\varphi_0''(x)\big)' = {} & i\alpha f_0(1) + i\big(S(1)+\kappa_2\big)f_0'(1) + i\int_x^1 \rho(w)f_1(w)\,dw \\ & + i\int_x^1 a_0(w)f_0(w)\,dw - ia_1(1)f_0'(1) + ia_1(x)f_0'(x).\end{aligned} \tag{3.21}$$

Integrating this equation once again from x to 1, we get

$$\begin{aligned}& EI(1)\varphi_0''(1) - EI(x)\varphi_0''(x) \\ & \quad = i\big[\alpha f_0(1) + \big(S(1)+\kappa_2\big)f_0'(1)\big](1-x) \\ & \qquad + i\int_x^1 d\xi\int_\xi^1 \rho(w)f_1(w)\,dw + i\int_x^1 d\xi\int_\xi^1 a_0(w)f_0(w)\,dw \\ & \qquad + i\int_x^1 a_1(w)f_0'(w)\,dw - ia_1(1)f_0'(1)(1-x).\end{aligned}$$

Taking into account the β – boundary condition from (3.4) we rewrite this equation as

$$\begin{aligned}& -i\big(a_1(1)+\kappa_1\big)f_0(1) - i\beta f_0'(1) - EI(x)\varphi_0''(x) \\ & \quad = i\big[\alpha f_0(1) + \big(S(1)+\kappa_2\big)f_0'(1) - 2a_1(1)f_0'(1)\big](1-x) \\ & \qquad + i\int_x^1 d\xi\int_\xi^1 \rho(w)f_1(w)\,dw \\ & \qquad + i\int_x^1 d\xi\int_\xi^1 a_0(w)f_0(w)\,dw + i\int_x^1 a_1(w)f_0'(w)\,dw.\end{aligned} \tag{3.22}$$

From (3.22) we obtain the following formula for φ_0'':

$$\begin{aligned}\varphi_0''(x) = -\frac{i}{EI(x)}\bigg\{ & \big[\big(a_1(1)+\kappa_1\big)f_0(1) + \beta f_0'(1)\big] \\ & + \big[\alpha f_0(1) + \big(S(1)+\kappa_2+2a_1(1)\big)f_0'(1)\big](1-x) + \int_x^1 d\xi\int_\xi^1 \rho(w)f_1(w)\,dw \\ & + \int_x^1 d\xi\int_\xi^1 a_0(w)f_0(w)\,dw + \int_x^1 a_1(w)f_0'(w)\,dw\bigg\}.\end{aligned} \tag{3.23}$$

Integrating Eq. (3.23) twice from 0 to x and taking into account that $\varphi_0'(0) = 0$, we obtain the explicit formula for φ_0

$$\begin{aligned}
\varphi_0(x) = &-i\big[\big(a_1(1)+\kappa_1\big)f_0(1)+\beta f_0'(1)\big]\int_0^x d\eta\int_0^\eta \frac{dw}{EI(w)}\\
&- i\big[\alpha f_0(1)+\big(S(1)+\kappa_2+a_1(1)\big)f_0'(1)\big]\int_0^x d\eta\int_0^\eta \frac{1-w}{EI(w)}\,dw\\
&- i\int_0^x d\tau\int_0^\tau \frac{d\eta}{EI(\eta)}\int_\eta^1 d\xi\int_\xi^1 a_0(w)f_0(w)\,dw\\
&- i\int_0^x d\tau\int_0^\tau \frac{d\xi}{EI(\xi)}\int_\eta^1 a_1(w)f_0'(w)\,dw\\
&- i\int_0^x d\tau\int_0^\tau \frac{d\eta}{EI(\eta)}\int_\eta^1 d\xi\int_\xi^1 \rho(w)f_1(w)\,dw.
\end{aligned}\tag{3.24}$$

If g and h are the functions defined in (3.17) and $\mathfrak{R}_0$, $\mathfrak{R}_1$, and $\mathfrak{R}_2$ are the Volterra integral operators defined in (3.14), then φ_0 can be written in the form

$$\begin{aligned}
\varphi_0(x) = &\big[\big(a_1(1)+\kappa_1\big)f_0(1)+\beta f_0'(1)\big]g(x)+\big[\alpha f_0(1)+\big(S(1)+\kappa_2+a_1(1)\big)f_0'(1)\big]h(x)\\
&+[\mathfrak{R}_0 f_0](x)+\big[\mathfrak{R}_1 f_0'\big](x)+[\mathfrak{R}_2 f_1](x).
\end{aligned}\tag{3.25}$$

Let $\mathbb{A}$ be an operator that maps a continuous function into its value at $x = 1$, i.e. $(\mathbb{A}f)(x) = f(1)$ and $\mathbb{B}$ be an operator that maps continuously differentiable function f into $f'(1)$, i.e. $[\mathbb{B}f](x) = f'(1)$. Then $\varphi_0(x)$ of (3.25) can be given as

$$\begin{aligned}
\varphi_0(x) = &\big\{\big(a_1(1)+\kappa_1\big)[\mathbb{A}f_0]+\beta[\mathbb{B}f_0]\big\}g(x)+\big\{\alpha[\mathbb{A}f_0]+\big(S(1)+\kappa_2+a_1(1)\big)[\mathbb{B}f_0]\big\}h(x)\\
&+[\mathfrak{R}_0 f_0](x)+\big[\mathfrak{R}_1 f_0'\big](x)+[\mathfrak{R}_2 f_1](x).
\end{aligned}\tag{3.26}$$

Using formulae (3.19) for φ_1 and (3.26) for φ_0, one can check that the operator, which is inverse to $\mathcal{L}^0$ and is defined by (3.9) on the domain (3.10), can be written as the following sum:

$$\big(\mathcal{L}^0\big)^{-1} = (\mathcal{L}_0)^{-1}+\mathcal{T},\tag{3.27}$$

$\mathcal{L}_0$ is an unbounded non-selfadjoint operator, which is defined by the differential expression (3.9) with free boundary conditions at the end $x = 1$, ($\alpha = \beta = \kappa_1 = \kappa_2 = 0$), i.e. $\mathcal{L}_0$ corresponds to the cantilever beam model; $\mathcal{T}$ is a finite-rank operator. The operators $(\mathcal{L}_0)^{-1}$ and $\mathcal{T}$ are given by the following formulae:

$$(\mathcal{L}_0)^{-1} = \begin{bmatrix} \mathfrak{R}_0\cdot+\mathfrak{R}_1(\frac{d}{dx}\cdot) & \mathfrak{R}_2\cdot \\ i & 0 \end{bmatrix},\tag{3.28}$$

$$
\begin{aligned}
&\mathcal{T}\cdot = \begin{bmatrix} [T_1\cdot]g(x) + [T_2\cdot]h(x) & 0 \\ 0 & 0 \end{bmatrix}, \\
&T_1\cdot = \bigl(a_1(1) + \kappa_1\bigr)[\mathbb{A}\cdot] + \beta[\mathbb{B}\cdot], \\
&T_2\cdot = \alpha[\mathbb{A}\cdot] + \bigl(S(1) + \kappa_2 + a_1(1)\bigr)[\mathbb{B}\cdot].
\end{aligned} \tag{3.29}
$$

The proof is complete. □

Finally, we present the main result of this section.

Theorem 3.3. *The non-selfadjoint operator, $\mathcal{L}$, has a countable set of normal eigenvalues that can accumulate at infinity. (An isolated eigenvalue is said to be normal if its multiplicity is finite (Gohberg and Krein [13]).)*

Proof. Using the operators $\mathcal{L}^0$ and $\mathfrak{M}$ introduced in (3.9)–(3.11), we obtain the following decomposition:

$$
\mathcal{L}^{-1} = \bigl[\mathcal{L}^0\bigl(I + \bigl(\mathcal{L}^0\bigr)^{-1}\mathfrak{M}\bigr)\bigr]^{-1} = \bigl(I + \bigl(\mathcal{L}^0\bigr)^{-1}\mathfrak{M}\bigr)^{-1}\bigl(\mathcal{L}^0\bigr)^{-1}. \tag{3.30}
$$

As is shown in Theorem 3.2, $(\mathcal{L}^0)^{-1} \in \mathfrak{S}_\infty(\mathcal{H})$ and $\mathfrak{M} \in \mathcal{R}(\mathcal{H})$ ($\mathfrak{S}_\infty(\mathcal{H})$ denotes the set of all compact operators in $\mathcal{H}$ and $\mathcal{R}(\mathcal{H})$ denotes the set of all bounded operators in $\mathcal{H}$). Thus, to prove that $\mathcal{L}^{-1} \in \mathfrak{S}_\infty(\mathcal{H})$, we have to show the only fact that 0 is not an eigenvalue of $\mathcal{L}$ (see Gohberg and Krein [13]). Using the contradiction argument, assume that there exists $\Phi \in \mathcal{D}(\mathcal{L})$ such that $\mathcal{L}\Phi = 0$. From formulae (2.14)–(2.16) for $\mathcal{L}$, we get that $\varphi_1(x) = 0$ and that φ_0 has to satisfy the equation

$$
-\bigl(EI(x)\varphi_0''(x)\bigr)'' + \bigl(S(x)\varphi_0'(x)\bigr)' - \gamma(x)\varphi_0(x) = 0, \tag{3.31}
$$

and the boundary conditions

$$
\varphi_0''(1) = 0, \qquad \bigl(EI(x)\varphi_0''(x)\bigr)'\Big|_{x=1} = 0. \tag{3.32}
$$

We multiply Eq. (3.31) by $\overline{\varphi_0(x)}$ and integrate it to have

$$
\begin{aligned}
&-\bigl(EI(x)\varphi_0''(x)\bigr)'\overline{\varphi(x)}\Big|_0^1 + EI(x)\varphi_0''(x)\overline{\varphi'(x)}\Big|_0^1 - \int_0^1 EI(x)\bigl|\varphi_0''(x)\bigr|^2\,dx \\
&\quad + S(x)\varphi_0'(x)\overline{\varphi_0(x)}\Big|_0^1 - \int_0^1 S(x)\bigl|\varphi_0'(x)\bigr|^2\,dx - \int_0^1 \gamma(x)\bigl|\varphi_0(x)\bigr|^2\,dx = 0.
\end{aligned} \tag{3.33}
$$

Taking into account the boundary conditions at $x = 0$ and conditions (3.32) we obtain

$$
S(1)\varphi_0'(1)\overline{\varphi_0(1)} - \int_0^1 \bigl[EI(x)\bigl|\varphi_0''(x)\bigr|^2 + S(x)\bigl|\varphi_0'(x)\bigr|^2 + \gamma(x)\bigl|\varphi_0(x)\bigr|^2\bigr]\,dx = 0. \tag{3.34}
$$

Since φ_0 satisfies Eq. (3.31) and conditions (3.32), this function is in fact an eigenfunction of an Euler–Bernoulli clamped-free beam model and, therefore, φ_0 is defined up to a multiplicative constant. Without loss of generality, we can choose φ_0 in such a way that $\varphi_0'(1)\overline{\varphi_0(1)} \leqslant 0$ (Benaroya [1]; Gladwell [12]; Shubov and Kindrat [30]). Since $EI(x) > 0$, $S(x) > 0$, and $\gamma(x) > 0$, we obtain that $\varphi_0(x) = ax + b$, which reduces Eq. (3.34) to $\int_0^1 [a^2 S(x) + \gamma(x)|ax + b|^2]\,dx = 0$, from which we get $a = 0$ and hence, $b = 0$.

The proof is complete. □

4. Spectral equation

In this section we derive the equation, whose solutions coincide with the eigenvalues of the operator $\mathcal{L}$. Namely, we consider the eigenvalue – eigenfunction equation for the operator $\mathcal{L}$, i.e. $\mathcal{L}\Phi = \lambda\Phi$, $\Phi \in \mathcal{D}(\mathcal{L})$. From this moment on, we will carry out the asymptotic analysis for the case of constant parameters of the model. The case of variable coefficients will be presented in our forthcoming work.

It is convenient for technical reason to represent the damping terms as $a_0(x) = 2\widetilde{a}_0(x)$ and $a_1(x) = 2\widetilde{a}_1(x)$. Without loss of generality, we set $EI = 1$ and obtain the following description of the domain of the operator $\mathcal{L}$:

$$\begin{aligned} &\varphi_0''(1) = -\big(2\widetilde{a}_1(1) + \kappa_1\big)\varphi_1(1) - \beta\varphi_1'(1), \\ &\varphi_0'''(1) = \alpha\varphi_1(1) + (S + \kappa_2)\varphi_1'(1). \end{aligned} \tag{4.1}$$

Eliminating φ_1 from the system of two equations, resulting from the equation $\mathcal{L}\Phi = \lambda\Phi$, we obtain that the first component of vector–function Φ must satisfy the following equation:

$$\varphi''''(x,\lambda) - (2i\lambda\widetilde{a}_1 + S)\varphi''(x,\lambda) - \big(\rho\lambda^2 - 2i\lambda\widetilde{a}_0 - \gamma\big)\varphi(x,\lambda) = 0. \tag{4.2}$$

Let us modify Eq. (4.2) by introducing $\widetilde{\lambda}$, $\widehat{a}_0$, and $\widehat{a}_1$ according to the formulae: $\widetilde{\lambda} = \lambda\sqrt{\rho}$, $\widehat{a}_0 = \widetilde{a}_0/\sqrt{\rho}$, and $\widehat{a}_1 = \widetilde{a}_1/\sqrt{\rho}$. In the new notations, Eq. (4.2) has the following form:

$$\varphi''''(x,\cdot) - (2i\widetilde{\lambda}\widehat{a}_1 + S)\varphi''(x,\cdot) - \big(\widetilde{\lambda}^2 - 2i\widetilde{\lambda}\widehat{a}_0 - \gamma\big)\varphi(x,\cdot) = 0. \tag{4.3}$$

Comparing (4.2) and (4.3) shows that if one knows the solution of Eq. (4.2) with $\rho = 1$, then the solution for the case $\rho \neq 1$ can be found without difficulties.

For the rest of this section we focus on Eq. (4.2) with $\rho = 1$. The characteristic equation associated with the differential equation (4.2) with $\rho = 1$ can be written in the form

$$z^4 - (2i\lambda\widetilde{a}_1 + S)z^2 - \big(\lambda^2 - 2i\lambda\widetilde{a}_0 - \gamma\big) = 0. \tag{4.4}$$

Solving this biquadratic equation we obtain the representation for its roots

$$z_{1,2}^2 = \left(i\lambda\widetilde{a}_1 + \frac{S}{2}\right) \pm \sqrt{\lambda^2\big(1 - \widetilde{a}_1^2\big) + i\lambda(S\widetilde{a}_1 - 2\widetilde{a}_0) + \left(\frac{S^2}{4} - \gamma\right)}. \tag{4.5}$$

It can be readily checked that for $\widetilde{a}_1 < 1$ the following asymptotic approximations are valid as $|\lambda| \to \infty$:

$$z_{1,2}^2 = \left(i\lambda\widetilde{a}_1 + \frac{S}{2}\right) \pm \lambda\sqrt{1-\widetilde{a}_1^2}\left(1 + \frac{i(S\widetilde{a}_1 - 2\widetilde{a}_0)}{\lambda(1-\widetilde{a}_1^2)} + \frac{\frac{S^2}{4}-\gamma}{\lambda^2(1-\widetilde{a}_1^2)}\right)^{1/2}$$
$$= i\lambda\widetilde{a}_1 \pm \lambda\sqrt{1-\widetilde{a}_1^2} + \left(\frac{S}{2} \pm \frac{i(S\widetilde{a}_1 - 2\widetilde{a}_0)}{2\sqrt{1-\widetilde{a}_1^2}}\right) + O\left(\frac{1}{\lambda}\right). \tag{4.6}$$

Notice, the damping parameter $\widetilde{a}_0$ does not enter the leading asymptotical terms in (4.6).

It is convenient to use the following notations for $z_{1,2}^2$ of (4.5):

$$\begin{aligned}\mu^2 &= \left(i\lambda\widetilde{a}_1 + \frac{S}{2}\right) + \sqrt{\lambda^2\left(1-\widetilde{a}_1^2\right) + i\lambda(S\widetilde{a}_1 - 2\widetilde{a}_0) + \left(\frac{S^2}{4}-\gamma\right)},\\ \nu^2 &= \left(i\lambda\widetilde{a}_1 + \frac{S}{2}\right) - \sqrt{\lambda^2\left(1-\widetilde{a}_1^2\right) + i\lambda(S\widetilde{a}_1 - 2\widetilde{a}_0) + \left(\frac{S^2}{4}-\gamma\right)}.\end{aligned} \tag{4.7}$$

Using (4.6) we immediately get asymptotic approximations for μ^2 and ν^2. As the fundamental set for Eq. (4.2), one can take a "standard" set of four solutions

$$\left\{\cosh(\mu x), \sinh(\mu x), \cosh(\nu x), \sinh(\nu x)\right\}.$$

In the sequel, we will use an equivalent basis composed from functions $C_\pm$ and $S_\pm$ defined below. Let us introduce new functions convenient for dealing with the beam equation

$$C_\pm(x) = \frac{1}{2}\left[\cosh(\mu x) \pm \cosh(\nu x)\right], \qquad S_\pm(x) = \frac{1}{2}\left[\frac{\sinh(\mu x)}{\mu} \pm \frac{\sinh(\nu x)}{\nu}\right]. \tag{4.8}$$

These functions satisfy the following set of the boundary conditions at $x = 0$:

$$\begin{aligned}&(i)\quad C_-(0) = 0, \qquad C_-'(0) = 0, \qquad C_+(0) = 1, \qquad C_+'(0) = 0;\\ &(ii)\quad S_-(0) = 0, \qquad S_-'(0) = 0, \qquad S_+(0) = 0, \qquad S_+'(1) = 1;\\ &(iii)\quad S_-'(1) = C_-(1), \qquad S_-''(1) = C_-'(1), \qquad S_-'''(1) = C_-''(1).\end{aligned} \tag{4.9}$$

One can readily verify that the solution of Eq. (4.2) satisfying the left-end boundary conditions $(\varphi(0,\lambda) = \varphi'(0,\lambda) = 0)$ can be represented in the form

$$\varphi(x,\lambda) = \mathcal{A}(\lambda)C_-(x) + \mathcal{B}(\lambda)S_-(x), \tag{4.10}$$

with $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ being arbitrary functions of λ. If $\varphi(x,\lambda)$ satisfies the α – boundary conditions from (4.1), i.e., $\varphi'''(1) = i\lambda(S + \widehat{\kappa}_2)\varphi'(1) + i\alpha\lambda\varphi(1)$, then in terms of (4.10) we obtain the equation

$$\begin{aligned}&\mathcal{A}(\lambda)\left[C_-'''(1) - i\lambda\widehat{\kappa}_2 C_-'(1) - i\lambda\alpha C_-(1)\right]\\ &\quad + \mathcal{B}(\lambda)\left[S_-'''(1) - i\lambda\widehat{\kappa}_2 S_-'(1) - i\lambda\alpha S_-(1)\right] = 0,\end{aligned} \tag{4.11}$$

where $\widehat{\kappa}_2 = S + \kappa_2$. If $\varphi(x,\lambda)$ satisfies the β–boundary condition, i.e., $\varphi''(1) + i\lambda\beta\varphi'(1) + i\lambda\widehat{\kappa}_1\varphi(1) = 0$, where $\widehat{\kappa}_1 = a_1 + \kappa_1 = 2\widetilde{a}_1 + \kappa_1$, then in terms of (4.10) we obtain the equation:

$$\begin{aligned}&\mathcal{A}(\lambda)\big[C''_-(1) + i\lambda\beta C'_-(1) + i\lambda\widehat{\kappa}_1 C_-(1)\big]\\ &\quad + \mathcal{B}(\lambda)\big[S''_-(1) + i\lambda\beta S'_-(1) + i\lambda\widehat{\kappa}_1 S_-(1)\big] = 0.\end{aligned} \tag{4.12}$$

To obtain a non–trivial solution of the system (4.11) and (4.12) for $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$, we set to zero its determinant denoted by $\mathcal{D}(\lambda)$:

$$\begin{aligned}&\big[C'''_-(1) - i\lambda\widehat{\kappa}_2 C'_-(1) - i\lambda\alpha C_-(1)\big]\big[S''_-(1) + i\lambda\beta S'_-(1) + i\lambda\widehat{\kappa}_1 S_-(1)\big]\\ &\quad - \big[C''_-(1) + i\lambda\beta C'_-(1) + i\lambda\widehat{\kappa}_1 C_-(1)\big]\big[S'''_-(1) - i\lambda\widehat{\kappa}_2 S'_-(1) - i\lambda\alpha S_-(1)\big] = 0.\end{aligned} \tag{4.13}$$

Using formulae (4.9)(iii), we represent the determinant, $\mathcal{D}(\lambda)$, as the following sum:

$$\mathcal{D}(\lambda) = \sum_{n=1}^{7} I_n(\lambda), \tag{4.14}$$

where

$$\begin{aligned}
&(i)\quad I_1(\lambda) = \big[C'''_-(1)S''_-(1) - S'''_-(1)C''_-(1)\big] = C'''_-(1)C'_-(1) - \big(C''_-(1)\big)^2,\\
&(ii)\quad I_2(\lambda) = i\lambda\beta\big[C'''_-(1)S'_-(1) - S'''_-(1)C'_-(1)\big] = i\lambda\beta\big[C'''_-(1)C_-(1) - C''_-(1)C'_-(1)\big],\\
&(iii)\quad I_3(\lambda) = i\lambda\widehat{\kappa}_1\big[C'''_-(1)S_-(1) - S'''_-(1)C_-(1)\big],\\
&(iv)\quad I_4(\lambda) = -i\lambda\widehat{\kappa}_2\big[C'_-(1)S''_-(1) - S'_-(1)C''_-(1)\big] = i\lambda\widehat{\kappa}_2\big[\big(C'_-(1)\big)^2 - C_-(1)C''_-(1)\big],\\
&(v)\quad I_5(\lambda) = -(i\lambda)^2\widehat{\kappa}_1\widehat{\kappa}_2\big[C'_-(1)S_-(1) - S'_-(1)C_-(1)\big]\\
&\qquad\qquad = -(i\lambda)^2\widehat{\kappa}_1\widehat{\kappa}_2\big[C'_-(1)S_-(1) - \big(C_-(1)\big)^2\big],\\
&(vi)\quad I_6(\lambda) = -(i\lambda)\alpha\big[C_-(1)S''_-(1) - S_-(1)C''_-(1)\big] = i\lambda\alpha\big[C_-(1)C'_-(1) - S_-(1)C''_-(1)\big],\\
&(vii)\quad I_7(\lambda) = -(i\lambda)^2\alpha\beta\big[C_-(1)S'_-(1) - S_-(1)C'_-(1)\big]\\
&\qquad\qquad = (i\lambda)^2\alpha\beta\big[\big(C_-(1)\big)^2) - S_-(1)C'_-(1)\big].
\end{aligned} \tag{4.15}$$

Taking into account the expressions for $C_-(1)$ and $S_-(1)$ of (4.8), we obtain for $I_1(\lambda)$:

$$\begin{aligned}I_1(\lambda) &= \frac{1}{4}\big\{\big(\mu^3\sinh\mu - \nu^3\sinh\nu\big)(\mu\sinh\mu - \nu\sinh\nu) - \big(\mu^2\cosh\mu - \nu^2\cosh\nu\big)^2\big\}\\ &= \frac{1}{4}\big\{-\mu^4 - \nu^4 - \mu\nu\big(\mu^2+\nu^2\big)\sinh\mu\sinh\nu + 2\mu^2\nu^2\cosh\mu\cosh\nu\big\}.\end{aligned} \tag{4.16}$$

For $I_2(\lambda)$ we get

$$\begin{aligned} I_2(\lambda) &= \frac{i\lambda\beta}{4}\left\{\left(\mu^3\sinh\mu-\nu^3\sinh\nu\right)(\cosh\mu-\cosh\nu)\right.\\ &\quad\left.-(\mu\sinh\mu-\nu\sinh\nu)\left(\mu^2\cosh\mu-\nu^2\cosh\nu\right)\right\}\\ &= \frac{i\lambda\beta}{4}\left\{\mu\left(\nu^2-\mu^2\right)\sinh\mu\cosh\nu+\nu\left(\mu^2-\nu^2\right)\cosh\mu\sinh\nu\right\}. \end{aligned} \tag{4.17}$$

For $I_3(\lambda)$ we get

$$\begin{aligned} I_3(\lambda) &= \frac{i\lambda\widehat{\kappa_1}}{4}\left\{\left(\mu^3\sinh\mu-\nu^3\sinh\nu\right)\left(\frac{\sinh\mu}{\mu}-\frac{\sinh\nu}{\nu}\right)\right.\\ &\quad\left.-(\cosh\mu-\cosh\nu)\left(\mu^2\cosh\mu-\nu^2\cosh\nu\right)\right\}\\ &= \frac{i\lambda\widehat{\kappa_1}}{4}\left\{-\mu^2-\nu^2+\left(\mu^2+\nu^2\right)\cosh\mu\cosh\nu-\left(\frac{\mu^3}{\nu}+\frac{\nu^3}{\mu}\right)\sinh\mu\sinh\nu\right\}. \end{aligned} \tag{4.18}$$

For $I_4(\lambda)$ we get

$$\begin{aligned} I_4(\lambda) &= -\frac{i\lambda\widehat{\kappa_2}}{4}\left\{(\mu\sinh\mu-\nu\sinh\nu)^2-(\cosh\mu-\cosh\nu)\left(\mu^2\cosh\mu-\nu^2\cosh\nu\right)\right\}\\ &= -\frac{i\lambda\widehat{\kappa_2}}{4}\left\{-\mu^2-\nu^2-2\mu\nu\sinh\mu\sinh\nu+\left(\mu^2+\nu^2\right)\cosh\mu\cosh\nu\right\}. \end{aligned} \tag{4.19}$$

For $I_5(\lambda)$ we get

$$\begin{aligned} I_5(\lambda) &= -\frac{(i\lambda)^2\widehat{\kappa_1}\widehat{\kappa_2}}{4}\left\{(\mu\sinh\mu-\nu\sinh\nu)\left(\frac{\sinh\mu}{\mu}-\frac{\sinh\nu}{\nu}\right)-(\cosh\mu-\cosh\nu)^2\right\}\\ &= -\frac{(i\lambda)^2\widehat{\kappa_1}\widehat{\kappa_2}}{4}\left\{-2-\left(\frac{\nu}{\mu}+\frac{\mu}{\nu}\right)\sinh\mu\sinh\nu+2\cosh\mu\cosh\nu\right\}. \end{aligned} \tag{4.20}$$

For $I_6(\lambda)$ we get

$$\begin{aligned} I_6(\lambda) &= -\frac{i\lambda\alpha}{4}\left\{(\cosh\mu-\cosh\nu)(\mu\sinh\mu-\nu\sinh\nu)\right.\\ &\quad\left.-\left(\mu^2\cosh\mu-\nu^2\cosh\nu\right)\left(\frac{\sinh\mu}{\mu}-\frac{\sinh\nu}{\nu}\right)\right\}\\ &= -\frac{i\lambda\alpha}{4}\left\{\left(\frac{\nu^2}{\mu}-\mu\right)\cosh\nu\sinh\mu+\left(\frac{\mu^2}{\nu}-\nu\right)\cosh\mu\sinh\nu\right\}. \end{aligned} \tag{4.21}$$

For $I_7(\lambda)$ we get

$$I_7(\lambda) = -\frac{(i\lambda)^2\alpha\beta}{4}\left\{(\cosh\mu - \cosh\nu)^2 - \left(\frac{\sinh\mu}{\mu} - \frac{\sinh\nu}{\nu}\right)(\mu\sinh\mu - \nu\sinh\nu)\right\}$$
$$= -\frac{(i\lambda)^2\alpha\beta}{4}\left\{2 - 2\cosh\mu\cosh\nu + \left(\frac{\mu}{\nu} + \frac{\nu}{\mu}\right)\sinh\mu\sinh\nu\right\}. \tag{4.22}$$

Substituting (4.17)–(4.22) into Eq. (4.13) we obtain the general form of *the spectral equation*

$$\begin{aligned}
&\left[-\left(\mu^4 + \nu^4\right) - \mu\nu\left(\mu^2 + \nu^2\right)\sinh\mu\sinh\nu + 2\mu^2\nu^2\cosh\mu\cosh\nu\right] \\
&\quad + i\lambda\beta\left[\mu\left(\nu^2 - \mu^2\right)\sinh\mu\cosh\nu + \nu\left(\mu^2 - \nu^2\right)\sinh\nu\cosh\mu\right] \\
&\quad + i\lambda\widehat{\kappa}_1\left[-\left(\mu^2 + \nu^2\right) + \left(\mu^2 + \nu^2\right)\cosh\mu\cosh\nu - \left(\frac{\mu^3}{\nu} + \frac{\nu^3}{\mu}\right)\sinh\mu\sinh\nu\right] \\
&\quad - i\lambda\widehat{\kappa}_2\left[-\left(\mu^2 + \nu^2\right) + \left(\mu^2 + \nu^2\right)\cosh\mu\cosh\nu - 2\mu\nu\sinh\mu\sinh\nu\right] \\
&\quad + \lambda^2\widehat{\kappa}_1\widehat{\kappa}_2\left[-2 - \left(\frac{\mu}{\nu} + \frac{\nu}{\mu}\right)\sinh\mu\sinh\nu + 2\cosh\mu\cosh\nu\right] \\
&\quad - i\lambda\alpha\left[\left(\frac{\nu^2}{\mu} - \mu\right)\cosh\nu\sinh\mu + \left(\frac{\mu^2}{\nu} - \nu\right)\cosh\mu\sinh\nu\right] \\
&\quad + \lambda^2\alpha\beta\left[2 - 2\cosh\mu\cosh\nu + \left(\frac{\mu}{\nu} + \frac{\nu}{\mu}\right)\sinh\mu\sinh\nu\right] = 0.
\end{aligned} \tag{4.23}$$

Remark 4.1. In our paper (Shubov and Kindrat [31]), we have derived the spectral equation for the case of the beam equation, in which the distributed damping, axial force, and modulus of elasticity were equal to zero, i.e., $\widetilde{a}_0 = \widetilde{a}_1 = S = \gamma = 0$. Let us show that the spectral equation (2.27) obtained in the aforementioned paper can be derived from Eq. (4.23). For $\widetilde{a}_0 = \widetilde{a}_1 = S = \gamma = 0$, we have $\nu = i\mu$, i.e., $\mu^2 + \nu^2 = 0$ and $\mu^2 - \nu^2 = 2\mu^2$. For this case, Eq. (4.23) can be reduced to the following:

$$\begin{aligned}
&\left[-2\mu^4 - 2\mu^4\cosh\mu\cosh\nu\right] + i\lambda\beta\left[-2\mu^3\sinh\mu\cosh\nu + 2i\mu^3\cosh\mu\sinh\nu\right] \\
&\quad + i\lambda\widehat{\kappa}_1\left[2i\mu^2\sinh\mu\sinh\nu\right] - i\lambda\widehat{\kappa}_2\left[-2i\mu^2\sinh\mu\sinh\nu\right] \\
&\quad - (i\lambda)^2\widehat{\kappa}_1\widehat{\kappa}_2[-2 + 2\cosh\mu\cosh\nu] - i\lambda\alpha[-2\mu\cosh\nu\sinh\mu - 2i\mu\cosh\mu\sinh\nu] \\
&\quad - (i\lambda)^2\alpha\beta[2 - 2\cosh\mu\cosh\nu] = 0.
\end{aligned} \tag{4.24}$$

Taking into account that $\cosh\nu = \cos\mu$ and $\sinh\nu = i\sin\mu$, we modify (4.24) as

$$\begin{aligned}
&-2\mu^4[1 + \cosh\mu\cos\mu] - 2i\lambda\mu^3\beta[\sinh\mu\cos\mu + \cosh\mu\sin\mu] \\
&\quad - 2i\lambda\mu^2\widehat{\kappa}_1[\sinh\mu\sin\mu] - 2i\lambda\mu^2\widehat{\kappa}_2[\sinh\mu\sin\mu] + 2(i\lambda)^2\widehat{\kappa}_1\widehat{\kappa}_2[1 - \cosh\mu\cos\mu] \\
&\quad + 2i\lambda\mu\alpha[\cos\mu\sinh\mu - \cosh\mu\sin\mu] - 2(i\lambda)^2\alpha\beta[1 - \cosh\mu\cos\mu] = 0.
\end{aligned} \tag{4.25}$$

Recalling (4.7), i.e., $\lambda = \mu^2$, and nothing that $\widehat{\kappa_1}$ and $\widehat{\kappa_2}$ correspond to κ_1 and κ_2 from Shubov and Kindrat [31], we rewrite this equation in the form

$$\begin{aligned}&\mu\{\kappa_1\kappa_2[1-\cosh\mu\cos\mu]+i(\kappa_1+\kappa_2)\sinh\mu\sin\mu+(1+\cosh\mu\cos\mu)\}\\&\quad=-i\mu^2\beta[\sinh\mu\cos\mu+\cosh\mu\sin\mu]+i\alpha[\sinh\mu\cos\mu-\cosh\mu\sin\mu]\\&\qquad+\alpha\beta\mu[1-\cosh\mu\cos\mu],\end{aligned}$$

which coincides with the spectral equation (2.27) from paper Shubov and Kindrat [31].

5. Asymptotic analysis of the spectral equation

In the first statement of this section, we collect all technical results needed in the sequel.

Lemma 5.1. *1) The following explicit formulae hold:*

$$\begin{aligned}&(i)\quad \mu^2\nu^2=-\lambda^2+2i\widetilde{a}_0\lambda+\gamma,\qquad \mu^2+\nu^2=S+2i\widetilde{a}_1\lambda,\\&(ii)\quad \mu^4+\nu^4=2\big(1-2\widetilde{a}_1^2\big)\lambda^2+4i(S\widetilde{a}_1-\widetilde{a}_0)\lambda+S^2-2\gamma,\\&(iii)\quad \mu^2-\nu^2=2\sqrt{\lambda^2\big(1-\widetilde{a}_1^2\big)+i\lambda(S\widetilde{a}_1-2\widetilde{a}_0)+\left(\frac{S^2}{4}-\gamma\right)}.\end{aligned}\tag{5.1}$$

2) For the case $0<\tilde{a}_1<1$, the following asymptotic approximations are valid as $|\lambda|\to\infty$:

$$\begin{aligned}&(i)\quad \mu=\sqrt{\sqrt{1-\widetilde{a}_1^2}+i\widetilde{a}_1}\sqrt{\lambda}+\frac{S\sqrt{1-\widetilde{a}_1^2}+i(S\widetilde{a}_1-2\widetilde{a}_0)}{4\sqrt{1-\widetilde{a}_1^2}\sqrt{\sqrt{1-\widetilde{a}_1^2}+i\widetilde{a}_1}}\frac{1}{\sqrt{\lambda}}+O\left(\frac{1}{\lambda^{1.5}}\right),\\&(ii)\quad \nu=i\sqrt{\sqrt{1-\widetilde{a}_1^2}-i\widetilde{a}_1}\sqrt{\lambda}-\frac{iS\sqrt{1-\widetilde{a}_1^2}+(S\widetilde{a}_1-2\widetilde{a}_0)}{4\sqrt{1-\widetilde{a}_1^2}\sqrt{\sqrt{1-\widetilde{a}_1^2}-i\widetilde{a}_1}}\frac{1}{\sqrt{\lambda}}+O\left(\frac{1}{\lambda^{1.5}}\right),\\&(iii)\quad \frac{\mu}{\nu}+\frac{\nu}{\mu}=2\widetilde{a}_1+\frac{i(2\widetilde{a}_0\widetilde{a}_1-S)}{\lambda}+O\left(\frac{1}{\lambda^2}\right),\\&(iv)\quad \mu^2-\nu^2=2\sqrt{1-\widetilde{a}_1^2}\lambda+\frac{i(S\widetilde{a}_1-2\widetilde{a}_0)}{\sqrt{1-\widetilde{a}_1^2}}\\&\qquad+\left[\frac{S^2-4\gamma}{4\sqrt{1-\widetilde{a}_1^2}}+\frac{(S\widetilde{a}_1-2\widetilde{a}_0)^2}{4(1-\widetilde{a}_1^2)^{3/2}}\right]\frac{1}{\lambda}+O\left(\frac{1}{\lambda^2}\right),\end{aligned}\tag{5.2}$$

$$(v)\quad \frac{\mu^3}{\nu}+\frac{\nu^3}{\mu}=2i\big(2\widetilde{a}_1^2-1\big)\lambda+2\big(-2\widetilde{a}_0\widetilde{a}_1^2+2S\widetilde{a}_1-\widetilde{a}_0\big)$$
$$+i\big(2\widetilde{a}_1^2\gamma+\gamma-6\widetilde{a}_0^2\widetilde{a}_1^2+4S\widetilde{a}_0\widetilde{a}_1-\widetilde{a}_0^2-S^2\big)\frac{1}{\lambda}+O\left(\frac{1}{\lambda^2}\right).$$

Proof. Using the explicit formulae of μ^2 and ν^2 given in (4.7), we keep the first three terms in the corresponding Taylor expansions as $|\lambda|\to\infty$. We obtain

$$\mu^2=\left(i\widetilde{a}_1+\sqrt{1-\widetilde{a}_1^2}\right)\lambda+\left(i\frac{S\widetilde{a}_1-2\widetilde{a}_0}{2\sqrt{1-\widetilde{a}_1^2}}+\frac{S}{2}\right)$$
$$+\left(\frac{S^2-4\gamma}{8\sqrt{1-\widetilde{a}_1^2}}+\frac{(S\widetilde{a}_1-2\widetilde{a}_0)^2}{8(1-\widetilde{a}_1^2)^{3/2}}\right)\frac{1}{\lambda}+O\left(\frac{1}{\lambda^2}\right), \tag{5.3}$$

and

$$\nu^2=\left(i\widetilde{a}_1-\sqrt{1-\widetilde{a}_1^2}\right)\lambda+\left(-i\frac{S\widetilde{a}_1-2\widetilde{a}_0}{2\sqrt{1-\widetilde{a}_1^2}}+\frac{S}{2}\right)$$
$$-\left(\frac{S^2-4\gamma}{8\sqrt{1-\widetilde{a}_1^2}}+\frac{(S\widetilde{a}_1-2\widetilde{a}_0)^2}{8(1-\widetilde{a}_1^2)^{3/2}}\right)\frac{1}{\lambda}+O\left(\frac{1}{\lambda^2}\right). \tag{5.4}$$

We calculate the square roots of right-hand sides of Eqs (5.3) and (5.4) to get formulae (5.2(i)) and (5.2(ii)). Using similar approach we obtain the remaining set of the formulae from (5.2).

The lemma is proven. □

Remark 5.2. For the case when $\widetilde{a}_1\geqslant 1$, the corresponding asymptotic approximations can be derived using a similar approach.

Now we are in a position to derive the asymptotic form of the spectral equation (4.23). Namely, let us represent this equation in the form

$$\mathbf{A}(\lambda)\sinh\mu\sinh\nu+\mathbf{B}(\lambda)\cosh\mu\cosh\nu$$
$$+\mathbf{C}(\lambda)\sinh\mu\cosh\nu+\mathbf{D}(\lambda)\cosh\mu\sinh\nu+\mathbf{F}(\lambda)=0, \tag{5.5}$$

where the coefficients are given by the following expressions:

$$\mathbf{A}(\lambda)=-\mu\nu\big(\mu^2+\nu^2\big)-i\lambda\left[\widehat{\kappa}_1\frac{\mu^4+\nu^4}{\mu\nu}-2\mu\nu\widehat{\kappa}_2\right]+\lambda^2\frac{\mu^2+\nu^2}{\mu\nu}(\alpha\beta-\widehat{\kappa}_1\widehat{\kappa}_2), \tag{5.6}$$

$$\mathbf{B}(\lambda)=2\mu^2\nu^2-i\lambda\big(\mu^2+\nu^2\big)(\widehat{\kappa}_1-\widehat{\kappa}_2)+2\lambda^2(\widehat{\kappa}_1\widehat{\kappa}_2-\alpha\beta), \tag{5.7}$$

$$\mathbf{C}(\lambda)=i\lambda\big(\nu^2-\mu^2\big)\left(\beta\mu-\frac{\alpha}{\mu}\right),\qquad \mathbf{D}(\lambda)=i\lambda\big(\mu^2-\nu^2\big)\left(\beta\nu-\frac{\alpha}{\nu}\right), \tag{5.8}$$

$$\mathbf{F}(\lambda) = -(\mu^4 + \nu^4) - i\lambda(\mu^2 + \nu^2)(\widehat{\kappa_1} - \widehat{\kappa_2}) - 2\lambda^2\widehat{\kappa_1}\widehat{\kappa_2} + 2\lambda^2\alpha\beta. \tag{5.9}$$

Using Lemma 5.1, we obtain the asymptotic approximation for all functions (5.6)–(5.9) when $|\lambda| \to \infty$. To simplify the presentation below, we will omit "∼" over a_0 and a_1. We have collected the assymptotical results in the statement below.

Lemma 5.3. (*i*) *Asymptotic approximation for the coefficient for* $\sinh\mu\sinh\nu$ *of Eq. (5.5) can be given in the form*

$$\begin{aligned}\mathbf{A}(\lambda) = {} & \bigl[2a_1 + 2(2a_1^2 - 1)\widehat{\kappa_1} - 2\widehat{\kappa_2} + 2a_1(\alpha\beta - \widehat{\kappa_1}\widehat{\kappa_2})\bigr]\lambda^2 \\ & + \bigl[-i(2a_0a_1 + S) + 2i(2a_0a_1^2 - 2Sa_1 + a_0)\widehat{\kappa_1} \\ & + 2i\widehat{\kappa_2}a_0 + i(2a_0a_1 - S)(\alpha\beta - \widehat{\kappa_1}\widehat{\kappa_2})\bigr]\lambda + O(1).\end{aligned} \tag{5.10}$$

(*ii*) *Asymptotic approximation for the coefficient for* $\cosh\mu\cosh\nu$ *of Eq. (5.5) can be given in the form*

$$\mathbf{B}(\lambda) = \bigl[-2 + 2a_1(\widehat{\kappa_1} - \widehat{\kappa_2}) + 2(\widehat{\kappa_1}\widehat{\kappa_2} - \alpha\beta)\bigr]\lambda^2 - \bigl[-4ia_0 + iS(\widehat{\kappa_1} - \widehat{\kappa_2})\bigr]\lambda + O(1). \tag{5.11}$$

(*iii*) *Asymptotic approximation for the coefficient for* $\sinh\mu\cosh\nu$ *of Eq. (5.5) can be given in the form*

$$\begin{aligned}\mathbf{C}(\lambda) = {} & -2i\sqrt{1-a_1^2}\sqrt{\sqrt{1-a_1^2} + ia_1}\,\beta\,\lambda^{2.5} \\ & - \left\{\frac{i\beta[S(1-3a_1^2) + 3i(Sa_1 - 2a_0)\sqrt{1-a_1^2} + 4a_0a_1]}{2\sqrt{1-a_1^2}\sqrt{\sqrt{1-a_1^2} + ia_1}} - \frac{2i\alpha\sqrt{1-a_1^2}}{\sqrt{\sqrt{1-a_1^2} + ia_1}}\right\}\lambda^{1.5} \\ & + O(\sqrt{\lambda}).\end{aligned} \tag{5.12}$$

(*iv*) *Asymptotic approximation for the coefficient for* $\sinh\nu\cosh\mu$ *of Eq. (5.5) can be given in the form*

$$\begin{aligned}\mathbf{D}(\lambda) = {} & -2\sqrt{1-a_1^2}\sqrt{\sqrt{1-a_1^2} - ia_1}\,\beta\,\lambda^{2.5} \\ & + \left\{\frac{\beta[S(1-3a_1^2) - 3i(Sa_1 - 2a_0)\sqrt{1-a_1^2} + 4a_0a_1]}{2\sqrt{1-a_1^2}\sqrt{\sqrt{1-a_1^2} - ia_1}} - \frac{2\alpha\sqrt{1-a_1^2}}{\sqrt{\sqrt{1-a_1^2} - ia_1}}\right\}\lambda^{1.5} \\ & + O(\sqrt{\lambda}).\end{aligned} \tag{5.13}$$

(*v*) *Asymptotic approximation for the free term of Eq. (5.5) can be given in the form*

$$\begin{aligned}\mathbf{F}(\lambda) = {} & \bigl[-2(1 - 2a_1^2) + 2a_1(\widehat{\kappa_1} - \widehat{\kappa_2}) + 2(\alpha\beta - \widehat{\kappa_1}\widehat{\kappa_2})\bigr]\lambda^2 \\ & - \bigl[4i(Sa_1 - a_0) + iS(\widehat{\kappa_1} - \widehat{\kappa_2})\bigr]\lambda + 2\gamma - S^2.\end{aligned} \tag{5.14}$$

Proof. We present the detailed proof only for formula (5.10) for $\mathbf{A}(\lambda)$. The formulae (5.11)–(5.14) can be proven by using similar techniques. According to (5.6), $\mathbf{A}(\lambda)$ can be represented as the sum $\sum_{k=1}^{4} j_k$, where

$$\begin{aligned} j_1 &= -\mu\nu\big(\mu^2+\nu^2\big), \qquad j_2 = -i\lambda\kappa_1\frac{\mu^4+\nu^4}{\mu\nu}, \\ j_3 &= 2i\lambda\kappa_2\mu\nu, \qquad j_4 = \lambda^2(\alpha\beta-\kappa_1\kappa_2)\frac{\mu^2+\nu^2}{\mu\nu}. \end{aligned} \tag{5.15}$$

Using formulae (5.1)(i) and (5.1)(ii), we obtain the following approximations when $|\lambda|\to\infty$:

$$\begin{aligned} &(i)\quad j_1 = -\sqrt{-\lambda^2+2ia_0\lambda+\gamma}\,(S+2ia_1\lambda) = 2a_1\lambda^2 - i(2a_0a_1+S)\lambda + O(1), \\ &(ii)\quad j_2 = -i\lambda\,\frac{2(1-2a_1^2)\lambda^2+4i(Sa_1-a_0)\lambda+S^2-2\gamma}{\sqrt{-\lambda^2+2ia_0\lambda+\gamma}}\kappa_1 \\ &\qquad = 2\big(2a_1^2-1\big)\kappa_1\lambda^2 + 2i\big(2a_0a_1^2-2Sa_1+a_0\big)\kappa_1\lambda + O(1), \\ &(iii)\quad j_3 = 2i\lambda\sqrt{-\lambda^2+2ia_0\lambda+\gamma}\,\widehat{\kappa}_2 = -2\widehat{\kappa}_2\lambda^2 + 2i\widehat{\kappa}_2a_0\lambda + O(1), \\ &(iv)\quad j_4 = \lambda^2\,\frac{S+2ia_1\lambda}{\sqrt{-\lambda^2+2ia_0\lambda+\gamma}}(\alpha\beta-\widehat{\kappa}_1\widehat{\kappa}_2) \\ &\qquad = \big[2a_1\lambda^2+i(2a_0a_1-S)\lambda\big](\alpha\beta-\widehat{\kappa}_1\widehat{\kappa}_2)+O(1). \end{aligned} \tag{5.16}$$

Combining these four terms we get formula (5.10) for $\mathbf{A}(\lambda)$.

The proof is complete. □

Remark 5.4. Examining the expressions for the coefficients, one can see that the parameter β enters the leading asymptotical terms in (5.10)–(5.14). This is the reason for our choice to consider the cases $\beta=0$ and $\beta\neq 0$ separately. In the present paper we discuss the spectral asymptotics for the case $\beta=0$. The case $\beta>0$ will be considered in the forthcoming work.

To simplify further calculations, we notice that (i) for $\beta=0$, the coefficients $\mathbf{A}(\lambda)$ and $\mathbf{B}(\lambda)$ in (5.6) and (5.7) do not depend on α, and (ii) the parameter α enters the lower order asymptotical terms in the coefficient $\mathbf{C}(\lambda)$, $\mathbf{D}(\lambda)$, and $\mathbf{F}(\lambda)$ in (5.8) and (5.9). It is likely that α will not have a strong effect on the final result; however, calculations will be technically simpler for $\alpha=0$.

Therefore, asymptotical form of the spectral equation corresponding to the case $\alpha=\beta=0$ can be given in the form

$$\begin{aligned} &\bigg\{\big[a_1+\big(2a_1^2-1\big)\widehat{\kappa}_1-\widehat{\kappa}_2-a_1\widehat{\kappa}_1\widehat{\kappa}_2\big]+\frac{i}{\lambda}\bigg[-\bigg(a_0a_1+\frac{S}{2}\bigg)+\big(2a_0a_1^2-2Sa_1+a_0\big)\widehat{\kappa}_1 \\ &\quad +\widehat{\kappa}_2a_0+\bigg(a_0a_1+\frac{S}{2}\bigg)\widehat{\kappa}_1\widehat{\kappa}_2\bigg]+O\bigg(\frac{1}{\lambda^2}\bigg)\bigg\}\sinh\mu\sinh\nu \\ &\quad +\bigg\{\big[-1+a_1(\widehat{\kappa}_1-\widehat{\kappa}_2)+\widehat{\kappa}_1\widehat{\kappa}_2\big]-\frac{i}{\lambda}\bigg[-2a_0+\frac{S}{2}(\widehat{\kappa}_1+\widehat{\kappa}_2)\bigg]+O\bigg(\frac{1}{\lambda^2}\bigg)\bigg\}\cosh\mu\cosh\nu \end{aligned}$$

$$
\begin{aligned}
&+\Big\{\big[-\big(1-2a_1^2\big)+a_1(\widehat{\kappa}_1-\widehat{\kappa}_2)-\widehat{\kappa}_1\widehat{\kappa}_2\big]\\
&-\frac{i}{\lambda}\Big[2(Sa_1-a_0)+\frac{S}{2}(\widehat{\kappa}_1-\widehat{\kappa}_2)\Big]+O\Big(\frac{1}{\lambda^2}\Big)\Big\}=0.
\end{aligned}
\tag{5.17}
$$

In the sequel, we focus on Eq. (5.17).

6. Derivation of the spectral asymptotics

In Sections 6 and 7, without misunderstanding we omit "hat" over κ_1 and κ_2. Let us rewrite Eq. (5.17) in the following form:

$$
\begin{aligned}
&\Big[\mathcal{A}_0+\frac{i\mathcal{B}_0}{\lambda}+O\Big(\frac{1}{\lambda^2}\Big)\Big]\sinh\mu\sinh\nu+\Big[\mathcal{A}_1+\frac{i\mathcal{B}_1}{\lambda}+O\Big(\frac{1}{\lambda^2}\Big)\Big]\cosh\mu\cosh\nu\\
&\quad+\Big[\mathcal{A}_2+\frac{i\mathcal{B}_2}{\lambda}+O\Big(\frac{1}{\lambda^2}\Big)\Big]=0,
\end{aligned}
\tag{6.1}
$$

where $\mathcal{A}_j$ and $\mathcal{B}_j$, $j=0,1,2$, are given by

$$
\begin{aligned}
&\mathcal{A}_0=a_1(1-\kappa_1\kappa_2)-\big(1-2a_1^2\big)\kappa_1-\kappa_2,\\
&\mathcal{B}_0=\Big(\frac{S}{2}+a_0a_1\Big)\kappa_1\kappa_2+\big(2a_0a_1^2+a_0-2Sa_1\big)\kappa_1+a_0\kappa_2-\Big(\frac{S}{2}+a_0a_1\Big);
\end{aligned}
\tag{6.2}
$$

$$
\mathcal{A}_1=\kappa_1\kappa_2+a_1(\kappa_1-\kappa_2)-1,\qquad \mathcal{B}_1=-\Big[\frac{S}{2}(\kappa_1+\kappa_2)-2a_0\Big];
\tag{6.3}
$$

$$
\mathcal{A}_2=-\big[\kappa_1\kappa_2-a_1(\kappa_1-\kappa_2)+\big(1-2a_1^2\big)\big],\qquad \mathcal{B}_2=-\Big[2(Sa_1-a_0)+\frac{S}{2}(\kappa_1-\kappa_2)\Big].
\tag{6.4}
$$

For the rest of the paper, we assume that the parameters are such that $\mathcal{A}_0+\mathcal{A}_1\neq 0$. As follows from formulae (5.2(i)) and (5.2(ii)), the following approximations hold for μ and ν:

$$
\mu=m_0\sqrt{\lambda}+\frac{m_1}{\sqrt{\lambda}}+O\Big(\frac{1}{\lambda^{3/2}}\Big),\qquad \nu=i\overline{m}_0\sqrt{\lambda}+\frac{i\overline{m}_1}{\sqrt{\lambda}}+O\Big(\frac{1}{\lambda^{3/2}}\Big),
\tag{6.5}
$$

where

$$
m_0=\sqrt{\sqrt{1-a_1^2}+ia_1},\qquad m_1=\frac{S\sqrt{1-a_1^2}+i(Sa_1-2a_0)}{4\sqrt{1-a_1^2}\sqrt{\sqrt{1-a_1^2}+ia_1}}.
\tag{6.6}
$$

Using (6.5) and (6.6), we derive approximations for $\sinh\mu$, $\sinh\nu$, and $\cosh\mu$, $\cosh\nu$ as $|\lambda| \to \infty$. We have

$$\begin{aligned}
&(i)\quad \sinh\mu = \sinh\left(m_0\sqrt{\lambda} + \frac{m_1}{\sqrt{\lambda}}\right)\left(1 + O\left(\frac{1}{\lambda^3}\right)\right) + \cosh\left(m_0\sqrt{\lambda} + \frac{m_1}{\sqrt{\lambda}}\right)O\left(\frac{1}{\lambda^{3/2}}\right),\\
&(ii)\quad \cosh\mu = \cosh\left(m_0\sqrt{\lambda} + \frac{m_1}{\sqrt{\lambda}}\right)\left(1 + O\left(\frac{1}{\lambda^3}\right)\right) + \sinh\left(m_0\sqrt{\lambda} + \frac{m_1}{\sqrt{\lambda}}\right)O\left(\frac{1}{\lambda^{3/2}}\right).
\end{aligned} \tag{6.7}$$

$$\begin{aligned}
(i)\quad \sinh\nu &= \sinh\left(i\overline{m}_0\sqrt{\lambda} + \frac{i\overline{m}_1}{\sqrt{\lambda}}\right)\left(1 + O\left(\frac{1}{\lambda^3}\right)\right) + \cosh\left(i\overline{m}_0\sqrt{\lambda} + \frac{i\overline{m}_1}{\sqrt{\lambda}}\right)O\left(\frac{1}{\lambda^{3/2}}\right)\\
&= i\sin\left(\overline{m}_0\sqrt{\lambda} + \frac{\overline{m}_1}{\sqrt{\lambda}}\right)\left(1 + O\left(\frac{1}{\lambda^3}\right)\right) + \cos\left(\overline{m}_0\sqrt{\lambda} + \frac{\overline{m}_1}{\sqrt{\lambda}}\right)O\left(\frac{1}{\lambda^{3/2}}\right),\\
(ii)\quad \cosh\nu &= \cos\left(\overline{m}_0\sqrt{\lambda} + \frac{\overline{m}_1}{\sqrt{\lambda}}\right)\left(1 + O\left(\frac{1}{\lambda^3}\right)\right) + i\sin\left(\overline{m}_0\sqrt{\lambda} + \frac{\overline{m}_1}{\sqrt{\lambda}}\right)O\left(\frac{1}{\lambda^{3/2}}\right).
\end{aligned} \tag{6.8}$$

Using formulae (6.7)(i) and (6.8)(i), we evaluate the approximation for the product of $\sinh\mu\sinh\nu$

$$\begin{aligned}
\sinh\mu\sinh\nu = {}& i\sinh\left(m_0\sqrt{\lambda} + \frac{m_1}{\sqrt{\lambda}}\right)\sin\left(\overline{m}_0\sqrt{\lambda} + \frac{\overline{m}_1}{\sqrt{\lambda}}\right)\left(1 + O\left(\frac{1}{\lambda^3}\right)\right)\\
&+ \sinh\left(m_0\sqrt{\lambda} + \frac{m_1}{\sqrt{\lambda}}\right)\cos\left(\overline{m}_0\sqrt{\lambda} + \frac{\overline{m}_1}{\sqrt{\lambda}}\right)O\left(\frac{1}{\lambda^{3/2}}\right)\\
&+ \cosh\left(m_0\sqrt{\lambda} + \frac{m_1}{\sqrt{\lambda}}\right)\sin\left(\overline{m}_0\sqrt{\lambda} + \frac{\overline{m}_1}{\sqrt{\lambda}}\right)O\left(\frac{1}{\lambda^{3/2}}\right)\\
&+ \cosh\left(m_0\sqrt{\lambda} + \frac{m_1}{\sqrt{\lambda}}\right)\cos\left(\overline{m}_0\sqrt{\lambda} + \frac{\overline{m}_1}{\sqrt{\lambda}}\right)O\left(\frac{1}{\lambda^3}\right).
\end{aligned} \tag{6.9}$$

Using formulae (6.7)(ii) and (6.7)(iii) we evaluate the approximations for the product of $\cosh\mu\cosh\nu$

$$\begin{aligned}
\cosh\mu\cosh\nu = {}& \cosh\left(m_0\sqrt{\lambda} + \frac{m_1}{\sqrt{\lambda}}\right)\cos\left(\overline{m}_0\sqrt{\lambda} + \frac{\overline{m}_1}{\sqrt{\lambda}}\right)\left(1 + O\left(\frac{1}{\lambda^3}\right)\right)\\
&+ \cosh\left(m_0\sqrt{\lambda} + \frac{m_1}{\sqrt{\lambda}}\right)\sin\left(\overline{m}_0\sqrt{\lambda} + \frac{\overline{m}_1}{\sqrt{\lambda}}\right)O\left(\frac{1}{\lambda^{3/2}}\right)\\
&+ \sinh\left(m_0\sqrt{\lambda} + \frac{m_1}{\sqrt{\lambda}}\right)\cos\left(\overline{m}_0\sqrt{\lambda} + \frac{\overline{m}_1}{\sqrt{\lambda}}\right)O\left(\frac{1}{\lambda^{3/2}}\right)\\
&+ \sinh\left(m_0\sqrt{\lambda} + \frac{m_1}{\sqrt{\lambda}}\right)\sin\left(\overline{m}_0\sqrt{\lambda} + \frac{\overline{m}_1}{\sqrt{\lambda}}\right)O\left(\frac{1}{\lambda^3}\right).
\end{aligned} \tag{6.10}$$

Substituting (6.9) and (6.10) into Eq. (6.1), we obtain the following modification):

$$
\begin{aligned}
& i\left[\mathcal{A}_0 + \frac{i\mathcal{B}_0}{\lambda} + O\left(\frac{1}{\lambda^2}\right)\right] \sinh\left(m_0\sqrt{\lambda} + \frac{m_1}{\sqrt{\lambda}}\right) \sin\left(\overline{m}_0\sqrt{\lambda} + \frac{\overline{m}_1}{\sqrt{\lambda}}\right)\left(1 + O\left(\frac{1}{\lambda^3}\right)\right) \\
& \quad + \left[\mathcal{A}_1 + \frac{i\mathcal{B}_1}{\lambda} + O\left(\frac{1}{\lambda^2}\right)\right] \cosh\left(m_0\sqrt{\lambda} + \frac{m_1}{\sqrt{\lambda}}\right) \cos\left(\overline{m}_0\sqrt{\lambda} + \frac{\overline{m}_1}{\sqrt{\lambda}}\right)\left(1 + O\left(\frac{1}{\lambda^3}\right)\right) \\
& \quad + \left[\mathcal{A}_2 + \frac{i\mathcal{B}_2}{\lambda} + O\left(\frac{1}{\lambda^2}\right)\right] + \sinh\left(m_0\sqrt{\lambda} + \frac{m_1}{\sqrt{\lambda}}\right) \cos\left(\overline{m}_0\sqrt{\lambda} + \frac{\overline{m}_1}{\sqrt{\lambda}}\right) O\left(\frac{1}{\lambda^{3/2}}\right) \\
& \quad + \cosh\left(m_0\sqrt{\lambda} + \frac{m_1}{\sqrt{\lambda}}\right) \sin\left(\overline{m}_0\sqrt{\lambda} + \frac{\overline{m}_1}{\sqrt{\lambda}}\right) O\left(\frac{1}{\lambda^{3/2}}\right) = 0.
\end{aligned}
\tag{6.11}
$$

Our next result is the following statement.

Lemma 6.1. *The set of solutions of the spectral equation (4.23) is symmetric with respect to the imaginary axis on the complex λ–plane, i.e. if λ is the solution of Eq. (4.23), then $(-\overline{\lambda})$ is the solution as well.*

Proof. We prove the result for the case $\alpha = \beta = 0$. The proof for the case of arbitrary real parameters, α, β, κ_1, and κ_2, can be done in a similar fashion. Let us introduce the following function:

$$
\begin{aligned}
\mathbb{F}(\lambda) = {} & \left[\left(\mu^4 + \nu^4\right) + \mu\nu\left(\mu^2 + \nu^2\right)\sinh\mu\sinh\nu - 2\mu^2\nu^2\cosh\mu\cosh\nu\right] \\
& + i\lambda\kappa_1\left[\left(\mu^2 + \nu^2\right)(1 - \cosh\mu\cosh\nu) + \frac{\mu^4 + \nu^4}{\mu\nu}\sinh\mu\sinh\nu\right] \\
& + i\lambda\kappa_2\left[\left(\mu^2 + \nu^2\right)(-1 + \cosh\mu\cosh\nu) - 2\mu\nu\sinh\mu\sinh\nu\right] \\
& + \lambda^2\kappa_1\kappa_2\left[2(1 - \cosh\mu\cosh\nu) + \frac{\mu^2 + \nu^2}{\mu\nu}\sinh\mu\sinh\nu\right].
\end{aligned}
\tag{6.12}
$$

It is clear that Eq. (4.23) with $\alpha = \beta = 0$ can be written in the form $\mathbb{F}(\lambda) = 0$. Let us show that $\mathbb{F}(-\overline{\lambda}) = \overline{F(\lambda)}$, which certainly means that if λ is a root of the afore equation, then $(-\overline{\lambda})$ is the root as well. Using formulae (4.7) for μ and ν, we obtain

$$
\begin{aligned}
& (i) \quad \mu(-\overline{\lambda}) = \sqrt{\left(-i\overline{\lambda}a_1 + \frac{S}{2}\right) + \sqrt{(\overline{\lambda})^2\left(1 - a_1^2\right) - i\overline{\lambda}(Sa_1 - 2a_0) + \left(\frac{S^2}{4} - \gamma\right)}} \\
& \qquad\qquad = \overline{\mu(\lambda)}; \\
& (ii) \quad \nu(-\overline{\lambda}) = \overline{\nu(\lambda)}; \qquad (iii) \quad \left(\mu^2 + \nu^2\right)(-\overline{\lambda}) = S - 2i\widetilde{a}_1\overline{\lambda} = \overline{\left(\mu^2 + \nu^2\right)(\lambda)}; \\
& (iv) \quad (\sinh\mu)(-\overline{\lambda}) = -\sinh\overline{\mu(\lambda)} = -\overline{(\sinh\mu)(\lambda)}, \qquad (\sinh\nu)(-\overline{\lambda}) = -\overline{(\sinh\nu)(\lambda)}; \\
& (v) \quad (\cosh\mu)(-\overline{\lambda}) = \cosh\overline{\mu(\lambda)} = \overline{(\cosh\mu)(\lambda)}, \qquad (\cosh\nu)(-\overline{\lambda}) = \overline{(\cosh\nu)(\lambda)}.
\end{aligned}
\tag{6.13}
$$

Using these formulae, we obtain

$$\left(\frac{\mu^2+\nu^2}{\mu\nu}\right)(-\overline{\lambda}) = \overline{\left(\frac{\mu^2+\nu^2}{\mu\nu}\right)(\lambda)}, \qquad \left(\frac{\mu^4+\nu^4}{\mu\nu}\right)(-\overline{\lambda}) = \overline{\left(\frac{\mu^4+\nu^4}{\mu\nu}\right)(\lambda)}. \tag{6.14}$$

Based on (6.13) and (6.14), we immediately obtain that $\mathbb{F}(-\overline{\lambda}) = 0$.

The lemma is shown. □

Below we present the main result of this section, which is concerned with the location of the set of eigenvalues of the operator $\mathcal{L}$ on the complex plane. Since the set of the eigenvalues is symmetric with respect to the imaginary axis, it is convenient to denote this set by $\{\lambda_n\}_{n\in\mathbb{Z}'}$, where $\mathbb{Z}' = \mathbb{Z}\setminus\{0\}$. Let us introduce a new complex parameter $w = \sqrt{\lambda}$, with the branch being fixed by the requirement that $\sqrt{\lambda} > 0$ for $\lambda > 0$. It is clear that if $\lambda \in \mathbb{C}$, then $w \in \overline{\mathbb{C}^+}$. (The set of points $\{w_n = \sqrt{\lambda_n}\}_{n\in\mathbb{Z}'}$ is in fact the set of solutions of the spectral equation (4.23).)

In what follows, it will be convenient to represent the set of eigenvalues of the operator $\mathcal{L}$ in the form $\{\lambda_n^+\}_{n\in\mathbb{Z}'} \cup \{\lambda_n^-\}_{n\in\mathbb{Z}'}$, where λ_n^+ is the notation for an eigenvalue located in the closed upper half–plane, $\lambda_n^+ \in \overline{\mathbb{C}^+}$, and λ_n^- is the notation for an eigenvalue located in the open lower half–plane, $\lambda_n^- \in \mathbb{C}^-$. Due to the symmetry of the set of eigenvalues of $\mathcal{L}$ with respect to the imaginary axis on the λ-plane, if $\lambda_n^\pm$ is an eigenvalue then $(-\overline{\lambda_n^\pm})$ is an eigenvalue as well. It can be easily seen that the set of eigenvalues $\{\lambda_n^+\}_{n\in\mathbb{Z}'}$ generates the set of complex points $\{w_n^+ = \sqrt{\lambda_n^+}\}_{n\in\mathbb{Z}'}$ in the first coordinate angle of the w-plane $(w = \sqrt{\lambda})$, which is symmetric with respect to its bisector. The set of eigenvalues $\{\lambda_n^-\}_{n\in\mathbb{Z}'}$ generates the set $\{w_n^- = \sqrt{\lambda_n^-}\}_{n\in\mathbb{Z}'}$ in the second coordinate angle of the w-plane, which is symmetric with respect to its bisector.

In what follows, it is convenient to use new notation.

Definition 6.2. Let $\{\psi_n\}_{n\in\mathbb{N}^+}$ and $\{\chi_n\}_{n\in\mathbb{N}^+}$ be two sequences of complex numbers. Then the relation

$$\left\{\begin{matrix}\psi_n\\ \chi_n\end{matrix}\right\} = g_n\bigl(1+o(1)\bigr), \quad n\to\infty, \tag{6.15}$$

means that two different sequences $\{\psi_n\}_{n\in\mathbb{N}^+}$ and $\{\chi_n\}_{n\in\mathbb{N}^+}$ can be approximated asymptotically by one and the same sequence $\{g_n\}_{n\in\mathbb{N}^+}$, i.e.,

$$\lim_{n\to\infty}\left|\frac{\psi_n}{g_n}-1\right| = 0 \quad \text{and} \quad \lim_{n\to\infty}\left|\frac{\chi_n}{g_n}-1\right| = 0$$

(see Fedoryuk [9]; Murray [23]).

We conclude this section with the following statement.

Lemma 6.3. *Under the condition $\mathcal{A}_0 + \mathcal{A}_1 \neq 0$, there could be only a finite number of the purely imaginary eigenvalues, i.e., there could a finite number of points $\{w_m\}$ located either on the bisector of the first coordinate angle or on the bisector of the second coordinate angle.*

Proof. Using contradiction argument, assume that there exists a sequence of roots of Eq. (6.11) located on the bisector $\{\widetilde{w}_n\}_{n=1}^\infty$ such that $\widetilde{w}_n = x_n + iy_n = (1+i)x_n$, and $x_n \to \infty$ as $n\to\infty$. Taking into account that $m_0 = \sqrt{\sqrt{1-a_1^2} + ia_1} = m_0^r + im_0^j$, with $m_0^r = \cos(\frac{1}{2}\tan^{-1}(\frac{a_1}{\sqrt{1-a_1^2}}))$ and $m_0^j = \sin(\frac{1}{2}\tan^{-1}(\frac{a_1}{\sqrt{1-a_1^2}}))$, we obtain the following approximation:

$$m_0\widetilde{w}_n = \left(m_0^r + im_0^j\right)(1+i)x_n = \sqrt{2}x_n \exp\{i\delta\} = \sqrt{2}x_n(\cos\delta + i\sin\delta),$$

with $\delta > 0$ being defined as

$$\delta = \frac{1}{2}\tan^{-1}\left(\frac{a_1}{\sqrt{1-a_1^2}}\right) + \frac{\pi}{4} < \frac{\pi}{2}. \tag{6.16}$$

One can readily check that the following asymptotic formulae are valid:

$$\begin{Bmatrix} \sinh(m_0\widetilde{w}_n + \frac{m_1}{\widetilde{w}_n}) \\ \cosh(m_0\widetilde{w}_n + \frac{m_1}{\widetilde{w}_n}) \end{Bmatrix} = \frac{1}{2}\exp\{\sqrt{2}x_n\cos\delta + i\sqrt{2}x_n\sin\delta\}\bigl(1+o(1)\bigr), \tag{6.17}$$

and also

$$\begin{Bmatrix} i\sin(\overline{m}_0\widetilde{w}_n + \frac{\overline{m}_1}{\widetilde{w}_n}) \\ \cos(\overline{m}_0\widetilde{w}_n + \frac{\overline{m}_1}{\widetilde{w}_n}) \end{Bmatrix} = \frac{1}{2}\exp\{\sqrt{2}x_n\sin\delta + i\sqrt{2}x_n\cos\delta\}\bigl(1+o(1)\bigr). \tag{6.18}$$

Combining (6.17) and (6.18), we immediately obtain

$$\begin{Bmatrix} \sinh(m_0\widetilde{w}_n + \frac{m_1}{\widetilde{w}_n})\sin(\overline{m}_0\widetilde{w}_n + \frac{\overline{m}_1}{\widetilde{w}_n}) \\ \sinh(m_0\widetilde{w}_n + \frac{m_1}{\widetilde{w}_n})\cos(\overline{m}_0\widetilde{w}_n + \frac{\overline{m}_1}{\widetilde{w}_n}) \\ \cosh(m_0\widetilde{w}_n + \frac{m_1}{\widetilde{w}_n})\sin(\overline{m}_0\widetilde{w}_n + \frac{\overline{m}_1}{\widetilde{w}_n}) \\ \cosh(m_0\widetilde{w}_n + \frac{m_1}{\widetilde{w}_n})\cos(\overline{m}_0\widetilde{w}_n + \frac{\overline{m}_1}{\widetilde{w}_n}) \end{Bmatrix} = \begin{Bmatrix} -i \\ 1 \\ -i \\ 1 \end{Bmatrix} E_n\bigl(1+o(1)\bigr), \tag{6.19}$$

where

$$E_n = \frac{1}{4}\exp\bigl\{\sqrt{2}(1+i)(\cos\delta + \sin\delta)x_n\bigr\}. \tag{6.20}$$

Substituting approximations (6.19) into Eq. (6.11), we obtain the following equation

$$\begin{aligned} &\left\{\left[\mathcal{A}_0 + \frac{i\mathcal{B}_0}{\widetilde{w}_n^2} + O\left(\frac{1}{\widetilde{w}_n^4}\right)\right]\bigl(1+o(1)\bigr) + \left[\mathcal{A}_1 + \frac{i\mathcal{B}_1}{\widetilde{w}_n^2} + O\left(\frac{1}{\widetilde{w}_n^4}\right)\right]\bigl(1+o(1)\bigr)\right\}E_n \\ &\quad + E_n O\left(\frac{1}{\widetilde{w}_n^3}\right) + \left[\mathcal{A}_2 + \frac{i\mathcal{B}_2}{\widetilde{w}_n^2} + O\left(\frac{1}{\widetilde{w}_n^4}\right)\right] = 0. \end{aligned} \tag{6.21}$$

However, since $\mathcal{A}_0 + \mathcal{A}_1 \neq 0$ and $m_0^r > 0$, Eq. (6.21) does not hold for large enough n. The proof is complete. $\square$

As a consequence we obtain that the set $\{w_n^+\}_{n\in\mathbb{Z}'}$ splits into two subsets symmetric with respect to the bisector. The first subset is located strictly below the bisector and the second is strictly above.

7. Eigenvalues

Based on Lemma 5.3, we can see that the set $\{w_n^+\}_{n\in\mathbb{Z}'}$ splits into two infinite subsets and possibly one finite subset located on the bisector of the first coordinate angle. The first of the aforementioned infinite subsets is located strictly below the bisector, and the second one is strictly above. Since we are interested in asymptotic approximations for the set $\{w_n^+\}_{n\in\mathbb{Z}'}$ as $|n| \to \infty$, without loss of generality, we assume that there are only infinite subsets. It can be readily seen that the numeration can be introduced in such a way, that the set $\{w_n^+\}_{n=-\infty}^{-1}$ corresponds to the subset located above the bisector and the set $\{w_n^+\}_{n=1}^{\infty}$ corresponds to the subset located below the bisector. From Lemma 6.1, one gets that these subsets are symmetric with respect to the bisector.

Let us derive the spectral asymptotics for the subset of the eigenvalues located in the first quadrant below the bisector $x = y$. Let Ω be the notation for this part of the first quadrant, i.e. if $w \in \Omega$, then $w = x + iy$ and $0 < y < x$.

Lemma 7.1. *Let $\xi = m_0 w$ and $\tau = \overline{m}_0 w$ with m_0 being defined in (6.6). Then for $w \in \Omega$ the following asymptotic approximation holds the spectral equation (6.11) as $|w| \to \infty$:*

$$\begin{aligned} &i\mathcal{A}_0 \sinh(\xi) \sin\left(\tau + \frac{\overline{m_0 m_1}}{\tau}\right)\left[1 + O\left(\frac{1}{w^2}\right)\right] \\ &\quad + \mathcal{A}_1 \cosh(\xi) \cos\left(\tau + \frac{\overline{m_0 m_1}}{\tau}\right)\left[1 + O\left(\frac{1}{w^2}\right)\right] \\ &\quad + \mathcal{A}_2 + \frac{i\mathcal{B}_2}{\lambda} + O\left(\frac{1}{\lambda^2}\right) + \cosh(\xi) \sin\left(\tau + \frac{\overline{m_0 m_1}}{\tau}\right) O\left(\frac{1}{w^3}\right) \\ &\quad + \sinh(\xi) \cos\left(\tau + \frac{\overline{m_0 m_1}}{\tau}\right) O\left(\frac{1}{w^3}\right) = 0, \end{aligned} \tag{7.1}$$

where m_1 is given in (6.6), $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{B}_2$ are given in (6.2) and (6.4) respectively.

Proof. Let us modify Eq. (6.11) for the case when $w \in \Omega$. The following approximations for the hyperbolic functions hold when $|w| \to \infty$, $w \in \Omega$:

$$\begin{aligned} &\begin{Bmatrix} \sinh(m_0 w + \frac{m_1}{w}) \\ \cosh(m_0 w + \frac{m_1}{w}) \end{Bmatrix} \\ &\quad = \begin{Bmatrix} \sinh(m_0 w) \cosh(\frac{m_1}{w}) + \cosh(m_0 w) \sinh(\frac{m_1}{w}) \\ \cosh(m_0 w) \cosh(\frac{m_1}{w}) + \sinh(m_0 w) \sinh(\frac{m_1}{w}) \end{Bmatrix} \\ &\quad = \begin{Bmatrix} 1 \\ \frac{m_1}{w} \end{Bmatrix} \sinh(m_0 w)\left[1 + O\left(\frac{1}{w^2}\right)\right] + \begin{Bmatrix} \frac{m_1}{w} \\ 1 \end{Bmatrix} \cosh(m_0 w)\left[1 + O\left(\frac{1}{w^2}\right)\right]. \end{aligned} \tag{7.2}$$

The following approximations for the trigonometric functions hold when $|w| \to \infty$, $w \in \Omega$:

$$\begin{Bmatrix} \sin(\overline{m}_0 w + \frac{\overline{m}_1}{w}) \\ \cos(\overline{m}_0 w + \frac{\overline{m}_1}{w}) \end{Bmatrix}$$
$$= \begin{Bmatrix} \sin(\overline{m}_0 w)[1 + O(\frac{1}{w^2})] + \cos(\overline{m}_0 w)[\frac{\overline{m}_1}{w} + O(\frac{1}{w^3})] \\ \cos(\overline{m}_0 w)[1 + O(\frac{1}{w^2})] - \sin(\overline{m}_0 w)[\frac{\overline{m}_1}{w} + O(\frac{1}{w^3})] \end{Bmatrix}$$
$$= \begin{Bmatrix} 1 \\ -\frac{\overline{m}_1}{w} \end{Bmatrix} \sin(\overline{m}_0 w)\left[1 + O\left(\frac{1}{w^2}\right)\right] + \begin{Bmatrix} \frac{\overline{m}_1}{w} \\ 1 \end{Bmatrix} \cos(\overline{m}_0 w)\left[1 + O\left(\frac{1}{w^2}\right)\right]. \tag{7.3}$$

It is convenient to rewrite formulae (7.2) and (7.3) in terms of $\xi = m_0 w$, and $\tau = \overline{m}_0 w$. We have

$$\begin{Bmatrix} \sinh(\xi + \frac{m_0 m_1}{\xi}) \\ \cosh(\xi + \frac{m_0 m_1}{\xi}) \end{Bmatrix}$$
$$= \begin{Bmatrix} 1 \\ \frac{m_0 m_1}{\xi} \end{Bmatrix} \sinh(\xi)\left[1 + O\left(\frac{1}{\xi^2}\right)\right] + \begin{Bmatrix} \frac{m_0 m_1}{\xi} \\ 1 \end{Bmatrix} \cosh(\xi)\left[1 + O\left(\frac{1}{\xi^2}\right)\right], \tag{7.4}$$

$$\begin{Bmatrix} \sin(\tau + \frac{\overline{m_0 m_1}}{\tau}) \\ \cos(\tau + \frac{\overline{m_0 m_1}}{\tau}) \end{Bmatrix}$$
$$= \begin{Bmatrix} 1 \\ -\frac{\overline{m_0 m_1}}{\tau} \end{Bmatrix} \sin(\tau)\left[1 + O\left(\frac{1}{\tau^2}\right)\right] + \begin{Bmatrix} \frac{\overline{m_0 m_1}}{\tau} \\ 1 \end{Bmatrix} \cos(\tau)\left[1 + O\left(\frac{1}{\tau^2}\right)\right]. \tag{7.5}$$

Based on the formulae (7.4) and (7.5), we obtain the asymptotic approximations for the products

$$\begin{aligned}
1)\quad & \sinh\left(\xi + \frac{m_0 m_1}{\xi}\right) \sin\left(\tau + \frac{\overline{m_0 m_1}}{\tau}\right) \\
&= \sinh(\xi)\sin(\tau)\left[1 + O\left(\frac{1}{w^2}\right)\right] + \frac{m_0 m_1}{\xi}\cosh(\xi)\sin(\tau)\left[1 + O\left(\frac{1}{w^2}\right)\right] \\
&\quad + \frac{\overline{m_0 m_1}}{\tau}\sinh(\xi)\cos(\tau)\left[1 + O\left(\frac{1}{w^2}\right)\right] \\
&\quad + \frac{|m_0 m_1|^2}{\xi\tau}\cosh(\xi)\cos(\tau)\left[1 + O\left(\frac{1}{w^2}\right)\right] \\
&= \sinh(\xi)\sin(\tau)\left[1 + O\left(\frac{1}{w^2}\right)\right] + \frac{m_0 m_1}{\xi}\cosh(\xi)\sin(\tau)\left[1 + O\left(\frac{1}{w^2}\right)\right] \\
&\quad + \frac{\overline{m_0 m_1}}{\tau}\sinh(\xi)\cos(\tau)\left[1 + O\left(\frac{1}{w^2}\right)\right] + \cosh(\xi)\cos(\tau)O\left(\frac{1}{w^2}\right),
\end{aligned} \tag{7.6}$$

2) $$\cosh\left(\xi+\frac{m_0m_1}{\xi}\right)\cos\left(\tau+\frac{\overline{m_0m_1}}{\tau}\right)$$
$$=\cosh(\xi)\cos(\tau)\left[1+O\left(\frac{1}{w^2}\right)\right]-\frac{\overline{m_0m_1}}{\tau}\cosh(\xi)\sin(\tau)\left[1+O\left(\frac{1}{w^2}\right)\right]$$
$$+\frac{m_0m_1}{\xi}\sinh(\xi)\cos(\tau)\left[1+O\left(\frac{1}{w^2}\right)\right]+\sinh(\xi)\sin(\tau)O\left(\frac{1}{w^2}\right), \tag{7.7}$$

3) $$\sinh\left(\xi+\frac{m_0m_1}{\xi}\right)\cos\left(\tau+\frac{\overline{m_0m_1}}{\tau}\right)$$
$$=\sinh(\xi)\cos(\tau)\left[1+O\left(\frac{1}{w^2}\right)\right]-\frac{\overline{m_0m_1}}{\tau}\sinh(\xi)\sin(\tau)\left[1+O\left(\frac{1}{w^2}\right)\right]$$
$$+\frac{m_0m_1}{\xi}\cosh(\xi)\cos(\tau)\left[1+O\left(\frac{1}{w^2}\right)\right]+\cosh(\xi)\sin(\tau)O\left(\frac{1}{w^2}\right), \tag{7.8}$$

4) $$\cosh\left(\xi+\frac{m_0m_1}{\xi}\right)\sin\left(\tau+\frac{\overline{m_0m_1}}{\tau}\right)$$
$$=\cosh(\xi)\sin(\tau)\left[1+O\left(\frac{1}{w^2}\right)\right]+\frac{\overline{m_0m_1}}{\tau}\cosh(\xi)\cos(\tau)\left[1+O\left(\frac{1}{w^2}\right)\right]$$
$$+\frac{m_0m_1}{\xi}\sinh(\xi)\sin(\tau)\left[1+O\left(\frac{1}{w^2}\right)\right]+\sinh(\xi)\cos(\tau)O\left(\frac{1}{w^2}\right). \tag{7.9}$$

Using formula (7.6) we evaluate the asymptotic approximation for the term of Eq. (6.11) containing the product of sinh and sin functions and have:

$$\left[\mathcal{A}_0+\frac{i\mathcal{B}_0}{\lambda}+O\left(\frac{1}{\lambda^2}\right)\right]\sinh\left(m_0w+\frac{m_1}{w}\right)\sin\left(\overline{m}_0w+\frac{\overline{m}_1}{w}\right)\left[1+O\left(\frac{1}{w^6}\right)\right]$$
$$=\left[\mathcal{A}_0+\frac{i\mathcal{B}_0}{\lambda}\right]\sinh\left(\xi+\frac{m_0m_1}{\xi}\right)\sin\left(\tau+\frac{\overline{m_0m_1}}{\tau}\right)\left[1+O\left(\frac{1}{w^4}\right)\right]. \tag{7.10}$$

Taking into account that the accuracy of all formulae (7.6)–(7.9) is $O(w^{-2})$, we proceed with the assumed level of accuracy and represent (7.10) as follows:

$$\mathcal{A}_0\sinh\left(\xi+\frac{m_0m_1}{\xi}\right)\sin\left(\tau+\frac{\overline{m_0m_1}}{\tau}\right)\left[1+O\left(\frac{1}{w^2}\right)\right]. \tag{7.11}$$

In a similar fashion, using formula (7.7) we obtain the asymptotic approximation for term of Eq. (6.11) containing the product of cosh and cos functions

$$\mathcal{A}_1\cosh\left(\xi+\frac{m_0m_1}{\xi}\right)\cos\left(\tau+\frac{\overline{m_0m_1}}{\tau}\right)\left[1+O\left(\frac{1}{w^2}\right)\right]. \tag{7.12}$$

Substituting (7.6)–(7.11) into Eq. (6.11) we obtain representation (7.1) for the spectral equation.

The lemma is shown. □

Finally, we present the main result of the paper.

Theorem 7.2. *The set of the eigenvalues $\{\lambda_n\}_{n=-\infty}^{\infty}$ is symmetric with respect to the imaginary axis on complex λ-plane: $\lambda_{-|n|} = -\overline{\lambda}_{|n|}$. Let $\{\lambda_n\}_{n=1}^{\infty}$ be the subset of the set of the eigenvalues located in the right half-plane and $\{\lambda_n\}_{n=-\infty}^{-1}$ be a symmetric subset. Under the assumption that $\mathcal{A}_0^2 - \mathcal{A}_1^2 > 0$, the following asymptotic approximation is valid for λ_n as $n \to \infty$:*

$$\lambda_n = \left(\frac{\pi m_0}{|m_0|^2}\right)^2 n^2 + i\pi\left(\frac{m_0}{|m_0|^2}\right)^2 \ln\left(\frac{\mathcal{A}_0+\mathcal{A}_1}{\mathcal{A}_0-\mathcal{A}_1}\right) n \\ - \left[\left(\frac{m_0}{2|m_0|^2}\right)^2 \ln^2\left(\frac{\mathcal{A}_0+\mathcal{A}_1}{\mathcal{A}_0-\mathcal{A}_1}\right) + \frac{2m_0\overline{m_1}}{|m_0|^2}\right] + O\left(\frac{1}{n}\right), \quad n \to \infty, \tag{7.13}$$

where m_0 and m_1 are defined in (6.6) and

$$\frac{\mathcal{A}_0+\mathcal{A}_1}{\mathcal{A}_0-\mathcal{A}_1} = \frac{(a_1-1)(1-\kappa_1\kappa_2) + (2a_1^2+a_1-1)\kappa_1 - (a_1+1)\kappa_2}{(a_1+1)(1-\kappa_1\kappa_2) + (2a_1^2-a_1-1)\kappa_1 + (a_1-1)\kappa_2}. \tag{7.14}$$

Proof. Let $G(x, y)$ be a function defined by

$$G(x,y) = \frac{1}{2}\exp\left[\left(m_0^r x - m_0^j y\right) + i\left(m_0^j x + m_0^r y\right)\right]. \tag{7.15}$$

Then since $\xi = (m_0^r x - m_0^j y) + i(m_0^j x + m_0^r y)$, we obtain that

$$\begin{pmatrix}\cosh(\xi)\\ \sinh(\xi)\end{pmatrix} = G(x,y)\left(1 + O\left(\exp\left[-2\left(m_0^r x - m_0^j y\right)\right]\right)\right). \tag{7.16}$$

Substituting (7.16) into (7.11) we obtain the following equation:

$$i\mathcal{A}_0 G(x,y)\sin\left(\tau + \frac{\overline{m_0 m_1}}{\tau}\right)\left[1 + O\left(\frac{1}{w^2}\right)\right] \\ + \mathcal{A}_1 G(x,y)\cos\left(\tau + \frac{\overline{m_0 m_1}}{\tau}\right)\left[1 + O\left(\frac{1}{w^2}\right)\right] \\ + \mathcal{A}_2 + G(x,y)\left[\cos(\tau) O\left(\frac{1}{w^3}\right) + \sin(\tau) O\left(\frac{1}{w^3}\right)\right] = 0. \tag{7.17}$$

Taking into account that $|G(x, y)| \to \infty$ as $x \to \infty$ (or $|w| \to \infty$) one can see that Eq. (7.17) can be represented in the form

$$i\mathcal{A}_0 \sin\left(\tau + \frac{\overline{m_0 m_1}}{\tau}\right)\left[1 + O\left(\frac{1}{w^2}\right)\right] + \mathcal{A}_1 \cos\left(\tau + \frac{\overline{m_0 m_1}}{\tau}\right)\left[1 + O\left(\frac{1}{w^2}\right)\right] \\ + \cos(\tau) O\left(\frac{1}{w^3}\right) + \sin(\tau) O\left(\frac{1}{w^3}\right) = O\left(\exp\left[-\left(m_0^r x - m_0^j y\right)\right]\right). \tag{7.18}$$

Since $\mathcal{A}_0 \neq \mathcal{A}_1$, this equation can be reduced to

$$[-\mathcal{A}_0 + \mathcal{A}_1] \exp\left[-i\left(\tau + \frac{\overline{m_0 m_1}}{\tau}\right)\right] + \left[\mathcal{A}_0 + \mathcal{A}_1 + O\left(\frac{1}{w^2}\right)\right] \exp\left[i\left(\tau + \frac{\overline{m_0 m_1}}{\tau}\right)\right]$$
$$= O\bigl(\exp\bigl[-\bigl(m_0^r x - m_0^j y\bigr)\bigr]\bigr). \tag{7.19}$$

Using an explicit formula $-i\tau = -m_0^j x + m_0^r y$, we obtain that

$$\exp(-i\tau) O\bigl(\exp\bigl[-\bigl(m_0^r x - m_0^j y\bigr)\bigr]\bigr) = O\bigl(\exp\bigl[-\bigl(m_0^r + m_0^j\bigr)(x - y)\bigr]\bigr),$$

which yields the following form of Eq. (7.19) (called the model equation):

$$\exp\left[-2i\left(\tau + \frac{\overline{m_0 m_1}}{\tau}\right)\right] = \frac{\mathcal{A}_0 + \mathcal{A}_1}{\mathcal{A}_0 - \mathcal{A}_1}\left[1 + O\left(\frac{1}{\tau^2}\right)\right] + O\bigl(\exp\bigl[-\bigl(m_0^r + m_0^j\bigr)(x - y)\bigr]\bigr). \tag{7.20}$$

Solving the model equation Fedoryuk [9]; Evgrafov [8]:

$$\exp\left[-2i\left(\tau + \frac{\overline{m_0 m_1}}{\tau}\right)\right] = \frac{\mathcal{A}_0 + \mathcal{A}_1}{\mathcal{A}_0 - \mathcal{A}_1}\left[1 + O\left(\frac{1}{\tau^2}\right)\right]. \tag{7.21}$$

Since $|\tau| \to \infty$ as $x \to \infty$, we represent Eq. (7.21) in asymptotical form as

$$e^{-2i\tau} = \frac{\mathcal{A}_0 + \mathcal{A}_1}{\mathcal{A}_0 - \mathcal{A}_1}\left(1 + \frac{2i\overline{m_0 m_1}}{\tau}\right)\left[1 + O\left(\frac{1}{\tau^2}\right)\right]. \tag{7.22}$$

Consider a sequence of explicitly defined points

$$\left\{\tau_n^0 = \pi n + i \ln\sqrt{\frac{\mathcal{A}_0 + \mathcal{A}_1}{\mathcal{A}_0 - \mathcal{A}_1}} - \frac{\overline{m_0 m_1}}{\pi n}\right\}_{n=1}^{\infty}. \tag{7.23}$$

Using Rouche's theorem, we show that the roots of Eq. (7.22) are located in small vicinities of the points (7.23). To this end we introduce two functions $\mathbb{F}(\tau)$ and $\mathbb{G}(\tau)$, which are analytic in Ω. The function $\mathbb{F}(\tau)$ is defined by

$$\mathbb{F}(\tau) = \exp\{-2i\tau\} - \frac{\mathcal{A}_0 + \mathcal{A}_1}{\mathcal{A}_0 - \mathcal{A}_1}\left(1 + \frac{2i\overline{m_0 m_1}}{\tau}\right) \tag{7.24}$$

and $\mathbb{G}(\tau)$ satisfies the estimate $|\tau^2\mathbb{G}(\tau)| \to \text{Const}$ as $|\tau| \to \infty$ within Ω. Equation (7.22) can be written in form $\mathbb{F}(\tau) + \mathbb{G}(\tau) = 0$. Let us fix large enough positive integer n and introduce a circle centered at the point τ_n^0 of a small radius $\varepsilon_n : C_{\varepsilon_n}(\tau_n^0)$. Let us evaluate $\mathbb{F}(\tau)$ for $\tau \in C_{\varepsilon_n}(\tau_n^0)$, i.e. $\tau = \tau_n^0 + \varepsilon_n e^{i\varphi}$,

$-\pi < \varphi \leqslant \pi$. Evaluating $\exp\{-2i\tau\}$ we obtain

$$\begin{aligned}\exp\{-2i\tau\} &= \exp\left\{-2i\left[\pi n + i\ln\sqrt{\frac{\mathcal{A}_0+\mathcal{A}_1}{\mathcal{A}_0-\mathcal{A}_1}} - \frac{\overline{m_0 m_1}}{\pi n} + \varepsilon_n e^{i\varphi}\right]\right\}\\ &= \exp\left\{\ln\frac{\mathcal{A}_0+\mathcal{A}_1}{\mathcal{A}_0-\mathcal{A}_1} + \frac{2i\overline{m_0 m_1}}{\pi n}\right\}\exp\{-2i\varepsilon_n e^{i\varphi}\}\\ &= \frac{\mathcal{A}_0+\mathcal{A}_1}{\mathcal{A}_0-\mathcal{A}_1}\left[1 + \frac{2i\overline{m_0 m_1}}{\pi n} + O\left(\frac{1}{n^2}\right)\right]\left[1 - 2i\varepsilon_n e^{i\varphi} + O(\varepsilon_n^2)\right]. \end{aligned}\tag{7.25}$$

Based on the origins of the estimates $O(n^{-2})$ and $O(\varepsilon_n^2)$ from (7.25), we claim that there exist two absolute positive constants, C_0 and C_1, such that

$$\frac{C_0}{n^2} \leqslant O\left(\frac{1}{n^2}\right) \leqslant \frac{C_1}{n^2} \quad \text{and} \quad C_0\varepsilon_n^2 \leqslant O(\varepsilon_n^2) \leqslant C_1\varepsilon_n^2. \tag{7.26}$$

Substituting (7.25) into (7.24) and taking into account (7.26) we obtain

$$\mathbb{F}(\tau)\Big|_{\tau\in C_{\varepsilon_n}(\tau_n^0)} = -2i\varepsilon_n e^{i\varphi}\frac{\mathcal{A}_0+\mathcal{A}_1}{\mathcal{A}_0-\mathcal{A}_1}\left[1 + \frac{2i\overline{m_0 m_1}}{\pi n}\right] + O\left(\frac{1}{n^2}\right) + O(\varepsilon_n^2). \tag{7.27}$$

Estimating the function $\mathbb{G}(\tau)$ for $\tau \in C_{\varepsilon_n}(\tau_n^0)$, we claim that there exists an absolute constant C_2 such that $|\mathbb{G}(\tau)| \leqslant C_2/n^2$. Now we choose ε_n in such a way that

$$\frac{2(C_1+C_2)}{n^2} \leqslant 2\varepsilon_n\left|\frac{\mathcal{A}_0-\mathcal{A}_1}{\mathcal{A}_0+\mathcal{A}_1}\right| \leqslant \frac{3(C_1+C_2)}{n^2}.$$

With this choice of ε_n, one gets the relation between $\mathbb{F}(\tau)$ and $\mathbb{G}(\tau)$ needed for application of the Rouche's theorem, i.e., $|\mathbb{F}(\tau)| > |\mathbb{G}(\tau)|$ for $\tau \in C_{\varepsilon_n}(\tau_n^0)$, which means that for the roots of the model equation (7.21), the following representation is valid:

$$\tau_n = \pi n + i\ln\sqrt{\frac{\mathcal{A}_0+\mathcal{A}_1}{\mathcal{A}_0-\mathcal{A}_1}} - \frac{\overline{m_0 m_1}}{\pi n} + O\left(\frac{1}{n^2}\right), \quad n \to \infty. \tag{7.28}$$

Incorporation into the proof an exponentially decaying term from (7.20) can be done without difficulties.

Finally, let us represent the asymptotic distribution of the eigenvalues of the original problem $\{\lambda_n\}$. Since $\tau = \overline{m_0}w$, we have

$$w_n = \frac{m_0}{|m_0|^2}\pi n + \frac{i m_0}{|m_0|^2}\ln\sqrt{\frac{\mathcal{A}_0+\mathcal{A}_1}{\mathcal{A}_0-\mathcal{A}_1}} - \frac{\overline{m_1}}{\pi n} + O\left(\frac{1}{n^2}\right), \quad n \to \infty, \tag{7.29}$$

which yields the following formula for λ_n:

$$\lambda_n = w_n^2 = \left[\frac{m_0}{|m_0|^2}\left\{\pi n + \frac{i}{2}\ln\frac{\mathcal{A}_0+\mathcal{A}_1}{\mathcal{A}_0-\mathcal{A}_1}\right\}\right]^2 - 2\frac{\overline{m_1}}{\pi n}\frac{m_0}{|m_0|^2}\left\{\pi n + \frac{i}{2}\ln\frac{\mathcal{A}_0+\mathcal{A}_1}{\mathcal{A}_0-\mathcal{A}_1}\right\} + O\left(\frac{1}{n}\right).$$

It can be easily checked that this formula implies (7.13).

The theorem is proven. □

Remark 7.3. For the case when $a_1 = S = 0$, one gets $m_0 = 1$, $m_1 = 0$, and formula (7.13) coincides with formula (3.17) of paper (Shubov and Kindrat [31]).

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