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A Local Limit Theorem for Linear Random Fields

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Abstract

In this paper, we establish a local limit theorem for linear fields of random variables constructed from independent and identically distributed innovations each with finite second moment. When the coefficients are absolutely summable we do not restrict the region of summation. However, when the coefficients are only square-summable we add the variables on unions of rectangle and we impose regularity conditions on the coefficients depending on the number of rectangles considered. Our results are new also for the dimension 1, i.e. for linear sequences of random variables. The examples include the fractionally integrated processes for which the results of a simulation study is also included.

1 Introduction

Consider independent and identically distributed (i.i.d.) standard normal random variables $\{Z_k\}_{k=1}^n$ and their sum $S_n = \sum_{k=1}^n Z_k$. In this context, we can define a sequence of measures given by

$$\mu_n(a, b) = \sqrt{2\pi n} P(S_n \in (a, b)) = \int_a^b e^{-\frac{1}{2n}x^2} dx, \quad (1)$$

and with this specific form, one can easily see that the integrand converges to one as $n \rightarrow \infty$. This sequence of measures therefore converges to Lebesgue measure. The result is also true for the situation when $\{Z_k\}_{k=1}^n$ is merely a sequence of i.i.d. random variables satisfying the central limit theorem (CLT). A result such as this is called a local limit theorem. A local limit theorem is much more delicate than the associated CLT.

Local limit theorems have been studied intensively for the case of lattice random variables and the case of non-lattice random variables. The lattice case means that there exists $v > 0$ and $a \in \mathbb{R}$ such that the values of Z_0 are concentrated on the lattice $\{a + kv : k \in \mathbb{Z}\}$, whereas the non-lattice case means that no such a and v exists. In this paper, we consider the non-lattice case.

For sequences of i.i.d. random variables, the local limit theorem in the non-lattice case is due to Shepp (1964) and the case of i.i.d. random vectors is considered by Stone (1965). We also refer the reader to

the books by Ibragimov and Linnik (1971), Petrov (1975), and Gnedenko (1962). Some papers containing classes of independent non identically distributed random variables include Mineka and Silverman (1970), Shore (1978) and Maller (1978). For more recent results we mention the paper by Dolgopyat (2016) and the references therein.

Linear random fields (also known in the statistical literature as spatial linear processes) have been extensively studied in probability and statistics. For example, Mallik and Woodroffe (2011) studied the CLT for linear random fields, and Sang and Xiao (2018) established exact moderate and large deviation asymptotics for linear random fields under moment or regularly varying tail conditions by extending the methods for linear processes in Peligrad et al. (2014). With a conjugate method, Beknazaryan, Sang, and Xiao (2019) studied the Cramér type moderate deviation for partial sums of linear random fields. We refer to Sang and Xiao (2018) for a brief review of the study of asymptotic properties of linear random fields and to Koul, Mimoto, and Surgailis (2016), Lahiri and Robinson (2016) and the references therein for recent developments in statistics. However, to the best of our knowledge, the local limit results for linear random fields, or even for one dimensional indexed linear processes, have not yet been established in the literature.

In this paper, we consider linear random fields of the form

$$X_j = \sum_{i \in \mathbb{Z}^d} a_i \varepsilon_{j-i} \quad (2)$$

defined on \mathbb{Z}^d , where the innovations ε_i are i.i.d. random variables with mean zero ($\mathbb{E} \varepsilon_i = 0$), finite variance ($\mathbb{E} \varepsilon_i^2 = \sigma_\varepsilon^2$), and non-lattice distribution and where the collection $\{a_i : i \in \mathbb{Z}^d\}$ of real coefficients satisfies

$$\sum_{i \in \mathbb{Z}^d} a_i^2 < \infty. \quad (3)$$

As a matter of fact, the field X_j given in (2) exists in $L^2(\mathbb{R})$ and almost surely if and only if (3) is satisfied. We say that the process has *long memory* (*long range dependence*) if $\sum_{i \in \mathbb{Z}^d} |a_i| = \infty$.

Let Γ_n^d be a sequence of finite subsets of \mathbb{Z}^d , and define the sum

$$S_n = \sum_{j \in \Gamma_n^d} X_j \quad (4)$$

with variance

$$B_n^2 = \text{Var}(S_n). \quad (5)$$

We may express (2) as

$$X_j = \sum_{i \in \mathbb{Z}^d} a_{j-i} \varepsilon_i,$$

from which it is easily apparent that

$$\text{var}(X_j) = \sigma_\varepsilon^2 \sum_{i \in \mathbb{Z}^d} a_i^2.$$

The sum $S_n = \sum_{j \in \Gamma_n^d} X_j$, expressed as an infinite linear combination of the innovations, is given by

$$S_n = \sum_{i \in \mathbb{Z}^d} b_{n,i} \varepsilon_i, \quad (6)$$

where

$$b_{n,i} = \sum_{j \in \Gamma_n^d} a_{j-i},$$

and similar to our earlier observation,

$$B_n^2 = \text{Var}(S_n) = \sigma_\varepsilon^2 \sum_{i \in \mathbb{Z}^d} b_{n,i}^2.$$

Without loss of generality, throughout the paper we assume that $\sigma_\varepsilon^2 = 1$. Note that, by the representation (6), S_n can be expressed as a sum of independent variables. However, the local limit theorems available

for sums of independent random variables that are not identically distributed involve rather strong degrees of stationarity which are not satisfied by (6). Building on the previous work of Shore (1978), we are able to show that the local limit theorem holds for all the situations including the long memory linear random fields, assuming reasonable requirements of the innovations and of the sets Γ_n^d .

As a matter of fact, we shall establish the following uniform local limit theorem: For all continuous complex-valued functions $h(x)$ with $|h| \in L^1(\mathbb{R})$ and with Fourier transform \hat{h} real and with compact support,

$$\lim_{n \rightarrow \infty} \sup_{u \in \mathbb{R}} \left| \sqrt{2\pi} B_n E h(S_n - u) - [\exp(-u^2/2B_n^2)] \int h(x) \lambda(dx) \right| = 0, \quad (7)$$

where λ is the Lebesgue measure. Here we require that $B_n \rightarrow \infty$ as $n \rightarrow \infty$. By arguments in Section 4 of Hafouta and Kifer (2016) this result implies that (7) also holds for the class of real continuous functions with compact support and by the Theorem 10.7 in Breiman (1992) it follows that

$$\lim_{n \rightarrow \infty} \sup_{u \in \mathbb{R}} \left| \sqrt{2\pi} B_n P(a + u \leq S_n \leq b + u) - [\exp(-u^2/2B_n^2)](b - a) \right| = 0,$$

for any $a < b$. In particular, since $B_n \rightarrow \infty$ as $n \rightarrow \infty$, then for fixed $A > 0$,

$$\lim_{n \rightarrow \infty} \sup_{|u| \leq A} \left| \sqrt{2\pi} B_n P(a + u \leq S_n \leq b + u) - (b - a) \right| = 0.$$

If we further take $u = 0$, then,

$$\lim_{n \rightarrow \infty} \sqrt{2\pi} B_n P(S_n \in [a, b]) = b - a.$$

In other words, the sequence of measures $\sqrt{2\pi} B_n P(S_n \in [a, b])$ of the interval $[a, b]$ converges to Lebesgue measure.

It should be noted that the local limit theorem, as formulated in (7), is useful to the study of recurrence conditions for S_n , as explained in Orey (1966) and Mineka and Silverman (1970).

The paper is organized as follows. In Section 2 we state and comment on the results, which include the long memory case. Section 3 is dedicated to examples of long memory time series to which we can apply the local limit theorem stated in the previous section. In Section 4 we summarize the result of a simulation study, designed to analyze the performance of our asymptotic local theorem for a finite sample. Finally, Section 5 contains the proof of the main result.

A few remarks about notation and terms used in the paper follow. In constructing the sum S_n that we analyze in this paper, we make use of a sequence Γ_n^d of subsets of \mathbb{Z}^d . For use with the long memory case, for each n , we will construct the sequence Γ_n^d of sets using a union of rectangles, whose dimensions could depend on n . For $\underline{n}(w) = \underline{n}(w, n) = (\underline{n}_1(w), \underline{n}_2(w), \dots, \underline{n}_d(w)) \in \mathbb{Z}^d$ and $\bar{n}(w) = (\bar{n}_1(w), \bar{n}_2(w), \dots, \bar{n}_d(w)) \in \mathbb{Z}^d$ with $\underline{n}(w) \leq \bar{n}(w)$, where $1 \leq w \leq J_n$, put $\Gamma_n^d(w) = \prod_{\ell=1}^d [\underline{n}_\ell(w), \bar{n}_\ell(w)] \cap \mathbb{Z}^d$. Any set of this form will be called a *discrete rectangle*. In general, we require the index sets to be of the form

$$\Gamma_n^d = \bigcup_{w=1}^{J_n} \Gamma_n^d(w), \quad (8)$$

where $\{\Gamma_n^d(w)\}_{w=1}^{J_n}$ is a pairwise disjoint family of discrete rectangles. Throughout the paper, we demand that $|\Gamma_n^d| \rightarrow \infty$ as $n \rightarrow \infty$. Here, for $\Gamma \subset \mathbb{Z}^d$, we denote the cardinality of Γ by $|\Gamma|$. For $n = (n_1, \dots, n_d)$ the Euclidian norm will be denoted by $\|n\| = (n_1^2 + n_2^2 + \dots + n_d^2)^{1/2}$. Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be real-valued sequences. To indicate relative growth rates at infinity, we use $a_n \propto b_n$ to indicate that $a_n/b_n \rightarrow C \in \mathbb{R}^+$, and the particular case when $C = 1$ is denoted $a_n \sim b_n$. By $a_n = o(b_n)$ we understand that $a_n/b_n \rightarrow 0$ and $a_n = O(b_n)$ means that $\limsup |a_n/b_n| < C$ for some positive number C . Throughout the paper, an indicator function will be denoted as \mathcal{I} . A function $l : [0, \infty) \rightarrow \mathbb{R}$ is referred to as *slowly varying* (at ∞) if it is positive and measurable on $[A, \infty)$ for some $A \in \mathbb{R}^+$ such that $\lim_{x \rightarrow \infty} l(\lambda x)/l(x) = 1$ holds for each $\lambda \in \mathbb{R}^+$. The integer part of a real number x will be denoted by $\lfloor x \rfloor$.

2 Main Results

In this work, we investigate the conditions under which the local limit theorem holds for the partial sums of the linear random fields given by (2). Before we can treat the local limit theorem of this paper, we mention the following CLT for linear random fields which is a variant of Corollary 2 and Corollary 4 of Mallik and Woodrooffe (2011). For $d = 1$ and $J_n = 1$ with $\Gamma_n^1 = \{1, 2, \dots, n\}$ the result is Theorem 18.6.5 in Ibragimov and Linnik (1971).

Theorem 2.1 (Mallik and Woodrooffe, 2011) *Let S_n and B_n be defined as in (4) and (5). Assume that $B_n \rightarrow \infty$. When the field has long range dependence we additionally require that the sets Γ_n^d are constructed as a disjoint union of J_n discrete rectangles, where $J_n = o(B_n^2)$, while otherwise no such restriction is required. Under these conditions, S_n/B_n converges in distribution to the standard normal distribution.*

Remark 2.1 *In case $\sum_{i \in \mathbb{Z}^d} |a_i| < \infty$, this theorem was proved in Corollary 2 of Mallik and Woodrooffe (2011). When the field has long range dependence the result of this theorem is a version of their Corollary 4. Indeed, from relation (11) in the proof of Proposition 2 of the same paper, the condition $\sup_{i \in \mathbb{Z}^d} |b_{n,i}|/B_n \rightarrow 0$ is satisfied if $J_n = o(B_n^2)$.*

Remark 2.2 *If $J_n = 1$, then Γ_n^d consists of only one rectangle $\Gamma_n^d(w) = \prod_{\ell=1}^d [\underline{n}_\ell(w), \bar{n}_\ell(w)] \cap \mathbb{Z}^d$. The condition $B_n \rightarrow \infty$, implies that $\max_{1 \leq \ell \leq d} |\bar{n}_\ell(w) - \underline{n}_\ell(w)| \rightarrow \infty$ as $n \rightarrow \infty$. Note that if more than one difference among $(\bar{n}_\ell(w) - \underline{n}_\ell(w))_{1 \leq \ell \leq d}$ tend to infinity, they can grow at independent rates.*

Remark 2.3 *Given that R_0 is an open connected subset of $(-1/2, 1/2]^d$ satisfying some regularity conditions and $\{\mu_n\}$ is a sequence of positive numbers such that $\mu_n \rightarrow \infty$ as $n \rightarrow \infty$, Lahiri and Robinson (2016) studied the central limit theorems for the sums of linear random fields over dilated regions $\Gamma_n^d = R_n \cap \mathbb{Z}^d$, where $R_n = \mu_n R_0$. In particular, when the coefficients are of the form $a_i = l(||i||)/||i||^\alpha$ with $d/2 < \alpha < d$, l a slowly varying function at infinity, then, as shown in Lahiri and Robinson (2016), $B_n^2 \propto \mu_n^{3d-2\alpha} l^2(\mu_n)$. However, since the volume of $R_n = O(\mu_n^d)$, the sample size $|\Gamma_n^d| = O(\mu_n^d)$. We can separate Γ_n^d into J_n disjoint rectangles with $J_n = O(\mu_n^d)$. Since $B_n^2 \propto \mu_n^{3d-2\alpha} l^2(\mu_n)$ and $3d - 2\alpha > d$, it is easy to see that $J_n = o(B_n^2)$. Hence their central limit theorem (Theorem 3.2 there) in the long memory case is a direct consequence of Theorem 2.1 here.*

Denote the characteristic function of ε_0 by $\varphi_\varepsilon(t) := \mathbb{E}(\exp\{it\varepsilon_0\})$. It is well known that ε_0 not having a lattice distribution is equivalent to $|\varphi_\varepsilon(t)| < 1$ for all $t \neq 0$. On the other hand, the Cramér condition means that $\limsup_{|t| \rightarrow \infty} |\varphi_\varepsilon(t)| < 1$. Thanks to the Riemann-Lebesgue lemma, the Cramér condition is automatically satisfied if the distribution function of ε_0 is absolutely continuous with respect to the Lebesgue measure. It should be mentioned that ε_0 has a non-lattice distribution whenever $\varphi_\varepsilon(t)$ satisfies the Cramér condition. See Lemma 5.1.

[The ‘‘Cramér condition’’ defined in the preceding paragraph is different from, and has no particular connection with, another condition (involving the existence of moment generating functions on certain domains) that has absolutely no role in this paper but has elsewhere in the probability theory literature sometimes been referred to as the ‘‘Cramér condition’’.]

Theorem 2.2 *Let S_n and B_n be defined as in (4) and (5) and assume that $B_n \rightarrow \infty$. In the case $\sum_{i \in \mathbb{Z}^d} |a_i| < \infty$, we assume that ε_0 is non-lattice. If the field has long range dependence, we assume that the innovations satisfy the Cramér condition and that the sets Γ_n^d are constructed as a disjoint union of J_n discrete rectangles and we require that*

$$\frac{J_n^{2/d} \log(B_n)}{\sup_{i \in \mathbb{Z}^d} |b_{n,i}|^{2/d}} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (9)$$

Under these conditions, (7) holds.

Remark 2.4 Because (X_k) is stationary we always have

$$\text{var}(S_n) = B_n^2 \leq |\Gamma_n^d|^2 \mathbb{E}(X_0^2). \quad (10)$$

So, if $J_n^{2/d} \log |\Gamma_n^d| / \sup_{i \in \mathbb{Z}^d} |b_{n,i}|^{2/d} \rightarrow 0$, then (9) is satisfied.

Remark 2.5 One may ponder whether condition (9) always holds for the long memory case. To settle such concerns, we offer the following counterexample. Take a linear random field of the form (2) with $d = 1$ —i.e., a linear process. In particular, consider the one-sided linear process with alternating harmonic coefficients. That is, put $a_i = (-1)^{i+1}/i$ for $i \in \mathbb{N}$, and $a_i = 0$ for $i \in \mathbb{Z} \setminus \mathbb{N}$. In this example, we take $J_n = 1$ and the index set Γ_n^1 to be the set $\{1, 2, \dots, n\}$. Note that $\sup_{i \in \mathbb{Z}} |b_{n,i}|$ does not go to infinity as $n \rightarrow \infty$, and therefore, the aforementioned condition is not satisfied. Even though the local limit theorem is not guaranteed by our Theorem 2.2 for this case, we note that the central limit theorem holds, since $B_n \rightarrow \infty$ as $n \rightarrow \infty$.

Remark 2.6 In Theorem 2.2 we provide a local limit theorem for linear random fields when the coefficients are absolutely summable with no restriction on the sequence of regions other than $B_n \rightarrow \infty$. We also provide a local limit theorem for the sum of a long memory linear random field over a sequence of regions Γ_n^d which are a disjoint union of discrete rectangles and with no other specification on the individual coefficients $a_i, i \in \mathbb{Z}^d$ besides the global conditions (3) and (9). In practical application it allows us to have disjoint discrete rectangles as spatial sampling regions, and the number of these disjoint spatial rectangular sampling regions may increase as the sample size increases. The discrete spatial rectangular sampling regions also include $(\prod_{k=1}^d [\underline{n}_k, \bar{n}_k]) \cap \mathbb{Z}^d$ where $\underline{n}_k = \bar{n}_k$ for some k 's. We may have a single point region if the equality holds for all k 's. We would also like to mention that our local limit results are new also for $d = 1$. Furthermore, we have the freedom to take the sum over J_n blocks of random variables as long as $J_n = o(B_n^2)$ for central limit theorem and $J_n = o(\sup_{i \in \mathbb{Z}^d} |b_{n,i}| / (\log(B_n))^{d/2})$ for local limit theorem.

Remark 2.7 El Machkouri et al. (2013) worked with nonlinear random fields, and in their work on central limit theorem, they required the condition that $|\partial\Gamma_n^d|/|\Gamma_n^d| \rightarrow 0$ as $n \rightarrow \infty$, where $\partial\Gamma_n^d$ is the boundary of the region Γ_n^d . We would just like to mention that our results demonstrate that this condition is not necessary in the linear random field setting. For example, in the case $d = 2$ with $\Gamma_n^2 = ([1, n] \times [1, 3]) \cap \mathbb{Z}^2$, we have $|\partial\Gamma_n^2|/|\Gamma_n^2| = (2n + 2)/3n$.

3 Examples

There are many situations of interest when (9) holds. In particular it is satisfied by the fractionally integrated processes which play an important role for analyzing various models in econometrics. They are a particular case of linear processes with regularly varying coefficients for which we provide a few examples. Of course, examples of this type, where the coefficients are absolutely summable, will certainly satisfy the local theorem as given in the first part of Theorem 2.2. In what follows, we shall discuss only the long memory case.

Example 1. Suppose we work on one rectangle $\Gamma_n^d = \prod_{\ell=1}^d [1, n_\ell] \cap \mathbb{Z}^d$, where $n_\ell = n_\ell(n)$ is a sequence of natural numbers for each ℓ . Let X_n and B_n be defined as in (2) and (5). For $j = (j_1, j_2, \dots, j_d)$, let $(a_j)_{j \in \mathbb{Z}^d}$ with $\sum_{j \in \mathbb{Z}^d} a_j^2 < \infty$ and assume that for some constant C ,

$$a_j \geq C \prod_{\ell=1}^d (1/|j_\ell|)^{\beta_\ell} \text{ with } \beta_\ell > 1/2, \quad 1 \leq \ell \leq d. \quad (11)$$

Here we take $1/|j_\ell| = 1$ if $j_\ell = 0$. Assume that at least one β_ℓ is strictly smaller than 1. Then $(a_j)_{j \in \mathbb{Z}^d}$ is not absolutely summable and the linear random field has long memory. Let us assume now that $\beta_k < 1$, for all positive integers k , $1 \leq k \leq m$, for some m with $1 \leq m \leq d$ and $\beta_\ell > 1$, $m + 1 \leq \ell \leq d$. Assume that $n_k \rightarrow \infty$, for all k , $1 \leq k \leq m$ and, for some $M < \infty$, $n_k \leq M$, $m + 1 \leq k \leq d$. Then, starting from (11),

by simple analytical manipulations, we have that

$$\begin{aligned} b_{n,0} &= \sum_{j \in \Gamma_n^d} a_j \geq C \sum_{j \in \Gamma_n^d} \prod_{\ell=1}^d (1/|j_\ell|)^{\beta_\ell} = C \prod_{\ell=1}^d \sum_{1 \leq j_\ell \leq n_\ell} (1/j_\ell)^{\beta_\ell} \\ &\propto \prod_{\ell=1}^m n_\ell^{1-\beta_\ell} \prod_{\ell=m+1}^d \sum_{1 \leq j_\ell \leq n_\ell} (1/j_\ell)^{\beta_\ell} \geq C_1 \prod_{\ell=1}^m n_\ell^{1-\beta_\ell}. \end{aligned}$$

On the other hand, using (10), $B_n^2 \leq |\Gamma_n^d|^2 \mathbb{E}(X_0^2) = \prod_{\ell=1}^d n_\ell^2 \mathbb{E}(X_0^2)$. Based on these computations, we obtain that

$$\begin{aligned} \frac{\log B_n}{\sup_{\ell \in \mathbb{Z}^d} |b_{n,\ell}|^{2/d}} &\leq C_2 \frac{\sum_{\ell=1}^d \log n_\ell}{b_{n,0}^{2/d}} \\ &\leq C_3 \frac{\sum_{\ell=1}^m \log n_\ell}{\prod_{\ell=1}^m n_\ell^{2(1-\beta_\ell)/d}} \rightarrow 0 \quad \text{as } \min_{1 \leq j \leq m} (n_j) \rightarrow \infty. \end{aligned}$$

This latter limit, shows that condition (9) is satisfied. Hence, the local limit theorem in Theorem 2.2 holds, provided Cramér condition is satisfied.

In the context of this example, note that we can also consider sets of the form $\Gamma_n^d = \bigcup_{w=1}^{J_n} \Gamma_n^d(w)$, where $\{\Gamma_n^d(w) : 1 \leq w \leq J_n\}$ are disjoint rectangles $\Gamma_n^d(w) = \prod_{\ell=1}^d [c_w, c_w + n_\ell] \cap \mathbb{Z}^d$ of equal size. For simplicity let us take $m = d$. For this case we have $B_n^2 \leq |\Gamma_n^d|^2 \mathbb{E}(X_0^2) = J_n^2 \prod_{\ell=1}^d n_\ell^2 \mathbb{E}(X_0^2)$ and $\sup_{\ell \in \mathbb{Z}^d} |b_{n,\ell}| \geq \prod_{\ell=1}^d n_\ell^{1-\beta_\ell}$. Hence,

$$\frac{J_n^{2/d} \log B_n}{\sup_{\ell \in \mathbb{Z}^d} |b_{n,\ell}|^{2/d}} \leq C_4 \frac{J_n^{2/d} (\log J_n + \sum_{\ell=1}^d \log n_\ell)}{\prod_{\ell=1}^d n_\ell^{2(1-\beta_\ell)/d}},$$

which converges to 0 when $\min_{1 \leq j \leq d} (n_j) \rightarrow \infty$, as soon as $J_n^{2/d} = o(\prod_{\ell=1}^d n_\ell^{2(1-\beta_\ell)/d} / \sum_{\ell=1}^d \log n_\ell)$. If we impose this condition on J_n then we can also obtain the conclusion of Theorem 2.2 for this situation.

Example 2. This example is a variant of Example 1, with the same index sets Γ_n^d . Take now

$$a_j = \prod_{\ell=1}^d (1/|j_\ell|)^{\alpha_\ell} h_\ell(|j_\ell|), \quad (12)$$

with $\alpha_\ell > 1/2$ and $h_\ell(\cdot)$ are positive slowly varying functions, $1 \leq \ell \leq d$. Again, we let $1/|j_\ell| = 1$ if $j_\ell = 0$. Then $\sum_{\ell \in \mathbb{Z}^d} a_\ell^2 < \infty$. If $\alpha_k < 1$ for some $1 \leq k \leq d$, then $\sum_{\ell \in \mathbb{Z}^d} |a_\ell| = \infty$ and we are in the long memory case. For some m with $1 \leq m \leq d$, assume now that $1/2 < \alpha_k < 1$, for all k , $1 \leq k \leq m$, and $\alpha_\ell > 1$, $m+1 \leq \ell \leq d$. Recall now that, for a positive slowly varying function $h(x)$, we have that for every $\varepsilon > 0$, $\lim_{x \rightarrow \infty} x^\varepsilon h(x) = \infty$ and $\lim_{x \rightarrow \infty} x^{-\varepsilon} h(x) = 0$ (see Seneta, 1976). Then we can find constants $1/2 < \beta_k < 1$, for all k , $1 \leq k \leq m$, and $\beta_\ell > 1$, $m \leq \ell \leq d$ such that (11) holds. If we assume that $n_k \rightarrow \infty$, for all k , $1 \leq k \leq m$ and, for some $M < \infty$, $n_k < M$, $m+1 \leq k \leq d$, then the conditions in Example 1 hold. Therefore, for this case, the conclusion of Theorem 2.2 holds with

$$B_n^2 = \prod_{\ell=1}^m c(\alpha_\ell) n_\ell^{3-2\alpha_\ell} h_\ell^2(n_\ell),$$

with constants $c(\alpha_\ell)$ specified in Wang, Lin and Gulati (2001).

Example 3. We work this time on one rectangle $\Gamma_n^d = \prod_{\ell=1}^d [1, k_i n] \cap \mathbb{Z}^d$, where $k_i \in \mathbb{R}^+$, $1 \leq i \leq d$. For $j = (j_1, j_2, \dots, j_d)$, let $(a_j)_{j \in \mathbb{Z}^d}$ with $\sum_{\ell \in \mathbb{Z}^d} a_\ell^2 < \infty$ and assume that for some constant $C > 0$,

$$a_j \geq C \|j\|^{-\beta} \quad \text{with } \beta \in (d/2, d) \text{ and } j \neq 0_d = (0, 0, \dots, 0). \quad (13)$$

It is easy to see that $\sum_{\ell \in \mathbb{Z}^d} a_\ell = \infty$ and we also have $a_j \geq C(j_1 + j_2 + \dots + j_d)^{-\beta}$. Straightforward computations show that we can find a positive constant C_1 and $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have

$$b_{n,0} = \sum_{j \in \Gamma_n^d} a_j \geq C_1 n^{d-\beta}.$$

Therefore, using (10), we can find a positive constant C_2 such that

$$\begin{aligned} \frac{\log(B_n)}{\sup_{i \in \mathbb{Z}^d} b_{n,i}^{2/d}} &\leq \frac{C_2 \log(n^d)}{b_{n,0}^{2/d}} \\ &\leq \frac{C_2 d \log n}{C_1 n^{2(d-\beta)/d}} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

This shows that condition (9) of Theorem 2.2 is satisfied and the local limit theorem holds if Cramér condition is satisfied.

Furthermore, we can also mention that for this case the local limit theorem also holds if we actually consider an union of J_n rectangles of equal size $(k_1 n, \dots, k_d n)$ such that $J_n^{2/d} = o(n^{2(d-\beta)/d} / \log n)$.

As a particular example of this kind we shall give an example treated by Surgailis (1982) and also by Beknazaryan et al. (2019).

Example 4. Assume that Γ_n^d are cubic, i.e., $\Gamma_n^d = [-n, n]^d \cap \mathbb{Z}^d$, and put $a_i = l(\|i\|)G(i/\|i\|)\|i\|^{-\alpha}$ with $\alpha \in (d/2, d)$, where $l(x)$ is slowly varying at ∞ and $G : \mathbb{S}_{d-1} \rightarrow \mathbb{R}^+$ is continuous on its domain (the unit sphere in d -dimensional space). For this example we know that $B_n \propto n^{\frac{3d}{2}-\alpha} l(n)$ (see Surgailis, 1982, Theorem 2) and from Beknazaryan et al. (2019) we can easily deduce that $\sup_{i \in \mathbb{Z}^d} |b_{n,i}| \propto (n^{d-\alpha} l(n))$. We could also see directly that condition (9) of Theorem 2.2 is satisfied by using the proof of Example 3. Indeed, by the properties of slowly varying functions, we can find $\beta \in (d/2, d)$ such that $a_j \geq C\|j\|^{-\beta}$. Since we are in the long memory case, if the innovations satisfy the Cramér condition, then (7) holds. ■

4 Simulation Study

In this section, we perform a simulation study for the local limit theorem in Example 4, applied to the one-dimensional case. The linear processes we used here are the fractionally integrated processes FARIMA(0, 1 - α , 0) which play an important role in financial time series modeling, and they are widely studied. Such processes are defined for $1/2 < \alpha < 1$ by

$$X_j = (1 - B)^{\alpha-1} \varepsilon_j = \sum_{i \geq 0} a_i \varepsilon_{j-i} \text{ with } a_i = \frac{\Gamma(i+1-\alpha)}{\Gamma(1-\alpha)\Gamma(i+1)},$$

where B is the backward shift operator, $B\varepsilon_j = \varepsilon_{j-1}$. By the well-known fact that $\lim_{n \rightarrow \infty} \Gamma(n+x)/n^x \Gamma(n) = 1$ for any real x , we have $\lim_{n \rightarrow \infty} a_n/n^{-\alpha} = 1/\Gamma(1-\alpha)$. The variance of the partial sum $S_n = \sum_{j=1}^n X_j$ is

$$B_n^2 \sim c_\alpha n^{3-2\alpha} \mathbb{E} \varepsilon^2 / [(1-\alpha)(3-2\alpha)\Gamma^2(1-\alpha)] \quad (14)$$

where

$$c_\alpha = \int_0^\infty x^{-\alpha} (1+x)^{-\alpha} dx.$$

The variance formula for the partial sum of FARIMA(0, 1 - α , 0) is well known. See, for example, Wang, Lin and Gulati (2001).

Using the FARIMA(0, 1 - α , 0) model, linear processes with innovations following the Student's t distribution with 5 degrees of freedom were generated. Employing the MATLAB code of Fay et al. (2009),

N replicates of linear processes were generated, each of length n . Specifically, we generated cases with $N = 5,000$ and $N = 10,000$ cross-referenced with $n = 2^{10}$, $n = 2^{12}$, and $n = 2^{14}$, and this was done for each of the values $\alpha = 0.95$, $\alpha = 0.70$, and $\alpha = 0.55$. Once the data were obtained, the local limit measures of various intervals were estimated by using relative frequency to estimate $P(S_n \in (a, b))$ and using the approximation of B_n given in (14).

The simulation study supports the validity of Example 4 for the one-dimensional case. See Tables 1 and 2 below. Of particular interest is the general tendency of results to be better for larger N , which is likely explained by the fact that we estimate $P(S_n \in (a, b))$ using relative frequency. Also, we notice that the results generally get better with larger values of the sample size n .

Table 1: Local limit measures of the intervals $(-100, 0)$, $(-50, 50)$, and $(0, 100)$ - one per row - using N one-dimensional linear processes, each of length n , employing various long memory cases using the $\text{FARIMA}(0, 1 - \alpha, 0)$ model with t_5 innovations.

N	$n = 2^{10}$			$n = 2^{12}$			$n = 2^{14}$		
	$\alpha = 0.95$	$\alpha = 0.70$	$\alpha = 0.55$	$\alpha = 0.95$	$\alpha = 0.70$	$\alpha = 0.55$	$\alpha = 0.95$	$\alpha = 0.70$	$\alpha = 0.55$
5×10^3	66	105	117	92	99	98	95	91	122
	90	99	115	101	95	108	100	88	110
	67	99	97	90	96	108	98	106	110
1×10^4	67	97	105	91	98	101	96	97	104
	89	98	95	99	103	105	101	97	98
	65	103	101	87	104	108	98	98	92

Table 2: Local limit measures of the intervals $(-50, 0)$, $(-25, 25)$, and $(0, 50)$ - one per row - using N one-dimensional linear processes, each of length n , employing various long memory cases using the $\text{FARIMA}(0, 1 - \alpha, 0)$ model with t_5 innovations.

N	$n = 2^{10}$			$n = 2^{12}$			$n = 2^{14}$		
	$\alpha = 0.95$	$\alpha = 0.70$	$\alpha = 0.55$	$\alpha = 0.95$	$\alpha = 0.70$	$\alpha = 0.55$	$\alpha = 0.95$	$\alpha = 0.70$	$\alpha = 0.55$
5×10^3	46	51	67	51	52	62	49	45	61
	50	47	54	49	49	43	48	40	61
	46	48	48	49	43	46	51	43	49
1×10^4	46	48	50	49	52	43	50	51	55
	48	50	51	50	52	44	50	45	49
	43	51	54	50	51	62	51	47	43

5 Proofs

For the proof of Theorem 2.2, we need several lemmas.

Lemma 5.1 *Let $\varphi(t)$ be the characteristic function of some random variable, and let b and $c < 1$ be positive real numbers. If $|\varphi(t)| \leq c$ for $b \leq |t| \leq 2b$, then*

$$|\varphi(t)| \leq 1 - \frac{1 - c^2}{8b^2}t^2 \quad \text{for all } |t| < b.$$

Proof. This is a version of Theorem 1 on page 10 in Petrov (1975), which is obtained by using the same proof. ■

Lemma 5.2 *If $\varphi(t)$ is the characteristic function of some random variable satisfying the Cramér condition, then for any $\delta > 0$ there is $\beta = \beta(\delta) \in (0, 1)$ such that*

$$|\varphi(t)| \leq \beta \quad \text{for all } |t| \geq \delta.$$

Proof. Since $\limsup_{|t| \rightarrow \infty} |\varphi(t)| < 1$, there exists $0 < \gamma < 1$ and $T > 0$ such that for all $|t| > T$ we have that $|\varphi(t)| \leq \gamma$. For any $\delta > 0$ such that $\delta > T$, the result holds with $\beta = \gamma$. By Lemma 5.1, on the other hand, $\limsup_{|t| \rightarrow \infty} |\varphi(t)| < 1$ implies that $|\varphi(t)| < 1$ for all $t \neq 0$. If $\delta < T$, we appeal to the continuity of $\varphi(t)$ to guarantee that $\eta = \max_{\delta \leq |t| \leq T} |\varphi(t)| \in (0, 1)$, whence $|\varphi(t)| \leq \eta$ for any t with $|t| \in [\delta, T]$. Therefore, the result holds with $\beta = \gamma \vee \eta$. ■

Lemma 5.3 *If S_n and B_n are as defined in (4) and (5) respectively, if the innovations have a non-lattice distribution, and if $\{\omega_n\}$ is a sequence of positive real numbers for which there exists some $M > 0$ so that $|\omega_n b_{n,i}| \leq M$ for all $n \in \mathbb{N}$ and for all $i \in \mathbb{Z}^d$, then the function*

$$\varphi_{\frac{S_n}{B_n}}(t) \mathcal{I}(|t| < \omega_n B_n)$$

is dominated by some integrable function $g(t)$.

Proof. Since we assume that $|\varphi_\varepsilon(t)| < 1$ for all $t \neq 0$, because $\varphi_\varepsilon(t)$ is continuous, there exists $c = c(M) \in (0, 1)$ such that $|\varphi_\varepsilon(u)| \leq c$ for $M \leq |u| \leq 2M$. By Lemma 5.1 and because of the inequality $1 - x \leq e^{-x}$ for all $x \in \mathbb{R}$, we deduce that $|u| \leq M$ implies that

$$|\varphi_\varepsilon(u)| \leq 1 - \frac{1 - c^2}{8M^2} u^2,$$

and therefore

$$|\varphi_\varepsilon(u)| \leq \exp\left(-\frac{1 - c^2}{8M^2} u^2\right).$$

Now, by independence, we have

$$\varphi_{\frac{S_n}{B_n}}(t) = \varphi_{\sum_{i \in \mathbb{Z}^d} b_{n,i} \varepsilon_i}\left(\frac{t}{B_n}\right) = \prod_{i \in \mathbb{Z}^d} \varphi_\varepsilon\left(\frac{b_{n,i} t}{B_n}\right).$$

For $|t| < \omega_n B_n$, we observe that

$$\left|\frac{b_{n,i} t}{B_n}\right| < |b_{n,i} \omega_n| \leq M.$$

Overall, we have

$$\begin{aligned} \left|\varphi_{\frac{S_n}{B_n}}(t)\right| \mathcal{I}(|t| < \omega_n B_n) &= \prod_{i \in \mathbb{Z}^d} \left|\varphi_\varepsilon\left(\frac{b_{n,i} t}{B_n}\right)\right| \mathcal{I}(|t| < \omega_n B_n) \\ &\leq \prod_{i \in \mathbb{Z}^d} \exp\left(-\frac{1 - c^2}{8M^2} \frac{b_{n,i}^2}{B_n^2} t^2\right) \\ &= \exp\left(-\frac{1 - c^2}{8M^2} \frac{1}{\sigma_\varepsilon^2} t^2\right), \end{aligned}$$

which we take to be our desired dominating integrable function $g(t)$. ■

For use in the following lemma, we shall introduce the following notation. For a countable collection of real numbers $\{b_j : j \in \mathbb{Z}^d\}$, where $j = (j_1, \dots, j_d)$, we denote an increment in the direction k by

$$\Delta_k b_{j_1, \dots, j_k, \dots, j_d} = b_{j_1, \dots, j_k, \dots, j_d} - b_{j_1, \dots, j_k - 1, \dots, j_d}$$

and their composition is denoted by Δ :

$$\Delta b_j = \Delta_1 \circ \Delta_2 \circ \dots \circ \Delta_d b_j. \quad (15)$$

For instance, if $d = 1$ we have $\Delta b_j = b_j - b_{j-1}$. For $d = 2$,

$$\Delta b_{i,j} = \Delta_1 \circ \Delta_2 b_{i,j} = b_{i,j} - b_{i,j-1} - b_{i-1,j} + b_{i-1,j-1}. \quad (16)$$

Denote $\sum_{i \in \mathbb{Z}^d} a_i^2 = D^2 < \infty$. Define as before $b_i = b_{n,i} = b_i(n) = \sum_{j \in \Gamma_n^d} a_{j-i}$. For $k \in \mathbb{N}$, and for $j \in \mathbb{Z}^d$ we denote by $V_k(j)$ the vertices of the cube $\prod_{1 \leq \ell \leq d} [j_\ell - k, j_\ell]$.

For the proof of the long memory case in Theorem 2.2, we need the following lemma about the size of the coefficients $b_i = b_{n,i}$.

Lemma 5.4 For any $\ell \in \mathbb{Z}^d$ and any $k \geq 1$ we have

$$\sum_{u \in V_k(\ell), u \neq \ell} |b_u| \geq |b_\ell| - 2^d D J_n k^{\frac{d}{2}}.$$

Proof. In order to avoid complicated notation, we shall (without loss of generality) prove in detail the case $d = 2$, and the general case will follow by a similar argument. For this case, the increment Δ is defined by (16), $b_{i,j} = \sum_{(s,t) \in \Gamma_n^2} a_{s-i,t-j}$, where Γ_n^2 is as defined in (8). Since

$$\sum_{u=i-k+1}^i \sum_{v=j-k+1}^j \Delta b_{u,v} = b_{i,j} - b_{i,j-k} - b_{i-k,j} + b_{i-k,j-k},$$

we employ the triangle inequality to obtain

$$|b_{i,j}| \leq |b_{i,j-k}| + |b_{i-k,j}| + |b_{i-k,j-k}| + \sum_{u=i-k+1}^i \sum_{v=j-k+1}^j |\Delta b_{u,v}|. \quad (17)$$

By the linearity of Δ and the definition of Γ_n^2 , we notice that

$$\Delta b_{u,v} = \Delta \left[\sum_{(s,t) \in \Gamma_n^2} a_{s-u,t-v} \right] = \Delta \left[\sum_{w=1}^{J_n} \sum_{(s,t) \in \Gamma_n^2(w)} a_{s-u,t-v} \right] = \sum_{w=1}^{J_n} \Delta \left[\sum_{(s,t) \in \Gamma_n^2(w)} a_{s-u,t-v} \right].$$

For fixed $w \in \{1, 2, \dots, J_n\}$, let us investigate the expression $\Delta \left[\sum_{(s,t) \in \Gamma_n^2(w)} a_{s-u,t-v} \right]$. Indeed, after some cancellations, we get

$$\begin{aligned} & \Delta \left[\sum_{(s,t) \in \Gamma_n^2(w)} a_{s-u,t-v} \right] \\ &= \sum_{(s,t) \in \Gamma_n^2(w)} [a_{s-u,t-v} - a_{s-u,t-(v-1)} - a_{s-(u-1),t-v} + a_{s-(u-1),t-(v-1)}] \\ &= \sum_{s=\underline{n}_1(\omega)}^{\bar{n}_1(\omega)} \sum_{t=\underline{n}_2(\omega)}^{\bar{n}_2(\omega)} [a_{s-u,t-v} - a_{s-u,(t+1)-v} - a_{(s+1)-u,t-v} + a_{(s+1)-u,(t+1)-v}] \\ &= \sum_{s=\underline{n}_1(\omega)}^{\bar{n}_1(\omega)} [a_{s-u,\underline{n}_2(\omega)-v} - a_{s-u,(\bar{n}_2(\omega)+1)-v} - a_{(s+1)-u,\underline{n}_2(\omega)-v} + a_{(s+1)-u,(\bar{n}_2(\omega)+1)-v}] \\ &= a_{\underline{n}_1(\omega)-u,\underline{n}_2(\omega)-v} - a_{(\bar{n}_1(\omega)+1)-u,\underline{n}_2(\omega)-v} + a_{(\bar{n}_1(\omega)+1)-u,(\bar{n}_2(\omega)+1)-v} - a_{\underline{n}_1(\omega)-u,(\bar{n}_2(\omega)+1)-v}. \end{aligned}$$

This identity together with the Cauchy-Schwarz inequality, demonstrate that

$$\sum_{u=i-k+1}^i \sum_{v=j-k+1}^j |\Delta b_{u,v}| \leq (4Dk) J_n.$$

Therefore, combining this latter inequality with (17), we obtain

$$|b_{i,j-k}| + |b_{i-k,j}| + |b_{i-k,j-k}| \geq |b_{i,j}| - 4D J_n k,$$

thereby establishing the result for $d = 2$. For general d , the difference is that we use the formula (15) instead of (16) and we take into account that, in this case, the number of vertices of the cube $\prod_{1 \leq \ell \leq d} [j_\ell - k, j_\ell]$ is 2^d . ■

Lemma 5.5 Assume that conditions of Theorem 2.2 are satisfied for the long memory case. Then there exists $0 < \rho < 1$ independent of n , such that for all n sufficiently large

$$|\varphi_{S_n}(t)| \mathcal{I}(|t| \geq \gamma_n^{-1}) \leq \rho^{\lfloor (\gamma_n/J_n)^{2/d} \rfloor},$$

where $\gamma_n = \sup_{i \in \mathbb{Z}^d} |b_{n,i}|$.

Proof. To simplify the notation, we will drop the index n and simply write b_i in place of $b_{n,i}$. Assume $|b_{j_0}| = \sup_{i \in \mathbb{Z}^d} |b_i|$. Such a j_0 exists, because $\sum_{i \in \mathbb{Z}^d} a_i^2 = D^2 < \infty$. Fix $\alpha \in (0, 1)$, and denote by k_0 the integer part of $[(1 - \alpha)|b_{j_0}|/(2^d D J_n)]^{2/d}$, namely

$$k_0 = \left\lfloor \left(\frac{(1 - \alpha)|b_{j_0}|}{2^d D J_n} \right)^{\frac{2}{d}} \right\rfloor. \quad (18)$$

Since $B_n \rightarrow \infty$, condition (9) in Theorem 2.2 implies that $|b_{j_0}| \rightarrow \infty$ and $J_n = o(\sup_{i \in \mathbb{Z}^d} |b_{n,i}|)$ as $n \rightarrow \infty$. Therefore the k_0 in (18) satisfies $k_0 \rightarrow \infty$ as $n \rightarrow \infty$. So, for n sufficiently large, $k_0 \geq 1$.

By Lemma 5.4, for $1 \leq k \leq k_0$,

$$\sum_{u \in V_k(j_0), u \neq j_0} |b_u| \geq |b_{j_0}| - 2^d D J_n k^{\frac{d}{2}} \geq \alpha |b_{j_0}|,$$

which immediately gives

$$\max_{u \in V_k(j_0), u \neq j_0} |b_u| \geq \alpha |b_{j_0}| / (2^d - 1).$$

So, on the set $|t| \geq \gamma_n^{-1} = |b_{j_0}|^{-1}$

$$\max_{u \in V_k(j_0), u \neq j_0} |tb_u| \geq \alpha / (2^d - 1). \quad (19)$$

With these preliminaries in place, let us define

$$\eta_k = \sum_{u \in V_k(j_0), u \neq j_0} b_u \varepsilon_u$$

and for $u \in V_k(j_0), u \neq j_0$

$$\varphi_u(t) = \varphi_\varepsilon(tb_u).$$

By independence

$$|\mathbb{E}(\exp(it\eta_k))| = \prod_{u \in V_k(j_0), u \neq j_0} |\varphi_u(t)| \leq \min_u |\varphi_u(t)|,$$

where the product and the minimum are over $u \in V_k(j_0), u \neq j_0$. Since the characteristic function of ε_0 satisfies the Cramér condition, by Lemma 5.2, for $\alpha/(2^d - 1) > 0$ we can find $0 < \beta = \beta(\alpha) < 1$ such that $|\varphi_\varepsilon(s)| \leq \beta$ for all $|s| \geq \alpha/(2^d - 1)$. As a consequence, for $k \leq k_0$, by (19), at least one of $|\varphi_u(t)|$ is smaller than β , i.e.,

$$\min_u |\varphi_u(t)| \leq \beta \text{ for all } |t| \geq |b_{j_0}|^{-1} \text{ and } k \leq k_0.$$

It implies that

$$|\mathbb{E}(\exp(it\eta_k))| \leq \beta < 1 \text{ for all } |t| \geq |b_{j_0}|^{-1} \text{ and } k \leq k_0. \quad (20)$$

And since

$$|\varphi_{S_n}(t)| = \prod_{j \in \mathbb{Z}^d} |\varphi_\varepsilon(b_j t)| \leq \prod_{1 \leq k \leq k_0} |\mathbb{E}(\exp(it\eta_k))|,$$

by inequality (20) and the definition of k_0 in (18), it follows that

$$|\varphi_{S_n}(t)| \mathcal{I}(|t| \geq \gamma_n^{-1}) \leq \beta^{k_0} \leq \rho^{\lfloor (\gamma_n/J_n)^{2/d} \rfloor}$$

for some $\rho \in (0, 1)$. ■

Proof of Theorem 2.2 The proof is based, as usual, on the study of the characteristic function of the sum S_N . As in Hafouta and Kifer (2016), we prove (7) for all continuous complex-valued functions h defined on \mathbb{R} , $|h| \in L^1(\mathbb{R})$ such that

$$\hat{h}(t) = \int_{\mathbb{R}} e^{-itx} h(x) dx$$

is real-valued and has compact support contained in some finite interval $[-L, L]$. By the inversion formula

$$h(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{itx} \hat{h}(x) dx.$$

Employing a change of variables, we see that

$$\mathbb{E}[h(S_n - u)] = \frac{1}{2\pi B_n} \int_{\mathbb{R}} \hat{h}\left(\frac{t}{B_n}\right) \varphi_{\frac{S_n}{B_n}}(t) \exp\left(-\frac{itu}{B_n}\right) dt. \quad (21)$$

By the Fourier inversion formula we also have

$$\exp\left(-\frac{u^2}{2B_n^2}\right) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-\frac{itu}{B_n}\right) \exp\left(-\frac{t^2}{2}\right) dt. \quad (22)$$

By (21) and (22) and some simple algebraic manipulations, we obtain

$$\begin{aligned} & \sup_{u \in \mathbb{R}} \left| \sqrt{2\pi} B_n \mathbb{E}[h(S_n - u)] - \exp\left(-\frac{u^2}{2B_n^2}\right) \int_{\mathbb{R}} h(x) dx \right| \\ & \leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left| \hat{h}\left(\frac{t}{B_n}\right) \varphi_{\frac{S_n}{B_n}}(t) - \exp\left(-\frac{t^2}{2}\right) \int_{\mathbb{R}} h(x) dx \right| dt. \end{aligned} \quad (23)$$

As in Lemma 5.5 denote $\gamma_n = \sup_{i \in \mathbb{Z}^d} |b_{n,i}|$. At this point, note that because we have $\gamma_n \leq B_n$, condition (9) implies that $J_n = o(B_n^2)$ in the long memory case. By the CLT in Theorem 2.1, for all $T > 0$, it follows that

$$\int_{|t| \leq T} \left| \varphi_{\frac{S_n}{B_n}}(t) - \exp\left(-\frac{t^2}{2}\right) \right| dt \rightarrow 0 \text{ as } n \rightarrow \infty.$$

On the other hand

$$\int_{|t| \geq T} \exp\left(-\frac{t^2}{2}\right) dt \rightarrow 0 \text{ as } T \rightarrow \infty.$$

Since h is integrable, \hat{h} is continuous, and $B_n \rightarrow \infty$, for all t

$$\lim_{n \rightarrow \infty} \hat{h}\left(\frac{t}{B_n}\right) = \int_{\mathbb{R}} h(x) dx.$$

Combining these facts with (23), we note that, in order to obtain the conclusion of Theorem 2.2, it suffices to show that

$$\lim_{T \rightarrow \infty} \limsup_n \int_{T \leq |t| \leq LB_n} \left| \varphi_{\frac{S_n}{B_n}}(t) \right| dt = 0. \quad (24)$$

First, we deal with the situation when coefficients are absolutely summable. Since $Lb_{n,i}$ is uniformly bounded by $L \sum_{i \in \mathbb{Z}^d} |a_i|$, Lemma 5.3, applied with $\omega_n = L$ and $M = L \sum_{i \in \mathbb{Z}^d} |a_i|$, guarantees that the integrand of (24) is dominated by some integrable function. In order to verify (24) we have just to apply the Lebesgue dominated convergence theorem.

Henceforth, we confine our attention to the long memory case. We decompose the region of integration in (24), yielding

$$\begin{aligned} \int_{T \leq |t| \leq LB_n} \left| \varphi_{\frac{S_n}{B_n}}(t) \right| dt & \leq \int_{T \leq |t| \leq \gamma_n^{-1} B_n} \left| \varphi_{\frac{S_n}{B_n}}(t) \right| dt + \int_{\gamma_n^{-1} B_n \leq |t| \leq LB_n} \left| \varphi_{\frac{S_n}{B_n}}(t) \right| dt \\ & =: I_{1,n} + I_{2,n}, \end{aligned}$$

so that we may deal with $I_{1,n}$ and $I_{2,n}$ separately. In what follows, our objective is to show that both $I_{1,n} \rightarrow 0$ and $I_{2,n} \rightarrow 0$ as $n \rightarrow \infty$.

Since $\gamma_n^{-1} b_{n,i}$ is uniformly bounded by one, Lemma 5.3 applied with $\omega_n = \gamma_n^{-1}$ and $M = 1$ guarantees that the integrand of $I_{1,n}$ is dominated by some integrable function $g(t)$. Ergo, by the Lebesgue dominated convergence theorem, we have

$$\limsup_n I_{1,n} \leq \int_{T \leq |t|} g(t) dt \rightarrow 0 \text{ as } T \rightarrow \infty,$$

which is exactly what we wished to show about $I_{1,n}$.

Now we proceed to show that $I_{2,n} \rightarrow 0$. By a change of variable, we deduce that

$$I_{2,n} = B_n \int_{\gamma_n^{-1} < |t| \leq L} |\varphi_{S_n}(t)| dt.$$

By Lemma 5.5

$$|\varphi_{S_n}(t)| \mathcal{I}(|t| \geq \gamma_n^{-1}) \leq \rho^{\lfloor (\gamma_n/J_n)^{2/d} \rfloor},$$

and so

$$I_{2,n} \leq B_n (2L) \rho^{\lfloor (\gamma_n/J_n)^{2/d} \rfloor}.$$

It is easy to see that $|I_{2,n}| \rightarrow 0$ if we impose (9). ■

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