GENERIC UNIQUENESS AND STABILITY FOR THE MIXED RAY TRANSFORM

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ABSTRACT. We consider the mixed ray transform of tensor fields on a three-dimensional compact simple Riemannian manifold with boundary. We prove the injectivity of the transform, up to natural obstructions, and establish stability estimates for the normal operator on generic three dimensional simple manifold in the case of $1+1$ and $2+2$ tensors fields.

We show how the anisotropic perturbations of averaged isotopic travel-times of qS-polarized elastic waves provide partial information about the mixed ray transform of $2+2$ tensors fields. If in addition we include the measurement of the shear wave amplitude, the complete mixed ray transform can be recovered. We also show how one can obtain the mixed ray transform from an anisotropic perturbation of the Dirichlet-to-Neumann map of an isotropic elastic wave equation on a smooth and bounded domain in three dimensional Euclidean space.

1. INTRODUCTION

In this paper we study an inverse problem of recovering a 4-tensor field from a family of certain line integrals.

This family

is called the *mixed ray transform*, and it was first considered in [42, Chapter 7]. We characterize the kernel of the mixed ray transform for $1+1$ and $2+2$ tensor fields for generic simple 3-dimensional Riemannian manifolds and provide a stability estimate for the corresponding L^2-normal operator.

We begin by introducing the mixed ray transform in the Euclidean space. Let f be a smooth compactly supported two tensor field on $\mathbf{R}^3$. We choose a point $x \in \mathbf{R}^3$ and a unit vector ξ. Thus x and ξ define a line $\{x + t\xi \in \mathbf{R}^3 : t \in \mathbf{R}\}$. Then we choose a vector η that is orthogonal to ξ. The mixed ray transform $L_{1,1}f$

of f for (x, ξ, η) is given by

$$(1.1) \qquad L_{1,1}f(x,\xi,\eta) := \int_{-\infty}^{\infty} f_{ij}(x + t\xi)\eta^i \xi^j \, \mathrm{d}t.$$

We note that if we had chosen $\eta = \xi$ in (1.1), then we would have obtained the (longitudinal) ray transform of f. We recall that any 2-tensor f has a unique decomposition $f_{ij}(x) = f_{ij}^{\mathcal{B}}(x) + c(x)\delta_{ij}$, with a zero *trace* $\mu f^{\mathcal{B}} := \sum_{i=1}^{3} f_{ii}^{\mathcal{B}} = 0$. Since ξ and η were chosen to be orthogonal to each other we get from (1.1) that $L_{1,1}(c(x)\delta_{ij}) = 0$. Thus for the mixed ray transform, the only relevant tensor fields are the *trace-free* ones, for which $\mu f = 0$. Notice that if $f = (\nabla v)^{\mathcal{B}}$ for some 1-form $v \in C_0^{\infty}(\mathbf{R}^3)$ then the fundamental theorem of calculus implies that $L_{1,1}f = 0$. Therefore $L_{1,1}$ always has a non-trivial *natural kernel*, consisting of *potential tensor fields* $(\nabla v)^{\mathcal{B}}$, $v \in C_0^{\infty}(\mathbf{R}^3)$. In this paper, we will consider the mixed ray transform on certain Riemannian manifolds and study its injectivity up to the natural obstruction.

Let (M, g) be a simple 3-dimensional Riemannian manifold with boundary ∂M. We recall that a compact Riemannian manifold is simple if it has a strictly convex boundary and any two points $x, y \in M$ can be connected by a unique geodesic, contained in M, depending smoothly on x and y. We use the notation TM for the tangent bundle of M, T^*M for the cotangent bundle, and SM for the unit sphere bundle, defined as $SM = \{(x, \xi) \in TM; |\xi|_g = 1\}$. Let $\partial_+(SM) = \{(x, \xi) \in SM; x \in \partial M, \langle \xi, \nu \rangle_g < 0\}$ be the inward pointing unit sphere bundle on ∂M, where ν is the outward pointing unit normal vector field to the boundary. We use the notation $S^k \tau'_M \otimes S^\ell \tau'_M$, $k, \ell \geq 1$ for the space of $k + \ell$ tensor fields on M that are symmetric with respect to first k and last ℓ indices. Note that *a priori* we do not pose any regularity properties for the tensor fields. To emphasize the regularity we use the standard notations C^m, C^∞, L^2, or H^m in front of the vector space of the corresponding tensor fields.

The mixed ray transform $L_{k,\ell}f$ of a smooth tensor field $f \in C^\infty(S^k \tau'_M \otimes S^\ell \tau'_M)$ is given by the following formula

$$(1.2) \quad (L_{k,\ell}f)(x,\xi,\eta) = \int_0^{\tau(x,\xi)} f_{i_1 \ldots i_k j_1 \ldots j_\ell}(\gamma(t)) \eta(t)^{i_1} \cdots \eta(t)^{i_k} \dot{\gamma}(t)^{j_1} \cdots \dot{\gamma}(t)^{j_\ell} \mathrm{d}t,$$

where $(x, \xi) \in \partial_+(SM)$ and $\gamma(t) = \gamma_{x,\xi}(t)$ is the unit-speed geodesic given by the initial conditions (x, ξ). The vector $\eta \in T_x M$ is perpendicular to ξ, and $\eta(t)$ is the parallel translation of η along the geodesic $\gamma(t)$. We note that $\eta(t) \perp \dot{\gamma}(t)$ for any t (see Figure 1 for an illustration). By $\tau(x, \xi)$ we mean the exit time of γ, which is the first positive time in which γ hits the boundary again. Since (M, g) is simple the exit time function τ is smooth on $\partial_+ SM$ [42, Lemma 4.1.1.].

If $k = 0$ in (1.2), the operator $L_{0,\ell}$ is the (longitudinal) *geodesic ray transform* I_ℓ for a symmetric ℓ-tensor field f. The most interesting case is $\ell = 2$ which arises from the linearization of the *boundary rigidity problem*, that concerns the recovery of the Riemannian metric from its boundary distance function.

It was conjectured by Michel [30] that simple metrics are *boundary rigid*, which means that they are uniquely determined, up to a diffeomorphism fixing the boundary, by the boundary distance function.

Significant progress has been made in studying this problem [7, 12, 29, 30, 34, 38, 45, 46]. The linearization of the boundary rigidity problem leads to an integral geometry problem of recovering a symmetric 2-tensor field f from its geodesic ray
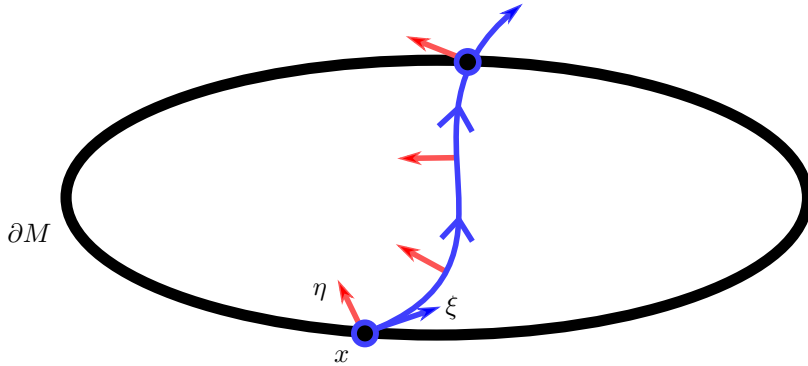
FIGURE 1. In this figure we illustrate the notations used in the definition of the mixed ray transform (1.2). We choose an initial point $x \in \partial M$ and an initial velocity $\xi \in S_x M$, blue arrow. The blue line is the geodesic γ given by these initial conditions. Finally we choose $\eta \in T_x M$, $\eta \perp \xi$ and compute its parallel translation along γ, this is illustrated by red arrows on γ.

transform $I_2 f$ (see, for instance, [41]). The problem of reconstructing a symmetric 4-tensor field f from $I_4 f$ arises from the linearization of elastic qP-wave travel-times [9, 18].

Using the fundamental theorem of calculus, it is straightforward to see that if $f = \mathrm{Sym} \nabla u$ with $u \in S^{\ell-1} \tau'_M$ and $u|_{\partial M} = 0$, then $I_\ell f = 0$. Here Sym is the symmetrization operator and ∇ is the Levi-Civita connection. We recall that the operator I_ℓ is called s-injective if its kernel coincides with the image of the operator $\mathrm{Sym} \nabla \colon H_0^1(S^{\ell-1} \tau'_M) \to L^2(S^\ell \tau'_M)$. We list here some cases where s-injectivity of I_ℓ is known:

- (M, g) simple, dim ≥ 2, $\ell = 0$ [31, 32], $\ell = 1$ [2].
- (M, g) simple, dim ≥ 2, $\ell \geq 2$ under curvature conditions [14, 36, 37, 42].
- (M, g) simple, dim $= 2$, ℓ arbitrary [35].
- (M, g) simple, dim ≥ 2, $\ell = 2$: generic s-injectivity [44].
- (M, g) admits a strictly convex foliation, dim ≥ 3, $\ell = 0$ [49], $\ell = 1, 2$ [47], $\ell = 4$ [18].

In this paper we consider the mixed ray transform $L_{k,\ell}$ as a generalization of the geodesic ray transform I_ℓ, and study its kernel. As for the Euclidean case, we only need to consider $L_{k,\ell}$ acting on "trace-free" tensors. First, we introduce the operator (*symmetrized tensor product with the metric*) $\lambda \colon S^{k-1} \tau'_M \otimes S^{\ell-1} \tau'_M \to S^k \tau'_M \otimes S^\ell \tau'_M$ defined by

$$(1.3) \qquad (\lambda w)_{i_1 \dots i_k j_1 \dots j_\ell} := \mathrm{Sym}(i_1 \dots i_k) \mathrm{Sym}(j_1 \dots j_\ell)(g_{i_k j_l} w_{i_1 \dots i_{k-1} j_1 \dots j_{\ell-1}}),$$

where $\mathrm{Sym}(\cdot)$ is the symmetrization with respect to indices listed in the argument. The algebraic dual of the operator λ is the *trace operator*
$$(1.4)$$
$$\mu \colon S^k \tau'_M \otimes S^\ell \tau'_M \to S^{k-1} \tau'_M \otimes S^{\ell-1} \tau'_M, \quad (\mu u)_{i_1 \dots i_{k-1} j_1 \dots j_{\ell-1}} := u_{i_1 \dots i_k j_1 \dots j_\ell} g^{i_k j_\ell}.$$

Therefore we see that

$$S^k \tau'_M \otimes S^\ell \tau'_M = \ker \mu \oplus \mathrm{Im} \lambda.$$

The tensors in $\ker \mu$ are called *trace free*. We use the notation $\mathcal{B}$ for the projection onto the trace-free class and write

$$S^k \tau'_M \otimes^{\mathcal{B}} S^\ell \tau'_M := \mathcal{B}(\{S^k \tau'_M \otimes S^\ell \tau'_M\}) = \ker \mu.$$

We note here that $L_{k,\ell}(\operatorname{Im}\lambda) = 0$ and $L_{k,\ell}(\mathcal{B}f) = L_{k,\ell}(f)$. Therefore, from now on we assume $f \in S^k \tau'_M \otimes^{\mathcal{B}} S^\ell \tau'_M$.

To describe the natural kernel of $L_{k,\ell}$, acting on $S^k \tau'_M \otimes^{\mathcal{B}} S^\ell \tau'_M$, we introduce the *symmetrized gradient operator* $\mathrm{d}' \colon S^k \tau'_M \otimes S^{\ell-1} \tau'_M \to S^k \tau'_M \otimes S^\ell \tau'_M$ defined by

$$(1.5) \qquad (\mathrm{d}'v)_{i_1\ldots i_k j_1 \ldots j_\ell} := \operatorname{Sym}(j_1\ldots j_\ell) v_{i_1\ldots i_k j_1\ldots j_{\ell-1};j_\ell}.$$

In (1.5) the index after the semicolon stands for the corresponding index of the covariant derivative of a tensor field v. It was shown in [42, Chapter 7] that

$$L_{k,\ell}(\mathcal{B}\mathrm{d}'u) = L_{k,\ell}(\mathrm{d}'u) = 0, \quad \text{for } u \in S^k \tau'_M \otimes^{\mathcal{B}} S^{\ell-1} \tau'_M, \quad \text{with } u|_{\partial M} = 0.$$

After these preparations we are ready to set the following definition of solenoidal-injectivity (s-injectivity) for the mixed ray transform: $L_{k,\ell}$ is called s-injective if $L_{k,\ell}f = 0$ and $f \in L^2(S^k \tau'_M \otimes^{\mathcal{B}} S^\ell \tau'_M)$ imply $f = \mathrm{d}^{\mathcal{B}}v := \mathcal{B}\mathrm{d}'v$ with some tensor field $v \in H_0^1(S^k \tau'_M \otimes^{\mathcal{B}} S^{\ell-1} \tau'_M)$. Here

$$\mathrm{d}^{\mathcal{B}} \colon H^1(S^k \tau'_M \otimes^{\mathcal{B}} S^{\ell-1} \tau'_M) \to L^2(S^k \tau'_M \otimes^{\mathcal{B}} S^\ell \tau'_M).$$

We also introduce the formal adjoint of d'

$$-\delta' \colon S^k \tau'_M \otimes S^\ell \tau'_M \to S^k \tau'_M \otimes S^{\ell-1} \tau'_M,$$

where δ' is the *divergence operator*

$$(\delta'u)_{i_1\ldots i_k j_1\ldots j_{\ell-1}} := g^{j_\ell j_{\ell+1}} u_{i_1\ldots i_k j_1\ldots j_\ell;j_{\ell+1}}.$$

We define $\delta^{\mathcal{B}} := \delta'|_{S^k \tau'_M \otimes^{\mathcal{B}} S^\ell \tau'_M}$. One can readily check that $\operatorname{Im}(\delta^{\mathcal{B}}) \subset S^k \tau'_M \otimes^{\mathcal{B}} S^{\ell-1} \tau'_M$, and therefore

$$\delta^{\mathcal{B}} \colon S^k \tau'_M \otimes^{\mathcal{B}} S^\ell \tau'_M \to S^k \tau'_M \otimes^{\mathcal{B}} S^{\ell-1} \tau'_M.$$

However we will verify later in Lemma 2.2 that $\mathrm{d}^{\mathcal{B}}$ and $-\delta^{\mathcal{B}}$ are well defined and formally adjoint to each other.

The following tensor decomposition plays an essential role in the analysis of the mixed ray transform.

Theorem 1.1. *For any $f \in H^m(S^k \tau'_M \otimes^{\mathcal{B}} S^\ell \tau'_M)$, $k = \ell \in \{1, 2\}$, $m \in \{0, 1, 2, \ldots\}$, there exists a unique decomposition*

$$(1.6) \qquad\qquad f = f^s + \mathrm{d}^{\mathcal{B}}v,$$

with $f^s \in H^m(S^k \tau'_M \otimes^{\mathcal{B}} S^\ell \tau'_M)$, $\delta^{\mathcal{B}}f^s = 0$, and $v \in H^{m+1}(S^k \tau'_M \otimes^{\mathcal{B}} S^{\ell-1} \tau'_M)$, $v|_{\partial M} = 0$. In addition there exists a constant $C > 0$ such that

$$(1.7) \qquad \|f^s\|_{H^m} \le C\|f\|_{H^m}, \quad \|v\|_{H^{m+1}} \le C\|\delta^{\mathcal{B}}f\|_{H^{m-1}}.$$

This theorem will be proved in Section 2. We note that a decomposition equivalent to (1.6) has been provided earlier by Sharafutdinov [42, Lemma 7.2.1]: for any $f \in L^2(S^k \tau'_M \otimes S^\ell \tau'_M)$, $k, \ell \ge 1$, there is a decomposition

$$(1.8) \qquad\qquad f = f^s + \mathrm{d}'v + \lambda w,$$

with $\mu f^s = 0$, $\delta' f^s = 0$, for some $v \in H_0^1(S^k \tau_M' \otimes S^{\ell-1} \tau_M')$, $w \in L^2(S^{k-1} \tau_M' \otimes S^{\ell-1} \tau_M')$. Moreover if $\ell \geq 2$ we can choose v such that $\mu v = 0$. The equivalence of the decompositions (1.6) and (1.8) can be observed by noticing that

$$d^{\mathcal{B}} v - d' v \in \text{Im} \lambda, \quad \text{and} \quad \mathcal{B} f - f \in \text{Im} \lambda,$$
$$v \in L^2(S^k \tau_M' \otimes^{\mathcal{B}} S^{\ell-1} \tau_M'), \; f \in L^2(S^k \tau_M' \otimes S^\ell \tau_M'),$$

and rearranging terms. We remark that it was shown in [42] that the solenoidal part f^s, in (1.8) is uniquely determined by f, but the uniqueness of v and w was not proven. However the uniqueness of a quite similar decomposition has been proved in [13].

Main result. The main purpose of this paper is to establish the s-injectivity of $L_{1,1}$ and $L_{2,2}$ for g in a generic subset of all simple metrics on M. We also provide a stability estimate for the corresponding normal operators. The analogous result for $I_2 = L_{0,2}$ is given in [44, Theorem 1.5]. We will present a detailed proof for $L_{1,1}$. The proof is similar for $L_{2,2}$, modulo some key calculations which we will also provide.

We then introduce some necessary notations in order to state the main result of this paper. We write $L_g = L_{k,\ell}$ to emphasize the dependence on the metric g. We denote the L^2-normal operator $L_g^* L_g$ of the mixed ray transform by $\mathcal{N}_g$ (see Section 3 for the rigorous definitions). Since (M, g) is simple we can without loss of generality assume that $M \subset \mathbf{R}^3$ with a simple metric g that is smoothly extended in whole $\mathbf{R}^3$. Thus we can find a small open neighborhood M_1 of M, such that $(\overline{M_1}, g)$ is simple, (see [43, page 454]). A tensor field f defined on M will be extended by a zero field to $M_1 \setminus M$. We note that this creates jumps at the boundary ∂M. To tackle this, the $\tilde{H}_2$-norm was introduced in [43] (see also Section 4). As the decomposition (1.6) depends on the domain, we use the notation f_M^s for the solenoidal part of f on M to emphasize this. Our main result is

Theorem 1.2. *Let $(k, \ell) = (1, 1)$ or $(2, 2)$. There exists an integer m_0 such that for each $m \geq m_0$, the set $\mathcal{G}^m(M)$ of simple C^m-regular metrics in M, for which L_g is s-injective, is open and dense in the C^m-topology. Moreover, for any $g \in \mathcal{G}^m$,*

$$\|f_M^s\|_{L^2(M)} \leq C \|\mathcal{N}_g f\|_{\tilde{H}_2(M_1)}, \qquad f \in H^1(S^k \tau_M' \otimes^{\mathcal{B}} S^\ell \tau_M'),$$

with a constant $C > 0$ that can be chosen locally uniformly in $\mathcal{G}^m(M)$ in the $C^m(M)$-topology.

The s-injectivity of $L_{k,\ell}$, $k, \ell \geq 1$, has been proved for two-dimensional simple manifolds in [17]. On higher dimensional manifolds, the s-injective was established in [42, Theorem 7.2.2] under restrictions on the sectional curvature of (M, g). In both of these aforementioned papers, the sharper tensor decomposition (1.6) is not needed. However, in this paper the decomposition (1.6) is a key component of the proof of Theorem 1.2.

We also refer to [28] for the study of a related problem in dimension two. In a recent paper [22] the authors showed that on globally hyperbolic stationary Lorentzian manifolds, *the light ray transform* is injective up to a similar natural obstruction that $L_{1,1}$ has. In Appendix A we relate the mixed ray transform $L_{2,2}$ to the averaged travel-times of qS-polarized elastic waves. However we note that the travel-time data alone only gives us partial information about the mixed ray

transform. If in addition we include the measurement of the shear wave amplitude, the complete mixed ray transform can be recovered. In Appendix B, we will show how one can obtain the mixed ray transform from a linearization of the Dirichlet-to-Neumann map of an elastic wave equation on a smooth bounded domain $M \subset \mathbf{R}^3$. Here we rely on the observation that both the travel-time and the amplitude are encoded in the Dirichlet-to-Neumann map. We refer to [42, Chapter 7] for an alternative approach to obtain $L_{2,2}$.

Outline of the proof. In the beginning of Section 2 we find an explicit representation for the projection $\mathcal{B}$ onto the space of trace-free tensors. Then we prove Theorem 1.1 in the case $k = \ell = 2$. The rest of the paper is devoted to prove Theorem 1.2. We give detailed proof for the case $k = \ell = 1$ in Sections 2–6, and discuss the required modifications for the $k = \ell = 2$ in the final section.

In Sections 3–6 we study the mixed ray transform on 1+1 tensor fields f satisfying the trace-free condition. Section 3 is dedicated to the study of the normal operator $\mathcal{N}_g$ of the mixed ray transform on $1 + 1$ tensor fields. First we show that $\mathcal{N}_g$ is an integral operator and find its Schwartz kernel. In the second part of the section we prove that the normal operator is a pseudo-differential operator (ΨDO) of order -1. We also give an explicit coordinate-dependent formula for the principal symbol of this operator.

Since in Theorem 1.2 we assumed that the metric is only finitely smooth we start Section 4 by recalling some basics of the theory of ΨDO's whose amplitudes are only finitely smooth. This is needed to establish the continuity of $\mathcal{N}_g$, and several other operators, with respect to metric g in C^m-topology. We prove that $\mathcal{N}_g$ is elliptic acting on the solenoidal tensor fields. This manifests the Fredholm nature of the normal operator on some extended simple manifold. Then we can recover the solenoidal part (on the extended manifold) of the tensor field f from $\mathcal{N}_g f$ modulo a finitely smooth term. In the second part of Section 4 we compare the solenoidal parts on original manifold and on the extended manifold. Then we establish a reconstruction formula for the solenoidal part of tensor fields on the original manifold. We also give a stability estimate for the normal operator (see Theorem 4.6).

In Section 5, we prove the s-injectivity of the mixed ray transform on analytic simple Riemannian manifolds (see Theorem 5.5). Since analytic metrics are C^m-dense in the space of all simple metrics, Theorem 5.5 can be used to prove Theorem 1.2 in Section 6.

2. Decomposition of the trace-free tensor fields

We begin this section by finding an explicit formula for the projection $\mathcal{B}$ from $S^k \tau'_M \otimes S^\ell \tau'_M$ onto $\ker \mu = S^k \tau'_M \otimes^\mathcal{B} S^\ell \tau'_M$.

2.1. Domain of the mixed ray transform. We choose some $f \in S^k \tau'_M \otimes S^\ell \tau'_M$ and write

$$(2.1) \qquad f = \mathcal{B}f + \lambda w, \quad w \in S^{k-1}\tau'_M \otimes S^{\ell-1}\tau'_M.$$

In Lemma 2.1 we find a representation for w in (2.1) under the assumption $k \geq \ell \geq 1$.

Lemma 2.1. *Any tensor field* $f \in S^k \tau'_M \otimes S^\ell \tau'_M$, $k \geq \ell \geq 1$ *admits the decomposition* (2.1), *where* $w \in S^{k-1} \tau'_M \otimes S^{\ell-1} \tau'_M$ *is given by the formula*

$$w = \mathcal{A}f := \left(\sum_{K=1}^{\ell} \left((-1)^{K+1} \frac{\prod_{h=1}^{K-1} b_h}{\prod_{h=1}^{K} a_h} \right) \lambda^{K-1} \mu^K \right) f,$$

and

$$\mathcal{B}f = (\mathrm{Id} - \lambda \mathcal{A})f,$$

where

$$a_h = \frac{h(k+\ell+2-h)}{k\ell}, \quad b_h = \frac{(k-h)(\ell-h)}{k\ell}.$$

Proof. To begin we derive the following formula for the commutator of λ and μ

$$\mu \lambda w = \left(\mathrm{Sym}(i_1 \ldots i_k) \mathrm{Sym}(j_1 \ldots j_\ell)(w_{i_1 \ldots i_{k-1} j_1 \ldots j_{\ell-1}}) g_{i_k j_l} \right) g^{i_k j_l}$$

$$= \frac{1}{k\ell} (w_{i_1 \ldots i_{k-1} j_1 \ldots j_{\ell-1}}) g_{i_k j_l} g^{i_k j_l}$$

(2.2)
$$+ \frac{k-1}{k\ell} w_{i_1 \ldots i_{k-2}, i_k j_1 \ldots j_{\ell-1}} g_{i_{k-1} j_\ell} g^{i_k j_\ell}$$

$$+ \frac{\ell-1}{k\ell} w_{i_1 \ldots i_{k-1} j_1 \ldots j_{\ell-2} j_\ell} g_{i_k j_{\ell-1}} g^{i_k j_\ell}$$

$$+ \mathrm{Sym}(i_1 \ldots i_{k-1}) \mathrm{Sym}(j_1 \ldots j_{\ell-1}) w_{i_1 \ldots i_{k-2} i_k j_1 \ldots j_{\ell-2} j_\ell} g_{i_{k-1} j_{\ell-1}} g^{i_k j_\ell}$$

$$= \frac{k+\ell+1}{k\ell} w + \frac{(k-1)(\ell-1)}{k\ell} \lambda \mu w.$$

In the case $k = \ell = 1$, w is a function, and formulas (2.1) and (2.2) imply

$$(2.3) \qquad\qquad\qquad w = \frac{\mu f}{3}.$$

To proceed for the higher order tensors we assume $\max\{k, \ell\} \geq 2$ and use the commutator formula (2.2) to prove that for $m \in \{2, \ldots, \min\{k, \ell\}\}$ we have

$$(2.4) \qquad \mu^{m-1} \lambda w = a_h \mu^{m-2} w + b_h \mu^{m-1-h} \lambda \mu^h w, \quad h \in \{1, \ldots, m-1\},$$

where

$$(2.5) \qquad a_h = \frac{1}{k\ell} \sum_{r=1}^{h} x_r, \quad x_r := k + \ell + 3 - 2r, \quad b_h = \frac{(k-h)(\ell-h)}{k\ell}.$$

We note that the case $m = 2$ is the same as (2.2). If $m > 2$ we do an induction over h. The initial step of the induction follows from (2.2). For the induction we note that $\mu^h w \in S^{k-h-1} \tau'_M \otimes S^{\ell-h-1} \tau'_M$. Due to (2.2) we have

$$\mu^{m-1-h} \lambda \mu^h w = \mu^{m-2-h} \left(\frac{k+\ell-2h+1}{(k-h)(\ell-h)} \mu^h w + \frac{(k-h-1)(\ell-h-1)}{(k-h)(\ell-h)} \lambda \mu^{h+1} w \right).$$

Therefore if (2.5) holds for $h \in \{1, \ldots, m-2\}$, it also holds for $h+1$.

Next we note that for any $m \leq \min\{k, \ell\}$ the formulas (2.1) and (2.4) imply

$$\mu^{m-1} f = b_{m-1} \lambda \mu^{m-1} w + a_{m-1} \mu^{m-2} w.$$

We denote $K = m - 1$. Thus for any $K \in \{1, \ldots, \min\{k, \ell\} - 1\}$, it holds

(2.6)
$$\mu^K f = b_K \lambda \mu^K w + a_K \mu^{K-1} w, \quad a_K = \frac{K(k+\ell+2-K)}{k\ell}, \quad b_K = \frac{(k-K)(\ell-K)}{k\ell}.$$

Now we assume that $\ell \leq k$ since we are mostly interested in the case $k = \ell = 2$. The case $\ell > k$ can be dealt with similarly.

We choose $K = \ell - 1$ and apply μ to both sides of equation (2.6) to get

$$\mu^\ell f = b_{\ell-1}\mu\lambda\mu^{\ell-1}w + a_{\ell-1}\mu^{\ell-1}w.$$

We note that for any $v \in S^m\tau'_M$, $\mu\lambda v = \frac{m+3}{m+1}v$. This implies

$$\mu\lambda\mu^{\ell-1}w = \frac{k - \ell + 3}{k - \ell + 1}\mu^{\ell-1}w,$$

and we have found the formula

$$\mu^{\ell-1}w = \left(b_{\ell-1}\frac{k - \ell + 3}{k - \ell + 1} + a_{\ell-1}\right)^{-1}\mu^\ell f = \frac{k}{k + 2}\mu^\ell f = \frac{1}{a_\ell}\mu^\ell f.$$

By the recursion formula (2.6) we get

(2.7)
$$\begin{aligned}
w &= \frac{\mu f}{a_1} - \frac{b_1}{a_1}\lambda\mu w = \frac{\mu f}{a_1} - \frac{b_1}{a_1}\lambda\left(\frac{\mu^2 f}{a_2} - \frac{b_2}{a_2}\lambda\mu^2 w\right) \\
&= \frac{\mu f}{a_1} - \frac{b_1}{a_1 a_2}\lambda\mu^2 f + \frac{b_1 b_2}{a_1 a_2}\lambda^2\mu^2 w \\
&= \cdots \\
&= \sum_{K=1}^{\ell-1}\left((-1)^{K+1}\frac{\prod_{h=1}^{K-1}b_h}{\prod_{h=1}^{K}a_h}\right)\lambda^{K-1}\mu^K f + \left((-1)^{\ell+1}\frac{\prod_{h=1}^{\ell-1}b_h}{\prod_{h=1}^{\ell-1}a_h}\right)\lambda^{\ell-1}\mu^{\ell-1}w \\
&= \left(\sum_{K=1}^{\ell}\left((-1)^{K+1}\frac{\prod_{h=1}^{K-1}b_h}{\prod_{h=1}^{K}a_h}\right)\lambda^{K-1}\mu^K\right)f.
\end{aligned}$$

The last row is the representation we were looking for. $\qquad\square$

We recall that in Section 1 we had given the formal definitions for the gradient operator

$$\mathrm{d}^{\mathcal{B}} := \mathcal{B}\mathrm{d}' : H^1(S^k\tau'_M \otimes^{\mathcal{B}} S^{\ell-1}\tau'_M) \to L^2(S^k\tau'_M \otimes^{\mathcal{B}} S^\ell\tau'_M),$$

and divergence operator

$$-\delta^{\mathcal{B}} := -\delta' : H^1(S^k\tau'_M \otimes^{\mathcal{B}} S^\ell\tau'_M) \to L^2(S^k\tau'_M \otimes^{\mathcal{B}} S^{\ell-1}\tau'_M)$$

on the trace-free class.

Lemma 2.2. *The differential operators $\mathrm{d}^{\mathcal{B}}$ and $-\delta^{\mathcal{B}}$ are well defined, formally adjoint to each other and*

(2.8) $$\mathrm{d}^{\mathcal{B}}v = \mathrm{d}'v - \frac{1}{a_1}\lambda\mu\mathrm{d}'v, \quad v \in S^k\tau'_M \otimes^{\mathcal{B}} S^{\ell-1}\tau'_M, \quad \text{when } k \geq \ell \geq 1.$$

Proof. The operator $\mathrm{d}^{\mathcal{B}}$ is clearly well defined by its definition, and the operator $\delta^{\mathcal{B}}$ is well defined since μ and δ' commute.

We note that for any $u \in S^k\tau'_M \otimes^{\mathcal{B}} S^{\ell-1}\tau'_M$ we have $\mu^2\mathrm{d}'u = 0$. Therefore operator $\mathrm{d}^{\mathcal{B}}$ has the representation (2.8). The proof of this claim is a direct consequence of the fact that the Levi-Civita connection commutes with any contraction.

The operators $\mathrm{d}^{\mathcal{B}}$ and $-\delta^{\mathcal{B}}$ are formal adjoints to each other since d' and $-\delta'$, and also λ and μ, are formal adjoints respectively. $\qquad\square$

2.2. **Tensor decomposition in the kernel of μ.** In the L^2-space of m-tensor fields on M we use the standard definition of the inner product

$$\langle f, h \rangle_g = \int_M f_{i_1 \cdots i_m} \overline{h}_{j_1 \cdots j_m} g^{i_1 j_1} \cdots g^{i_m j_m} (\det g)^{1/2} \mathrm{d}x.$$

Assuming the result of Lemma 2.3 we are ready to present the proof of Theorem 1.1.

Lemma 2.3. *Let (M, g) be a smooth Riemannian manifold. There exists a unique solution*

$$u \in H^m(S^k \tau'_M \otimes^{\mathcal{B}} S^{\ell-1} \tau'_M), \ k = \ell \in \{1, 2\}, \ m \in 1, 2, \ldots$$

to the boundary value problem

$$(2.9) \qquad \Delta^{\mathcal{B}} u := \delta^{\mathcal{B}} \mathrm{d}^{\mathcal{B}} u = h \quad \text{in } M^{int}, \quad u|_{\partial M} = w,$$

for any $h \in H^{m-2}(S^k \tau'_M \otimes^{\mathcal{B}} S^{\ell-1} \tau'_M)$ and $w \in H^{m-\frac{1}{2}}(S^k \tau'_M \otimes^{\mathcal{B}} S^\ell \tau'_M|_{\partial M})$. Moreover there exists $C > 0$ such that the following energy estimate is valid

$$(2.10) \qquad \|u\|_{H^m(M)} \le C \left(\|h\|_{H^{m-2}(M)} + \|w\|_{H^{m-\frac{1}{2}}(\partial M)} \right).$$

Proof of Theorem 1.1. We consider the boundary value problem (2.9) with the zero boundary value $w \equiv 0$. Let $(\Delta^{\mathcal{B}})^{-1}$ be the corresponding solution operator. We denote $v := (\Delta^{\mathcal{B}})^{-1} \delta^{\mathcal{B}} f$. Thus v solves the problem

$$(2.11) \qquad \Delta^{\mathcal{B}} v = \delta^{\mathcal{B}} f, \qquad v|_{\partial M} = 0,$$

and the energy estimate (2.10) implies that the *projection operator onto the potential fields* $\mathcal{P}_M := \mathrm{d}^{\mathcal{B}} (\Delta^{\mathcal{B}})^{-1} \delta^{\mathcal{B}}$ is a bounded operator in $H^m(S^k \tau'_M \otimes^{\mathcal{B}} S^\ell \tau'_M)$. We define a second bounded operator by setting $\mathcal{S}_M := I - \mathcal{P}_M$ and call this the *projection operator onto the solenoidal tensor fields*. Finally we denote $f^s := \mathcal{S}_M f$ and obtain

$$f = f^s + \mathrm{d}^{\mathcal{B}} v, \quad \text{with} \quad \delta^{\mathcal{B}} f^s = 0.$$

The estimate (1.7) follows from the boundedness of the operators $\mathcal{S}_M$ and $(\Delta^{\mathcal{B}})^{-1}$. $\square$

Remark 2.4. The operators $\mathcal{S}_M$ and $\mathcal{P}_M$ are both projections, i.e., $\mathcal{S}_M^2 = \mathcal{S}_M$, $\mathcal{P}_M^2 = \mathcal{P}_M$. These projections are formally self-adjoint since $\Delta^{\mathcal{B}}$ is formally self-adjoint and thus its inverse $(\Delta^{\mathcal{B}})^{-1}$ is also formally self-adjoint; see [27, Theorem 10.2-2].

The rest of this section is devoted to the proof of Lemma 2.3. We first recall some facts about the solvability of boundary value problems for elliptic systems. See for instance [51, Section 9] for a thorough review. Recall that we can without loss of generality assume that $M^{int} \subset \mathbf{R}^3$ is a domain with a smooth boundary. We use the notations $T^* M$ and $T^* \mathbf{R}^3$ for the cotangent bundles of M and $\mathbf{R}^3$ respectively.

Let $\alpha \in \mathbf{N}^3$ be a multi-index and $D^\alpha = (-\mathrm{i})^{|\alpha|} \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$. We say that a differential operator $A = (\sum_{|\alpha| \le 2} A_{ij}^\alpha(x) D^\alpha)_{i,j=1}^3$ is a second order (homogeneously) elliptic operator if the order of the operator $\sum_{|\alpha| \le 2} A_{ij}^\alpha(x) D^\alpha$ is two for any i, j and the *characteristic polynomial* $\chi(x, \xi)$ of the operator L does not vanish outside the

set $\mathbf{R}^3 \times \{\xi = 0\} \subset T^*\mathbf{R}^3$. Recall that the characteristic polynomial of A is defined by

$$\chi(x, \xi) := \det\left(\sigma_A(x, \xi)\right), \quad \sigma_A(x, \xi) := \left(\sum_{|\alpha|=2} A_{ij}^{\alpha}(x)\xi^{\alpha}\right)_{i,j=1}^3, \quad (x, \xi) \in T^*\mathbf{R}^3.$$

We note that this is equivalent for the *principal symbol* $\sigma_A(x, \xi)$ of A to be a bijective linear operator for every cotangent vector $(x, \xi) \in T^*\mathbf{R}^3 \setminus \{0\}$.

Next we define the Lopatinskij condition. Let $z \in \partial M$ and (x', t) be boundary coordinates near z, that is $t^{-1}\{0\} \subset \partial M$.

Definition 2.5. We say that the operator A satisfies the Lopatinskij at a point $z \in \partial M$ if the constant coefficient initial value problem

$$\sigma_A\left(z, 0, \xi', -i\frac{d}{dt}\right) v(t) = 0, \quad t \in \mathbf{R}_+, \quad v(0) = 0 \in \mathbf{R}^3, \quad \xi' \in T_z^* \partial M \setminus \{0\},$$

has only the trivial solution in $\left\{u \in C^2(\mathbf{R}_+) : u(0) = 0, \lim_{t \to \infty} u(t) = 0\right\}$.

For the rest of the paper we use the notations $\sigma(A)$ to denote the principal symbol of an operator A. Often we do not emphasize the point in which the principal symbol is evaluated.

Definition 2.6. We say that the boundary value problem

(2.12)
$$Au = f, \text{ in } M, \quad u|_{\partial M} = w, \quad u \in H^m(M),$$
$$f \in H^{m-2}(M), \ w \in H^{m-\frac{1}{2}}(\partial M), \ m \in 1, 2 \ldots$$

is *elliptic* if:

(I) The operator A is elliptic.
(II) The Lopatinskij condition holds for any $z \in \partial M$.

We aim to use techniques for the elliptic problems to prove Lemma 2.3. To do so we first find the principal symbols of the operators $d^{\mathcal{B}}$, $\delta^{\mathcal{B}}$ and $\Delta^{\mathcal{B}}$. We introduce the notation

$$S^k T_x' M \otimes^{\mathcal{B}} S^{\ell} T_x' M := \{f(x) |\, f \in S^k \tau_M' \otimes^{\mathcal{B}} S^{\ell} \tau_M'\},$$

for the evaluation of $f \in S^k \tau_M' \otimes S^{\ell} \tau_M'$ at $x \in M$. This is just the space of all tensors acting on the fiber $T_x M$ that are symmetric with respect to the first k and the last ℓ indices and trace free.

Lemma 2.7. *Let $(x, \xi) \in T^*M$. Define operators*

(2.13)
$$i_{\xi}^{\mathcal{B}}\colon S^k T_x' M \otimes^{\mathcal{B}} S^{\ell-1} T_x' M \to S^k T_x' M \otimes^{\mathcal{B}} S^{\ell} T_x' M, \quad i_{\xi}^{\mathcal{B}} := \mathcal{B}i_{\xi},$$
$$(i_{\xi}u)_{i_1\ldots i_k j_1\ldots j_{\ell}} := \text{Sym}(j_1 \ldots j_{\ell})u_{i_1\ldots i_k j_1\ldots j_{\ell-1}}\xi_{j_{\ell}}$$

and
(2.14)
$$j_{\xi}^{\mathcal{B}}\colon S^k T_x' M \otimes^{\mathcal{B}} S^{\ell} T_x' M \to S^k T_x' M \otimes^{\mathcal{B}} S^{\ell-1} T_x' M, \quad (j_{\xi}^{\mathcal{B}}v)_{i_1\ldots i_k j_1,\ldots,j_{\ell}} := v_{i_1\ldots i_k j_1\ldots j_{\ell}}\xi^{j_{\ell}}.$$

In the case $k = \ell = 2$ the principal symbols of $\frac{1}{i}d^{\mathcal{B}}$ and $\frac{1}{i}\delta^{\mathcal{B}}$ are $i_{\xi}^{\mathcal{B}}$ and $j_{\xi}^{\mathcal{B}}$ respectively. The principal symbol of $\Delta^{\mathcal{B}}$ is

$$\sigma(\Delta^{\mathcal{B}})\colon S^2 T_x' M \otimes^{\mathcal{B}} S^1 T_x' M \to S^2 T_x' M \otimes^{\mathcal{B}} S^1 T_x' M$$

such that

(2.15)

$$-(\sigma(\Delta^{\mathcal{B}})u)_{i_1 i_2 j_1} = \frac{1}{2}\left(u_{i_1 i_2 j_1}|\xi|_g^2 + u_{i_1 i_2 h}\xi^h\xi_{j_1}\right)$$
$$-\frac{1}{10}\left(u_{ri_2 j_1}\xi^r\xi_{i_1} + u_{ri_1 j_1}\xi^r\xi_{i_2} + g_{i_1 j_1}\xi^r\xi^h u_{ri_2 h} + g_{i_2 j_1}\xi^r\xi^h u_{ri_1 h}\right).$$

Proof. We refer to [51, Section 8] for the definitions of the principal symbols of ΨDOs over vector bundles. Recall that in local coordinates differential $d'u$ of a tensor field $u \in S^2 T_x'M \otimes S^1 T_x'M$ has a representation

$$(\mathrm{d}'u)_{i_1 i_2 j_1 j_2} = \mathrm{Sym}(j_1, j_2)\left(\frac{\partial u_{i_1 i_2 j_1}}{\partial x_{j_2}} - \left(u_{ri_2 j_1}\Gamma^r_{i_1 j_2} + u_{i_1 r j_1}\Gamma^r_{i_2 j_2} + u_{i_1 i_2 r}\Gamma^r_{j_1 j_2}\right)\right).$$

Here Γ^k_{ij} are the Christoffel symbols of the metric g. Therefore the principal symbol of d' is exactly the map $(x, \xi) \mapsto i_\xi$. Since $\mu^2 i_\xi u = 0$ for any $u \in S^2 T_x'M \otimes^{\mathcal{B}} S^1 T_x'M$ we have due to (2.8) that

$$\frac{1}{i}\sigma(\mathrm{d}^{\mathcal{B}}) = i_\xi - \frac{4}{5}\lambda\mu i_\xi = i_\xi^{\mathcal{B}}.$$

Similarly for $\delta'u$, $u \in S^2 T_x'M \otimes S^2 T_x'M$ we have

$$(\delta'u)_{i_1 i_2 j_1}$$
$$= \left(\frac{\partial u_{i_1 i_2 j_1 j_2}}{\partial x_h} - \left(u_{ri_2 j_1 j_2}\Gamma^r_{i_1 h} + u_{i_1 r j_1 j_2}\Gamma^r_{i_2 h} + u_{i_1 i_2 r j_2}\Gamma^r_{j_1 h} + u_{i_1 i_2 j_1 r}\Gamma^r_{j_2 h}\right)\right)g^{hj_2}.$$

Thus the principal symbol of δ' is given by

$$\frac{1}{i}\sigma(\delta') = \frac{1}{i}\sigma(\delta^{\mathcal{B}}) = j_\xi = j_\xi^{\mathcal{B}}.$$

By [51, Theorem 8.44] we have $-\sigma(\Delta^{\mathcal{B}}) = j_\xi^{\mathcal{B}} i_\xi^{\mathcal{B}}$. The proof of (2.15) is a direct computation recalling that $\mu u = 0$. However we give it here as we need the computations later.

$$-\sigma(\Delta^{\mathcal{B}}u)_{i_1 i_2 j_1}$$
$$= (j_\xi^{\mathcal{B}} i_\xi^{\mathcal{B}} u)_{i_1 i_2 j_1} = \frac{1}{2}\xi^h\left(u_{i_1 i_2 j_1}\xi_h + u_{i_1 i_2 h}\xi_{j_1} - \frac{4}{5}\lambda\mu\left(u_{i_1 i_2 j_1}\xi_h + u_{i_1 i_2 h}\xi_{j_1}\right)\right)$$
$$= \frac{1}{2}\xi^h\left(u_{i_1 i_2 j_1}\xi_h + u_{i_1 i_2 h}\xi_{j_1} - \frac{4}{5}\lambda\left(u_{ri_2 t}\xi_h + u_{ri_2 h}\xi_t\right)g^{rt}\right)$$
$$= \frac{1}{2}\xi^h\left(u_{i_1 i_2 j_1}\xi_h + u_{i_1 i_2 h}\xi_{j_1} - \frac{4}{5}\mathrm{Sym}(i_1 i_2)\mathrm{Sym}(j_1 h)\left(g_{i_1 j_1} u_{ri_2 h}\xi^r\right)\right)$$
$$= \frac{1}{2}\xi^h\left(u_{i_1 i_2 j_1}\xi_h + u_{i_1 i_2 h}\xi_{j_1} - \frac{1}{5}\left(g_{i_1 j_1} u_{ri_2 h} + g_{i_1 h} u_{ri_2 j_1} + g_{i_2 j_1} u_{ri_1 h} + g_{i_2 h} u_{ri_1 j_1}\right)\xi^r\right)$$
$$= \frac{1}{2}\left(u_{i_1 i_2 j_1}|\xi|_g^2 + u_{i_1 i_2 h}\xi^h\xi_{j_1}\right)$$
$$- \frac{1}{10}\left(u_{ri_2 j_1}\xi^r\xi_{i_1} + u_{ri_1 j_1}\xi^r\xi_{i_2} + g_{i_1 j_1}\xi^r\xi^h u_{ri_2 h} + g_{i_2 j_1}\xi^r\xi^h u_{ri_1 h}\right).$$

$\square$

In the following lemma we show that the problem (2.9) is elliptic.

Lemma 2.8. *The problem* (2.9) *is elliptic in the sense of Definition* 2.6.

Proof. We check first the ellipticity of the principal symbol of $\Delta^{\mathcal{B}}$. If we denote

$$(p_1(\xi)u)_{i_1 i_2 j_1} = u_{i_1 i_2 h}\xi^h \xi_{j_1}, \quad (p_2(\xi)u)_{i_1 i_2 j_1} = |\xi|_g^2 u_{i_1 i_2 j_1} - u_{i_1 i_2 h}\xi^h \xi_{j_1},$$

$$(p_3(\xi)u)_{i_1 i_2 j_1} = |\xi|_g^2 u_{i_1 i_2 j_1} - g_{i_2 j_1}\xi^r \xi^h u_{ri_1 h},$$

then straightforward calculations show that $p_\alpha(\xi)$, $\alpha = 1, 2, 3$ are non-negative in the sense of

$$\langle p_\alpha(\xi)u, u\rangle_{L^2} \geq 0, \quad u \in L^2\left(S^2 T'_x M \otimes^{\mathcal{B}} S^1 T'_x M\right).$$

Equation (2.15) implies

$$\left\langle -\sigma(\Delta^{\mathcal{B}})u - \frac{1}{10}|\xi|_g^2 u, u\right\rangle_{L^2} \geq \frac{1}{2}\langle p_1(\xi)u, u\rangle_{L^2} + \frac{1}{5}\langle p_2(\xi)u, u\rangle_{L^2} + \frac{1}{5}\langle p_3(\xi)u, u\rangle_{L^2} \geq 0.$$

Hence we obtain

$$\langle -\sigma(\Delta^{\mathcal{B}})u, u\rangle_{L^2} \geq \frac{1}{10}|\xi|_g^2 \langle u, u\rangle_{L^2},$$

which proves the ellipticity of $\sigma(\Delta^{\mathcal{B}})$.

Next, we verify the Lopatinskij condition. For that, we choose local coordinates $(x^1, x^2, x^3 = t)$, $t \geq 0$ in a neighborhood of a point $x_0 \in \partial M$, such that the boundary ∂M is locally represented by $t = 0$, and $g_{ij}(x_0) = \delta_{ij}$. We set a differential operator $D = (D_j)_{j=1}^3$, $D_j = -\mathrm{i}\frac{\partial}{\partial x^j}$, $j \in \{1, 2\}$ and $D_3 = D_t = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}$. Then we denote

$$\mathrm{d}_0^{\mathcal{B}}(D) = \sigma(\mathrm{d}^{\mathcal{B}})(x_0, D), \quad \delta_0^{\mathcal{B}}(D) = \sigma(\delta^{\mathcal{B}})(x_0, D).$$

We need to show that the only solution for the system of ordinary differential equations

$$(2.16) \qquad \begin{aligned} \delta_0^{\mathcal{B}}(\xi', D_t)\mathrm{d}_0^{\mathcal{B}}(\xi', D_t)v(t) &= 0, \quad t \in \mathbf{R}_+ \\ v(0) &= 0, \end{aligned}$$

which satisfies $v(t) \to 0$ as $t \to +\infty$, is the zero field.

Let

$$u \in \mathcal{S}(S^2 \tau'_{\mathbf{R}_+} \otimes^{\mathcal{B}_0} S^2 \tau'_{\mathbf{R}_+}), \quad v \in \mathcal{S}(S^2 \tau'_{\mathbf{R}_+} \otimes^{\mathcal{B}_0} S^1 \tau'_{\mathbf{R}_+}),$$

where $\mathcal{S}$ means that the function $u(t)$ has a rapid decrease when t tends to $+\infty$ and $\mathcal{B}_0$ means that u belongs to the kernel of the operator μ associated to the Euclidean metric. If $v(0) = 0$ then Lemma 2.2 implies the following Green's formula

$$(2.17) \qquad \int_0^\infty \langle \delta_0^{\mathcal{B}}(\xi', D_t)u, v\rangle \mathrm{d}t = -\int_0^\infty \langle u, \mathrm{d}_0^{\mathcal{B}}(\xi', D_t)v\rangle \mathrm{d}t.$$

Due to denseness of rapidly decreasing tensor fields, the formula (2.17) holds for any u and v both vanishing in the infinity and $v(0) = 0$.

Let $v(t)$ be a solution of (2.16). Taking $u(t) := \mathrm{d}_0^{\mathcal{B}}(\xi', D_t)v(t)$ in (2.17) we obtain

$$(2.18) \qquad \mathrm{d}_0^{\mathcal{B}}(\xi', D_t)v(t) = 0, \quad v(0) = 0.$$

Finally we show that only zero field solves this initial value problem.

We note that (2.18) implies the following equation in coordinates

$$\begin{aligned} \frac{1}{\mathrm{i}}(\mathrm{d}_0^{\mathcal{B}}(\xi)v)_{i_1 i_2 j_1 j_2} = &\frac{1}{2}v_{i_1 i_2 j_1}\xi_{j_2} + \frac{1}{2}v_{i_1 i_2 j_2}\xi_{j_1} \\ &- \frac{1}{10}\left(\delta_{i_1 j_1}v_{ri_2 j_2} + \delta_{i_1 j_2}v_{ri_2 j_1} + \delta_{i_2 j_1}v_{i_1 r j_2} + \delta_{i_2 j_2}v_{i_1 r j_1}\right)\xi^r = 0. \end{aligned}$$

In the previous formula we set $j_2 = 3$ and $\xi_3 = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}$. Then we obtain the following system of ordinary differential equations.

$$\frac{1}{\mathrm{i}}\left(\mathrm{d}_0^{\mathcal{B}}(\xi', D_t)v(t)\right)_{i_1 i_2 33} = D_t v_{i_1 i_2 3} - \frac{1}{5}\left(D_t v_{i_1 33}\delta_{i_2 3} + D_t v_{i_2 33}\delta_{i_1 3}\right)$$

(2.19)
$$- \frac{1}{5}\sum_{r \neq 3}(v_{i_1 r 3}\xi^r \delta_{i_2 3} + v_{i_2 r 3}\xi^r \delta_{i_1 3})$$

$$= 0$$

and, for $j_1 \neq 3$,

$$\frac{1}{\mathrm{i}}\left(\mathrm{d}_0^{\mathcal{B}}(\xi', D_t)v(t)\right)_{i_1 i_2 j_1 3}$$

$$= \frac{1}{2}D_t v_{i_1 i_2 j_1} - \frac{1}{10}(D_t v_{i_1 3 j_1}\delta_{i_2 3} + D_t v_{i_2 3 j_1}\delta_{i_1 3}$$

$$+ D_t v_{i_1 33}\delta_{i_2 j_1} + D_t v_{i_2 33}\delta_{i_1 j_1})$$

(2.20)
$$+ \frac{1}{2}v_{i_1 i_2 3}\xi_{j_1} - \frac{1}{10}\sum_{m \neq 3}(v_{i_1 m j_1}\xi^m \delta_{i_2 3} + v_{i_2 m j_1}\xi^m \delta_{i_1 3}$$

$$+ v_{i_1 m 3}\xi^m \delta_{i_2 j_1} + v_{i_2 m 3}\xi^m \delta_{i_1 j_1})$$

$$= 0.$$

Finally we solve these equations with initial value $v(0) = 0$, which is done in the following sequence:

- If $i_1, i_2 \neq 3$, equation (2.19) gives

$$D_t v_{i_1 i_2 3} = 0, t > 0, \quad v(0) = 0.$$

  This implies $v_{i_1 i_2 3} = 0$ for $i_1, i_2 \neq 3$.
- If $i_1 \neq 3, i_2 = 3$ or $i_2 \neq 3, i_1 = 3$, equation (2.19) gives

$$\frac{4}{5}D_t v_{i_1 33} = \frac{4}{5}D_t v_{3 i_2 3} = 0, t > 0, \quad v(0) = 0,$$

  which implies $v_{i 33}$ and $v_{3 i 3}$ for $i \neq 3$.
- Finally equation (2.19) gives

$$\frac{3}{5}D_t v_{333} = 0, t > 0, \quad v(0) = 0,$$

  and thus $v_{333} = 0$.

We have proved that $v_{i_1 i_2 3} = 0$ and therefore equation (2.20) simplifies to

$$D_t v_{i_1 i_2 j_1} - \frac{1}{5}\left(D_t v_{i_1 3 j_1}\delta_{i_2 3} + D_t v_{i_2 3 j_1}\delta_{i_1 3}\right.$$

(2.21)
$$\left. - \sum_{m \neq 3}(v_{i_1 m j_1}\xi^m \delta_{i_2 3} + v_{i_2 m j_1}\xi^m \delta_{i_1 3})\right) = 0,$$

$$j_1 \neq 3.$$

- We take $i_1, i_2 \neq 3$ in (2.21), and get

$$D_t v_{i_1 i_2 j_1} = 0, t > 0, \quad v(0) = 0.$$

  Thus $v_{i_1 i_2 j_1} = 0$ for $i_1, i_2, j_1 \neq 3$.

- The choices $i_1 \neq 3, i_2 = 3$ and $i_2 \neq 3, i_1 = 3$ in (2.21), imply

$$\frac{4}{5} D_t v_{i_1 3 j_1} = \frac{4}{5} D_t v_{3 i_2 j_1} = 0, t > 0, \quad v(0) = 0,$$

  and then $v_{i_1 j_2 j_1} = 0$ if at most one of i_1, i_2 equals 3.
- Finally we take $i_1 = i_2 = 3$ in (2.21), to get

$$\frac{3}{5} D_t v_{33 j_2} = 0, t > 0, \quad v(0) = 0,$$

  and then $v_{33 j_2} = 0$ for $j_2 \neq 3$.

Thus we have proven that the zero field is the only solution of (2.18) and we have verified the Lopatsinkij condition for (2.9). Also we have proved the ellipticity of the boundary value problem (2.9). □

Now we are ready to give a proof for Lemma 2.3.

Proof of Lemma 2.3. We use the notation $\mathrm{Tr} \colon H^m(M) \to H^{m-\frac{1}{2}}(\partial M)$ for the trace operator. As we have verified in Lemma 2.8 that the problem (2.9) is elliptic, it holds due to [51, Theorem 9.32] that the operator $(\Delta^{\mathcal{B}}, \mathrm{Tr}) \colon H^m(M) \to (H^{m-2}(M) \times H^{m-\frac{1}{2}}(\partial M))$ is a Fredholm operator (a bounded operator with finite dimensional kernel and co-kernel). Moreover there exists a uniform constant $C > 0$ such that the following *a priori* estimate holds for any $u \in H^m(M)$

$$(2.22) \qquad \|u\|_{H^m(M)} \leq C \left(\|\Delta^{\mathcal{B}} u\|_{H^{m-2}(M)} + \|\mathrm{Tr} u\|_{H^{m-\frac{1}{2}}(\partial M)} + \|u\|_{H^{m-1}(M)} \right).$$

As the embedding $H^m(M) \hookrightarrow H^{m-1}(M)$ is compact it holds due to [43, Lemma 2] that we can write (2.22) in the form

$$\|u\|_{H^m(M)} \leq C \left(\|\Delta^{\mathcal{B}} u\|_{H^{m-2}(M)} + \|\mathrm{Tr} u\|_{H^{m-\frac{1}{2}}(\partial M)} \right)$$

for some uniform constant C, if (2.9) is uniquely solvable. This is the estimate (2.10). In order to verify the unique solvability of the boundary value problem (2.9), and to conclude the proof, we show that $(\Delta^{\mathcal{B}}, \mathrm{Tr})$ has a trivial kernel and co-kernel.

We show first the kernel of $(\Delta^{\mathcal{B}}, \mathrm{Tr})$ is trivial. Let u solve (2.9) with a source $h \equiv 0$ and boundary value $w \equiv 0$. Since we have proved that $\Delta^{\mathcal{B}}$ is elliptic, it holds that u is smooth. As $\mathrm{d}^{\mathcal{B}}$ and $-\delta^{\mathcal{B}}$ are formally adjoint we get

$$\int_M \langle \mathrm{d}^{\mathcal{B}} u, \mathrm{d}^{\mathcal{B}} u \rangle_g \mathrm{d}V = \int_M \langle -\Delta^{\mathcal{B}} u, u \rangle_g \mathrm{d}V = 0,$$

and

$$\mathrm{d}^{\mathcal{B}} u = \mathrm{d}' u - \frac{4}{5} \lambda \mu \mathrm{d}' u = 0.$$

Next we note that for any $x \in M$ and $v \in S^2 \tau'_M \otimes^{\mathcal{B}} S^1 \tau'_M$ holds

$$(\mathrm{d}^{\mathcal{B}} v)_{ijkl} \eta^i \eta^j \xi^k \xi^l = (\mathrm{d}' v)_{ijkl} \eta^i \eta^j \xi^k \xi^l$$

if $\xi, \eta \in T_x M$ are orthogonal. In the following we use the notation γ for the geodesic $\gamma_{x,\xi}$, $\xi \in S_x M$ and $\eta(t)$ stands for the parallel transport of η along γ. By straightforward computations we obtain

$$(2.23) \qquad \frac{\mathrm{d}}{\mathrm{d}t}[v_{ijk}(\gamma(t)) \eta^i(t) \eta^j(t) \dot{\gamma}^k(t)] = (\mathrm{d}^{\mathcal{B}} v)_{ijkl} \eta^i(t) \eta^j(t) \dot{\gamma}^k(t) \dot{\gamma}^l(t).$$

Let $x_0 \in M \setminus \partial M$ and z_0 be a closest boundary point to x_0. We use the notation $\xi_0 \in S_{x_0} M \setminus \{0\}$ for the direction of the unit speed geodesic γ_0 connecting x_0

to z_0. Since the geodesic γ_0 intersects ∂M transversally there exists a neighborhood $U \subset S_{x_0}M$ of ξ_0 such that the exit time function τ is finite and smooth in U. Then for any $\xi \in U$ and $\eta \perp \xi$, equation (2.23) and the fundamental theorem of calculus imply

$$u_{ijk}(x_0)\eta^i\eta^j\xi^k = -\int_0^{\tau(\xi)} [\mathrm{d}^{\mathcal{B}}u]_{ijkl}(\gamma_{x_0,\xi}(t))\eta^i(t)\eta^j(t)\xi^k(t)\xi^l(t)\mathrm{d}t = 0.$$

Here $\eta(t)$ is a parallel field along the geodesic $\gamma_{x_0,\xi}$ with $\eta(0) = \eta$ and $\xi(t) = \dot{\gamma}_{x_0,\xi}(t)$. In the following we use a short hand notation $u = u(x_0)$. Therefore

$$(2.24) \qquad u_{ijk}\eta^i\eta^j\xi^k = 0, \quad \xi \in U,\ \eta \perp \xi.$$

We choose an orthonormal basis $B = \{\xi, \eta, \widetilde{\eta}\}$ for the three dimensional space $T_{x_0}M$, where $\xi \in U$. By polarization, (2.24) implies

$$(2.25) \qquad u_{ijk}\eta^i\widetilde{\eta}^j\xi^k = 0.$$

For any $\epsilon > 0$ that is small enough, equation (2.24) gives

$$(2.26) \qquad u_{ijk}(\eta + \epsilon\xi)^i(\eta + \epsilon\xi)^j(\xi - \epsilon\eta)^k = 0.$$

Therefore the coefficients of the $\epsilon^3, \epsilon^2, \epsilon, 1$ of the expansion of (2.26) have to vanish. Clearly the same holds if η is replaced by $\widetilde{\eta}$ in (2.26). Now we have proven

$$(2.27)$$
$$u_{ijk}\xi^i\xi^j\eta^k = 0, \quad u_{ijk}\left(\xi^i\xi^j\xi^k - 2\eta^i\xi^j\eta^k\right) = 0, \quad u_{ijk}\left(2\eta^i\xi^j\xi^k - \eta^i\eta^j\eta^k\right) = 0,$$
$$u_{ijk}\xi^i\xi^j\widetilde{\eta}^k = 0, \quad u_{ijk}\left(\xi^i\xi^j\xi^k - 2\widetilde{\eta}^i\xi^j\widetilde{\eta}^k\right) = 0, \quad u_{ijk}\left(2\widetilde{\eta}^i\xi^j\xi^k - \widetilde{\eta}^i\widetilde{\eta}^j\widetilde{\eta}^k\right) = 0.$$

Also we have $u_{ijk}\eta^i\eta^j(\xi + \epsilon\widetilde{\eta})^k = 0$, from which we derive

$$(2.28) \qquad u_{ijk}\eta^i\eta^j\widetilde{\eta}^k = 0.$$

Next, we note

$$u_{ijk}(\widetilde{\eta} + \eta + \epsilon\xi)^i(\widetilde{\eta} + \eta + \epsilon\xi)^j(\xi - \epsilon\eta)^k = 0,$$

and the roles of η and $\widetilde{\eta}$ can be interchanged. Collecting the coefficients for $1, \epsilon, \epsilon^2$ terms we get by (2.25)–(2.28)

$$(2.29) \qquad \begin{aligned} u_{ijk}\widetilde{\eta}^i\xi^j\xi^k - u_{ijk}\widetilde{\eta}^i\eta^j\eta^k = 0, &\qquad u_{ijk}\widetilde{\eta}^i\xi^j\eta^k = 0, \\ u_{ijk}\eta^i\xi^j\xi^k - u_{ijk}\eta^i\widetilde{\eta}^j\widetilde{\eta}^k = 0, &\qquad u_{ijk}\eta^i\xi^j\widetilde{\eta}^k = 0. \end{aligned}$$

To continue we note that since B is an orthogonal basis, it follows that

$$(\mu u)_j(x_0) = \delta^{ik}u_{ijk} = \sum_{k=1}^3 u_{kjk} = \sum_{k=1}^3 u_{jkk} = 0, \quad j \in \{1, 2, 3\},$$

or equivalently

$$(2.30) \qquad \begin{cases} u_{ijk}\eta^i\xi^j\xi^k + u_{ijk}\eta^i\eta^j\eta^k + u_{ijk}\eta^i\widetilde{\eta}^j\widetilde{\eta}^k = 0, \\ u_{ijk}\widetilde{\eta}^i\xi^j\xi^k + u_{ijk}\widetilde{\eta}^i\eta^j\eta^k + u_{ijk}\widetilde{\eta}^i\widetilde{\eta}^j\widetilde{\eta}^k = 0, \\ u_{ijk}\xi^i\xi^j\xi^k + u_{ijk}\xi^i\eta^j\eta^k + u_{ijk}\xi^i\widetilde{\eta}^j\widetilde{\eta}^k = 0. \end{cases}$$

It remains to show that each term in (2.30) vanishes. As (2.27) and (2.29) give 6 additional equations, the following linear systems hold true:

$$
\begin{cases}
u_{ijk}\xi^i\xi^j\xi^k - 2u_{ijk}\xi^i\eta^j\eta^k = 0, \\
u_{ijk}\xi^i\xi^j\xi^k - 2u_{ijk}\xi^i\tilde{\eta}^j\tilde{\eta}^k = 0, \\
u_{ijk}\xi^i\xi^j\xi^k + u_{ijk}\xi^i\eta^j\eta^k + u_{ijk}\xi^i\tilde{\eta}^j\tilde{\eta}^k = 0;
\end{cases}
$$

(2.31)
$$
\begin{cases}
2u_{ijk}\xi^i\eta^j\xi^k - u_{ijk}\eta^i\eta^j\eta^k = 0, \\
u_{ijk}\eta^i\xi^j\xi^k + u_{ijk}\eta^i\eta^j\eta^k + u_{ijk}\eta^i\tilde{\eta}^j\tilde{\eta}^k = 0, \\
u_{ijk}\eta^i\xi^j\xi^k - u_{ijk}\eta^i\tilde{\eta}^j\tilde{\eta}^k = 0;
\end{cases}
$$

$$
\begin{cases}
2u_{ijk}\xi^i\tilde{\eta}^j\xi^k - u_{ijk}\tilde{\eta}^i\tilde{\eta}^j\tilde{\eta}^k = 0, \\
u_{ijk}\tilde{\eta}^i\xi^j\xi^k + u_{ijk}\tilde{\eta}^i\eta^j\eta^k + u_{ijk}\tilde{\eta}^i\tilde{\eta}^j\tilde{\eta}^k = 0, \\
u_{ijk}\tilde{\eta}^i\xi^j\xi^k - u_{ijk}\tilde{\eta}^i\eta^j\eta^k = 0.
\end{cases}
$$

Each system consists of three linearly independent equations for three variables, and thus can only have trivial solutions.

Finally, we show that the boundary value problem (2.9) has a trivial cokernel. If $f \in H^{-1}$ we can choose a series of $f_j \in L^2$, $j \in \mathbf{N}$ that converges to f in H^{-1} sense. For the existence of such sequence see for instance [1, Section 3]. Let ϕ_{j_k} be a sequence of smooth tensor fields approximating f_j in L^2. Let $\epsilon > 0$ and $k, j \in \mathbf{N}$ be so large that

$$\|f - f_k\|_{H^{-1}} < \epsilon/2, \quad \|f_k - \phi_{k_j}\|_{L^2} < \epsilon/2.$$

Then

$$\|f - \phi_{k_j}\|_{H^{-1}} \le \|f - f_k\|_{H^{-1}} + \|f_k - \phi_{k_j}\|_{L^2} < \epsilon.$$

Thus smooth tensor fields are dense in H^{-1}.

Suppose then that

$$(f, h) \in C^\infty(S^2\tau'_M \otimes^{\mathcal{B}} S^1\tau'_M) \times C^\infty(S^2\tau'_{\partial M} \otimes^{\mathcal{B}} S^1\tau'_{\partial M})$$

is in the co-kernel of the operator $(\Delta^{\mathcal{B}}, \mathrm{Tr})$. We show that $(f, h) \equiv (0, 0)$. To verify this we first note that the choice of (f, h) implies in particularly that

$$(2.32) \qquad \int_M \langle \Delta^{\mathcal{B}}u, f \rangle_g \mathrm{d}V = 0, \quad \text{for any } u \in C^\infty(S^2\tau'_M \otimes^{\mathcal{B}} S^1\tau'_M), \ u|_{\partial M} = 0.$$

To show that (2.32) implies $f \equiv 0$ is very similar to the proof of an analogous claim in [42, Theorem 3.3.2], and thus omitted here. The second claim $h \equiv 0$ follows from the fact that the trace map is onto. Finally due to denseness of smooth tensor fields we conclude that $(\Delta^{\mathcal{B}}, \mathrm{Tr})$ has a trivial co-kernel. $\qquad\square$

3. The normal operator of mixed ray transform of $1+1$ tensors

In this section, we show that the L^2-normal operator $\mathcal{N}_L$ of the mixed ray transform $L = L_{1,1}$ is an integral operator and find its Schwartz kernel. We also show that $\mathcal{N}_L$ is a pseudo-differential operator (ΨDO) of order -1 and give a representation for the principal symbol. In order to do this we will assume without loss of generality that $M \subset \mathbf{R}^3$ is a smooth domain and the metric tensors g extends to $\mathbf{R}^3$ in such a way that any geodesic exiting M never returns to M. We make a standing assumption, for the rest of this paper, that any tensor field, excluding the metric g, defined in M is extended to any larger domain with zero extension.

We begin this section by giving a formal definition of the mixed ray transform, equivalent to (1.2), on the trace-free tensors. For a vector $(x, v) \in TM$, we define the *contraction map*

$$(3.1) \quad \Lambda_v \colon S^k T'_x M \otimes^{\mathcal{B}} S^\ell T'_x M \to S^k T'_x M, \quad (\Lambda_v f)_{i_1 \ldots i_m} = f_{i_1 \ldots i_m j_1 \ldots j_\ell} v^{j_1} \cdots v^{j_\ell}.$$

Let

$$p_v \colon T_x M \to v^\perp, \quad (p_v)^k_i \xi^i = \left(\delta^k_i - \frac{v_i v^k}{|v|^2_g} \right) \xi^i, \quad k \in \{1, 2, 3\},$$

be the *projection map* onto the orthocomplement of the vector v. The second linear operator is the *restriction map*

$$(3.2) \qquad P_v \colon S^m T'_x M \to S^m T'_x M, \quad (P_v f)_{i_1 \ldots i_m} = f_{j_1 \ldots j_m} (p_v)^{j_1}_{i_1} \ldots (p_v)^{j_m}_{i_m}.$$

Then we can give the following definition of the mixed ray transform by the following "distributional" representation (see for instance [42, Chapter 7.2])

$$(3.3) \qquad L_{k,\ell} f(x, \xi) = \int_0^{\tau(x,\xi)} \mathcal{T}^{0,t}_\gamma (P_{\dot\gamma(t)} \Lambda_{\dot\gamma(t)} f_{\gamma(t)}) \mathrm{d}t,$$

$$f \in C^\infty(S^k \tau'_M \otimes^{\mathcal{B}} S^\ell \tau'_M), \ (x, v) \in \partial_+ SM.$$

Here we used the notation $\mathcal{T}^{t,s}_\gamma$ for the parallel translation along γ from the point $\gamma(s)$ to $\gamma(t)$. Symbol $\tau(x, \xi)$ stands for the exit time of the geodesic $\gamma = \gamma_{x,\xi}$ and $f_{\gamma(t)}$ is the evaluation of the tensor field f at $\gamma(t)$. Let us still clarify the action of the mixed ray transform. Let $f \in C^\infty(S^k \tau'_M \otimes^{\mathcal{B}} S^\ell \tau'_M)$ and $(x, \xi) \in \partial_+ SM$. We set the action of $L_{k,\ell} f(x, \xi)$ on vector $v = a\xi + \eta \in T_x M$, where $\eta \perp \xi$, $a \in \mathbf{R}$ and $\eta(t)$ is the parallel transport of η, to be given by integrating the following quantity over the interval $[0, \tau(x, \xi)]$

$$\langle \mathcal{T}^{0,t}_\gamma P_{\dot\gamma(t)} \Lambda_{\dot\gamma(t)} f_{\gamma(t)}, v^k \rangle = \langle P_{\dot\gamma(t)} \Lambda_{\dot\gamma(t)} f_{\gamma(t)}, (\mathcal{T}^{t,0}_\gamma v)^k \rangle$$

$$= f_{i_1 \ldots i_k j_1 \ldots j_\ell}(\gamma(t)) \eta(t)^{i_1} \cdots \eta(t)^{i_k} \dot\gamma(t)^{j_1} \cdots \dot\gamma(t)^{j_\ell},$$

where $v^k = \underbrace{(v, \ldots, v)}_{k}$ and in the last equation we used the formulas (3.1) and (3.2). This implies the equivalence for (1.2) and (3.3).

Finally we define the target space of the mixed ray transform. Let $\pi \colon \partial_+(SM) \to M$ be the restriction of the natural projection map from tangent bundle to the base manifold. Using the pullback map π^* we construct the symmetric pullback bundle of k-sensors on $\partial_+(SM)$, and reserve the notation $\beta_k(\partial_+(SM))$ for the sections of this bundle. These sections act as follows: Let $f \in \beta_k(\partial_+(SM))$ and $(x, \xi) \in \partial_+(SM)$, then, using the pairing notation, the following map is symmetric and k-linear

$$\langle f(x, \xi); v_1, \ldots, v_k \rangle \in \mathbf{R}, \quad v_1 \ldots v_k \in T_x M.$$

Since the exit-time function τ is smooth in $(x, \xi) \in \partial_+ SM$ by [42, Lemma 4.1.1.], the formula (3.3) implies that the mixed ray transform maps the elements of $C^\infty(S^k \tau'_M \otimes^{\mathcal{B}} S^\ell \tau'_M)$ into $C^\infty(\beta_k(\partial_+(SM)))$.

3.1. The normal operator of $L_{1,1}$ is an integral operator.

For now on we denote $L_{1,1} = L$ for brevity and work only in the space of $S\tau'_M \otimes^{\mathcal{B}} S\tau'_M$.

Let us now describe the measure we use on the target space of the mixed ray transform. For the measure $\mathrm{d}\sigma$ we mean the Riemannian volume of ∂SM, however

we choose a more suitable measure for $\partial_+(SM)$,

$$\mathrm{d}\mu(z,\omega) = |\langle \omega, \nu(z)\rangle_g|\mathrm{d}\sigma = |\langle \omega, \nu(z)\rangle_g|\mathrm{d}S_z\mathrm{d}S_\omega,$$

where $\mathrm{d}S_z$ is the surface measure of ∂M and $\mathrm{d}S_\omega$ is the surface measure of S_zM. That is if (z', z^3) is a boundary coordinate system we have

$$\mathrm{d}S_z = (\det g(z))^{1/2}\mathrm{d}z^1\mathrm{d}z^2 \quad \text{and} \quad \mathrm{d}S_\omega = (\det g(z))^{1/2}\mathrm{d}S_{\omega_0},$$

where $\mathrm{d}S_{\omega_0}$ is the Euclidean measure of the unit sphere $\mathbb{S}^2 \subset \mathbf{R}^3$. The L^2-inner product on $\beta_1(\partial_+(SM))$ is given by

$$\int_{\partial_+(SM)} f_i(z,\omega)\overline{h}_j(z,\omega)g^{ij}\mathrm{d}\mu(z,\omega).$$

It is shown in [42, Chapter 7] that L, originally defined on smooth tensor fields, has a bounded extension

$$L\colon H^m(S\tau_M' \otimes^\mathcal{B} S\tau_M') \to H^m(\beta_1(\partial_+(SM))), \quad m \geq 0.$$

In this section we consider L as an operator

$$L\colon L^2(S\tau_M' \otimes^\mathcal{B} S\tau_M') \to L^2(\beta_1(\partial_+(SM)); \mathrm{d}\mu),$$

and compute its normal operator.

Remark 3.1. Since L is a bounded operator, its adjoint

$$L^*\colon L^2(\beta_1(\partial_+(SM)); \mathrm{d}\mu) \to L^2(S\tau_M' \otimes^\mathcal{B} S\tau_M')$$

exists and is bounded. Thus the normal operator $\mathcal{N}_L := L^*L$ is bounded on $L^2(S\tau_M' \otimes^\mathcal{B} S\tau_M')$.

In the following we denote by $(x(t), \omega(t)) \in SM$, $t \in \mathbf{R}$, the lift of the geodesic $x(t)$ in SM, that is issued from $(z,\omega) \in \partial_+(SM)$. Let $f, h \in S\tau_M' \otimes^\mathcal{B} S\tau_M'$. Then we have

$$\langle Lf, Lh\rangle_{L^2(\beta_1(\partial_+(SM)))}$$

$$= \int_{\partial_+(SM)} \left(\int_0^{\tau(z,\omega)} (\mathcal{T}_\gamma^{0,t})_o^u (P_{\omega(t)})_u^i f_{ij}(x(t))\omega^j(t)\mathrm{d}t \right)$$

$$\left(\int_0^{\tau(z,\omega)} (\mathcal{T}_\gamma^{0,s})_{o'}^{u'} (P_{\omega(s)})_{u'}^{i'} \overline{h}_{i'j'}(x(s))\omega^{j'}(s)\mathrm{d}s \right) g^{oo'}(z)\mathrm{d}\mu(z,\omega)$$

$$=: I_+ + I_-.$$

Here we wrote

$$I_\pm = \int_{\partial_+(SM)} \int_{\mathbf{R}} \int_0^\infty g^{ou'}(x(s))(P_{\omega(s)})_{u'}^{i'} \overline{h}_{i'j'}(x(s))\omega^{j'}(s)(\mathcal{T}_\gamma^{s,s\pm t})_o^u$$

$$(P_{\omega(s\pm t)})_u^i f_{ij}(x(s\pm t))\omega^j(s\pm t) \, \mathrm{d}t\mathrm{d}s\mathrm{d}\mu(z,\omega),$$

and used the fact that g^{-1} is parallel.

We introduce new variables $x := x(s, z, \omega) \in M$, that is the point obtained by following the geodesic $x(s)$ given by the initial conditions $(z,\omega) \in \partial_+ SM$ until the time s, and $\xi := \xi(s, t, z, \omega) := t\omega(s) \in T_xM$, which is the scaling of the velocity $\omega(s)$, of the geodesic $x(s)$, by the positive factor t. Since (M, g) is simple, the map

$$\mathbf{R} \times (0, \infty) \times \partial_+(SM) \ni (s, t, (z,\omega)) \mapsto (x, \xi) \in TM$$

can be restricted to a diffeomorphism onto a set $U \subset TM \setminus (0 \cup T\partial M)$, where $U = \bigsqcup_{x \in M} U_x$ and $U_x := (\exp_x)^{-1}M \subset T_x M$. Moreover since the geodesic flow preserves the measure $(\det g)\mathrm{d}\xi\mathrm{d}x$ of TM, we get

$$\mathrm{d}t\mathrm{d}s\mathrm{d}\mu(z,\omega) = (-1)^3 |\xi|_g^2 (\det g)\mathrm{d}\xi\mathrm{d}x.$$

For more details about this change of coordinates we refer to [25].

We denote

$$y = \exp_x \xi, \ x = x(s,z,\omega), \ \xi = \xi(s,\pm t,z,\omega) \quad \text{and} \quad \widehat{\xi} = \frac{\xi}{|\xi|_g}.$$

It is straightforward to see that

$$\omega^j(s) = \widehat{\xi}^j, \quad \text{and} \quad \omega^j(s \pm t) = (\mathrm{grad}_y^g \rho(x,y))^j = g^{ij}(y)\frac{\partial\rho}{\partial y^i},$$

where $\rho(x,y)$ is the Riemannian distance function of g on $M \times M$. Thus

$$I_\pm = \int_M \int_{U_x} g^{ou'}(x)(P_{\widehat{\xi}})_{u'}^{i'} \bar{h}_{i'j'}(x)\widehat{\xi}^{j'}$$
$$(\mathcal{T}_{\gamma_{x,\widehat{\xi}}}^{0,|\xi|_g})_o^u (P_{\partial_y\rho})_u^i f_{ij}(y)g^{kj}(y)\frac{\partial\rho}{\partial y^k}\frac{1}{|\xi|_g^2}(\det g)\mathrm{d}\xi\mathrm{d}x,$$

and we get

$$I_+ = I_-.$$

Therefore

$$(3.4) \quad \langle \mathcal{N}_L f, h \rangle_{L^2(S\tau'_M \otimes S\tau'_M)} = 2\int_M \int_{U_x} g^{ou'}(x)(P_{\widehat{\xi}})_{u'}^{i'} \bar{h}_{i'j'}(x)\widehat{\xi}^{j'} (\mathcal{T}_{\gamma_{x,\widehat{\xi}}}^{0,|\xi|_g})_o^u$$
$$(P_{\partial_y\rho})_u^i f_{ij}(y)g^{kj}(y)\frac{\partial\rho}{\partial y^k}\frac{1}{|\xi|_g^2}(\det g)\mathrm{d}\xi\mathrm{d}x.$$

Since (M,g) is simple the map $(T_x M \setminus \{0\}) \ni \xi \mapsto \exp_x \xi =: y$ is a diffeomorphism with inverse

$$\xi^i = -\frac{1}{2}(\mathrm{grad}_x^g(\rho(x,y))^2)^i = -\frac{1}{2}g^{ij}(x)\frac{\partial\rho(x,y)^2}{\partial x^j},$$

and moreover

$$|\xi|_g = \rho(x,y), \quad \widehat{\xi}_m = -\frac{\partial\rho(x,y)}{\partial x^m} \quad \text{and} \quad \mathrm{d}\xi = (\det g^{-1})\left|\det\frac{\partial^2\rho(x,y)^2/2}{\partial x\partial y}\right|\mathrm{d}y.$$

To simplify the integral (3.4) even more we compute

$$(P_{\widehat{\xi}})_{u'}^{i'} \bar{h}_{i'j'}(x) = \bar{h}_{j'}^{i'}(x)\left(g_{i'u'}(y) - \frac{\partial\rho}{\partial x^{u'}}\frac{\partial\rho}{\partial x^{i'}}\right)$$

and

$$(P_{\partial_y\rho})^i_u f_{ij}(y) g^{kj}(y) = (P_{\partial_y\rho})^i_u f^k_i(y) = f^{ik}(y)\left(g_{iu}(y) - \frac{\partial\rho}{\partial y^u}\frac{\partial\rho}{\partial y^i}\right).$$

Finally we have

(3.5)
$$\langle\mathcal{N}_L f, h\rangle_{L^2(S\tau'_M \otimes S\tau'_M)} =$$
$$-2\int_M \bar{h}^{k\ell}(x)\frac{1}{\sqrt{\det g(x)}}\int_M \left(g_{ku'}(x) - \frac{\partial\rho}{\partial x^{u'}}\frac{\partial\rho}{\partial x^k}\right)\frac{\partial\rho}{\partial x^\ell}g^{ou'}(x)\left(\mathcal{T}^{0,\rho(x,y)}_{\gamma_{x,-\mathrm{grad}^g_x\rho}}\right)^u_o$$
$$\times\frac{f^{ij}(y)}{\rho(x,y)^2}\left(g_{iu}(y) - \frac{\partial\rho}{\partial y^u}\frac{\partial\rho}{\partial y^i}\right)\frac{\partial\rho}{\partial y^j}\left|\det\frac{\partial^2\rho(x,y)^2/2}{\partial x\partial y}\right|dy\sqrt{\det g(x)}dx.$$

Therefore the normal operator of mixed ray transform can be written as

$$(\mathcal{N}_L f)_{k\ell}(x) = \frac{-2}{\sqrt{\det g(x)}}\int_M \left(g_{ku'}(x) - \frac{\partial\rho}{\partial x^{u'}}\frac{\partial\rho}{\partial x^k}\right)A^{uu'}(x,y)\left(g_{iu}(y) - \frac{\partial\rho}{\partial y^u}\frac{\partial\rho}{\partial y^i}\right)$$
$$\times\frac{f^{ij}(y)}{\rho(x,y)^2}\frac{\partial\rho}{\partial y^j}\frac{\partial\rho}{\partial x^\ell}\left|\det\frac{\partial^2\rho(x,y)^2/2}{\partial x\partial y}\right|dy.$$

Here

(3.6)
$$A(x,y)^{uu'} := g^{ou'}(x)\left(\mathcal{T}^{0,\rho(x,y)}_{\gamma_{x,-\mathrm{grad}^g_x\rho(x,y)}}\right)^u_o,$$

with $A^{uu'}(x,x) = g^{uu'}(x)$. We note here that $\mathcal{T}^{0,\rho(x,y)}_{\gamma_{x,-\mathrm{grad}^g_x\rho(x,y)}}$ is the parallel transport along the geodesic connecting y to x. Thus $\mathcal{N}_L$ is an integral operator with an integral kernel

(3.7)
$$K_{ijk\ell}(x,y) = \frac{-2\left|\det\frac{\partial^2\rho(x,y)^2/2}{\partial x\partial y}\right|}{\sqrt{\det g(x)}\rho(x,y)^2}\left(g_{ku'}(x) - \frac{\partial\rho}{\partial x^{u'}}\frac{\partial\rho}{\partial x^k}\right)A^{uu'}(x,y)$$
$$\times\left(g_{iu}(y) - \frac{\partial\rho}{\partial y^u}\frac{\partial\rho}{\partial y^i}\right)\frac{\partial\rho}{\partial y^j}\frac{\partial\rho}{\partial x^\ell}.$$

3.2. Normal operator as a ΨDO.

From now on, we rely on the fact that $M \subset \mathbf{R}^3$ and g is extended to whole $\mathbf{R}^3$ to apply the theory of pseudo-differential operators. In this subsection we show that the normal operator is a ΨDO of order -1 and find its principal symbol. Since M is closed we consider certain open neighborhoods of it.

Since $(M, g) \subset \mathbf{R}^3$ is simple and g is extended to whole $\mathbf{R}^3$, we can find open domains $M_1, M_2 \subset \mathbf{R}^3$ such that $M \subset M_1 \subset\subset M_2$ and $(\overline{M_i}, g)$, $\{1,2\}$ is simple (see [43, page 454]). We need an open extension of M in order to show that $\mathcal{N}_L$ is a ΨDO. In the next section we explain why we need the aforementioned double extension. We note that by this extension the normal operator $\mathcal{N}_L$ is defined for $1+1$ tensor fields over M_2, and $\mathcal{N}_L f(x)$ remains the same for $x \in M$ if supp $f \subset M$.

First we find more convenient representation for the kernel K near the diagonal of $M_2 \times M_2$. To do so we use the following relations introduced in [43, Lemma 1]:

Lemma 3.2. *There exists $\delta > 0$ such that in $U := \{(x,y) \in M_2 \times M_2 : |x-y|_e < \delta\}$ the following hold*

$$
\rho^2(x,y) = G^{(1)}_{ij}(x,y)(x-y)^i(x-y)^j,
$$

(3.8)
$$
\frac{\partial \rho^2(x,y)}{\partial x^j} = 2G^{(2)}_{ij}(x,y)(x-y)^i,
$$

$$
\frac{\partial^2 \rho^2(x,y)}{\partial x^i \partial y^j} = -2G^{(3)}_{ij}(x,y),
$$

where $G^{(1)}_{ij}(x,y)$, $G^{(2)}_{ij}(x,y)$, $G^{(3)}_{ij}(x,y)$ are smooth and on the diagonal

(3.9)
$$
G^{(1)}_{ij}(x,x) = G^{(2)}_{ij}(x,x) = G^{(3)}_{ij}(x,x) = g_{ij}(x).
$$

Proof. See [44, Lemma 3.1]. □

As $G^{(m)}(x,y)$ is a matrix depending on the points $(x,y) \in U$ we use the short-hand notations $G^{(m)} := G^{(m)}_{ij}(x,y)$, $\tilde{G}^{(2)}_{ij} := G^{(2)}_{ij}(y,x)$, $G^{(m)}z := G^{(m)}_{ij}(x,y)(x-y)^i$, $z := x - y$. These imply

(3.10)
$$
\rho^2(x,y) = G^{(1)}z \cdot z, \quad \frac{\partial \rho}{\partial x^j} = \frac{\left[G^{(2)}z\right]_j}{(G^{(1)}z \cdot z)^{1/2}}, \quad \frac{\partial \rho}{\partial y^j} = \frac{\left[\widetilde{G}^{(2)}z\right]_j}{(G^{(1)}z \cdot z)^{1/2}}.
$$

Thus the following formula holds for the integral kernel of $\mathcal{N}_L$ on the neighborhood U of the diagonal of the extension $M_2 \times M_2$.

$$
K_{ijk\ell}(x,y) = -2\left(g_{iu}(y) - \frac{\left[\tilde{G}^{(2)}z\right]_i\left[\tilde{G}^{(2)}z\right]_u}{G^{(1)}z \cdot z}\right)\frac{A^{uu'}(x,y)}{\left(G^{(1)}z \cdot z\right)^2}
$$

$$
\times \left(g_{ku'}(x) - \frac{\left[G^{(2)}z\right]_k\left[G^{(2)}z\right]_{u'}}{G^{(1)}z \cdot z}\right)\left[\widetilde{G}^{(2)}z\right]_j\left[G^{(2)}z\right]_\ell\frac{\left|\det G^{(3)}\right|}{\sqrt{\det g(x)}}.
$$

From this and (3.7) we see that the integral kernel $K_{ijk\ell}$ is smooth in $M_2 \times M_2$ outside the diagonal, at which it has a singularity of the type $|x-y|_e^{-2}$.

Let $\chi \in C_0^\infty(U)$ equal to 1 near the diagonal of $M_2 \times M_2$. We write

(3.11)
$$
K_{ijk\ell} = \chi K_{ijk\ell} + (1 - \chi K_{ijk\ell}) =: K^1_{ijk\ell} + K^2_{ijk\ell}.
$$

Since $K^2_{ijk\ell} \in C^\infty(M_2 \times M_2)$ the corresponding integral operator is a ΨDO of order $-\infty$, with an amplitude of order $-\infty$.

Lemma 3.3. *Let set U be as in Lemma 3.2. For any $((x,y),z) \in U \times \mathbf{R}^3$ we define*

(3.12)

$$
\widetilde{M}_{ijk\ell}(x,y,z) := -2\left(g_{iu}(y) - \frac{\left[\tilde{G}^{(2)}z\right]_i\left[\tilde{G}^{(2)}z\right]_u}{G^{(1)}z \cdot z}\right)\frac{A^{uu'}(x,y)}{\left(G^{(1)}z \cdot z\right)^2}
$$

$$
\times \left(g_{ku'}(x) - \frac{\left[G^{(2)}z\right]_k\left[G^{(2)}z\right]_{u'}}{G^{(1)}z \cdot z}\right)\left[\widetilde{G}^{(2)}z\right]_j\left[G^{(2)}z\right]_\ell\frac{\left|\det G^{(3)}\right|}{\sqrt{\det g(x)}}.
$$

The distribution $\widetilde{M}_{ijk\ell}$ belongs to $L^1_{loc}(\mathbf{R}^3)$ with respect to z variable and is positively homogeneous of order -2. Moreover $\widetilde{M}_{ijk\ell}$ is smooth in $U \times (\mathbf{R}^3 \setminus \{0\})$.

Proof. We note that equation (3.8) implies that $G^{(2)}z$ and $\widetilde{G}^{(2)}z$ are 1-homogeneous with respect to z. Let K_r be the compact set that is the image of the closed ball $B_r(0)$, $r > 0$ under the diffeomorphism $z' = H^{-1}z$, where H is the square root of $G^{(1)}(x, y)$. By a change to spherical coordinates we obtain

$$\int_{K_r} |\widetilde{M}_{ijk\ell}(x, y, z')|\, \mathrm{d}z' = \int_{B_r} |\widetilde{M}_{ijk\ell}(x, y, H^{-1}z)|\, |\det H^{-1}|\mathrm{d}z$$

$$= rC \int_{S^2} \left| \left(g_{iu}(y) - \left[\widetilde{G}^{(2)}H^{-1}\omega\right]_u \left[\widetilde{G}^{(2)}H^{-1}\omega\right]_i \right) A^{uu'}(x, y) \right.$$

$$\times \left(g_{ku'}(x) - \left[G^{(2)}H^{-1}\omega\right]_{u'} \left[G^{(2)}H^{-1}\omega\right]_k \right)$$

$$\times \left. \left[\widetilde{G}^{(2)}H^{-1}\omega\right]_j \left[G^{(2)}H^{-1}\omega\right]_\ell \right| \mathrm{d}\omega,$$

where $C = 2|\det H^{-1}| \dfrac{|\det G^{(3)}|}{\sqrt{\det g(x)}}$. Since the last integrand is continuous we have proven the first claim.

The second claim follows since $\widetilde{M}_{ijk\ell}(x, y, H^{-1}tz) = t^{-2}\widetilde{M}_{ijk\ell}(x, y, H^{-1}z)$, $t > 0$ implies

$$\int_{\mathbf{R}^3} \widetilde{M}_{ijk\ell}(x, y, z)\varphi(z)\, \mathrm{d}z = t \int_{\mathbf{R}^3} \widetilde{M}_{ijk\ell}(x, y, z)\varphi(tz)\, \mathrm{d}z$$

for every test function φ and $t > 0$. $\qquad\square$

Due to the previous lemma and (3.11) we can write for $(x, y) \in M_2 \times M_2$ that

$$K^1_{ijk\ell}(x, y) = \chi(x, y)\widetilde{M}_{ijk\ell}(x, y, x - y) = \mathcal{F}_\xi^{-1}\left(M_{ijk\ell}(x, y, \xi)\right)\Big|_{x-y},$$

where

$$(3.13) \qquad M_{ijk\ell}(x, y, \xi) = \chi(x, y)\int e^{-i\xi \cdot z}\widetilde{M}_{ijk\ell}(x, y, z)\mathrm{d}z.$$

Therefore $M_{ijk\ell}$ is homogeneous of order -1 in ξ. Since $M_{ijk\ell}$ is smooth in $M_2 \times M_2 \times (\mathbf{R}^3 \setminus \{0\})$ and homogeneous of order -1, it is an amplitude of order -1. Thus the decomposition (3.11) implies that $\mathcal{N}_L$ is a ΨDO of order -1. To conclude the study of the normal operator, we state the main result of this section.

Proposition 3.4. *The normal operator $\mathcal{N}_L$ of mixed ray transform is a pseudo-differential operator of order -1 in M_2 with the principal symbol*

$$\sigma_p(\mathcal{N}_L)^{ijk\ell}(x, \xi)$$

$$= -2\sqrt{\det g(x)} \int_{\mathbf{R}^3} e^{-i\xi \cdot z}\left(g^{ki}(x) - \frac{z^i z^k}{|z|_g^2}\right)\frac{z^j z^\ell}{|z|_g^4}\, \mathrm{d}z$$

$$(3.14) \quad = -\frac{\sqrt{\pi}}{3 \cdot 2^{5/2}}\left(-12g^{ik}|\xi|_g^{-1}(g^{jl} - |\xi|_g^{-2}\xi^j\xi^l)\right.$$

$$+ 9|\xi|_g^{-5}\xi^i\xi^j\xi^k\xi^l + 3|\xi|_g^{-1}(g^{jk}g^{il} + g^{ik}g^{jl} + g^{kl}g^{ij})$$

$$\left. - 3|\xi|_g^{-3}(g^{ij}\xi^k\xi^l + g^{ik}\xi^j\xi^l + g^{il}\xi^j\xi^k + g^{jk}\xi^i\xi^l + g^{jl}\xi^i\xi^k + g^{kl}\xi^i\xi^j)\right).$$

Here $z_i = g_{ij}(x)z^j$ and $|z|_g^2 = g_{ij}(x)z^i z^j$.

Remark 3.5. We note that terms $g^{kl}g^{ij}$ and $g^{kl}\xi^i\xi^j$ in (3.14) do not contribute to the action of the symbol as we are working on the kernel of the map μ. Thus we ignore those two terms in the following calculations.

Proof. We emphasize that $\mathcal{N}_L$ is not properly supported. However there exists properly supported ΨDO $A_{ijk\ell}$ of order -1 such that $(\mathcal{N}_L)_{ijk\ell} - A_{ijk\ell}$ is smoothing. Neglecting this technicality we obtain the principal symbol of $\mathcal{N}_L$ by setting $x = y$ in (3.13). Then formula (3.9) implies

$$G_{ij}^{(2)}(x,x)z^i = z_j, \quad \text{and} \quad G_{ij}^{(1)}(x,x)z^iz^j = |z|_g^2.$$

Therefore after raising indices formulas (3.12) and (3.13) imply the first equation of (3.14).

We proceed on to compute the Fourier transform in (3.14). We recall the following formula for the $n-$dimensional Fourier transform of the powers of a norm given by a positive definite matrix g:

$$\mathcal{F}[|x|_g^\alpha](\xi) = c_{n,\alpha}\frac{1}{\sqrt{\det g}}|\xi|_g^{-\alpha-n}, \quad c_{n,\alpha} = 2^{n/2-\alpha}\frac{\Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}, \quad \alpha \neq n+2k,\ k \in \mathbf{Z}.$$

Here Γ is the Gamma function. In dimension 3 we have

$$\sqrt{\det g}\,\mathcal{F}_x[|x|_g^{-6}](\xi) = 2^{-9/2}\frac{\Gamma(-\frac{3}{2})}{\Gamma(3)}|\xi|_g^3 = \frac{\sqrt{\pi}}{3\cdot 2^{7/2}}|\xi|_g^3,$$

$$\sqrt{\det g}\,\mathcal{F}_x[|x|_g^{-4}](\xi) = 2^{-5/2}\frac{\Gamma(-\frac{1}{2})}{\Gamma(2)}|\xi|_g = -\frac{\sqrt{\pi}}{2^{3/2}}|\xi|_g.$$

Thus the Fourier transform in (3.14) equals to

$$C\left(-12g^{ik}\frac{\partial^2|\xi|_g}{\partial\xi^j\partial\xi^l} + \frac{\partial^4|\xi|_g^3}{\partial\xi^i\partial\xi^j\partial\xi^k\partial\xi^l}\right) =: CN^{ijkl}, \quad C = -\frac{\sqrt{\pi}}{3\cdot 2^{5/2}}.$$

Finally we compute the derivatives in the formula above to find that CN^{ijkl} is the right hand side of equation (3.14). $\qquad\square$

Remark 3.6. If g is a constant coefficient metric, then (3.14) gives the full symbol of the normal operator. The proof is similar to [43, Section 4].

4. RECONSTRUCTION FORMULAS AND STABILITY ESTIMATES

For this section we set the assumption that the metric g is only C^k-smooth in $\mathbf{R}^3$ for some $k \in \mathbf{N}$ that is large enough. Nevertheless we can still assume that the closed set $M \subset \mathbf{R}^3$ is extended to simple open domains (M_1, g) and (M_2, g) such that $M \subset\subset M_1 \subset\subset M_2$.

We begin this section by recalling some basic theory of ΨDOs whose amplitudes are only finitely smooth. We also show that the solution operator of the boundary value problem (2.9) depends continuously on g with respect to C^k-topology. In the second part of this section we show that the normal operator $\mathcal{N}_L$ is elliptic on the subspace of solenoidal tensors on M_1 and construct a parametrix with respect to this subspace. In order to do this we need to have the second extension M_2 as we will use the projection operator $\mathcal{P}_{M_2} = d^{\mathcal{B}}(\Delta_{M_2}^{\mathcal{B}})^{-1}\delta^{\mathcal{B}}$ onto the potential tensors, with respect to the largest domain M_2, in the construction of the parametrix. In addition we define the projection operator $\mathcal{S}_{M_2} := \mathrm{Id} - \mathcal{P}_{M_2}$ onto solenoidal tensor fields on M_2. In the final part of the section we study the stability of the normal

operator. We also provide a reconstruction formula for the solenoidal component, with respect to M.

4.1. Pseudo-differential operators with finitely smooth amplitudes.

Since the metric is assumed to pose only finite smoothness we need to set some finite regularity conditions also for the amplitudes of the ΨDOs we are interested in. We use the notation A^m, $m \in \mathbf{R}$ for the space of C^k-smooth amplitudes of order m in M_2. That is an amplitude $a(x, y, \xi)$, $(x, y, \xi) \in M_2 \times M_2 \times \mathbf{R}^3$ in A^m are set to satisfy only a finite amount of seminorm estimates:

$$|D_\xi^\alpha D_x^\beta D_y^\gamma a(x, y, \xi)| \le C_{\alpha, \beta, \gamma}(\mathcal{K})\, (1 + |\xi|_e)^{m - |\alpha|},$$
$$(x, y) \in \mathcal{K} \subset M_2 \text{ is compact}, \quad C_{\alpha, \beta, \gamma}(\mathcal{K}) > 0,$$

where $\alpha, \beta, \gamma \in \mathbf{N}^3$ are multi-indices that satisfy $|\alpha|, |\beta|, |\gamma| \le k$. We repeat the proof of [48, Theorem 2.1] to observe that for any $m_0, s_0 > 0$ there exists $k \in \mathbf{N}$ such that for any $|m| \le m_0$ and $|s| \le s_0$ the linear operator

$$\mathcal{A} \colon H^s(\overline{M_1}) \to H^{s-m}(\overline{M_1})$$

is bounded if $\mathcal{A}$ is a ΨDO in M_2 with an amplitude in A^m having the regularity k. We also note that the operators with amplitudes in A^m, for any $m \in \mathbf{R}$ are *finitely pseudo-local* in the following sense: If $U \subset M_2$ is open and $u \in \mathcal{E}'(M_2)$, then for any $k' \in \mathbf{N}$ and $m \in \mathbf{R}$ there exists $k \in N$ such that if $\mathcal{A}$ is ΨDO with a C^k-smooth amplitude in A^m then the following holds:

$$\mathcal{A}u \in C^{k'}(U), \quad \text{if } u \in C^{k'}(U).$$

This follows from the proof of [48, Theorem 2.2.] after minor modifications.

Using the aforementioned machinery for finitely smooth amplitudes we note that the integral kernel $K_{ijk\ell}$ of $\mathcal{N}_L$, as given in (3.7), is well defined and depends continuously on the metric g in C^k-topology, if k is large enough. Thus the normal operator $\mathcal{N}_L$ depends continuously on the metric g and moreover the same holds also for the functions $G^{(m)}$ in Lemma 3.2. Thus the claim of Proposition 3.4 is unchanged if g is regular enough and the formula (3.14) shows that also the principal symbol $\sigma(\mathcal{N}_L)$ depends continuously on g in C^k-topology.

We note that the volume form $\mathrm{d}V_g$ depends on the metric tensor g. However for any $t \in \mathbf{R}$ there exists $k \in \mathbf{N}$ such that if we fix a simple C^k-metric g_0 of M, then there exists a C^k-neighborhood U of g_0, consisting of simple metrics, on which we can choose a uniform bi-Lipschitz constant for the norms $\| \cdot \|_{H^t_{g_i}}$, $i \in \{1, 2\}$ for any $g_1, g_2 \in U$. Recall that we are working with $1 + 1$-tensors that are in the kernel $S\tau_M' \otimes^\mathcal{B} S\tau_M'$ of the trace operator μ related to a metric tensor g. To avoid this issue, we introduce operator

$$\kappa_g^\sharp \colon S\tau_M' \otimes^\mathcal{B} S\tau_M' \to S\tau_M \otimes^\mathcal{B} S\tau_M',$$

such that

$$(\kappa_g^\sharp f)_j^i = g^{ai} f_{aj}.$$

Then the following subspace of 1-covariant 1-contravariant tensor fields

$$S\tau_M \otimes^\mathcal{B} S\tau_M' = \{f \in S\tau_M \otimes S\tau_M', f_i^i = 0\}$$

coincides with the image of $\kappa_g^\sharp$, but is defined independent of the metric g. We let $\kappa_g^\flat$ be the inverse of $\kappa_g^\sharp$. The continuity of the maps $\kappa_g^\flat, \kappa_g^\sharp$ with respect to metric g is evident.

Lemma 4.1. *Let $g_0 \in C^1(M)$. There exists a neighborhood $U \subset C^1(M)$ of g_0, such that the map*

$$\kappa_g^{\sharp} \circ (\Delta^{\mathcal{B}})_{M,g}^{-1} \circ \kappa_g^{\flat} \colon H^{-1}(S\tau_M) \to H_0^1(S\tau_M),$$

where $(\Delta^{\mathcal{B}})_{M,g}^{-1}$ is the solution operator of (2.9), with vanishing boundary value, and the projections

$$\kappa_g^{\sharp} \circ \mathcal{P}_M \circ \kappa_g^{\flat}, \; \kappa_g^{\sharp} \circ \mathcal{S}_M \circ \kappa_g^{\flat} \colon L^2(S\tau_M \otimes^{\mathcal{B}} S\tau_M') \to L^2(S\tau_M \otimes^{\mathcal{B}} S\tau_M')$$

depend continuously on $g \in U$.

Proof. We consider first a smooth metric g_0, and note that Lemma 2.3 implies that the solution operator $(\Delta^{\mathcal{B}})_{M,g_0}^{-1} \colon H^{-1}(S\tau_M') \to H_0^1(S\tau_M')$ is bounded. Let $\epsilon > 0$ and g be any smooth metric such that $\|g_0 - g\|_{C^1} < \epsilon$. We write

$$\kappa_g^{\sharp} \circ (\Delta^{\mathcal{B}})_{M,g}^{-1} \circ \kappa_g^{\flat} - \kappa_{g_0}^{\sharp} \circ (\Delta^{\mathcal{B}})_{M,g_0}^{-1} \circ \kappa_{g_0}^{\flat}$$

(4.1)
$$= \left(\kappa_g^{\sharp} \circ (\Delta^{\mathcal{B}})_{M,g}^{-1} \circ \kappa_g^{\flat} \right) \left(\kappa_{g_0}^{\sharp} \circ \Delta^{\mathcal{B}}{}_{M,g_0} \circ \kappa_{g_0}^{\flat} - \kappa_g^{\sharp} \circ \Delta^{\mathcal{B}}{}_{M,g} \circ \kappa_g^{\flat} \right)$$
$$\times \left(\kappa_{g_0}^{\sharp} \circ (\Delta^{\mathcal{B}})_{M,g_0}^{-1} \circ \kappa_{g_0}^{\flat} \right),$$

and choose $u, v \in H_0^1(S\tau_M)$. Then by Cauchy-Schwarz inequality we obtain

$$\left| \left\langle (\kappa_{g_0}^{\sharp} \circ \Delta^{\mathcal{B}}{}_{M,g_0} \circ \kappa_{g_0}^{\flat} - \kappa_g^{\sharp} \circ \Delta^{\mathcal{B}}{}_{M,g} \circ \kappa_g^{\flat}) u, v \right\rangle \right|$$

(4.2)
$$= \left| \langle \mathrm{d}_{g_0}^{\mathcal{B}} \kappa_{g_0}^{\flat} u, \mathrm{d}_{g_0}^{\mathcal{B}} \kappa_{g_0}^{\flat} v \rangle_{L_{g_0}^2} - \langle \mathrm{d}_g^{\mathcal{B}} \kappa_g^{\flat} u, \mathrm{d}_g^{\mathcal{B}} \kappa_g^{\flat} v \rangle_{L_g^2} \right|$$
$$\leq C \|g - g_0\|_{C^1} (\|g\|_{C^1} + \|g_0\|_{C^1}) \|u\|_{H_{g_0}^1} \|v\|_{H_{g_0}^1}.$$

In the last inequality we used the uniform Lipschitz equivalence of the H^1-norms.

If $\varepsilon \ll 1$, then $\|g - g_0\|_{C^1}(\|g\|_{C^1} + \|g_0\|_{C^1}) \leq C\varepsilon$, for some $C > 0$, which can be chosen uniformly whenever g is close enough to g_0. Consequently (4.2) implies

$$\|\kappa_{g_0}^{\sharp} \circ \Delta^{\mathcal{B}}{}_{M,g_0} \circ \kappa_{g_0}^{\flat} - \kappa_g^{\sharp} \circ \Delta^{\mathcal{B}}{}_{M,g} \circ \kappa_g^{\flat}\|_{H^1 \to H^{-1}} \leq C\varepsilon,$$

from which after using (4.1) it follows that

$$\|(\Delta^{\mathcal{B}})_{M,g}^{-1}\|_{H^{-1} \to H^1} \leq \|(\Delta^{\mathcal{B}})_{M,g_0}^{-1}\|_{H^{-1} \to H^1} \left(1 + \varepsilon \|(\Delta^{\mathcal{B}})_{M,g}^{-1}\|_{H^{-1} \to H^1} \right),$$

and moreover

$$\|(\Delta^{\mathcal{B}})_{M,g}^{-1}\|_{H^{-1} \to H^1} \leq \left(1 - C\varepsilon \|(\Delta^{\mathcal{B}})_{M,g_0}^{-1}\|_{H^{-1} \to H^1} \right)^{-1} \|(\Delta^{\mathcal{B}})_{M,g_0}^{-1}\|_{H^{-1} \to H^1}.$$

Finally we use (4.1) again, to conclude

(4.3) $\quad \|\kappa_g^{\sharp} \circ (\Delta^{\mathcal{B}})_{M,g}^{-1} \circ \kappa_g^{\flat} - \kappa_{g_0}^{\sharp} \circ (\Delta^{\mathcal{B}})_{M,g_0}^{-1} \circ \kappa_{g_0}^{\flat}\|_{H^{-1} \to H^1} \leq C_0 \|g - g_0\|_{C^1},$

where $C_0 > 0$ can be chosen uniformly in some small C^1-neighborhood U of g_0. Since smooth metrics are dense in $C^1(M)$, the Lipschitz estimate (4.3) implies that we can define the solution operator $(\Delta^{\mathcal{B}})_{M,g}^{-1}$ as bounded map $H^{-1}(S\tau_M') \to H_0^1(S\tau_M')$ for $g \in C^1(M)$ with the estimate (4.3) still valid. Thus the first claim of the lemma is proven. We note that the continuity of operators $\kappa_g^{\sharp} \circ \mathcal{P} \circ \kappa_g^{\flat}$, $\kappa_g^{\sharp} \circ \mathcal{S} \circ \kappa_g^{\flat}$ follows from this. $\qquad \square$

Lemma 4.1 has the following straightforward generalization.

Corollary 4.2. *Let $t > 0$. There exists $k_0 \in \mathbf{N}$, such that for any $k \geq k_0$ and $g_0 \in C^k(M)$ there exists a neighborhood $U \subset C^k(M)$ of g_0, such that the solution operator*

$$\kappa_g^\sharp \circ (\Delta^\mathcal{B})_{M,g}^{-1} \circ \kappa_g^\flat \colon H^{t-1}(S\tau_M) \to (H^{t+1} \cap H_0^1)(S\tau_M),$$

and the projections

$$\kappa_g^\sharp \circ \mathcal{P}_M \circ \kappa_g^\flat,\ \kappa_g^\sharp \circ \mathcal{S}_M \circ \kappa_g^\flat \colon H^t(S\tau_M \otimes^\mathcal{B} S\tau_M') \to H^t(S\tau_M \otimes^\mathcal{B} S\tau_M')$$

depend continuously on $g \in U$.

4.2. **Reconstruction formula.** We fix a simple metric $g_0 \in C^k(M)$, where $k \in \mathbf{N}$ and consider a simple metric $g \in C^k(M)$ in a small neighborhood of g_0 with respect to C^k-topology. We recall that any tensor field defined in M is extended to any larger domain with the zero extension. Moreover we note that by conjugating all the operators, that are to be used in this section, with $\kappa_g^\sharp$ from left and $\kappa_g^\flat$ from right, we can work in the space of trace-free tensor fields $S\tau_M \otimes^\mathcal{B} S\tau_M'$, that is invariant of any metric structure. To reduce the notations we omit the conjugation from here onwards.

Let $|D|_g$ be a ΨDO with the full symbol $|\xi|_g$. We begin by constructing a parametrix for the ΨDO

(4.4)
$$\mathcal{M}f = \begin{pmatrix} |D|_g \mathcal{N}_L f \\ \mathcal{P}_{M_2} f \end{pmatrix}, \quad \text{in } M_1.$$

We note that the right hand side of (4.4) extends $\mathcal{M}$ in M_2, and due to Corollary 4.2, the operator $\mathcal{M}$ depends continuously on g. In the following we use the notation $\circ$ for a finite asymptotic expansion for the symbol of a product of two ΨDOs. The principal symbol $\sigma(\mathcal{M})$ of $\mathcal{M}$ is given by

$$\sigma(\mathcal{M}) = \begin{pmatrix} |\xi|_g \circ \sigma(\mathcal{N}_L) \\ \sigma(\mathcal{P}_{M_2}) \end{pmatrix}.$$

We show now that $\sigma(\mathcal{M})$ is elliptic near M_1. To do so we lower the (ij)-indices in (3.14) to get:

$$\begin{aligned}
\sigma(\mathcal{N}_L)_{ij}^{kl} = C N_{ij}^{kl} = C\Big(&- 12|\xi|_g^{-1}\delta_i^k\delta_j^l + 12|\xi|_g^{-2}\delta_i^k\xi_j\xi^l \\
&+ 9|\xi|_g^{-5}\xi_i\xi_j\xi^k\xi^l + 3|\xi|_g^{-1}(\delta_j^k\delta_i^l + \delta_i^k\delta_j^l) \\
&- 3|\xi|_g^{-3}(g_{ij}\xi^k\xi^l + \delta_i^k\xi_j\xi^l + \delta_i^l\xi_j\xi^k + \delta_j^k\xi_i\xi^l + \delta_j^l\xi_i\xi^k)\Big).
\end{aligned}$$

Remark 4.3. A straightforward calculation shows that

$$g^{ij}N_{ij}^{kl} = 0, \quad \xi^j N_{ij}^{kl} = 0, \quad \xi^i N_{ij}^{kl} = 12|\xi|_g^{-1}\left(|\xi|_g^{-2}\xi^k\xi_j\xi^l - \xi^k\delta_j^l\right)$$

and

$$N_{ji}^{kl} - N_{ij}^{kl} = 12\left(|\xi|_g^{-1}(\delta_i^k\delta_j^l - \delta_j^k\delta_i^l) - |\xi|_g^{-3}(\delta_i^k\xi_j\xi^l - \delta_j^k\xi_i\xi^l)\right).$$

Therefore

$$|\xi|_g\left[N_{ij}^{kl} + \frac{1}{4}\left(N_{ji}^{kl} - N_{ij}^{kl} - |\xi|_g^{-2}\xi_j\xi^{j'}N_{j'i}^{kl} - |\xi|_g^{-2}\xi_i\xi^{i'}N_{i'j}^{kl}\right)\right]$$
$$= 6|\xi|_g^{-2}\delta_i^k\xi_j\xi^l - 6\delta_i^k\delta_j^l - 3|\xi|_g^{-2}g_{ij}\xi^k\xi^l + 3|\xi|_g^{-4}\xi_i\xi_j\xi^k\xi^l.$$

Motivated by Remark 4.3 we define

$$E_{ij}^{mn} = -\frac{1}{6C}\left(\delta_i^m\delta_j^n + \frac{1}{4}\left(\delta_i^n\delta_j^m - \delta_i^m\delta_j^n - |\xi|_g^{-2}\xi_j\xi^m\delta_i^n - |\xi|_g^{-2}\xi_i\xi^m\delta_j^n\right)\right).$$

Therefore

$$(4.5) \quad E_{ij}^{mn}|\xi|_g\sigma(\mathcal{N}_L)_{mn}^{kl} = \delta_i^k\delta_j^l - \delta_i^k|\xi|_g^{-2}\xi_j\xi^l + \frac{1}{2}|\xi|_g^{-2}g_{ij}\xi^k\xi^l - \frac{1}{2}|\xi|_g^{-4}\xi_i\xi_j\xi^k\xi^l.$$

Thus we need to characterize the remainder term in (4.5). Since $-\Delta^{\mathcal{B}}$ is elliptic near M_1, with principal symbol

$$-\sigma(\Delta^{\mathcal{B}}) = |\xi|_g^2\delta_\ell^i - \frac{1}{3}\xi_\ell\xi^i,$$

it has a parametrix $(\Delta^{\mathcal{B}})^{-1}$ with principal symbol

$$-\sigma((\Delta^{\mathcal{B}})^{-1})_i^k = \frac{1}{2}|\xi|_g^{-4}\xi_i\xi^k + |\xi|_g^{-2}\delta_i^k.$$

We note that near M_1 the parametrix $(\Delta^{\mathcal{B}})^{-1}$ and the solution operator $(\Delta_{M_2}^{\mathcal{B}})^{-1}$ of (2.9), with zero boundary value, coincide up to a finitely smoothing operator. This implies

$$i\sigma((\Delta_{M_2}^{\mathcal{B}})^{-1}\delta^{\mathcal{B}})_i^{kl} = \frac{1}{2}|\xi|_g^{-4}\xi_i\xi^k\xi^l + \delta_i^k|\xi|_g^{-2}\xi^l,$$

and

$$\sigma(\mathcal{P}_{M_2}) = \sigma(\mathrm{d}^{\mathcal{B}}(\Delta_{M_2}^{\mathcal{B}})^{-1}\delta^{\mathcal{B}})_{ij}^{kl} = \frac{1}{2}|\xi|_g^{-4}\xi_i\xi_j\xi^k\xi^\ell + \delta_i^k|\xi|_g^{-2}\xi_j\xi^l - \frac{1}{2}g_{ij}|\xi|_g^{-2}\xi^k\xi^l.$$

Therefore

$$\begin{pmatrix} E & \mathrm{Id} \end{pmatrix} \circ \begin{pmatrix} |\xi|_g \circ \sigma(\mathcal{N}_L) \\ \sigma(\mathcal{P}_{M_2}) \end{pmatrix} = \mathrm{Id},$$

and we have shown that $\mathcal{M}$ is elliptic near M_1.

From now on we study the mapping properties of $\mathcal{M}$. We use the notation $\tilde{\sigma}$ for the full symbol of a ΨDO and S^m for the space of C^k-regular symbols $a(x,\xi)$ of order m. Let $m > 0$ be given. We choose $k \in \mathbf{N}$ to be so large that there exists a ΨDO $\mathcal{A}$ of order -2, that is given by a finite asymptotic expansion of $(\Delta^{\mathcal{B}})^{-1}$ with homogeneous symbols in ξ-variable, and satisfies $\tilde{\sigma}(\mathcal{A}) \circ \tilde{\sigma}(\Delta^{\mathcal{B}}) = \mathrm{Id} \bmod S^{-m}$ near M_1. From here onwards we increase the regularity of the C^k-smooth metric g whenever needed without further mention. We get

$$\tilde{\sigma}(\mathcal{P}_{M_2}) = \tilde{\sigma}(\mathrm{d}^{\mathcal{B}}) \circ \tilde{\sigma}(\mathcal{A}) \circ \tilde{\sigma}(\delta^{\mathcal{B}}) \quad \bmod S^{-m},$$

$$\tilde{\sigma}(\mathcal{S}_{M_2}) = \mathrm{Id} - \tilde{\sigma}(\mathrm{d}^{\mathcal{B}}) \circ \tilde{\sigma}(\mathcal{A}) \circ \tilde{\sigma}(\delta^{\mathcal{B}}) \quad \bmod S^{-m}.$$

Since $\Delta^{\mathcal{B}}$ and $(\Delta^{\mathcal{B}})^{-1}$ depend continuously on g we can also choose $\mathcal{A}$ that depends continuously on g, with respect to C^k-topology, if g is near to g_0. Since $\mathcal{M}$ is elliptic near M_1 there exists a pseudo-differential operator $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2)$ of order 0, with principle symbol (E, Id), such that

$$(4.6) \quad \begin{aligned} & (\tilde{\sigma}(\mathcal{L}_1), \tilde{\sigma}(\mathcal{L}_2)) \circ \begin{pmatrix} |\xi|_g \circ \tilde{\sigma}(\mathcal{N}_L) \\ \tilde{\sigma}(\mathcal{P}_{M_2}) \end{pmatrix} \\ & = \tilde{\sigma}(\mathcal{L}_1) \circ |\xi|_g \circ \tilde{\sigma}(\mathcal{N}_L) + \tilde{\sigma}(\mathcal{L}_2) \circ \tilde{\sigma}(\mathcal{P}_{M_2}) = \mathrm{Id} \quad \bmod S^{-m}, \end{aligned}$$

near M_1. We set two operators

$$\Lambda = \mathrm{Id} - \mathrm{d}^{\mathcal{B}}\mathcal{A}\delta^{\mathcal{B}}, \quad \text{and} \quad \mathcal{T}_1 = \Lambda\mathcal{L}_1|D|_g\Lambda, \quad \text{in } M_2,$$

and note
$$\mathcal{S}_{M_2}\mathcal{N}_L = \mathcal{N}_L\mathcal{S}_{M_2} = \mathcal{N}_L, \quad \mathcal{S}_{M_2}\mathcal{P}_{M_2} = \mathcal{P}_{M_2}\mathcal{S}_{M_2} = 0.$$

Then we apply $\tilde{\sigma}(\Lambda)$ from right and left to (4.6) to obtain
$$\begin{aligned}
\tilde{\sigma}(\Lambda) &= \tilde{\sigma}(\Lambda)^2 = \tilde{\sigma}(\Lambda) \circ \tilde{\sigma}(\mathcal{L}_1) \circ |\xi|_g \circ \tilde{\sigma}(\mathcal{N}_L) \circ \tilde{\sigma}(\Lambda) \\
&= \tilde{\sigma}(\Lambda) \circ \tilde{\sigma}(\mathcal{L}_1) \circ |\xi|_g \circ \tilde{\sigma}(\Lambda) \circ \tilde{\sigma}(\mathcal{N}_L) \\
&= \tilde{\sigma}(\mathcal{T}_1) \circ \tilde{\sigma}(\mathcal{N}_L) \quad \mod S^{-m}.
\end{aligned}$$

We choose $t > 0$ and note that we have shown the existence of a bounded operator $\mathcal{K}_1 \colon L^2(S\tau'_M \otimes^{\mathcal{B}} S\tau'_M) \to H^t(S\tau'_{M_1} \otimes^{\mathcal{B}} S\tau'_{M_1})$, which satisfies
(4.7)
$$\mathcal{T}_1\mathcal{N}_L f = f - \mathrm{d}^{\mathcal{B}}\mathcal{A}\delta^{\mathcal{B}}f + \mathcal{K}_1 f = f^s_{M_1} - \mathrm{d}^{\mathcal{B}}w + \mathcal{K}_1 f, \quad \text{in } M_1 \quad f \in L^2(S\tau'_M \otimes^{\mathcal{B}} S\tau'_M),$$

where $w := \left((\Delta^{\mathcal{B}}_{M_1})^{-1} - \mathcal{A}\right)\delta^{\mathcal{B}}f$. Since $\mathcal{T}_1$, $\mathcal{N}_L$ and $\mathcal{A}$ depend continuously on g, the formula (4.7) implies that also $\mathcal{K}_1$ depends continuously on g.

We conclude this subsection by finding a reconstruction formula for the solenoidal part $f^s_{M_1}$ modulo H^t-regular fields in M_1. To do this we show that the linear map $L^2(S\tau'_M \otimes^{\mathcal{B}} S\tau'_M) \ni f \mapsto \mathrm{d}^{\mathcal{B}}w \in H^{t+1}(S\tau'_M \otimes^{\mathcal{B}} S\tau'_M)$ is bounded and depends continuously in g. First we note that the map $L^2(S\tau'_M \otimes^{\mathcal{B}} S\tau'_M) \ni f \mapsto w \in H^1(S\tau'_{M_1} \otimes^{\mathcal{B}} S\tau'_{M_1})$ is bounded since $\mathcal{A}$ is an operator of order -2. Since f vanishes outside M we have due to the finite pseudo-local property that the distribution $-\mathcal{A}\delta^{\mathcal{B}}f$ near ∂M_1 is of regularity $t+2$. Thus the map
$$L^2(S\tau'_{\partial M} \otimes^{\mathcal{B}} S\tau'_{\partial M}) \ni f \to w|_{\partial M_1} = -\mathcal{A}\delta^{\mathcal{B}}f|_{\partial M_1} \in H^{t+\frac{3}{2}}(S\tau'_{\partial M_1} \otimes^{\mathcal{B}} S\tau'_{\partial M_1})$$

is bounded and depends continuously about the metric g in C^k-topology. This implies that w solves the boundary value problem:
$$\Delta^{\mathcal{B}}w = \left(\mathrm{Id} - \Delta^{\mathcal{B}}\mathcal{A}\right)\delta^{\mathcal{B}}f, \text{ in } M_1, \quad w|_{\partial M_1} = -\mathcal{A}\delta f|_{\partial M_1}.$$

As the symbol of the ΨDO $\left(\mathrm{Id} - \Delta^{\mathcal{B}}\mathcal{A}\right)\delta^{\mathcal{B}}$ is in S^{-m} for $m > 0$ and f is L^2-regular we have that $\left(\mathrm{Id} - \Delta^{\mathcal{B}}\mathcal{A}\right)\delta^{\mathcal{B}}f \in H^t(S\tau'_{M_1} \otimes^{\mathcal{B}} S\tau'_{M_1})$. Corollary 4.2 implies that the map $L^2(S\tau'_M \otimes^{\mathcal{B}} S\tau'_M) \ni f \mapsto w \in H^{t+2}(S\tau'_{M_1} \otimes^{\mathcal{B}} S\tau'_{M_1})$ is bounded.

Therefore we have verified that the map
$$L^2(S\tau'_M \otimes^{\mathcal{B}} S\tau'_M) \ni f \mapsto \mathrm{d}^{\mathcal{B}}w \in H^{t+1}(S\tau'_{M_1} \otimes^{\mathcal{B}} S\tau'_{M_1})$$

is bounded and depends continuously on g with respect to C^k-topology if $k \in \mathbf{N}$ is large enough and g is close to g_0. After setting
$$\mathcal{K}_2 f := -\mathrm{d}^{\mathcal{B}}w + \mathcal{K}_1 f, \quad f \in L^2(S\tau'_M \otimes^{\mathcal{B}} S\tau'_M),$$

equation (4.7) implies the main result of this subsection:

Proposition 4.4. *Let $t > 0$. There exists $k_0 \in \mathbf{N}$ such that for any simple metric $g \in C^k(M), k \geq k_0$ there exists a first order ΨDO, $\mathcal{T}_1$ in M_2 and a bounded operator*
$$\mathcal{K}_2 \colon L^2(S\tau'_M \otimes^{\mathcal{B}} S\tau'_M) \to H^t(S\tau'_{M_1} \otimes^{\mathcal{B}} S\tau'_{M_1}),$$

such that the first reconstruction formula for the solenoidal part is valid:

(4.8) $\mathcal{T}_1\mathcal{N}_L f = f^s_{M_1} + \mathcal{K}_2 f \quad \text{in } M_1, \quad \text{for } f \in L^2(S\tau'_M \otimes^{\mathcal{B}} S\tau'_M).$

Moreover if we fix a simple metric $g_0 \in C^k(M)$, the operators $\mathcal{T}_1$ and $\mathcal{K}_2$ depend continuously about g in some neighborhood of g_0 with respect to C^k-topology.

If g is infinitely smooth, there exists a smoothing operator

$$\widetilde{\mathcal{K}}_2 \colon L^2(S\tau'_M \otimes^{\mathcal{B}} S\tau'_M) \to C^\infty(S\tau'_{M_1} \otimes^{\mathcal{B}} S\tau'_{M_1}),$$

such that

$$\mathcal{T}_1 \mathcal{N}_L f = f^s_{M_1} + \widetilde{\mathcal{K}}_2 f \quad in \ M_1,$$

for any $f \in L^2(S\tau'_M \otimes^{\mathcal{B}} S\tau'_M)$.

4.3. **Stability estimates for the normal operator.** In the previous section we found a reconstruction formula for the solenoidal part $f^s_{M_1}$ with respect to the extended domain M_1. In this section we prove a stability estimate for the normal operator and find a reconstruction formula for the solenoidal part f^s_M. However as it turns out we need higher regularity for f to do so.

Let $g \in C^k(M)$ be a simple metric and $f \in L^2(S\tau'_M \otimes^{\mathcal{B}} S\tau'_M)$. We write $f = f^s_{M_1} + \mathrm{d}^{\mathcal{B}} v_{M_1}$, where v_{M_1} solves a boundary value problem (2.11), on M_1. Since $f = 0$ on $M_1 \setminus M$, the finite pseudo-local property of $(\Delta^{\mathcal{B}}_{M_1})^{-1}\delta^{\mathcal{B}}$ yields $v_{M_1} \in C^1(M_1 \setminus M)$. Moreover we have by (4.8) that

$$(4.9) \qquad -\mathrm{d}^{\mathcal{B}} v_{M_1} = \mathcal{T}_1 \mathcal{N}_L f - \mathcal{K}_2 f \quad in \ M_1 \setminus M.$$

In the following we will find a L^2-estimate for v_{M_1} on $M_1 \setminus M$. Let $x_0 \in \partial M$. Then for any $x \in M_1 \setminus M$ in a small neighborhood U of x_0, we choose a unit vector ξ such that the geodesic $\gamma(t) = \gamma_{x,\xi}(t)$ in $M_1 \setminus M$ issued from x meets ∂M_1 before it meets ∂M. We use the notation $\tau = \tau(x, \xi) > 0$ for the time this geodesic hits ∂M_1. Since v_{M_1} vanishes at ∂M_1 we have as in the proof of Lemma 2.3 that

$$(4.10) \qquad [v_{M_1}(x)]_i \eta^i = -\int_0^\tau [\mathrm{d}^{\mathcal{B}} v_{M_1}(\gamma(t))]_{ij} \eta^i(t) \dot{\gamma}^j(t) \mathrm{d}t,$$

where $\eta(t)$ is a unit length vector field, parallel along γ and $\eta(0) = \eta$ is perpendicular to ξ. The substitution (4.9) and the continuity of the integrand give

$$\left| [v_{M_1}(x)]_i \eta^i \right| \le \int_0^{\tau(x,\xi)} \left| [\mathcal{T}_1 \mathcal{N}_L f - \mathcal{K}_2 f]_{ij} \eta^i(t) \dot{\gamma}^j(t) \right| \mathrm{d}t \le C \left| (\mathcal{T}_1 \mathcal{N}_L f - \mathcal{K}_2 f)(x) \right|_g,$$

where C depends only on the distance to ∂M_1. Perturbing the initial direction ξ, we see that the inequality above holds for linearly independent $\{\eta_{(k)}\}_{k=1}^3$. As $|v_{M_1}(x)|^2_g$ can be estimated by $C \sum_{k=1}^3 \left| [v_{M_1}(x)]_i \eta^i_{(k)} \right|^2$, where the constant C is uniform in a neighborhood of x_0, we get

$$\|v_{M_1}\|_{L^2(M_1 \setminus M)} \le C \|\mathcal{T}_1 \mathcal{N}_L f - \mathcal{K}_2 f\|_{L^2(M_1 \setminus M)},$$

first in U and then globally by using a finite covering for the pre-compact set $M_1 \setminus M$.

Next we estimate the H^1-norm of v_{M_1} in $M_1 \setminus M$. As we can again estimate $|\nabla v_{M_1}|^2_g$ by

$$C \sum_{k,\ell=1}^3 \left| \alpha^j_{(k)} \nabla_j [v_{M_1}]_i \alpha^i_{(\ell)} \right|^2, \quad \{\alpha_{(k)}\}_{k=1}^3 \text{ orthonormal},$$

it is enough to estimate $\alpha^j_{(k)} \nabla_j [v_{M_1}]_i \alpha^i_{(\ell)}$. Recall that

$$\xi^j \nabla_j [v_{M_1}]_i \eta^i = \xi^j \left[\mathrm{d}^{\mathcal{B}} v_{M_1} \right]_{ij} \eta^i,$$

for any η, ξ perpendicular to each other. Then (4.9) implies

$$(4.11) \qquad \left|\xi^j \nabla_j [v_{M_1}]_i \eta^i\right| \le C \left|\mathcal{T}_1 \mathcal{N}_L f - \mathcal{K}_2 f\right|_g,$$

and it remains to estimate $\alpha^i_{(k)} \nabla_j [v_{M_1}]_i \alpha^j_{(k)}$.

We choose $x_0 \in \partial M$, and local coordinates x' on ∂M near x_0. Let (x', x_3) be the boundary normal coordinates given in a neighborhood $U \subset M_1 \setminus M$ of x_0. That is each point $(x', x_3) = x \in U$ is uniquely expressed as $x = \gamma_{(x',0),\nu}(t)$, where ν is the exterior unit normal to ∂M and we have chosen $x_3 = t$ as the third coordinate. We denote $\xi = \dot{\gamma}_{(x',0),\nu}(t)$. For $\eta \in T_x M_1$, that is of unit length and perpendicular to ξ, the formula (4.10), in the given coordinates, has the form

$$(4.12) \qquad [v_{M_1}(x)]_i \eta^i = - \int_{x_3}^{\infty} [\mathrm{d}^{\mathcal{B}} v_{M_1}(\gamma(t))]_{i3} \eta^i(t) \mathrm{d}t.$$

We can replace the exit time by ∞ in upper bound of integration in (4.12) since v_{M_1} has a line integrable zero extension outside M_1.

We denote the coordinate vector fields with respect to x' variables as $\{X_{(k)}\}_{k=1}^2$. Note that these fields are orthogonal to the third coordinate frame $\frac{\mathrm{d}}{\mathrm{d}t} = \dot{\gamma}_{(x',0),\nu}(x_3)$ and η can be given by a linear combination of $\{X_{(k)}\}_{k=1}^2$. We extend η near γ in such a way that $\nabla_{X_{(k)}} \eta = 0$ at $\gamma(t)$. This can be done for instance with parallel transport using Fermi coordinates given by the coordinate frame $\{\frac{\mathrm{d}}{\mathrm{d}t}, X_{(1)}, X_{(2)}\}$ along γ. Then we apply $X_{(k)}$ to both sides of (4.12). Since $v_{M_1} \in H_0^1(M_1)$ we obtain

$$(4.13) \qquad X^j_{(k)} \nabla_j [v_{M_1}(x', x_3)]_i \eta^i = - \int_{x_3}^{\infty} X_{(k)} \left([\mathrm{d}^{\mathcal{B}} v_{M_1}(\gamma(t))]_{i3} \eta^i\right) \mathrm{d}t.$$

Let χ be a smooth cut-off function such that $\chi = 1$ near ∂M and $\chi = 0$ near ∂M_1 and outside M_1. Then $\mathcal{K}_3 \colon f \mapsto (1-\chi) X_{(k)} \mathrm{d}^{\mathcal{B}} v_{M_1}$ is finitely smoothing operator by the fact $f = 0$ in $M_1 \setminus M$ and the finite pseudo local property of the operator $X_{(k)} \mathrm{d}^{\mathcal{B}} (\Delta_{M_1}^{\mathcal{B}})^{-1} \delta^{\mathcal{B}}$. Equations (4.9) and (4.13) imply

$$X^j_{(k)} \nabla_j [v_{M_1}(x', x_3)]_i \eta^i = \int_{x_3}^{\infty} \chi X_{(k)} \left([\mathcal{T}_1 \mathcal{N}_L f]_{i3} \eta^i\right) \mathrm{d}t + \mathcal{K}_4 f,$$

for some $\mathcal{K}_4 \colon L^2(M_1) \to H^t(M_1)$, where $t > 0$ is as in Proposition 4.4. Therefore the continuity implies the existence of C, depending only on the distance to ∂M_1, such that the following pointwise estimate holds:

$$(4.14) \qquad \left|\eta^j \nabla_j [v_{M_1}]_i \eta^i\right| \le C \left(\sum_{k=1}^2 |\chi \nabla_{X_{(k)}} (\mathcal{T}_1 \mathcal{N}_L f)|_g + |\mathcal{K}_4 f|_g\right).$$

It remains to estimate $\xi^j \nabla_j [v_{M_1}(x, x_3)]_i \xi^i$. We take $\tilde{\eta}$ such that $\{\eta, \tilde{\eta}, \xi\}$ form an orthonormal parallel basis along γ. In this basis we can write

$$\mu(\mathrm{d}' v_{M_1}) = \eta^j \nabla_j [v_{M_1}]_i \eta^i + \tilde{\eta}^j \nabla_j [v_{M_1}]_i \tilde{\eta}^i + \xi^j \nabla_j [v_{M_1}]_i \xi^i.$$

Therefore we have

$$\xi^j \left(-\mathrm{d}^{\mathcal{B}} v_{M_1}\right) \xi^i$$
$$= \xi^j \left(-\nabla_j [v_{M_1}]_i + \frac{1}{3} \mu(\mathrm{d}' v_{M_1}) g_{ij}\right) \xi^i$$
$$= -\frac{2}{3} \xi^j \nabla_j [v_{M_1}]_i \xi^i + \frac{1}{3} \left(\eta^j \nabla_j [v_{M_1}]_i \eta^i + \tilde{\eta}^j \nabla_j [v_{M_1}]_i \tilde{\eta}^i\right).$$

By (4.11) and (4.14) we have proved
(4.15)
$$\|v_{M_1}\|_{H^1(U)} \le C \left(\sum_{k=1}^{2} \|\chi \nabla_{X_{(k)}}(\mathcal{T}_1 \mathcal{N}_L f)\|_{L^2(U)} + \|\mathcal{T}_1 \mathcal{N}_L f\|_{L^2(U)} + \|\mathcal{K}_4 f\|_{H^t(U)} \right).$$

To conclude this section we introduce a norm $\tilde{H}^2(M_1)$ originally given in [43, 44] to be implemented in the main result of this section. By shrinking M_1 if necessary we choose a finite open cover $(U_j)_{j=1}^J$ for $M_1 \setminus M$, such that in U_j we have boundary normal coordinates (x'_j, x_j^3), as above. Let $(\chi_j)_{j=1}^J$ be a collection of functions that satisfy $\chi_j \in C_0^\infty(U_j)$, $\chi := \sum_j^J \chi_j$ equals to 1 near ∂M and each χ_j vanishes near ∂M_1. We set

$$\|h\|_{\tilde{H}^1(M_1)}^2 = \int_{M_1} \sum_{j=1}^J \chi_j \left(\sum_{i=1}^2 |\nabla_{X_j^{(k)}} h|_g^2 + |x_j^3 \nabla_{V_j} h|_g^2 \right) + |h|_g^2 \, dx,$$

where V_j is the tangent vector to $\gamma_{(x'_j,0),\nu}(x_j^3)$. We note that here $x_3 > 0$ in $M_1 \setminus M$. The norm $\tilde{H}^2(M_1)$ is then defined by

(4.16)
$$\|h\|_{\tilde{H}^2(M_1)} = \sum_{i=1}^3 \|\nabla_{X_{(k)}} h\|_{\tilde{H}^1(M_1)} + \|h\|_{H^1(M_1)}.$$

Equations (4.9) and (4.15) imply the first estimate

(4.17)
$$\|v_{M_1}\|_{H^1(M_1 \setminus M)} \le C \left(\|\mathcal{T}_1 \mathcal{N}_L f\|_{\tilde{H}^1(M_1)} + \|\mathcal{K}_4 f\|_{H^t(M_1)} \right).$$

Finally we are ready to estimate the solenoidal part f_M^s. We write

$$f = f_{M_1}^s + d^{\mathcal{B}} v_{M_1} \quad \text{in } M_1, \quad f = f_M^s + d^{\mathcal{B}} v_M \quad \text{in } M,$$

and denote $u = v_{M_1} - v_M$. The construction of the potential parts implies

(4.18)
$$\Delta^{\mathcal{B}} u = 0 \quad \text{in } M^{int}, \quad u|_{\partial M} = v_{M_1}|_{\partial M}.$$

By Corollary 4.2, equations (4.9), (4.17) and the trace theorem we obtain the second estimate

$$\|v_{M_1} - v_M\|_{H^1(M)} \le C \|v_{M_1}\|_{H^{1/2}(\partial M)} \le C \|v_{M_1}\|_{H^1(M_1 \setminus M)}$$

$$\le C \left(\|\mathcal{T}_1 \mathcal{N}_L f\|_{\tilde{H}^1(M_1)} + \|\mathcal{K}_4 f\|_{H^t(M_1)} \right).$$

Then we use $f_M^s = f_{M_1}^s + d^{\mathcal{B}}(v_{M_1} - v_M)$, (4.7), and (4.15) to establish our main estimate

(4.19)
$$\|f_M^s\|_{L^2(M)} \le \|\mathcal{T}_1 \mathcal{N}_L f - \mathcal{K}_2 f\|_{L^2(M)} + \|d^{\mathcal{B}}(v_{M_1} - v_M)\|_{L^2(M)}$$

$$\le \|\mathcal{T}_1 \mathcal{N}_L f - \mathcal{K}_2 f\|_{L^2(M)} + \|v_{M_1} - v_M\|_{H^1(M)} + \|(v_{M_1})\|_{H^1(M_1 \setminus M)}$$

$$\le C \left(\|\mathcal{T}_1 \mathcal{N}_L f\|_{\tilde{H}^1(M_1)} + \|\mathcal{T}_1 \mathcal{N}_L f\|_{L^2(M_1)} + \|\mathcal{K}_4 f\|_{H^t(M_1)} \right)$$

$$\le C \left(\|\mathcal{N}_L f\|_{\tilde{H}^2(M_1)} + \|\mathcal{K}_4 f\|_{H^t(M_1)} \right).$$

We note that the last estimate is valid since $\mathcal{T}_1$ is an operator of order 1. Lemma 4.5 guarantees that H^1-regularity for f implies the finiteness of $\|\mathcal{N}_L f\|_{\tilde{H}^2(M_1)}$. This result has been presented earlier in [43].

Lemma 4.5. *If $f \in H^1(S\tau'_M \otimes^\mathcal{B} S\tau'_M)$, then $\|\mathcal{N}_L f\|_{\widetilde{H}^2(M_1)}$ is finite.*

In the following we use the notation $\mathcal{S}$ for the solenoidal projection on M. The main theorem of this section is:

Theorem 4.6. *Let $t > 0$. There exists $k_0 \in \mathbf{N}$ such that for any simple metric $g \in C^k(M)$, $k \geq k_0$, the following claims hold:*

(1) *There exists a bounded linear operator $\widetilde{\mathcal{K}} \colon L^2(S\tau'_M \otimes^\mathcal{B} S\tau'_M) \to H^t(S\tau'_{M_1} \otimes^\mathcal{B} S\tau'_{M_1})$ such that*

$$(4.20) \quad \|f^s_M\|_{L^2(M)} \leq C\left(\|\mathcal{N}_L f\|_{\widetilde{H}^2(M_1)} + \|\widetilde{\mathcal{K}} f\|_{H^t(M_1)}\right), \quad \text{if } f \in H^1(S\tau'_M \otimes^\mathcal{B} S\tau'_M),$$

for some $C > 0$.

(2) *There exist bounded linear operators*

$$\mathcal{Q} \colon \widetilde{H}^2(S\tau'_{M_1} \otimes^\mathcal{B} S\tau'_{M_1}) \to \mathcal{S}\left(L^2(S\tau'_M \otimes^\mathcal{B} S\tau'_M)\right),$$

$$\mathcal{K} \colon L^2(S\tau'_M \otimes^\mathcal{B} S\tau'_M) \to \mathcal{S}\left(H^t(S\tau'_{M_1} \otimes^\mathcal{B} S\tau'_{M_1})\right)$$

such that

$$(4.21) \quad \mathcal{Q}\mathcal{N}_L f = f^s_M + \mathcal{K} f, \quad \text{if } f \in H^1(S\tau'_M \otimes^\mathcal{B} S\tau'_M).$$

Moreover for any simple metric $g_0 \in C^k(M)$, there exists a neighborhood $U \subset C^k(M)$, consisting of simple metrics, such that the operators $\mathcal{Q}$ and $\mathcal{K}$, in (4.21), depend continuously on $g \in U$.

(3) *If $g \in C^\infty(M)$, then the vector space*

$$\ker L \cap \mathcal{S}\left(L^2(S\tau'_M \otimes^\mathcal{B} S\tau'_M)\right) \subset C^\infty(S\tau'_M \otimes^\mathcal{B} S\tau'_M)$$

is finite dimensional.

(4) *If $g \in C^\infty(M)$ and L is s-injective then*

$$\|f^s_M\|_{L^2(M)} \leq C\|\mathcal{N}_L f\|_{\widetilde{H}^2(M_1)}, \quad \text{for } f \in H^1(S\tau'_M \otimes^\mathcal{B} S\tau'_M),$$

for some $C > 0$.

Proof. We note that (4.20) is the same inequality as (4.19), if we set $\widetilde{\mathcal{K}} = \mathcal{K}_4$. The first claim follows from the construction done before this theorem.

Let $f \in H^1(S\tau'_M \otimes^\mathcal{B} S\tau'_M)$. To prove the reconstruction formula (4.21) for f^s_M, we proceed as in the proof of [44, Proposition 5.1]. We fix a simple metric $g_0 \in C^k(M)$ and a neighborhood U of g_0 that consists of simple metrics. During the proof we are implicitly shrinking U and increasing k without further mention. Let $g \in U$. If we vary the initial direction ξ in (4.10), we find three linearly independent $\eta_i \in T_x M$, $i \in \{1, 2, 3\}$ such that the right hand side of (4.10) gives $v_{M_1}(x)$, for $x \in M_1 \setminus M$. Due to the finite pseudo-local property v_{M_1} can be assumed to be C^1-smooth in $M_1 \setminus M$ and contained in $H^t(S\tau'_{M_1 \setminus M})$. Moreover the map $H^1(S\tau'_M \otimes^\mathcal{B} S\tau'_M) \ni f \mapsto v_{M_1} \in H^t(S\tau'_{M_1 \setminus M})$ is bounded and due to Corollary 4.2 it depends continuously on g in C^k-topology if k is large enough. On the other hand we can use formula (4.10) and the trace theorem to define a linear operator

$$\mathcal{T}_2 \colon H^{t-1}(S\tau'_{M_1 \setminus M} \otimes^\mathcal{B} S\tau'_{M_1 \setminus M}) \to H^{t-\frac{1}{2}}(S\tau'_{\partial M}),$$

which depends continuously on g, and satisfies

$$(4.22) \quad \mathrm{Tr}_M\, v_{M_1} = \mathcal{T}_2(\mathrm{d}^\mathcal{B} v_{M_1}) = \mathcal{T}_2\left(\mathcal{T}_1 \mathcal{N}_L - \mathcal{K}_2\right) f.$$

In the last equation we used the substitution (4.9). Thus the right hand side of (4.22) is a bounded map from $H^1(S\tau'_M \otimes^{\mathcal{B}} S\tau'_M)$ into $H^{t-\frac{1}{2}}(S\tau'_{\partial M})$. After converting the problem (4.18) into an elliptic problem with zero boundary value, Corollary 4.2 implies that the solution operator $\mathcal{R}$ of the boundary value problem (4.18), is a bounded operator $\mathcal{R}\colon H^{t-\frac{1}{2}}(S\tau'_{\partial M}) \to H^t(S\tau'_M)$, depending continuously on g in some neighborhood of g_0 with respect to C^k-topology. We get

$$u := v_{M_1} - v_M = \mathcal{R}(\mathrm{Tr}_M(v_{M_1})) = \mathcal{R}\mathcal{T}_2(\mathcal{T}_1\mathcal{N}_L - \mathcal{K}_2)f.$$

Then the first reconstruction formula (4.8) implies

$$\begin{aligned} f_M^s &= f_{M_1}^s + \mathrm{d}^{\mathcal{B}}u \\ &= (\mathcal{T}_1\mathcal{N}_L - \mathcal{K}_2)f + \mathrm{d}^{\mathcal{B}}\mathcal{R}\mathcal{T}_2(\mathcal{T}_1\mathcal{N}_L - \mathcal{K}_2)f \\ &= (\mathrm{Id} + \mathrm{d}^{\mathcal{B}}\mathcal{R}\mathcal{T}_2)\mathcal{T}_1\mathcal{N}_L f + \mathcal{K}_5 f, \end{aligned}$$

where $\mathcal{K}_5 = -(\mathrm{Id} + \mathrm{d}^{\mathcal{B}}\mathcal{R}\mathcal{T}_2)\mathcal{K}_2$. We conclude the proof of (4.21) by setting $\mathcal{Q} := \mathcal{S}(\mathrm{Id} + \mathrm{d}^{\mathcal{B}}\mathcal{R}\mathcal{T}_2)\mathcal{T}_1$ and $\mathcal{K} := \mathcal{S}\mathcal{K}_5$. We emphasize that by Theorem 1.1 the solenoidal projection $\mathcal{S}\colon H^t(S\tau'_M \otimes^{\mathcal{B}} S\tau'_M) \to H^t(S\tau'_M \otimes^{\mathcal{B}} S\tau'_M)$ is bounded, and due to Corollary 4.2 the operators $\mathcal{Q}$ and $\mathcal{K}$ depend continuously on the metric g in some small neighborhood of a fixed simple metric $g_0 \in C^k(M)$ in C^k-topology, for $k \in \mathbf{N}$ large enough.

The remaining parts of the theorem can be proven as in [43, Theorem 2]. $\square$

5. S-INJECTIVITY FOR ANALYTIC METRICS

5.1. **The analytic parametrix.** In this section we assume that (M, g) is a simple manifold with a (real) analytic metric g on M up to the boundary. As in Section 4 we extend M and g to simple open domains (M_1, g), (M_2, g) and $M \subset\subset M_1 \subset\subset M_2 \subset \mathbf{R}^3$. We note that this can be done in such a way that, g is analytic in a neighborhood of M_2 and M_i, $i \in \{1, 2\}$ have analytic boundaries. Since analytic functions are dense this does not require the original boundary ∂M to be analytic (see [44, Section 3]).

We construct an analytic parametrix for operator $\mathcal{M}$. We denote the set of analytic tensor fields on M by $\mathbb{A}(M)$. That is every $f \in \mathbb{A}(M)$ has an analytic extension to some open domain containing M. For the basic theory of analytic ΨDO we refer to [48, Chapter V]. Recall that a continuous linear operator from $\mathcal{E}'(M)$ to $\mathcal{D}'(M)$ is analytic regularizing, if its range is contained in $\mathbb{A}(M)$.

Our first result in this section is a re-formulation of Proposition 3.4 in the analytic setting.

Proposition 5.1. *The operators $\mathcal{N}_L$ and $\mathcal{M}$, from (4.4), are analytic ΨDOs in M_2.*

Proof. Our proof follows the proof of [44, Proposition 3.2].

Since g is analytic in M_2, there exists $\delta > 0$ such that the operator A defined by (3.6) and functions $G^{(m)}$, $m \in \{1, 2, 3\}$ from (3.8) are analytic in $U = \{(x, y) \in M_2 \times M_2; |x - y|_e < \delta\}$. Let V be an open set such that $V \times V \subset U$. Then $\widetilde{M}_{ijk\ell}$, given in (3.12), is analytic in $V \times V \times (\mathbf{R}^3 \setminus \{0\})$, and due to Lemma 3.3 distribution $\widetilde{M}_{ijk\ell}$ is positively homogeneous of order -2 in z variable. Here we use the fact that A is analytic in U, since the solution to an ODE with analytic coefficients is analytic.

Thus $M_{ijk\ell}(x, y, \xi)$ is analytic in $V \times V \times (\mathbf{R}^3 \setminus \{0\})$ as a Fourier transform of $\widetilde{M}_{ijk\ell}$ in z variable. To see this, one only need to notice that $\widetilde{M}(x, y, z)$ is even in z, and [26, Theorem 7.1.24] implies

(5.1)
$$M_{ijk\ell}(x, y, \xi) = \int_{\mathbf{R}^3} e^{-i\xi \cdot z} \widetilde{M}_{ijk\ell}(x, y, z)\, \mathrm{dz} = \pi \int_{\mathbb{S}^2} \widetilde{M}_{ijk\ell}(x, y, \omega) \delta_0(\omega \cdot \xi)\, \mathrm{d}\omega,$$

where $\widetilde{M}$, in the last integrand, is an analytic function of all its variables.

To prove that $M(x, y, \xi)$ is an analytic amplitude (see [48, V, Definition 2.2]) in $V \times V$ we proceed as follows. Let $\phi \in C_0^\infty(\mathbf{R}^3)$. We write

$$M_{ijk\ell}(x, y, \xi) = \phi(\xi) M_{ijk\ell}(x, y, \xi) + (1 - \phi(\xi)) M_{ijk\ell}(x, y, \xi),$$

and show that the first term is an amplitude of analytic regularizing operator and the second one is an analytic amplitude. This shows that $\mathcal{N}_L$ is an analytic ΨDO in V.

To prove that the operator of $\phi(\xi) M_{ijk\ell}(x, y, \xi)$ is analytic regularizing, we need to show that the corresponding integral kernel

$$\mathcal{H}_{ijk\ell}(x, y) = (2\pi)^{-3} \int_{\mathbf{R}^3} e^{i\xi \cdot (x-y)} \phi(\xi) M_{ijk\ell}(x, y, \xi)\, \mathrm{d}\xi, \quad (x, y) \in V \times V,$$

is analytic. To do this we use the fact that $M_{ijk\ell}(x, y, \xi)$ is positively homogeneous of order -1 in ξ. Then a change to spherical coordinates gives

$$\mathcal{H}_{ijk\ell}(x, y) = (2\pi)^{-3} \int_{\mathbb{S}^2} \int_0^\infty e^{i(r\omega) \cdot (x-y)} \phi(r\omega) M_{ijk\ell}(x, y, \omega) r\, \mathrm{d}r \mathrm{d}\omega.$$

Since a product of analytic functions is analytic and ϕ is compactly supported, this proves that $\mathcal{H}_{ijk\ell}$ is analytic.

Let $R_0 > 0$ be a radius of a ball containing the support of ϕ. Since $F(x, y, \xi) := (1 - \phi(\xi)) M_{ijk\ell}(x, y, \xi)$ equals to $M_{ijk\ell}(x, y, \xi)$, when $|\xi|_e > R_0$ and $M_{ijk\ell}(x, y, \xi)$ is homogeneous of order -1 in ξ, we can write

(5.2)
$$F(x, y, \xi) = M_{ijk\ell}\left(x, y, \frac{\xi}{|\xi|_e}\right) |\xi|_e^{-1}, \quad |\xi|_e > R_0.$$

Therefore we can use the right hand side of (5.2) to extend F analytically on $V^{\mathbf{C}} \times V^{\mathbf{C}} \times (\mathbf{C}^3 \setminus \overline{B}(0, R_0))$. Here $V^{\mathbf{C}}$ is an extension of V to $\mathbf{C}^3$. This implies that for any compact $K \subset V^{\mathbf{C}} \times V^{\mathbf{C}}$ there exist $C > 0$ such that

$$|F(x, y, z)| \leq C|\xi|_e^{-1}, \qquad (x, y) \in K, \ |\xi|_e > R_0.$$

We choose $R > 0$ so large that $\widetilde{B}_R(\xi) := \prod_{i=1}^3 B_{i,R}(\xi) \subset (\mathbf{C}^3 \setminus B_{R_0}(0))$ if $\xi \in \mathbf{R}^3$, $|\xi|_e > R$ and $B_{i,R}(\xi) := \{z \in \mathbf{C} : |z - \xi_i|_e \leq \frac{1}{2R}|\xi|_e\}$. Then we apply Cauchy's integral formula on $\widetilde{B}_R(\xi)$ to find

$$|D_\xi^\alpha F(x, y, \xi)| \leq \alpha! \left(\frac{C}{2R}\right)^{|\alpha|} |\xi|_e^{-|\alpha|-1}, \quad (x, y) \in K, \ |\xi|_e > R.$$

Therefore F is an analytic amplitude. Since $V \subset U$ was arbitrarily chosen, we have proven that for any $x_0 \in M_2^{int}$ there exists a neighborhood V_{x_0} contained in M_2^{int} in which $\mathcal{N}_L$ is an analytic ΨDO. From here we can follow the lines in the proof of [44, Proposition 3.2] to conclude that $\mathcal{N}_L$ is actually an analytic ΨDO in the whole M_2.

Next we proceed to prove that $\mathcal{P}_{M_2} = \delta^{\mathcal{B}}(\Delta^{\mathcal{B}}_{M_2})^{-1}\mathrm{d}^{\mathcal{B}}$ is an analytic ΨDO. Formula (2.8) implies that $\mathrm{d}^{\mathcal{B}}$ is an analytic operator. Also $\delta^{\mathcal{B}}$ is analytic. Therefore $\Delta^{\mathcal{B}}$ is an elliptic analytic operator in the domain Ω. Thus there exists an analytic parametrix T of $\Delta^{\mathcal{B}}$ in an open set M' containing $\overline{M_2}$. We need to show that $(\Delta^{\mathcal{B}}_{M_2})^{-1} - T$ is analytic regularizing on M_2. Let the distribution f be supported in M_2. We set $w := ((\Delta^{\mathcal{B}}_{M_2})^{-1} - T)f$. Then $\Delta^{\mathcal{B}}w \in \mathbb{A}(M_2)$ and the interior analytic regularity implies $w \in \mathbb{A}(M_2)$, and we have proven that $(\Delta^{\mathcal{B}}_{M_2})^{-1}$ is an analytic ΨDO in M_2.

Since $|D|_g$ is analytic, we have proven that operator $\mathcal{M}$ in (4.4) is analytic. $\quad\square$

In Lemma 5.2 we extend the result of Lemma 2.3 to an analytic case. The proof is similar to [44, Lemma 3.3].

Lemma 5.2. *Let $x_0 \in \partial M$, and assume that the metric g and the tensor fields u and v_0 are analytic in a (two-sided) neighborhood of x_0 and that ∂M is analytic near x_0. Let tensor field v solve*

$$(5.3) \qquad \Delta^{\mathcal{B}}v = u \quad in \ M, \quad v|_{\partial M} = v_0.$$

Then v extends as an analytic function in some (two-sided) neighborhood of x_0.

The main result of this subsection is the following.

Proposition 5.3. *There exists a bounded operator $\mathcal{W} : H^1(S\tau'_{M_1} \otimes^{\mathcal{B}} S\tau'_{M_1}) \to L^2(S\tau'_{M_1} \otimes^{\mathcal{B}} S\tau'_{M_1})$ such that for any 2-tensor $f \in L^2(S\tau'_M \otimes^{\mathcal{B}} S\tau'_M)$ we have*

$$f^s_{M_1} = \mathcal{W}\mathcal{N}_L f + \mathcal{K}f$$

with $\mathcal{K}f$ analytic in M_1.

Proof. Since $\mathcal{M}$ is an elliptic analytic ΨDO in M_1 we can construct a parametrix $\mathcal{L} = \{\mathcal{L}_1, \mathcal{L}_2\}$ of $\mathcal{M}$ in M_1 such that

$$(5.4) \qquad \mathcal{L}\mathcal{M} = \mathrm{Id} + \mathcal{K}_1,$$

with $\mathcal{L}$ an analytic ΨDO of order 0 in a neighborhood of M_1, and $\mathcal{K}_1$ analytically regularizing in M_1. Apply S_{M_2} to the left and right, to (5.4) and notice $\mathcal{N}_L S_{M_2} = S_{M_2}\mathcal{N}_L = \mathcal{N}_L$, $\mathcal{P}_{M_2}S_{M_2} = 0$. We denote $\mathcal{W} = S_{M_2}\mathcal{L}_1|D|_g$, and have

$$\mathcal{W}\mathcal{N}_L = S_{M_2} + \mathcal{K}_2 \quad in \ M_1.$$

Here $\mathcal{K}_2$ is analytic regularizing in M_1.

We need to compare $f^s_{M_1}$ and $f^s_{M_2}$ for $f \in L^2(S\tau'_M \otimes^{\mathcal{B}} S\tau'_M)$. We write $f^s_{M_i} = f - \mathrm{d}^{\mathcal{B}}v_{M_i}$. As in the previous section, we have $f^s_{M_1} = f^s_{M_2} + \mathrm{d}^{\mathcal{B}}u$ in M_1 for $u = v_{M_2} - v_{M_1}$. Then $u \in H^1(M_1)$ and solves

$$\Delta^{\mathcal{B}}u = 0 \quad in \ M_1, \quad u|_{\partial M_1} = v_{M_2}.$$

We note that since ∂M_1 is analytic we have $\mathrm{Tr}_{M_1}h \in \mathbb{A}(\partial M_1)$ for any h analytic near ∂M_1. As $\mathrm{supp}f$ is disjoint from ∂M_1, analytic pseudo-locality yields $v_{M_2} \in \mathbb{A}(\partial M_1)$. By Lemma 5.2, $u \in \mathbb{A}(M_1)$; thus $f \mapsto \mathrm{d}^{\mathcal{B}}u$ is a linear operator mapping $L^2(M)$ into $\mathbb{A}(M_1)$. Then we use the relation

$$f^s_{M_1} = f^s_{M_2} + \mathrm{d}^{\mathcal{B}}u = \mathcal{W}\mathcal{N}f - \mathcal{K}_2 f + \mathrm{d}^{\mathcal{B}}u$$

to complete the proof. $\quad\square$

5.2. S-injectivity of $1+1$ tensors for analytic metrics.
In this subsection we will prove s-injectivity for an analytic simple metric g.

Lemma 5.4. *Let g be a smooth, simple metric in M and let $f \in C^\infty(S\tau_M' \otimes^{\mathcal{B}} S\tau_M')$. If $Lf = 0$, then there exists a tensor field v vanishing on ∂M such that for $\tilde{f} = f - \mathrm{d}^{\mathcal{B}}v$ we have $\mathrm{Tr}_M(\partial^m \tilde{f}) = 0$ for any multi-index $m \in \mathbf{N}^3$.*

Moreover, if g and f are analytic in a (two-sided) neighborhood of ∂M, and ∂M is also analytic, then v can be chosen so that $\tilde{f} = 0$ near ∂M.

Proof. We fix $x_0 \in \partial M$ and take boundary normal coordinates $x = (x', x^3)$ in a neighborhood $U \subset M$ of x_0. Then in U we have $g_{i3} = \delta_{i3}$ for any $i = 1, 2, 3$. We aim first to find a tensor field v, vanishing on ∂M, such that for $\tilde{f} := f - \mathrm{d}^{\mathcal{B}}v$ we have

$$(5.5) \qquad \tilde{f}_{i3} = 0, \quad \text{in some open neighborhood } \widetilde{U} \subset U \text{ of } x_0.$$

Due to (2.8), this is equivalent to

$$(5.6) \quad f_{13} - \nabla_3 v_1 = 0, \quad f_{23} - \nabla_3 v_2 = 0, \quad f_{33} - \nabla_3 v_3 + \frac{1}{3}(g^{k\ell}\nabla_k v_\ell) = 0, \quad \text{in } U$$
$$v|_{x^3=0} = 0.$$

Remember that $\nabla_j v_i = \partial_j v_i - \Gamma_{ji}^k v_k$, and the Christoffel symbols in the boundary normal coordinates, satisfy $\Gamma_{33}^k = \Gamma_{k3}^3 = \Gamma_{3k}^3 = 0$. We first solve the system of initial value problems

$$\begin{cases} \partial_3 v_1 - \Gamma_{31}^1 v_1 - \Gamma_{31}^2 v_2 = \nabla_3 v_1 = f_{13}, \\ \partial_3 v_2 - \Gamma_{32}^1 v_1 - \Gamma_{32}^2 v_2 = \nabla_3 v_2 = f_{23}, \\ v_1(x', 0) = 0, \quad v_2(x', 0) = 0, \end{cases}$$

for v_1 and v_2, which are given along boundary normal geodesics $\gamma_{(x',0),\nu}(x_3)$. Then using $g^{i3} = \delta^{i3}$ we write the last equation of (5.6) in a form of the following initial value problem

$$\partial_3 v_3 = \frac{3}{2}(f_{33} - G), \quad v_3(x', 0) = 0,$$

where G depends only on v_i, $\partial_j v_i$, g^{jk}, $\Gamma_{j\ell}^k$ for $i \in \{1, 2\}$. We have found v near the boundary. Clearly, if g and f are analytic near ∂M, so is v.

We note that the convexity of the boundary implies that for $(x, \xi) \in \partial_+(SM)$ where $x \in \partial M \cap U$, is close to x_0, $|\xi|_g = 1$ and the normal component of ξ is small enough, the geodesic issued from (x, ξ) hits the boundary again in U. Then the boundary value of v and $Lf = 0$ imply $L\tilde{f}(x, \xi) = 0$. We choose the boundary coordinates x' such that $g_{ij}|_{x=x_0} = \delta_{ij}$. We claim

$$(5.7) \qquad \tilde{f}_{3\alpha}|_{x=x_0} = 0, \quad \tilde{f}_{\alpha\beta}|_{x=x_0} = 0, \quad (\tilde{f}_{11} - \tilde{f}_{22})|_{x=x_0} = 0,$$

for $\alpha = 1, 2$, $\beta = 1, 2$ and $\alpha \neq \beta$. If this is true, then (5.5) and $\sum_{i=1}^3 \tilde{f}_{ii}(x_0) = \mu\tilde{f}(x_0) = 0$, give $\tilde{f}(x_0) = 0$. To prove (5.7) we let $\xi \in T_{x_0}\partial M$, $|\xi|_g = 1$, and take a curve $\delta : (-\epsilon, \epsilon) \to \partial M$ adapted to (x_0, ξ). Let $\gamma = \gamma_\epsilon : [0, 1] \to M$ be the shortest geodesic of the metric g joining the points x_0 and $\delta(\epsilon)$, i.e., $\gamma(0) = x_0$ and $\gamma(1) = \delta(\epsilon)$. Let $\eta \in T_{x_0}M$ be perpendicular to ξ, and η_ϵ be the orthogonal projection of η to $\dot{\gamma}_\epsilon(0)$. We also set $\eta(t) = \eta_\epsilon(t)$ to be the parallel transport of η_ϵ

along γ_ϵ. Since the points $(\gamma(t), \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|_g})$ and $(\gamma(t), \eta(t))$ tend to (x_0, ξ) and (x_0, η), respectively, uniformly for $t \in [0,1]$ as $\epsilon \to 0$, we have

$$\eta^k \tilde{f}_{kj} \xi^j = \lim_{\epsilon \to 0} \int_0^1 \eta^k(t) \tilde{f}_{kj}(\gamma(t)) \frac{\dot{\gamma}^j(t)}{|\dot{\gamma}(t)|_g} dt = \lim_{\epsilon \to 0} \frac{1}{|\dot{\gamma}_\epsilon(0)|_g} \left\langle L\tilde{f}(x_0, \dot{\gamma}_\epsilon(0)), \eta_\epsilon \right\rangle = 0.$$

We set $e_3 = \nu(x_0)$, and $e_\alpha = \frac{\partial}{\partial x'^\alpha}|_{x_0}$ for $\alpha = 1, 2$. The previous equation implies

$$\tilde{f}_{m\alpha} = 0, \quad m \in \{1,2,3\}, \, \alpha \in \{1,2\}, \, m \neq \alpha.$$

To obtain the last equation in (5.7) we set $\xi = \frac{1}{\sqrt{2}}(e_1 + e_2)$, $\eta = \frac{1}{\sqrt{2}}(e_1 - e_2)$, thus

$$\frac{1}{2}(f_{11} - f_{22}) = \xi^i \tilde{f}_{ij} \eta^j = 0.$$

This completes the proof of (5.7). Since x_0 was an arbitrary point in $\partial M \cap U$ we have shown that $\tilde{f}$ vanishes at $\partial M \cap U$. It remains to show

(5.8) $$\partial_{x_3}^m \tilde{f}_{ij}|_{x=x_0} = 0, \quad m = 1, 2, \cdots$$

We do not prove this directly but note that, if
(5.9)
$$\partial_{x_3}^m \tilde{f}_{3\alpha}|_{x=x_0} = 0, \quad \partial_{x_3}^m \tilde{f}_{\alpha\beta}|_{x=x_0} = 0, \quad \partial_{x_3}^m (\tilde{f}_{11} - \tilde{f}_{22})|_{x=x_0} = 0, \quad \alpha, \beta \in \{1,2\}, \, \alpha \neq \beta$$

holds, then due to

$$\sum_{i=1}^3 \partial_{x_3} \tilde{f}_{ii}(x_0) = \mu \left(\partial_{x_3} \tilde{f} \right) \bigg|_{x=x_0} = (\partial_{x_3} \mu \tilde{f}) \bigg|_{x=x_0} = 0$$

(5.8) also holds. The equation above holds since the trace and the covariant derivative commute and Christoffel symbols vanish at x_0.

The proof of (5.9) is similar to the proof of [29, Theorem 2.1]. We give it here for the sake of completeness. Let $m > 0$ be the smallest integer for which (5.9) does not hold. We consider a 2-tensor $h_{ij} := \partial_{x_3}^m \tilde{f}_{ij}|_{x=x_0}$ acting on $T_{x_0} M$. Since (5.9) does not hold for m, there exists $\xi_0 \in T_{x_0} M$ of unit length, tangent to ∂M, and $\eta_0 \in T_{x_0} M$ that is perpendicular to ξ_0, such that $\eta_0^i h_{ij} \xi_0^j \neq 0$. Then the Taylor expansion of $\tilde{f}$ implies that $\eta^i \tilde{f}_{ij} \xi^j$ is either (strictly) positive or negative for $x^3 > 0$ and $|x' - x_0'|_e$ both sufficiently small and (ξ, η) close to (ξ_0, η_0). Therefore, $\langle L\tilde{f}(x, \xi), \eta \rangle$ is either (strictly) positive or negative for all $(x, \xi) \in \partial_+(SM)$ close enough to (x_0, ξ_0) and $\eta \perp \xi$ close to η_0. This is a contradiction.

We have completed the construction of v near x_0. As in the proof of [44, Lemma 4.1], we can extend the construction of v anywhere near ∂M.

If g and f are analytic, then v is analytic near ∂M. Then $\tilde{f}$ is analytic and thus $\tilde{f} = 0$ near ∂M. $\qquad \square$

In Theorem 5.5 we use global semi-geodesic coordinates for simple manifold (M_1, g), introduced in [44, Lemma 4.2], under which the metric g has the global representation

$$g_{i3} = \delta_{i3}, \quad i = 1, 2, 3.$$

We use the notations e_i, $i \in \{1,2,3\}$ for the corresponding coordinate vector fields.

Theorem 5.5. *Let g be a simple metric in M, that has an analytic extension. Then L is s-injective.*

Proof. Assume that $g \in \mathbb{A}(M)$, and the mixed ray transform of $f \in L^2(S\tau'_M \otimes^{\mathcal{B}} S\tau'_M)$ vanishes. Then, by Proposition 5.3, we have $f^s_{M_1} \in \mathbb{A}(M_1)$. Clearly, $Lf^s_{M_1} = 0$ as well.

By Lemma 5.4, there exists a smooth tensor field w that is analytic near ∂M and moreover $\tilde{f} := f^s_{M_1} - \mathrm{d}^{\mathcal{B}}w$ vanishes near ∂M_1. We denote the set of all points $x \in \partial M_1$, for which the coordinate vector field $e_3(x) \in \partial_{\pm}(SM_1)$, by $(\partial M_1)_{\pm}$. We aim first to find a second tensor field v that satisfies the following global equation

$$(\tilde{f} - \mathrm{d}^{\mathcal{B}}v)_{i3} = 0, \quad i \in \{1,2,3\}, \quad v|_{(\partial M_1)_+} = 0.$$

That is we solve the equations similar to (5.6):

(5.10)

$$\tilde{f}_{13} - \nabla_3 v_1 = 0, \quad \tilde{f}_{23} - \nabla_3 v_2 = 0, \quad \tilde{f}_{33} - \nabla_3 v_3 + \frac{1}{3}(g^{k\ell}\nabla_k v_\ell) = 0, \quad \text{in } M_1$$

$$v|_{(\partial M_1)_+} = 0.$$

Since (M, g) is simple it follows from the definition of the semi-geodesic coordinates that each point in M can be reached by a geodesic parallel to e_3 from a unique point of $(\partial M)_+$. Therefore the system (5.10) can be used to define v globally. As before, we first determine v_1 and v_2 from the system of linear boundary value problems

$$\begin{cases} \partial_3 v_1 - \Gamma^1_{31}v_1 - \Gamma^2_{31}v_2 = \tilde{f}_{13}, \quad \partial_3 v_2 - \Gamma^1_{32}v_1 - \Gamma^2_{32}v_2 = \tilde{f}_{23} \\ v_1|_{(\partial M_1)_+} = v_2|_{(\partial M_1)_+} = 0. \end{cases}$$

We note that this system has a unique global solution since M is compact. Due to analyticity, v_1, v_2 vanish in a neighborhood U of $\overline{(\partial M_1)_+}$. The last equation of (5.10) takes the form of the following boundary value problem

$$\partial_3 v_3 = \frac{3}{2}\left(\tilde{f}_{33} - G\right), \quad v_3|_{(\partial M_1)_+} = 0,$$

where G depends only on v_i, $\partial_j v_i$, g^{jk}, $\Gamma^k_{j\ell}$ for $i \in \{1,2\}$. Thus we have found v and shown that it vanishes in U.

Now we define $f^{\sharp} := f^s_{M_1} - \mathrm{d}^{\mathcal{B}}w - \mathrm{d}^{\mathcal{B}}v$. We have $f^{\sharp} = 0$ in U and $f^{\sharp}_{i3} = 0$ in M_1, $i = 1, 2, 3$. Moreover, $w + v = 0$ on $(\partial M_1)_+$. On the other hand, there is a unique $v^{\sharp} \in C(M_1)$ that solves (5.10) with $\tilde{f}$ replaced by $f^s_{M_1}$, and $v^{\sharp} = 0$ on $(\partial M_1)_+$. Therefore $f^{\sharp} = f^s_{M_1} - \mathrm{d}^{\mathcal{B}}v^{\sharp}$, with $v^{\sharp} = w + v$. Since the coefficients in the system (5.10) are analytic, and so are $f^s_{M_1}$ and ∂M_1, tensor field $v^{\sharp}$ is analytic in $M_1 \setminus E$, where $E \subset \partial M_1$ is the set where e_3 is tangential to ∂M. Thus $f^{\sharp}$ is analytic in $M_1 \setminus E$. Due to the fact that $f^{\sharp} = 0$ in U containing E, and by analytic continuation, $f^{\sharp} = 0$ in M_1.

We have proven $f^s_{M_1} = \mathrm{d}^{\mathcal{B}}v^{\sharp}$ in M_1, and $v^{\sharp} = 0$ on $(\partial M_1)_+$. Soon we show that $v^{\sharp} = 0$ also on the complement of $(\partial M_1)_+$. If this holds, then we have

$$\Delta^{\mathcal{B}}v^{\sharp} = \delta^{\mathcal{B}}f^s_{M_1} = 0, \quad \text{in } M_1, \quad v^{\sharp}|_{\partial M_1} = 0.$$

Thus Lemma 2.3 implies $v^{\sharp} = 0$, and $f^s_{M_1} = 0$. To prove that $v^{\sharp} = 0$ on ∂M_1 we proceed as follows: Let $x \in (\partial M_1)_+$ and $y \in (\partial M_1 \setminus (\partial M_1)_+)$. Since (M_1, g) is simple there exists a unique geodesic γ connecting y to x. Let $\eta \in T_y M_1$ be perpendicular to $\dot{\gamma}(0)$. Since $v^{\sharp} = 0$ in $(\partial M_1)_+$ and $Lf^s_{M_1} = 0$ we have the following equation by (4.10)

$$v^{\sharp}_i \eta^i = \langle L\mathrm{d}^{\mathcal{B}}v^{\sharp}, \eta \rangle = \langle Lf^s_{M_1}, \eta \rangle = 0.$$

Perturbing x in the open set $(\partial M_1)_+ \subset \partial M$ we can show that the previous equation holds for any η in the linearly independent set $(\eta_i)_{i=1}^3$. Thus $v^\sharp(y) = 0$.

So far we have shown $f = d^\mathcal{B} v_{M_1}$. Since supp $f \subset M$, we have that supp $v_{M_1} \subset M$. Therefore $v_{M_1} = v_M$, by Lemma 5.6 that is proven analogously to [44, Proposition 4.3]. This gives $f_M^s = 0$ and completes the proof of the theorem. □

Lemma 5.6. *Let* $f = d^\mathcal{B} v$, $v|_{\partial M} = 0$, *and* $v \in C^1(M)$. *Then* $v(y) = 0$ *for any* y *such that* $f(y) = 0$, *and* y *can be connected to a point on* ∂M *by a path that does not intersect* supp f.

6. Generic s-injectivity

In this section we prove Theorem 1.2 for $1 + 1$ tensor fields using the Fredholm property (4.21) of the normal operator, and the s-injectivity result for analytic metrics. We note that by possibly conjugating all the operators with $\kappa_g^\sharp$ from left and $\kappa_g^\flat$ from right, we can work with the space $S\tau_M \otimes^\mathcal{B} S\tau_M'$, of trace-free $(1,1)$-tensor fields that is defined independent of any metric structure.

We are ready to present the proof of Theorem 1.2.

Proof of Theorem 1.2 . Let $g \in C^m(M)$ be a simple metric. By formula (4.21), in Theorem 4.6 we have

$$\mathcal{QN} = \mathcal{S} + \mathcal{K}, \qquad \mathcal{N} := \mathcal{N}_L,\, \mathcal{S} := \mathcal{S}_{M_2},$$

with $\mathcal{SQ} = \mathcal{Q}$, $\mathcal{NS} = \mathcal{N}$. After applying $\mathcal{S}$ from the left to the above identity, we have

$$\mathcal{QN} = \mathcal{S} + \mathcal{SK}.$$

Thus $\mathcal{K} = \mathcal{SK}$ and similarly $\mathcal{KS} = \mathcal{K}$. As $\mathcal{S}$ is self adjoint we also have $\mathcal{K}^* = \mathcal{SK}^* = \mathcal{K}^*\mathcal{S}$. If we set $\widetilde{\mathcal{Q}} := \mathcal{S}(\mathrm{Id} + \mathcal{K}^*)\mathcal{Q}$, then previous observations yield

$$\widetilde{\mathcal{Q}}\mathcal{N} = \mathcal{S}(\mathrm{Id} + \mathcal{K}^*)\mathcal{QN} = \mathcal{S}(\mathrm{Id} + \mathcal{K}^*)(\mathrm{Id} + \mathcal{K}) = \mathcal{S} + \mathcal{K}^* + \mathcal{K} + \mathcal{K}^*\mathcal{K} = \mathcal{S} + \widetilde{\mathcal{K}},$$

where $\widetilde{\mathcal{K}} = \mathcal{K}^* + \mathcal{K} + \mathcal{K}^*\mathcal{K}$ is a compact self-adjoint operator $L^2(S\tau_M' \otimes^\mathcal{B} S\tau_M') \to \mathcal{S}L^2(S\tau_M' \otimes^\mathcal{B} S\tau_M')$. This implies

$$(6.1) \qquad\qquad \widetilde{\mathcal{Q}}\mathcal{N} + \mathcal{P} = \mathrm{Id} + \widetilde{\mathcal{K}} \quad \text{on } L^2(S\tau_M' \otimes^\mathcal{B} S\tau_M').$$

We are ready to show that the set of s-injective metrics is open in C^m-topology, for any $m \in \mathbf{N}$ that is large enough. In the following we will indicate the dependence on g by placing the subscript g on the operators $\mathcal{N}$, $\mathcal{S}$, etc. Suppose that L_{g_0} is s-injective for some simple metric $g_0 \in C^m(M)$. Then $\mathcal{N}_{g_0}$ is s-injective as well, and moreover the operator on right hand side of (6.1) has a finite dimensional kernel on the space of solenoidal tensor fields. By using the s-injectivity of $\mathcal{N}_{g_0}$, as in the proof of [44, Theorem 1.5], we can construct a finite rank operator $\mathcal{Q}_0 : L^2(S\tau_{M_1}' \otimes^\mathcal{B} S\tau_{M_1}') \to L^2(S\tau_M' \otimes^\mathcal{B} S\tau_M')$ such that

$$(6.2) \qquad \mathrm{Id} + \mathcal{K}_{g_0}^\sharp = (\widetilde{\mathcal{Q}}_{g_0} + \mathcal{Q}_0)\mathcal{N}_{g_0} + \mathcal{P}_{g_0} \quad \text{on } L^2(S\tau_M' \otimes^\mathcal{B} S\tau_M')$$

is one-to-one, where $\mathcal{K}_{g_0}^\sharp := \widetilde{\mathcal{K}}_{g_0} + \mathcal{Q}_0\mathcal{N}_{g_0}$ is compact. Thus according to Fredholm alternative $(\mathrm{Id} + \widetilde{\mathcal{K}}_{g_0})^{-1}$ is bounded. We choose $f \in H^1(S\tau_{M_1}' \otimes^\mathcal{B} S\tau_{M_1}')$ and apply the operator $\mathrm{Id} + \mathcal{K}_{g_0}^\sharp$ to the solenoidal part f_{M,g_0}^s of f to obtain

$$\|f_{M,g_0}^s\|_{L^2(M)} \leq C \left(\|\widetilde{\mathcal{Q}}_{g_0}\mathcal{N}_{g_0}f\|_{\tilde{H}^1(M_1)} + \|\mathcal{Q}_0\mathcal{N}_{g_0}f\|_{L^2(M_1)} \right) \leq C\|\mathcal{N}_{g_0}f\|_{\tilde{H}^2(M_1)}.$$

This is the stability estimate of Theorem 1.2 for $g = g_0$. Next verify the same estimate, with uniform C for $g \in C^m(M)$ that is close enough to g_0 with respect to C^m-topology, for any m large enough. To do this we first write analogously

$$(6.3) \qquad (\widetilde{\mathcal{Q}}_g + \mathcal{Q}_0)\mathcal{N}_g + \mathcal{P}_g = \mathrm{Id} + \mathcal{K}_g^\sharp.$$

We note here that the finite rank operator Q_0 is the same as in (6.2), and the compact operator $\mathcal{K}_g^\sharp := \widetilde{\mathcal{K}}_g + \mathcal{Q}_0\mathcal{N}_g$, as an operator in $L^2(S\tau'_M \otimes^{\mathcal{B}} S\tau'_M)$, depends continuously on g. Therefore if k is large enough, it holds that the operator $\mathrm{Id} + \mathcal{K}_g^\sharp$ remains invertible, with a uniform bound for its inverse, whenever g is close enough to g_0 in C^m-topology. After applying (6.3) to $f = f^s_{M,g} \in H^1(S\tau_{M_1} \otimes^{\mathcal{B}} S\tau'_{M_1})$, we have

$$\|f^s_{M,g}\|_{L^2(M)} \leq C \left(\|\mathcal{N}_g f\|_{\tilde{H}^2(M_1)} + \|\mathcal{N}_g f\|_{L^2(M_1)} \right) \leq C\|\mathcal{N}_g f\|_{\tilde{H}^2(M_1)},$$

with $C > 0$ independent of g in a small neighborhood of g_0 in C^m-topology. This implies that also g is s-injective.

The proof of the theorem can be completed by using s-injectivity of L_g for analytic metric g (Theorem 5.5), and the fact that analytic metrics are dense in $C^m(M)$. $\qquad\square$

7. Ellipticity of the normal operator and adaptation of the proofs for $2 + 2$ tensor fields

In this section, we will first show the ellipticity of the normal operator for $2+2$ tensors (restricted to the subspace of solenoidal tensors). To be more specific, we will show that the operator $\mathcal{M} = (|D|_g\mathcal{N}_L, \mathcal{P}_{M_2})^T$ is elliptic. Then we will sketch adaptions needed to prove Theorems 1.2, 4.6 and 5.5 for $2 + 2$ tensor fields.

7.1. Parametrix of the normal operator for solenoidal $2 + 2$ tensor fields.
In the following we study the action of the principal symbol $\sigma(\mathcal{N}_L)$. We note that in $1 + 1$ case, the principal symbol (3.14) can be written as

$$\sigma(\mathcal{N}_L)^{ijk\ell}(x, \xi)$$
$$= -2\sqrt{\det g(x)} \int_{\mathbf{R}^3} e^{-i\xi \cdot z} \left(\delta_u^k - \frac{z_u z^k}{|z|_g} \right) g^{uu'}(x) \left(\delta_{u'}^i - \frac{z^i z_{u'}}{|z|_g^2} \right) \frac{z^j z^\ell}{|z|_g^4} \, dz$$
$$= \frac{-2\pi}{|\xi|_g} \int_{S_x M \cap \xi^\perp} \left(\delta_u^k - \omega^k \omega_u \right) g^{uu'}(x) \left(\delta_{u'}^i - \omega^i \omega_{u'} \right) \omega^j \omega^\ell d\omega.$$

We recall the notation $(P_\omega)_j^i := \delta_j^i - \omega^i \omega_j$. Thus for $f \in T'_x M \otimes^{\mathcal{B}} T'_x M$ we have

$$\langle \sigma(\mathcal{N}_L)(x, \xi)f, f \rangle_g = \frac{-2\pi}{|\xi|_g} \int_{S_x M \cap \xi^\perp} |(P_\omega)_u^i \omega^j f_{ij}|_g^2 d\omega.$$

We do not derive an explicit formula for the principal symbol of the normal operator in the case of $2 + 2$ tensors, but sketch the main steps to conclude that for any $f \in S^2 T'_x M \otimes^{\mathcal{B}} S^2 T'_x M$, we have analogously to the $1 + 1$ case,

$$(7.1) \qquad \langle \sigma(\mathcal{N}_L)(x, \xi)f, f \rangle_g = \frac{2\pi}{|\xi|_g} \int_{S_x M \cap \xi^\perp} |(P_\omega)_a^i (P_\omega)_b^j \omega^k \omega^l f_{ijkl}|_g^2 d\omega.$$

Let $f, h \in S^2 \tau'_M \otimes^{\mathcal{B}} S^2 \tau'_M$. Then, for the geodesic γ with initial conditions $(z, \omega) \in \partial_+ SM$, we have

$$\langle Lf, Lh \rangle_{L^2(\beta_2(\partial_+(SM)))}$$

$$= \int_{\partial_+(SM)} \left(\int_0^{\tau(z,\omega)} (\mathcal{T}_\gamma^{0,t})_{ab}^{uv} (P_{\omega(t)})_v^i (P_{\omega(t)})_u^j f_{ijk\ell}(x(t)) \omega^k(t) \omega^\ell(t) \mathrm{d}t \right)$$

$$\left(\int_0^{\tau(z,\omega)} (\mathcal{T}_\gamma^{0,s})_{a'b'}^{u'v'} (P_{\omega(s)})_{v'}^{i'} (P_{\omega(s)})_{u'}^{j'} \bar{h}_{i'j'k'\ell'}(x(s)) \omega^{k'}(s) \omega^{\ell'}(s) \mathrm{d}s \right)$$

$$g^{aa'}(z) g^{bb'}(z) \mathrm{d}\mu(z,\omega).$$

By an analogous argument to one in Section 3 we show that $\mathcal{N}_L$ is an integral operator, whose Schwartz kernel near the diagonal can be written as

$$K_{ii'jj'kk'\ell'}(x,y)$$

$$= \frac{2A^{uu'vv'}(x,y)}{(G^{(1)}z \cdot z)^3} \left[\tilde{G}^{(2)}z\right]_\ell \left[\tilde{G}^{(2)}z\right]_{\ell'} \left[G^{(2)}z\right]_j \left[G^{(2)}z\right]_{j'} \frac{|\det G^{(3)}|}{\sqrt{\det g(x)}}$$

$$(7.2)$$
$$\times \left(g_{ku}(y) - \frac{\left[\tilde{G}^{(2)}z\right]_k \left[\tilde{G}^{(2)}z\right]_u}{G^{(1)}z \cdot z} \right) \left(g_{k'v}(y) - \frac{\left[\tilde{G}^{(2)}z\right]_{k'} \left[\tilde{G}^{(2)}z\right]_v}{G^{(1)}z \cdot z} \right)$$

$$\times \left(g_{iu'}(x) - \frac{\left[G^{(2)}z\right]_i \left[G^{(2)}z\right]_{u'}}{G^{(1)}z \cdot z} \right) \left(g_{i'v'}(x) - \frac{\left[G^{(2)}z\right]_{i'} \left[G^{(2)}z\right]_{v'}}{G^{(1)}z \cdot z} \right),$$

where $z = x - y$ and

$$A^{uu'vv'}(x,y) = g^{au'}(x) g^{bv'}(x) \left(\mathcal{T}_{\gamma_{x,-\nabla_x^g \rho(x,y)}}^{0,\rho(x,y)} \right)_{ab}^{uv}.$$

Therefore $\mathcal{N}_L$ is a ΨDO of order -1, and formula (7.1) is valid.

For now on we use the short hand notation $\mathcal{P} = \mathcal{P}_{M_2}$ and aim to show that

$$\sigma(\mathcal{M})f := \begin{pmatrix} |\xi|_g \circ \sigma(\mathcal{N}_L)f \\ \sigma(\mathcal{P})f \end{pmatrix} = 0, \quad (x,\xi) \in T^* M_2 \setminus \{0\},$$

implies $f = 0$, which proves that the zeroth order operator $\mathcal{M} = (|D|_g \mathcal{N}_L, \mathcal{P})^T$ is elliptic.

Let $\xi \in T_x M$. We choose $\omega, \tilde{\omega} \in S_x M$ such that $\{\hat{\xi} := \frac{\xi}{|\xi|_g}, \omega, \tilde{\omega}\}$ is an orthonormal basis of $T_x M$. We also simplify

$$Q_{ab}^{ijkl}(\omega) := (P_\omega)_a^i (P_\omega)_b^j \omega^k \omega^l.$$

If $(|\xi|_g \circ \sigma(\mathcal{N}_L)(x,\xi), \sigma(\mathcal{P}))^T f = 0$, then

$$j_\xi^{\mathcal{B}} \sigma(\mathcal{P}) = j_\xi^{\mathcal{B}} i_\xi^{\mathcal{B}} \sigma((\Delta^{\mathcal{B}})^{-1}) j_\xi^{\mathcal{B}} = j_\xi^{\mathcal{B}},$$

where $j_\xi^{\mathcal{B}}, i_\xi^{\mathcal{B}}$ are given in (2.13) and (2.14), implies

$$(7.3) \qquad\qquad \hat{\xi}^\ell f_{ijk\ell} = \hat{\xi}^k f_{ijk\ell} = 0.$$

Thus it suffices to prove that $f_{ijk\ell} \omega^\ell = 0$ and $f_{ijk\ell} \tilde{\omega}^\ell = 0$.

Next we note that (7.1) gives

$$Q(\omega)f := Q_{ab}^{ijkl}(\omega) f_{ijkl} = Q_{ab}^{ijkl}(\tilde{\omega}) f_{ijkl} = Q_{ab}^{ijkl} \left(\frac{\tilde{\omega} + \omega}{\sqrt{2}} \right) f_{ijkl} = 0.$$

Therefore we have

$$\hat{\xi}^j f_{ijkl}\omega^k\omega^l = \hat{\xi}Q(\omega)f = 0, \qquad \hat{\xi}^j f_{ijkl}\tilde{\omega}^k\tilde{\omega}^l = \hat{\xi}Q(\tilde{\omega})f = 0,$$

$$\hat{\xi}^j f_{ijkl}\omega^k\tilde{\omega}^l = \hat{\xi}Q\left(\frac{\omega+\tilde{\omega}}{\sqrt{2}}\right)f - \frac{1}{2}\hat{\xi}^j f_{ijkl}\omega^k\omega^l - \frac{1}{2}\hat{\xi}^j f_{ijkl}\tilde{\omega}^k\tilde{\omega}^l = 0,$$

and

$$(7.4) \qquad \tilde{\omega}^j f_{ijkl}\omega^k\omega^l = \tilde{\omega}Q(\omega)f = 0, \quad \omega^j f_{ijkl}\tilde{\omega}^k\tilde{\omega}^l = \omega Q(\tilde{\omega})f = 0.$$

Since we assumed $f \in S^2 T'_x M \otimes^{\mathcal{B}} S^2 T'_x M$, the trace-free condition $\mu f = 0$ and (7.3) yield

$$f_{ijkl}\omega^i\omega^k = f_{ijkl}\omega^j\omega^l = -f_{ijkl}\tilde{\omega}^j\tilde{\omega}^l = -f_{ijkl}\tilde{\omega}^i\tilde{\omega}^k.$$

After applying (7.4) to previous equation we get

$$f_{ijkl}\omega^i\omega^j\omega^k\tilde{\omega}^l = f_{ijkl}\omega^i\tilde{\omega}^j\tilde{\omega}^k\tilde{\omega}^l = f_{ijkl}\tilde{\omega}^i\tilde{\omega}^j\omega^k\tilde{\omega}^l = f_{ijkl}\tilde{\omega}^i\omega^j\tilde{\omega}^k\omega^l = 0.$$

Then we compute

$$2(\omega-\tilde{\omega})(\omega-\tilde{\omega})Q\left(\frac{\omega+\tilde{\omega}}{\sqrt{2}}\right)f$$

$$=f_{ijkl}(\omega-\tilde{\omega})^i(\omega-\tilde{\omega})^j(\omega+\tilde{\omega})^k(\omega+\tilde{\omega})^l$$

$$=f_{ijkl}\omega^i\omega^j\omega^k\omega^l + f_{ijkl}\tilde{\omega}^i\tilde{\omega}^j\tilde{\omega}^k\tilde{\omega}^l - 4f_{ijkl}\omega^i\tilde{\omega}^j\omega^k\tilde{\omega}^l$$

$$=f_{ijkl}\omega^i\omega^j\omega^k\omega^l + 5f_{ijkl}\tilde{\omega}^i\tilde{\omega}^j\tilde{\omega}^k\tilde{\omega}^l$$

$$=6f_{ijkl}\omega^i\omega^j\omega^k\omega^l$$

$$=0.$$

Therefore we can conclude that $f = 0$.

The rest of the proof of Proposition 4.4 for $2+2$ tensor fields is as presented earlier.

7.2. A sketch of proof for Theorem 4.6 in $2+2$ case.

For $x \in M_1 \setminus M$, choose ξ such that the geodesic $\gamma = \gamma_{x,\xi}$ hits the boundary ∂M_1 before ∂M and minimizes the distance between x and ∂M_1. As we have proven Proposition 4.4 for $2+2$ tensor fields, formula (4.9) holds and (4.10) is to be replaced by

$$(7.5) \qquad [v_{M_1}(x)]_{ijk}\eta^i\eta^j\xi^k = -\int_0^\tau [\mathrm{d}^{\mathcal{B}}v_{M_1}(\gamma(t))]_{ijk\ell}\eta^i(t)\eta^j(t)\dot{\gamma}^k(t)\dot{\gamma}^\ell(t)\mathrm{d}t,$$

where $\eta \perp \xi$. Choose $\eta, \tilde{\eta}$ such that $B = \{\eta, \tilde{\eta}, \xi\}$ form an orthonormal basis of $T_x(M_1 \setminus M)$.

Therefore we have

$$\left|[v_{M_1}(x)]_{ijk}\eta^i\eta^j\xi^k\right| \leq C\left|(\mathcal{T}_1\mathcal{N}_L f - \mathcal{K}_2 f)(x)\right|_g.$$

We need to show that there exists $C > 0$, uniform for any $x \in M_1 \setminus M^{int}$, such that

$$(7.6) \qquad \left|[v_{M_1}(x)]_{ijk}w_{m_1}^i w_{m_2}^j w_{m_3}^k\right| \leq C\left|(\mathcal{T}_1\mathcal{N}_L f - \mathcal{K}_2 f)(x)\right|_g, \quad w_{m_h} \in B.$$

As $|v_{M_1}(x)|_g^2$ can be estimated by the sum of all the terms $\left|[v_{M_1}(x)]_{ijk}w_{m_1}^i w_{m_2}^j w_{m_3}^k\right|^2$, the following L^2-estimate holds

$$\|v_{M_1}\|_{L^2(M_1\setminus M)} \leq C\|\mathcal{T}_1\mathcal{N}_L f - \mathcal{K}_2 f\|_{L^2(M_1\setminus M)}.$$

To prove (7.6) we need to repeat the steps between (2.25) and (2.31). First, we have

$$[v_{M_1}(x)]_{ijk} \left((\eta + \widetilde{\eta})^i (\eta + \widetilde{\eta})^j \xi^k - \eta^i \eta^j \xi^k - \widetilde{\eta}^i \widetilde{\eta}^j \xi^k \right) = 2[v_{M_1}(x)]_{ijk} \eta^i \widetilde{\eta}^j \xi^k.$$

Then

$$|[v_{M_1}(x)]_{ijk} \eta^i \widetilde{\eta}^j \xi^k| \leq C \left| (\mathcal{T}_1 \mathcal{N}_L f - \mathcal{K}_2 f)(x) \right|_g.$$

For x in a neighborhood of $x_0 \in \partial M$, there exists $\epsilon_0 > 0$ such that $\gamma_{x, \xi - \epsilon \eta}$ meets ∂M_1 before meeting ∂M for any $\epsilon < \epsilon_0$. Then we can obtain

$$|[v_{M_1}(x)]_{ijk} (\eta + \epsilon \xi)^i (\eta + \epsilon \xi)^j (\xi - \epsilon \eta)^k| \leq C \left| (\mathcal{T}_1 \mathcal{N}_L f - \mathcal{K}_2 f)(x) \right|_g.$$

Choosing four distinct real numbers $0 < \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 < \epsilon_0$, by invertibility of the Vandermonde matrix

$$\begin{pmatrix} 1 & \epsilon_1 & \epsilon_1^2 & \epsilon_1^3 \\ 1 & \epsilon_2 & \epsilon_2^2 & \epsilon_2^3 \\ 1 & \epsilon_3 & \epsilon_3^2 & \epsilon_3^3 \\ 1 & \epsilon_4 & \epsilon_4^2 & \epsilon_4^3 \end{pmatrix},$$

we have the estimates

$$|u_{ijk} \xi^i \xi^j \eta^k|, \left| u_{ijk} \left(\xi^i \xi^j \xi^k - 2\eta^i \xi^j \eta^k \right) \right|, \left| u_{ijk} \left(2\eta^i \xi^j \xi^k - \eta^i \eta^j \eta^k \right) \right|$$
$$\leq C \left| (\mathcal{T}_1 \mathcal{N}_L f - \mathcal{K}_2 f)(x) \right|_g.$$

Here, the constant C depends on $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$, which could be chosen such that C is uniform in a neighborhood of x_0. One can then just continue the steps and get the estimates (7.6) with C uniform in a neighborhood of x_0. We omit the details here. Finally, by a compactness argument, we have (7.6) with C uniform in $M_1 \setminus M$.

Next we estimate the H^1-norm of v_{M_1} in $M_1 \setminus M$. As earlier we can estimate $|\nabla v_{M_1}|_g^2$ by the sum of all terms

$$\left| w_{m_4}^\ell \nabla_\ell [v_{M_1}(x)]_{ijk} w_{m_1}^i w_{m_2}^j w_{m_3}^k \right|^2, \quad w_{m_h} \in B.$$

Recall that we have

$$w^\ell \nabla_\ell [v_{M_1}]_{ijk} w_{m_1}^i w_{m_2}^j w^k = \left[\mathrm{d}^{\mathcal{B}} v_{M_1} \right]_{ijk\ell} w_{m_1}^i w_{m_2}^j w^k w^\ell,$$

if $w \neq w_{m_1}, w_{m_2}$. We only need to estimate the terms

$$(7.7) \qquad \widetilde{w}^\ell \nabla_\ell [v_{M_1}(x)]_{ijk} w_{m_1}^i w_{m_2}^j w^k, \quad w \neq \widetilde{w}, \ w \neq w_{m_h},$$

and

$$(7.8) \qquad \widetilde{w}^\ell \nabla_\ell [v_{M_1}(x)]_{ijk} \widehat{w}^i w^j w^k.$$

We start with the term (7.7) and as earlier we work in boundary normal coordinates (x', x_3) of M near some fixed boundary point $x_0 \in \partial M$.

We have the following identity analogous to (4.12):

$$[v_{M_1}(x)]_{ij3} w_{m_1}^i w_{m_2}^j = - \int_{x_3}^\infty [\mathrm{d}^{\mathcal{B}} v_{M_1}(\gamma_{x,\xi}(t))]_{ij33} w_{m_1}^i(t) w_{m_2}^j(t) \mathrm{d}t, \quad w_{m_h} \in \{\eta, \widetilde{\eta}\},$$

and (4.13) has become

$$(7.9)$$
$$X_{(k)}^\ell \nabla_\ell [v_{M_1}(x)]_{ij3} w_{m_1}^i w_{m_2}^j = - \int_{x_3}^\infty X_{(k)} \left([\mathrm{d}^{\mathcal{B}} v_{M_1}(\gamma_{x,\xi}(t))]_{ij33} w_{m_1}^i(t) w_{m_2}^j(t) \right) \mathrm{d}t.$$

That is we have estimated (7.7) when $w = \xi$ and $\widetilde{w} \in \{\eta, \widetilde{\eta}\}$. To estimate for the remaining case of (7.7) we denote $\widetilde{w} = \xi$ and $w = \eta$. Then it must hold that $w_{m_h} \in \{\xi, \widetilde{\eta}\}$ and

$$[\mu \nabla v_{M_1}(x)]_{jk} = \xi^m \nabla_m [v_{M_1}(x)]_{pjk} \xi^p + \eta^m \nabla_m [v_{M_1}(x)]_{pjk} \eta^p + \widetilde{\eta}^m \nabla_m [v_{M_1}(x)]_{pjk} \widetilde{\eta}^p.$$

Straightforward computation yields

$$
\begin{aligned}
\left[d^{\mathcal{B}} v_{M_1} \right]_{ijk\ell} w_{m_1}^i w_{m_2}^j \eta^k \xi^\ell = & \left(\frac{1}{2} (\nabla_\ell [v_{M_1}(x)]_{ijk} + \nabla_k [v_{M_1}(x)]_{ij\ell}) \right) w_{m_1}^i w_{m_2}^j \eta^k \xi^\ell \\
& - \frac{1}{10} \Bigg(\delta_{i\ell} \left(\xi^m \nabla_m [v_{M_1}(x)]_{pjk} \xi^p + \eta^m \nabla_m [v_{M_1}(x)]_{pjk} \eta^p \right. \\
& + \widetilde{\eta}^m \nabla_m [v_{M_1}(x)]_{pjk} \widetilde{\eta}^p \Bigg) w_{m_1}^i w_{m_2}^j \eta^k \xi^\ell \\
& + \delta_{j\ell} \left(\xi^m \nabla_m [v_{M_1}(x)]_{pik} \xi^p + \eta^m \nabla_m [v_{M_1}(x)]_{pik} \eta^p \right. \\
& + \widetilde{\eta}^m \nabla_m [v_{M_1}(x)]_{pik} \widetilde{\eta}^p \Bigg) w_{m_1}^i w_{m_2}^j \eta^k \xi^\ell \Bigg).
\end{aligned}
$$

Taking $w_1, w_2 \in \{\xi, \widetilde{\eta}\}$ in the above formula, we can get desired estimates for $\xi^\ell \nabla_\ell [v_{M_1}(x)]_{ijk} w_{m_1}^i w_{m_2}^j \eta^k$.

It remains to estimate the terms appearing in (7.8). Set $\widetilde{w} = \xi$ in (7.8) and write

$$
\begin{aligned}
& \left[d^{\mathcal{B}} v_{M_1} \right]_{ijk\ell} \widehat{w}^i w^j w^k \xi^\ell \\
= & \left(\frac{1}{2} (\nabla_\ell [v_{M_1}(x)]_{ijk} + \nabla_k [v_{M_1}(x)]_{ij\ell}) \right) \widehat{w}^i w^j w^k \xi^\ell \\
& - \frac{1}{10} \Bigg(\left(\delta_{i\ell} \xi^m \nabla_m [v_{M_1}(x)]_{pjk} \xi^p + \eta^m \nabla_m [v_{M_1}(x)]_{pjk} \eta^p \right. \\
& \qquad\qquad\qquad\qquad + \widetilde{\eta}^m \nabla_m [v_{M_1}(x)]_{pjk} \widetilde{\eta}^p \Big) \widehat{w}^i w^j w^k \xi^\ell \\
& + \delta_{j\ell} \left(\xi^m \nabla_m [v_{M_1}(x)]_{pik} \xi^p + \eta^m \nabla_m [v_{M_1}(x)]_{pik} \eta^p \right. \\
& \qquad\qquad\qquad\qquad + \widetilde{\eta}^m \nabla_m [v_{M_1}(x)]_{pik} \widetilde{\eta}^p \Big) \widehat{w}^i w^j w^k \xi^\ell \\
& + \delta_{ik} \left(\xi^m \nabla_m [v_{M_1}(x)]_{pj\ell} \xi^p + \eta^m \nabla_m [v_{M_1}(x)]_{pj\ell} \eta^p \right. \\
& \qquad\qquad\qquad\qquad + \widetilde{\eta}^m \nabla_m [v_{M_1}(x)]_{pj\ell} \widetilde{\eta}^p \Big) \widehat{w}^i w^j w^k \xi^\ell \\
& + \left(\xi^m \nabla_m [v_{M_1}(x)]_{pi\ell} \xi^p + \eta^m \nabla_m [v_{M_1}(x)]_{pi\ell} \eta^p \right. \\
& \qquad\qquad\qquad\qquad + \widetilde{\eta}^m \nabla_m [v_{M_1}(x)]_{pi\ell} \widetilde{\eta}^p \Big) \widehat{w}^i \xi^\ell \Bigg).
\end{aligned}
\tag{7.10}
$$

We drop out all the terms in the right hand side of (7.10) that do not have the normal derivative $\xi^m \nabla_m$, to obtain

$$
\begin{aligned}
& \frac{1}{2} \nabla_\ell [v_{M_1}(x)]_{ijk} \widehat{w}^i w^j w^k \xi^\ell - \frac{1}{10} \Bigg(\delta_{i\ell} \xi^m \nabla_m [v_{M_1}(x)]_{pjk} \xi^p \widehat{w}^i w^j w^k \xi^\ell \\
& \qquad + \delta_{j\ell} \xi^m \nabla_m [v_{M_1}(x)]_{pik} \xi^p \widehat{w}^i w^j w^k \xi^\ell + \delta_{ik} \xi^m \nabla_m [v_{M_1}(x)]_{pj\ell} \xi^p \widehat{w}^i w^j w^k \xi^\ell \\
& \qquad + \xi^m \nabla_m [v_{M_1}(x)]_{pi\ell} \xi^p \widehat{w}^i \xi^\ell \Bigg).
\end{aligned}
$$

If $w = \xi$ the simplified version of the right hand side of (7.10) is

$$\frac{1}{2}\nabla_\ell[v_{M_1}(x)]_{ijk}\widehat{w}^i\xi^j\xi^k\xi^\ell - \frac{1}{10}\left(\delta_{i\ell}\xi^m\nabla_m[v_{M_1}(x)]_{pjk}\xi^p\xi^j\xi^k\widehat{w}^i\xi^\ell\right.$$

$$+ \xi^m\nabla_m[v_{M_1}(x)]_{pik}\xi^p\widehat{w}^i\xi^k + \delta_{ik}\xi^m\nabla_m[v_{M_1}(x)]_{pj\ell}\widehat{w}^i\xi^p\xi^j\xi^\ell\xi^k$$

$$\left.+ \xi^m\nabla_m[v_{M_1}(x)]_{pi\ell}\xi^p\widehat{w}^i\xi^\ell\right).$$

Which is always a nonzero multiple of $\nabla_\ell[v_{M_1}(x)]_{ijk}\widehat{w}^i\xi^j\xi^k\xi^\ell$, and thus can be estimated.

For $w = \eta$ or $w = \widetilde{\eta}$, the situations are analogous, and we only consider the first case. We get

$$\frac{1}{2}\nabla_\ell[v_{M_1}(x)]_{ijk}\widehat{w}^i\eta^j\eta^k\xi^\ell - \frac{1}{10}\left(\delta_{i\ell}\xi^m\nabla_m[v_{M_1}(x)]_{pjk}\xi^p\widehat{w}^i\eta^j\eta^k\xi^\ell\right.$$

$$\left.+ \delta_{ik}\xi^m\nabla_m[v_{M_1}(x)]_{pj\ell}\xi^p\widehat{w}^i\eta^j\eta^k\xi^\ell + \xi^m\nabla_m[v_{M_1}(x)]_{pi\ell}\xi^p\widehat{w}^i\xi^\ell\right),$$

for the simplified version of the right hand side of (7.10). Here the last two terms have already been estimated and the first term vanishes if $\widehat{w} \neq \xi$. Therefore we have also found a formula for $\nabla_\ell[v_{M_1}(x)]_{ijk}\widehat{w}^i\eta^j\eta^k\xi^\ell$ that contains the only tangential derivatives and $d^{\mathcal{B}}v_{M_1}$.

As earlier we can find $C > 0$ depending only on the distance to ∂M_1, which satisfies the following pointwise estimate:

$$\left|w^\ell_{m_4}\nabla_\ell[v_{M_1}(x)]_{ijk}w^i_{m_1}w^j_{m_2}w^k_{m_3}\right| \leq C\left(\sum_{k=1}^{2}|\chi\nabla_{X_{(k)}}(\mathcal{T}_1\mathcal{N}_Lf)|_g + |\mathcal{K}_4f|_g\right), \quad w_{m_h} \in B.$$

To complete the proof of the second claim of Theorem 4.6 we refine operator $\mathcal{T}_2$ using equation (7.5). The rest of the proof of Theorem 4.6 is analogous to what we did earlier.

7.3. **A sketch of proof for Theorem 5.5 in** $2 + 2$ **case.** We sketch here the required changes needed for the proofs of Theorems 5.5 and 1.2 for $2 + 2$ tensors fields.

First we note that the formula (7.2) implies the claim of Proposition 5.1 for $2+2$ tensor fields. We note that Proposition 5.3 is analogous to $1 + 1$ case, since we have proved Theorem 4.6 for $2 + 2$ tensor fields. Then we arrive at Lemma 5.4, which requires some modifications.

Proof of Lemma 5.4 in $2 + 2$ case. We fix $x_0 \in \partial M$ and take the boundary normal coordinates $x = (x', x^3)$ in a neighborhood $U \subset M$ of x_0. In these coordinates we have $g_{i3} = \delta_{i3}$, in U, for any $i = 1, 2, 3$. We aim to find a (trace-free) 3-tensor field v, vanishing on ∂M, such that for $\tilde{f} := f - d^{\mathcal{B}}v$ we have

$$(7.11) \qquad \tilde{f}_{ijk3} = 0, \quad \text{in some open } \tilde{U} \subset U, \text{ that contains } x_0,$$

which is, due to (2.8), equivalent to

$$(7.12) \qquad \begin{aligned} f_{ijk3} &- \frac{1}{2}\nabla_3v_{ijk} - \frac{1}{2}\nabla_kv_{ij3} + \frac{1}{10}\left(\delta_{j3}g^{mn}\nabla_nv_{imk} + \delta_{i3}g^{mn}\nabla_nv_{jmk}\right.\\ &\left.+ g_{jk}g^{mn}\nabla_nv_{im3} + g_{ik}g^{mn}\nabla_nv_{jm3}\right) = 0, \quad \text{in } \tilde{U}. \end{aligned}$$

The order for determining the components of v is quite similar to what is outlined in the proof of Lemma 2.3. Let us first consider the case $k = 3$. Then the above equation becomes

$$(7.13) \qquad f_{ij33} - \nabla_3 v_{ij3} + \frac{1}{5}\left(\delta_{j3}g^{mn}\nabla_n v_{im3} + \delta_{i3}g^{mn}\nabla_n v_{jm3}\right) = 0, \quad \text{in } \tilde{U}.$$

Remember that

$$\nabla_\ell v_{ijk} = \partial_\ell v_{ijk} - \left(\Gamma_{i\ell}^m v_{mjk} + \Gamma_{j\ell}^m v_{imk} + \Gamma_{k\ell}^m v_{ijm}\right),$$

and the Christoffel symbols, in the boundary normal coordinates, satisfy $\Gamma_{33}^k = \Gamma_{k3}^3 = \Gamma_{3k}^3 = 0$. If $i, j \neq 3$ we can write (7.13) as an ODE system, with respect to the travel-time variable x_3, for the unknowns $v_{\alpha\beta3}$, $\alpha, \beta \neq 3$, which can thus be determined.

If $i = 3$ and $j \neq 3$ we write (7.13) in the form

$$f_{3j33} - \frac{6}{5}\nabla_3 v_{3j3} + \frac{1}{5}g^{\alpha\beta}\nabla_\alpha v_{\beta j3} = 0, \quad \text{in } U, \quad \alpha, \beta \in \{1, 2\}.$$

Thus v_{3j3} can be found by solving the corresponding initial value problems. Finally we set $i = j = 3$ and the system (7.13) takes the form

$$f_{3333} - \frac{7}{5}\nabla_3 v_{333} + \frac{2}{5}g^{\alpha\beta}\nabla_\alpha v_{\beta33} = 0, \quad \text{in } U, \quad \alpha, \beta \in \{1, 2\}.$$

Now we have determined v_{ij3}, next we consider the case $k \neq 3$. For $i, j \neq 3$, equation (7.12) gives an ODE system for $v_{\alpha\beta k}$, $\alpha, \beta \in \{1, 2\}$. Then take $i = 3$ and $j \neq 3$, we get a system for $v_{3\alpha k}$, $\alpha \neq 3$. Finally, take $i = j = 3$, we get a system for v_{33k}.

Thus we have found a tensor field v that vanishes at the boundary and solves (7.11). We claim that constructed v is trace-free. To see this, we multiply g^{jk} to both sides of (7.12) and get

$$-\frac{1}{2}\nabla_3(v_{ijk}g^{jk}) + \frac{1}{10}\delta_{i3}\nabla^m(v_{mjk}g^{jk}) = 0.$$

First take $i \neq 3$, we have

$$\nabla_3(v_{ijk}g^{jk}) = \partial_3(\mu v)_i - \Gamma_{3i}^k(\mu v)_k = 0.$$

Since $\Gamma_{3i}^k = 0$ for $k = 3$, the above identity gives an ODE system for $(v_{1jk}g^{jl}, v_{2jk}g^{jl})$. Consequently, $v_{ijk}g^{jk} = 0$ for $i \neq 3$. Then we take $i = 3$ and conclude that $v_{3jk}g^{jk} = 0$. The claim is proved.

It is easy to see that if f and g are analytic near ∂M, so is v.

Similar to the proof of Lemma 5.4, we can show that

$$(7.14) \qquad\qquad \eta^i \eta^j \tilde{f}_{ijk\ell}\xi^k\xi^\ell = 0,$$

at the chosen boundary point x_0, whenever ξ is tangential to the boundary and $\eta \perp \xi$. We set $e_3 = \nu(x_0)$, and $e_\alpha = \frac{\partial}{\partial x'^\alpha}|_{x_0}$ for $\alpha = 1, 2$. Setting $\xi = e_\alpha$ and $\eta = e_i$, $\alpha \in \{1, 2\}$, $i \in \{1, 2, 3\}$, $i \neq \alpha$, then the previous equation implies

$$\tilde{f}_{ii\alpha\alpha} = 0, \quad i \in \{1, 2, 3\}, \, \alpha \in \{1, 2\}, \, i \neq \alpha.$$

This means that the following terms vanish

$$\tilde{f}_{1122}, \, \tilde{f}_{2211}, \, \tilde{f}_{3311}, \, \tilde{f}_{3322}.$$

Then take $\xi = e_1$, $\eta = \frac{1}{\sqrt{2}}(e_2 + e_3)$, we can conclude that $\tilde{f}_{2311} = 0$. Similarly $\tilde{f}_{1322} = 0$. Let us summarize what we have right now:

$$\tilde{f}_{ij\alpha\alpha} = 0, \quad i,j \in \{1,2,3\}, \, \alpha \in \{1,2\}, \, i,j \neq \alpha.$$

Take $\xi = \frac{1}{\sqrt{2}}(e_1 + e_2)$ and $\eta = e_3$, we obtain that $f_{3312} = 0$.

Let $\epsilon > 0$. If we set $\eta = e_1 + \epsilon e_2$ and $\xi = e_2 - \epsilon e_1$, then equation (7.14) implies that the coefficients of the powers of the ϵ satisfy
(7.15)
$$\tilde{f}_{\alpha\alpha\alpha\alpha} - 4\tilde{f}_{\alpha\beta\alpha\beta} + \tilde{f}_{\beta\beta\beta\beta} = 0, \quad \text{and} \quad \tilde{f}_{\beta\alpha\alpha\alpha} - \tilde{f}_{\beta\beta\beta\alpha} = 0, \quad \alpha \neq \beta, \, \alpha,\beta \in \{1,2\}.$$

By (7.11) and the trace-free condition, we have $\tilde{f}_{\alpha\alpha\alpha\alpha} = -\tilde{f}_{\alpha\beta\alpha\beta} = \tilde{f}_{\beta\beta\beta\beta}$ and $\tilde{f}_{\beta\alpha\alpha\alpha} = -\tilde{f}_{\beta\beta\beta\alpha}$. Together with (7.15), we have

$$\tilde{f}_{\alpha\alpha\alpha\alpha} = \tilde{f}_{\alpha\beta\alpha\beta} = \tilde{f}_{\beta\alpha\alpha\alpha} = 0 \quad \alpha \neq \beta, \, \alpha,\beta \in \{1,2\}.$$

Taking $\xi = e_1 + \epsilon e_2$ and $\eta = e_3 + e_2 - \epsilon e_1$ in (7.14) and collecting coefficients of $1, \epsilon, \epsilon^2, \epsilon^3$, we have

$$-\tilde{f}_{3222} + 2\tilde{f}_{3112} = 0,$$
$$-\tilde{f}_{3111} + 2\tilde{f}_{3221} = 0.$$

Together with the relation resulted from trace-free condition $\tilde{f}_{3222} + \tilde{f}_{3112} = \tilde{f}_{3111} + \tilde{f}_{3221} = 0$, we obtain $\tilde{f}_{3222} = \tilde{f}_{3112} = \tilde{f}_{3111} = \tilde{f}_{3221} = 0$. Therefore we can conclude that $\tilde{f}$ vanishes at x_0. Since x_0 was an arbitrary point in $\partial M \cap U$ we have shown that $\tilde{f}$ vanishes at $\partial M \cap U$.

Similar to the proof of Lemma 5.4, we can prove

(7.16) $$\partial_{x_3}^p \tilde{f}_{ijk\ell}|_{x=x_0} = 0, \quad p \in \mathbf{N}, \quad i,j,k,\ell \in \{1,2,3\},$$

and conclude the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The adaptations needed for the proof of Theorem 5.5 in the $2+2$ case are straightforward and therefore omitted. The rest of the proof for Theorem 1.2 is analogous to $1+1$ case.

APPENDIX A. LINEARIZATION OF ANISOTROPIC ELASTIC TRAVEL TOMOGRAPHY

In this appendix, we effectively study linearized travel-time tomography problems for polarized elastic waves. For our purposes this means the determination of some elastic parameters by measuring the travel times of qS-polarized waves – see the definition below. We use the typical notations and terminologies of the seismological literature; see for instance [8]. We let $\mathbf{C} = C_{ijkl}(x)$ be a smooth stiffness tensor on $\mathbf{R}^3$ which satisfies the symmetry

(A.1) $$C_{ijk\ell}(x) = C_{jik\ell}(x) = C_{k\ell ij}(x), \quad x \in \mathbf{R}^3.$$

We also assume that the density of mass $\rho(x)$ is a smooth function of x and define density–normalized elastic moduli

$$\mathbf{A} = A_{ijk\ell}(x) = \frac{C_{ijk\ell}(x)}{\rho(x)}.$$

The elastic wave operator P, associated with the elastic moduli $\mathbf{A}$, is a matrix valued second order partial differential operator given by

$$P_{ik} = \delta_{ik}\frac{\partial^2}{\partial t^2} - \sum_{j,k}\left(A_{ijk\ell}(x)\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^\ell}\right) + \text{lower order terms.}$$

For every $(x,p) \in T^*\mathbf{R}^3$ we define a square matrix $\Gamma(x,p)$, by

(A.2)
$$\Gamma_{ik}(x,p) := \sum_{j,\ell} A_{ijk\ell}(x)p_j p_\ell.$$

This is known as the *Christoffel matrix*. Due to (A.1) the matrix $\Gamma(x,p)$ is symmetric. We also assume that $\Gamma(x,p)$ is positive definite for every $(x,p) \in T^*\mathbf{R}^3 \setminus \{0\}$.

The principal symbol $\sigma(t,x,\omega,p)$ of the operator P is then a matrix-valued map given by

$$\sigma(t,x,\omega,p) = \omega^2 I - \Gamma(x,p), \quad (t,x,\omega,p) \in T^*\mathbf{R}^{1+3}.$$

Since the matrix $\Gamma(x,p)$ is positive definite and symmetric, it has three positive eigenvalues $G^m(x,p)$, $m \in \{1,2,3\}$, which are homogeneous of degree 2 in the momentum variable p.

We assume that

(A.3)
$$G^1(x,p) > G^m(x,p), \quad m \in \{2,3\}, (x,p) \in T^*\mathbf{R}^3 \setminus \{0\}.$$

It was shown in [15] that $\sqrt{G^1}$ is a Legendre transform of some Finsler metric F. Thus the bicharacteristic curves of $\omega^2 - G^1(x,p)$ are given by the co-geodesic flow of F. We recall that a bicharacteristic curve is a smooth curve on $T^*\mathbf{R}^{1+3}$, on which $\omega^2 - G^1(x,p)$ vanishes, that solves the Hamilton's equation for the Hamiltonian $\omega^2 - G^1(x,p)$. We consider a second order pseudo-differential operator $\square_P := \frac{\partial^2}{\partial t^2} - G^1(x,D)$, $D := \mathrm{i}(\partial_{x_1},\partial_{x_2},\partial_{x_3})$. Since G^1 is related to a Finsler metric the operator $\square_P$ is of real principal type (the bi-characteristic curves exit any compact set). The solutions u of the corresponding scalar ΨDE $\square_P u = f$ represent qP-waves (quasi-pressure waves) and moreover the wavefront set of $u(t,\cdot)$ propagates along the bicharacteristics of $\omega^2 - G^1(x,p)$ [19,23].

Next, we describe the propagation of the slower qS_1 and qS_2 waves (quasi-shear waves), that are given as solutions to the scalar equations with the operators $\square_{S_m} := \frac{\partial^2}{\partial t^2} - G^m(x,D)$, $m \in \{2,3\}$. In the anisotropic case the unit level sets $(G^m)^{-1}\{1\} \subset T^*\mathbf{R}^3$, $m \in \{2,3\}$, also referred to as slowness surfaces, typically will have points in common. See [11] for a study of different types of intersections. The size and codimension of their intersection set depends on the additional symmetries that the stiffness tensor may have. Thus in general the smaller eigenvalues G^2, G^3 are only continuous. We denote by $D_c = (G^2)^{-1}\{1\} \cap (G^3)^{-1}\{1\}$ the set of degenerate eigenvalues, and note that outside this set $G \in \{G^2, G^3\}$ yields a smooth Hamiltonian $H(x,p) = \frac{1}{2}G(x,p)$. Let $U \subset (T^*\mathbf{R}^3 \setminus \{0\}) \setminus D_c$ be an open set, then a local Hamiltonian flow $\theta\colon \mathcal{D} \to U$ of H exists, where $\mathcal{D}$ is the maximal flow domain of θ that satisfies

(A.4)
$$\theta(t,(x,p)) \in U, \quad (x,p) \in (U \cap G^{-1}\{1\}).$$

In general it is possible that the *momentum gradient* $D_p H$ vanishes at some point $(x,p) \in U$, which would cause problems in translating between Hamiltonian and Lagrangian formalisms. For this reason the operators $\square_{S_m}$ may not be of real principal type in U. See for instance [33, Section 1.2] for the connection between

different definitions for real principal type operators. We make a standing assumption that $D_p H$ does not vanish in U. In other words we exclude the occurrence of inflection points. We choose $(x_0, p_0) \in U \cap G^{-1}\{1\}$ and say that the *elastic travel-time* τ_c from (x_0, p_0) to $(x, p) \in \theta(\mathbf{R}_+, (x_0, p_0)) \cap U$ is the smallest t for which $\theta(t, (x_0, p_0)) = (x, p)$.

Under these two assumptions for the Hamiltonian H in U we are ready to set an inverse problem for anisotropic elastic travel-times. We suppose that there exists an open set $M \subset \mathbf{R}^3$ and open sets $\Sigma, \Sigma' \subset \partial M$ such that for any $x \in \Sigma, x' \in \Sigma$ there is a unique characteristic curve of H contained in $T^*M \cap U$ whose spatial projection γ connects x to x', where T^*M is the cotangent bundle of M. Thus for any $x \in \Sigma, x' \in \Sigma$ there exists a unique triplet

$$(\tau_c; (x, p); (x', p')) \in \mathbf{R}_+ \times U \times U \quad \text{which satisfies} \quad \theta(\tau_c, (x, p)) = (x', p') \in G^{-1}\{1\}.$$

Therefore τ_c is the elastic travel-time from (x, p) to (x', p') and we call $d_G(x, x') :=$ τ_c the *elastic distance between* x and x'. We arrive in an inverse problem of *anisotropic elastic travel-time tomography*:

Problem A.1. What can one infer about G in T^*M when *boundary distance data*

(A.5) $$\{d_G(x, x') \in \mathbf{R}_+ : x \in \Sigma, \ x' \in \Sigma'\}$$

is given?

We note that in general the sets Σ and Σ' can be very small, and (A.5) may not contain any information about G in some open set of T^*M. This is illustrated in Figure 2.
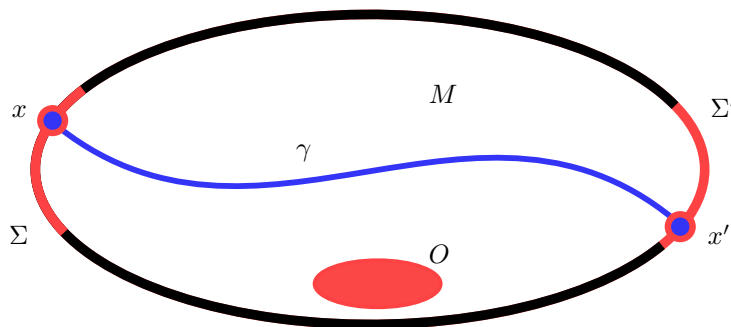

FIGURE 2. If there exists a set $O \subset M$ such that all characteristic curves whose terminal points are contained in $\Sigma \cup \Sigma'$ avoid the set T^*O, then data (A.5) does not provide information about G on T^*O.

Problem A.1 is highly non-linear and perturbations to anisotropic elasticity have been largely unresolved. In the following we consider linearizations of this problem. Since we have assumed that on the set U the eigenvalues G^2 and G^3 are distinct, they and the corresponding unit length eigenvector fields are smooth on U (see for instance [20, Chapter 11, Theorem 2]). In the following we recall a coordinate representation of the Hamilton's equation that the characteristic curve $\theta(t, (x_0, p_0))$, $(x_0, p_0) \in U \cap G^{-1}\{1\}$ satisfies. Let $q = q(x, p)$ be a polarization vector of G on U. In other words it is the unit (with respect to Euclidean metric) eigenvector of

the Christoffel matrix associated with the eigenvalue G. Then we can write the eigenvalue G as

$$G(x, p) = \Gamma_{ik} q^i q^k = A_{ijk\ell} q^i q^k p^j p^\ell, \quad (x, p) \in U.$$

From now on we denote the characteristic curve $\theta(t, (x_0, p_0))$ by $(x(t), p(t)) \in U$. Therefore the polarization vector $q = q(x(t), p(t))$ is also seen as a function of the time variable t implicitly. The following Hamilton's equation holds true on $(x(t), p(t))$

$$\dot{x}_m = \frac{\partial}{\partial p_m} H(x, p) = A_{ijkm} q^i q^k p^j$$

(A.6)

$$\text{and} \quad \dot{p}_m = -\frac{\partial}{\partial x_m} H(x, p) = -\frac{1}{2} \frac{\partial A_{ijk\ell}}{\partial x_m} q^i q^j p^j p^\ell.$$

Next, we recall the linearization scheme for the elastic distance d_G, that leads to an integral geometric problem of 4-tensor fields. This procedure has been introduced earlier in geophysical literature (see for instance [10]) using Fermat's principle. It is well known that characteristic curves of Hamiltonian flow satisfy this principle (see for instance [3]). For the convenience of the reader we give the proof and the exact claim below.

Since $D_p H$ does not vanish on U, the Legendre transform, that maps a co-vector to a vector, is well defined. Using the inverse of this map we define a Lagrangian function L on the image of U under this transform. That is

$$L(x, y) := p(x, y) \cdot y - H(x, p(x, y)).$$

If $(x(t), y(t)) = (x(t), \dot{x}(t))$ is the image of a characteristic curve of H, under the Legendre transform, then on this curve $L \equiv 1/2$ and the following Euler-Lagrange equations hold true

$$\frac{\partial}{\partial x_i} L(x, y) - \frac{d}{dt} \left(\frac{\partial}{\partial y_i} L(x, y) \right) = 0, \quad \text{for every } i \in \{1, 2, 3\}.$$

Let $x(t)$ be the base projection of a characteristic curve and $x_s(t)$ any smooth one-parameter variation of this curve that fixes the start point $x = x_s(0)$ and end point $x' = x_s(\tau_c)$. We choose the notation V for the variation field $V(t) = \frac{\partial}{\partial s} x_s(t)|_{s=0}$ of $x_s(t)$. Then using integration by parts, we obtain

$$\frac{d}{ds} \int_0^{\tau_c} L(x_s(t), \dot{x}_s(t)) dt \bigg|_{s=0} = \int_0^{\tau_c} \left(D_x L(x, \dot{x}) - \frac{d}{dt} (D_y L(x, \dot{x})) \right) \cdot V dt \bigg|_{s=0}.$$

Thus Euler-Lagrange equations imply that characteristic curves are the critical points of the energy functional

$$\mathcal{L}(\gamma) = \int_0^{\tau_c} L(\gamma(t), \dot{\gamma}(t)) dt,$$

where γ is any C^1-smooth curve. This is the version of Fermat's principle we need. We note that the characteristic curves are not necessarily local minimizers of the energy functional.

We now have the framework to linearize the travel-time tomography problem A.1. Let $\mathbf{A}_s$ be a smooth one-parameter family of elastic moduli. Suppose that we can choose sets U, M, Σ, Σ' such that the discussion above holds for any s, if $\mathbf{A}$

is replaced by $\mathbf{A}_s$. Since the Legendre transforms depend on the parameter s we obtain a family of Lagrangians

$$L_s(x, y) = p_s(x, y) \cdot y - H_s(x, p_s(x, y)),$$

which satisfy

(A.7)
$$\frac{\mathrm{d}}{\mathrm{d}s} L_s(x, y) = \frac{\mathrm{d}}{\mathrm{d}s} p_s \cdot y - \frac{\mathrm{d}}{\mathrm{d}s} H_s - D_p H_s \cdot \frac{\mathrm{d}}{\mathrm{d}s} p_s.$$

If the data (A.5) is independent of s, then due to Fermat's principle, equations (A.6) (A.7), and the assumption that q_s is a unit vector on U, we obtain

(A.8)
$$
\begin{aligned}
0 =& \frac{\mathrm{d}}{\mathrm{d}s} d_G(x, x') \Big|_{s=0} \\
=& \frac{\mathrm{d}}{\mathrm{d}s} \int_0^{\tau_c} L_s(x_s(t), \dot{x}_s(t)) \mathrm{d}t \Big|_{s=0} = \int_0^{\tau_c} \frac{\mathrm{d}}{\mathrm{d}s} L_s(x(t), \dot{x}(t)) \mathrm{d}t \Big|_{s=0} \\
=& -\int_0^{\tau_c} \frac{\mathrm{d}}{\mathrm{d}s} H_s(x, p) \mathrm{d}t \Big|_{s=0} = -\int_0^{\tau_c} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}s} \left[(A_s)_{ijk\ell} q_s^i q_s^k \right] p^j p^\ell \mathrm{d}t \Big|_{s=0} \\
=& -\int_0^{\tau_c} \frac{A'_{ijk\ell}}{2} q^i q^k p^j p^\ell \mathrm{d}t,
\end{aligned}
$$

where $A'_{ijk\ell} = \frac{\mathrm{d}}{\mathrm{d}s} (A_s)_{ijk\ell}|_{s=0}$, and $(x, p) = (x(t), p(t))$ is the characteristic curve of the reference model. Thus the linearization of anisotropic elastic travel-time d_G leads to an integral problem for 4-tensor fields.

In the following we will see that a similar linearization scheme applies in the case when $\mathbf{A}_s$ is isotropic for any s. Recall that in isotropic medium the elastic moduli can be written as

(A.9)
$$\mathbf{C}^0 = C^0_{ijk\ell} = \lambda \delta_{ij} \delta_{k\ell} + \mu \left(\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk} \right).$$

The functions $\lambda(x)$, $\mu(x) > 0$ are known as the Lamé parameters.

If the mass density ρ is given it follows from (A.9) that the isotropic Christoffel matrix is

(A.10)
$$\frac{\lambda}{\rho} p_i p_k + \frac{\mu}{\rho} \left(\delta_{ik} |p|_e^2 + p_i p_k \right),$$

whose eigenvectors are p and any unit vector q that is perpendicular to p. Moreover the corresponding eigenvalues are

(A.11)
$$G^1 = \Gamma_{ik} \hat{p}^i \hat{p}^k = \left(\frac{\lambda + 2\mu}{\rho} \right) |p|_e^2, \ \hat{p} = \frac{p}{|p|_e} \quad \text{and} \quad G := G^2 = G^3 = \Gamma_{ik} q^i q^k = \frac{\mu}{\rho} |p|_e^2.$$

In particular G^1 or G are both smooth and do not have inflection points. Thus any smooth one-parameter family $\mathbf{A}_s$ of isotropic elastic moduli satisfies all the additional assumptions we had to impose earlier for the general anisotropic case.

We recall that for isotropic elasticity, there are two different wave-speeds, namely, P-wave (Pressure, longitudinal wave) speed $c_P = \sqrt{\frac{\lambda + 2\mu}{\rho}}$ and S-wave (Shear, transverse wave) speed $c_S = \sqrt{\frac{\mu}{\rho}}$. Therefore we can consider M, in Problem A.1, as a Riemannian manifold with conformally Euclidean metric $g_P = c_P^{-2} \mathrm{d}s^2$ or $g_S = c_S^{-2} \mathrm{d}s^2$, where $\mathrm{d}s^2$ is the Euclidean metric.

We repeat the earlier linearization scheme with respect to smaller eigenvalue G and obtain

$$0 = \frac{\mathrm{d}}{\mathrm{d}s} d_G(x, x') \Big|_{s=0} = -\frac{1}{2} \int_0^{\tau_c} \left(\log \left(c_S^2 \right) \right)' \mathrm{d}t.$$

This equation follows from (A.11) and from the initial condition $(x, p) \in G^{-1}\{1\}$. Moreover in this case τ_c is the Riemannian distance between $x, x' \in \partial M$ with respect to the metric g_S. Therefore we have shown that the linearization of elastic travel-times in isotropic case leads to an integral geometry problem on functions.

In our final example we consider the linearization of the averaged quasi-shear wave travel-times in *weakly anisotropic medium*. Suppose that we are given a family $\mathbf{A}_s := \frac{\mathbf{C}^0}{\rho} + s\frac{\mathbf{C}}{\rho}$ of elastic moduli on some open and precompact domain of $\mathbf{R}^3$. Here $\mathbf{C}^0$ is an isotropic stiffness tensor having the form (A.9), $\mathbf{C}$ is an arbitrary anisotropic stiffness tensor, which satisfies the symmetry (A.1), and s is a real parameter close to zero. We note that for $|s|$ small enough the largest eigenvalue G_s^1 of the Christoffel matrix Γ_s of $\mathbf{A}_s$ is always distinct from the smaller ones. Therefore, $G_s^1(x, p)$ and the corresponding g_S-unit eigenvector field $q_s(x, p)$ are smooth in all variables (x, p, s). As the elastic moduli $\mathbf{A}_s$ is isotropic at $s = 0$, we have the degeneracy of eigenvalues, $G_0^2(x, p) = G_0^3(x, p)$ and, hence, $G_s^2(x, p)$ and $G_s^3(x, p)$ may not be smooth when s tends to zero. However, $G_s^2(x, p) + G_s^3(x, p) = \mathrm{Tr}\,(\Gamma_s)\,(x, p) - G^1(x, p)$ is smooth in (x, p, s), and strictly positive for $p \neq 0$. Therefore, we introduce the smooth one-parameter family of averaged qS-Hamiltonians,

$$(A.12) \qquad\qquad H_s := \frac{1}{4} \left(G_s^2 + G_s^3 \right),$$

inheriting the homogeneity of order 2 in p-variable. Since H_0 is conformally Euclidean, we note that, for $|s|$ sufficiently small, $\sqrt{H_s}$ is, in fact, a smooth family of co-Finsler metrics, with s-uniformly lower bounded injectivity radii while H_s does not have inflection points. We briefly analyze the Hamiltonian flow associated with H_s – which may be thought of as describing the propagation of singularities of an artificial wave – see below. In the following we will show that, up to the first order, the travel time along this flow can be identified with the average of travel times associated with the two qS-waves. This identification has been originally proposed in [9].

First we note that it follows from [9, equations (24) and (26)] that for $x, x' \in \mathbf{R}^3$ that are close enough we can write the average qS-travel time as

$$(A.13) \qquad \frac{d_{G_s^2}(x, x') + d_{G_s^3}(x, x')}{2}$$
$$= d_{g_S}(x, x') - s \int_{x_0(t)} \frac{1}{4} \sum_{ijk\ell} \left(\delta_{ik} - c_S^2 p_i p_k \right) p_j p_\ell \frac{C_{ijk\ell}}{\rho} \, \mathrm{d}t + \mathcal{O}(s^2),$$

where $x_0(t)$ is the g_S-geodesic connecting x to x' and $p = p(t)$ is the momentum of $x_0(t)$. We recall that the term $\delta_{ik} - c_S^2 p_i p_k$, in (A.13), is the projection onto the orthocomplement of p.

Next we study the linearization of H_s-travel times $d_{H_s}(x, x')$. Let $p_0(t) \in T_{x_0(t)}^* \mathbf{R}^3$ be the momentum of the g_S-geodesic connecting x to x'. Thus $(x_0(t), p_0(t))$ is a characteristic curve of H_0, with initial value $(x, p) \in T^* \mathbf{R}^3$, $|p|_{g_S} = 1$. In addition we denote $d_{H_s}(x, x') = \tau_c$ that is the g_S-distance between x and x'. We note that due to the uniform lower bound for the injectivity radii of $\sqrt{H_s}$, for every

$s \in (-\epsilon, \epsilon)$ there exists a $\sqrt{H_s}$-distance minimizing geodesic $x_s(t)$ from x to x', possibly after choosing x' closer to x and choosing $\epsilon > 0$ small enough.

We consider the following system of linear ODEs

$$\text{(A.14)} \qquad \frac{D}{\mathrm{d}t} e_s^I(t) = -\left\langle e_s^I, \frac{D}{\mathrm{d}t} q_s \right\rangle_{g_S} q_s(t), \quad e_s^I(0) = \eta_s^I \in T_x \mathbf{R}^3,$$

$$\langle \eta_s^I, \eta_s^J \rangle_{g_S} = \delta^{IJ}, \quad \text{and} \quad \langle \eta_s^I, q_s(0) \rangle_{g_S} = 0, \quad I, J \in \{1, 2\}.$$

Here $q_s(t) = q_s(x_s(t), p_s(t))$ and $\frac{D}{\mathrm{d}t}$ is the covariant derivative with respect to Riemannian metric g_S along the $\sqrt{H_s}$-geodesic $x_s(t)$ from x to x'. Both solutions e_s^I of (A.14) satisfy $\langle e_s^I, q_s \rangle_{g_S} \equiv 0$, due to the assumption $|q_s|_{g_S} \equiv 1$, hence it also holds that $\langle e_s^I, e_s^J \rangle_{g_S} \equiv \delta^{IJ}$ along x_s. Therefore we have shown that for any s the vector fields $\{q_s, e_s^1, e_s^2\}$ form a g_S-orthonormal frame moving along x_s. With respect to this basis we can write in $T^*_{x_s(t)} \mathbf{R}^3$

$$G_s^2 + G_s^3 = (c_S)^{-2} \mathrm{Tr}\left(\widehat{\Gamma}_s\right), \qquad \text{for } \widehat{\Gamma}_s = \begin{pmatrix} (\Gamma_s e_s^1) \cdot e_s^1 & (\Gamma_s e_s^1) \cdot e_s^2 \\ (\Gamma_s e_s^2) \cdot e_s^1 & (\Gamma_s e_s^2) \cdot e_s^2 \end{pmatrix}.$$

Since $e_0^1(t)$ and $e_0^2(t)$ are orthogonal to $\dot{x}_0(t)$, and we assumed that they are g_S-normalized, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[(\Gamma_s e_s^J) \cdot e_s^J \right]\Big|_{s=0} = \frac{C_{ijk\ell}}{\rho} p_0^j p_0^\ell \left(e_0^J\right)^i \left(e_0^J\right)^k, \quad J \in \{1, 2\}.$$

Finally we assume that H_s-travel-time $d_{H_s}(x, x')$ from x to x' is the constant τ_c. Then we run through the same linearization process as earlier and obtain

$$\text{(A.15)} \qquad \begin{aligned} 0 =& \frac{\mathrm{d}}{\mathrm{d}s} d_{H_s}(x, x') = -\int_0^{\tau_c} \frac{\mathrm{d}}{\mathrm{d}s} H_s(x_0(t), p_0(t)) \mathrm{d}t \Big|_{s=0} \\ =& -\int_0^{\tau_c} \frac{C_{ijk\ell}}{4 c_S^6 \rho} \dot{x}^j \dot{x}^\ell (e^1)^i (e^1)^k \mathrm{d}t - \int_0^{\tau_c} \frac{C_{ijk\ell}}{4 c_S^6 \rho} \dot{x}^j \dot{x}^\ell (e^2)^i (e^2)^k \mathrm{d}t, \end{aligned}$$

$$\dot{x} = \dot{x}_0(t), \ e^I := e_0^I(t), \ I \in \{1, 2\}.$$

In the last equation we also transformed the momentum variable into the velocity variable.

If we write the right hand side of (A.13) using the basis $\{\dot{x}_0, e_0^1, e_0^2\}$ we notice that the first order term equals to the right hand side of (A.15). Therefore we have verified that H_s-travel times and average of travel times associated with the two qS-waves coincide up to the first order. We also note that up to a constant the integrands in the right hand side of (A.15) are the same as in [42, Problem 7.1.1].

Finally we note that the polarization vector $q_0(t)$, coincides with the velocity field $\dot{x}_0(t)$ of the geodesic $x_0(t)$ in the reference medium g_S. Thus we have, due to (A.14), that $e_0^I(t)$ is given by a parallel translation of $\eta_0^I \in T_x \mathbf{R}^3$ along the reference ray $x_0(t)$. We have shown that an anisotropic perturbation of an averaged isotropic shear wave travel-time leads to an integral geometry problem on the 4-tensor field $f_{ijk\ell} := \frac{1}{2} \frac{C_{ki\ell j} + C_{kj\ell i}}{\rho c_S^6}$ which is related to the mixed ray transform $L_{2,2} f$ for the metric g_S. However we want to emphasize that the travel-time $d_{H_s}(x, x)$, of the aforementioned artificial wave, is given only by the averaged qS-Hamiltonian H_s, in (A.12), that is independent of the choice of the initial values η_s^I in (A.14), which yield the shear wave polarization vector fields $e_0^I(t)$ in formula (A.15).

To conclude the first part of the appendix we note that (A.15) implies that the travel-time data alone only gives us partial information about the mixed ray transform. However, if in addition we include the measurement of the shear wave amplitude, the complete mixed ray transform can be obtained. In Appendix B, we will show how one can recover the mixed ray transform from the linearization of the Dirichlet-to-Neumann map of an elastic wave equation on M, by probing with Gaussian beams. We also refer to [42, Chapter 7] for an alternative derivation of the mixed ray transform.

APPENDIX B. THE RELATION OF THE MRT AND THE DIRICHLET-TO-NEUMANN MAP

In this section, we give another derivation of the mixed ray transform from the inverse boundary value problem for elastic wave equations. We let $M \subset \mathbb{R}^3$ be a bounded domain with smooth boundary ∂M and $x = (x^1, x^2, x^3)$ be the Cartesian coordinates. The system of equations describing elastic waves can be written as

$$\rho \frac{\partial^2 u}{\partial t^2} - \operatorname{div}(\mathbf{C}\varepsilon(u)) = 0, \quad (t, x) \in (0, T) \times M,$$

(B.1)

$$u = h, \quad \text{on } (0, T) \times \partial M, \quad u(0, x) = \frac{\partial}{\partial t} u(0, x) = 0, \quad x \in M.$$

Here, u denotes the displacement vector and

$$\varepsilon(u) = (\varepsilon_{ij}(u)) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} \right)$$

is the linear strain tensor. Furthermore, $\mathbf{C} = (C_{ijk\ell}) = (C_{ijk\ell}(x))$ is the stiffness tensor and $\rho = \rho(x)$ is the density of mass. As in Appendix A we assume that $C_{ijk\ell}$ and ρ are all smooth functions, and the elastic tensor $\mathbf{C}$ is assumed to have the symmetries as in (A.1). In addition we assume the operator $-\operatorname{div}(\mathbf{C}\varepsilon(\cdot))$ to be elliptic, in the following sense: There exists $\delta > 0$ such that for any 3×3 real-valued symmetric matrix (ε_{ij}),

$$\sum_{i,j,k,\ell=1}^{3} C_{ijk\ell}\varepsilon_{ij}\varepsilon_{k\ell} \geq \delta \sum_{i,j=1}^{3} \varepsilon_{ij}^2.$$

Under these assumptions we let $\Lambda_{\mathbf{C}}$ to be the Dirichlet-to-Neumann map for the elastic wave equation (B.1) (see for instance [16]), given by

$$\Lambda_{\mathbf{C}} : C^2([0, T]; H^{1/2}(\partial M)) \ni h \mapsto \mathbf{C}\varepsilon(u)\nu|_{(0,T) \times \partial M} \in L^2([0, T]; H^{-1/2}(\partial M)).$$

Where $T > 0$ is large enough. The following inverse problem is of fundamental importance in seismology:

Problem B.1. Can we reconstruct the elastic tensor $\mathbf{C}$ and the density ρ from the Dirichlet-to-Neuman map $\Lambda_{\mathbf{C}}$?

We note that this problem is open for a general anisotropic $\mathbf{C}$. For isotropic medium, the uniqueness is shown under certain geometrical assumptions [6, 24, 39, 40, 46]. The uniqueness of transversely isotropic tensors is proved under piecewise analytic assumption in [16], as well as fully anisotropic tensors under piecewise homogeneous assumption. In contrast to the elastic problem, the corresponding inverse problem for scalar wave equation has been solved in [4, 5]. In this second appendix, instead of studying Inverse Problem B.1 for general anisotropic elastic

tensors, we consider a linearization of the problem around isotropic elasticity. We will see that the linearization leads to a family of ray transforms on four tensors.

From here we consider a one parameter family of anisotropic perturbations $s\mathbf{C}$ around the isotropic elasticity $\mathbf{C}^0$ of the form (A.9), that is we study an elastic tensor $\mathbf{C}_s = \mathbf{C}^0 + s\mathbf{C}$. We note that the map $\mathbf{C}_s \mapsto \Lambda_{\mathbf{C}_s}$ is Frechét differentiable at $\mathbf{C}^0$, and the Frechét derivative is $\dot{\Lambda}_{\mathbf{C}^0} : \mathbf{C} \mapsto \dot{\Lambda}_{\mathbf{C}^0}(\mathbf{C}) := \lim_{s \to 0} \frac{1}{s}(\Lambda_{\mathbf{C}^0 + s\mathbf{C}} - \Lambda_{\mathbf{C}^0})$. We will study the injectivity of the linear map $\dot{\Lambda}_{\mathbf{C}^0}$, whose action is given by

$$(B.2) \quad \langle \dot{\Lambda}_{\mathbf{C}^0}(\mathbf{C})h_1, h_2 \rangle_{(0,T) \times \partial M} = \int_{(0,T) \times M} C_{ijk\ell}(x)\partial_{x_i} w_j(x,t)\partial_{x_k} v_\ell(x,t)\mathrm{d}x\mathrm{d}t,$$

and w (v) solves the elastic wave equation (backward one) with the isotropic elastic tensor $\mathbf{C}^0$,

$$(B.3) \quad \begin{cases} \rho\frac{\partial^2 w}{\partial t^2} - \mathrm{div}(\mathbf{C}^0 w) = 0, & \text{in } (0,T) \times M, \\ w = h_1, & \text{on } (0,T) \times \partial M, \\ w(0,x) = \frac{\partial}{\partial t}w(0,x) = 0, & x \in M, \end{cases}$$
$$\begin{cases} \rho\frac{\partial^2 v}{\partial t^2} - \mathrm{div}(\mathbf{C}^0 v) = 0, & \text{in } (0,T) \times M, \\ v = h_2, & \text{on } (0,T) \times \partial M, \\ v(T,x) = \frac{\partial}{\partial t}v(T,x) = 0, & x \in M. \end{cases}$$

A similar linearization for the time-harmonic elastic wave equation can be found in [52].

Next we summarize the construction of Gaussian beam solutions to (B.3) used in [50, Section 3]. We also refer to [21] for more discussions on Gaussian beams solutions. Assume that $M \subset\subset \widetilde{M} \subset \mathbf{R}^3$, where $\widetilde{M}$ is open and bounded, the Riemannian metric $g_{P/S}$ with respect to $\mathbf{C}^0$ is known on $\widetilde{M}$ and the Riemannian manifold $(M, g_{P/S})$ is simple. We choose a maximal unit-speed geodesic γ in $(M, g_{P/S})$, and extend it to $\widetilde{M}$ assuming that once leaving M it will not return back to it. Then $\vartheta(t) = (t + \alpha, \gamma(t))$ is a null-geodesic in the Lorentzian manifold $((0,T) \times \widetilde{M}, -\mathrm{d}t^2 + g_{P/S})$ joining two points on $(0,T) \times \partial M$, as long as for $\alpha > 0$ and T large enough. Let us first take an asymptotic solution M_ϱ to the elastic wave equation on $(0,T) \times \widetilde{M}$,

$$\rho\frac{\partial^2 M_\varrho}{\partial t^2} - \mathrm{div}(\mathbf{C}^0 M_\varrho) = \mathcal{O}(\varrho^{-N}),$$

representing S-waves, of the form

$$M_\varrho = \chi\left(\sum_{j=0}^{N+1} \varrho^{-j}\mathbf{a}_j\right)e^{\mathrm{i}\varrho\varphi},$$

where ϱ is a large parameter and all the vector fields $(\mathbf{a}_j)_{j=0}^{N+1}$ and the phase function φ depend on time t and on location x. Here χ is a real valued cut-off function that is compactly supported and equal to 1 in a neighborhood of ϑ. The phase function φ satisfies $D\varphi|_{\vartheta(t)} = \dot{\gamma}(t)$, where D is the gradient with respect to the Euclidean metric on M. The imaginary part of the spatial Hessian of the phase function φ is positive definite, i.e. $\Im(D^2\varphi) > 0$. In addition we have

$$(B.4) \quad \mathbf{a}_0(\vartheta(t)) = A_S(\vartheta(t))\mathbf{e}(\vartheta(t)),$$

where $\mathbf{e}(\vartheta(t)) = \eta(t)$ is an arbitrary parallel vector field along $\gamma(t)$, perpendicular to $\dot\gamma(t)$, that is $\frac{D}{\mathrm{d}t}\eta(t) = 0$, $\eta(t) \perp \dot\gamma(t)$, and the amplitude A_S can be chosen such that

$$(B.5) \qquad A_S|_\vartheta = \det(Y_S)^{-1/2} c_S^{-1/2} \rho^{-1/2},$$

where $Y_S(x,t)$ is well defined on ϑ and is given as a solution of a second order ODE. Furthermore, we have

$$(B.6) \qquad \det(\Im(D^2\varphi))|\det(Y_S)|^2 \equiv c_0$$

on ϑ with c_0 a constant. Let $h_1 = M_\varrho|_{(0,T)\times\partial M}$, then one can determine the remainder R_ϱ satisfying zero boundary and initial conditions, such that

$$(B.7) \qquad w = M_\varrho + R_\varrho$$

is a solution to the first equation in (B.3). For any $m \in \mathbf{N}$ we can choose large enough N such that the remainder term R_ϱ satisfies the estimate $\|R_\varrho\|_{H^1(M\times(0,T))} = \mathcal{O}(\varrho^{-m})$. We also take

$$(B.8) \qquad v = \overline{M_\varrho} + R'_\varrho = \chi\left(\sum_{j=0}^{N+1} \varrho^{-j}\overline{\mathbf{a}_j}\right) e^{-\mathrm{i}\varrho\overline{\varphi}} + R'_\varrho,$$

for a solution of the backward elastic wave equation in (B.3) with $h_2 = \overline{M_\varrho}|_{(0,T)\times\partial M}$.

We multiply the identity (B.2) by $\varrho^{-\frac{1}{2}}$ and use the representations (B.7) for w and (B.8) for v, then due to the estimate [21, equation (3.33)] and the substitution $u_0 := \chi^2 \partial_{x_i}\varphi\,[\mathbf{a}_0]_j \overline{\partial_{x_k}\varphi\,[\mathbf{a}_0]_\ell}$ we obtain

$$(B.9)$$
$$\varrho^{-\frac{1}{2}}\langle\dot\Lambda_{\mathbf{C}^0}(\mathbf{C})h_1, h_2\rangle_{(0,T)\times\partial M} = \varrho^{\frac{3}{2}}\int_0^T\int_M e^{-2\varrho\Im\varphi}u_0\,\mathrm{d}x\mathrm{d}t + \mathcal{O}(\varrho^{-1}), \quad \varrho\to\infty.$$

Note that one can use the Fermi coordinates (τ, x'), as constructed in [50], under which the Euclidean volume form is $\mathrm{d}x\mathrm{d}t = c_S^3\mathrm{d}\tau \wedge \mathrm{d}x'$, moreover $\tau = \sqrt{2}t$ and $x' = 0$ on ϑ. Notice that $D\Im\varphi = 0$ on ϑ. Then after using the method of stationary phase to the integral

$$\int e^{-2\varrho\Im\varphi}u_0 c_S^3\mathrm{d}x'$$

with the phase function $f := \mathrm{i}2\Im\varphi$ and amplitude $u := u_0 c_S^3$ as in [26, Theorem 7.5.5.], we can write the right hand side of (B.9) into the form

$$(-\mathrm{i}\pi)^{\frac{3}{2}}\int_\vartheta \left|\det D^2\Im\varphi(\vartheta(\tau))\right|^{-\frac{1}{2}} u_0(\vartheta(\tau))c_S^3(\vartheta(\tau))\mathrm{d}\tau + \mathcal{O}(\varrho^{-1}), \quad \varrho\to\infty.$$

Next we use the properties (B.5) and (B.6) to observe that

$$u_0(\vartheta(\tau)) = |A_S|^2\partial_{x_i}\varphi\,\mathbf{e}_j\overline{\partial_{x_k}\varphi\,\mathbf{e}_\ell}\Big|_{\vartheta(t)} = |\det(Y_S)|^{-1}c_S^{-1}\rho^{-1}\dot\gamma_i(t)\eta_j(t)\dot\gamma_k(t)\eta_\ell(t)$$

$$= \frac{\left|\det D^2\Im\varphi(\vartheta(\tau))\right|^{\frac{1}{2}}}{\sqrt{c_0}}\rho^{-1}\dot\gamma_i(t)\eta_j(t)\dot\gamma_k(t)\eta_\ell(t).$$

Since c_0 is a known constant, we have verified that

$$(B.10) \quad 0 = \lim_{\varrho\to\infty}\varrho^{-\frac{1}{2}}\langle\dot\Lambda_{\mathbf{C}^0}(\mathbf{C})h_1, h_2\rangle_{(0,T)\times\partial M} = \int_\vartheta \frac{C_{ijk\ell}}{\rho}c_S^2\dot\gamma_i(t)\eta_j(t)\dot\gamma_k(t)\eta_\ell(t)\mathrm{d}t.$$

Finally we use the notation v^i for raising the indices of a co-vector v_i under the metric g_S, that is we have $v_i = c_S^{-2} v^i$. Then due to formula (B.10) we have recovered the mixed ray transform of the tensor field $f_{ijk\ell} := \frac{1}{2} \frac{C_{ki\ell j} + C_{kj\ell i}}{\rho c_S^6} \in S^2 \tau_M' \otimes S^2 \tau_M'$, from $\dot{\Lambda}_{\mathbf{C}^0}(\mathbf{C})$ along the arbitrarily chosen geodesic γ with respect to the metric g_S for any parallel vector field η along γ that is perpendicular to $\dot{\gamma}$.

For P-waves, we can construct solutions concentrating near a null geodesic $\vartheta(t) = (t + \alpha, \gamma(t))$ in the Lorentzian manifold $((0, T) \times M, -\mathrm{d}t^2 + g_P)$. For the solutions w, v constructed as (B.7), (B.8), we can take

$$\mathbf{a}_0 = A_P D\varphi,$$

where the P-wave amplitude satisfies

$$A_P|_\vartheta = \det(Y_P)^{-1/2} c_P^{-1/2} \rho^{-1/2}.$$

Similar as above, we end up with the (longitudinal) ray transform

$$0 = \int_\gamma \frac{C_{ijk\ell}}{\rho c_P^6} \dot{\gamma}^i(t) \dot{\gamma}^j(t) \dot{\gamma}^k(t) \dot{\gamma}^\ell(t) \mathrm{d}t.$$

Acknowledgments

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