Optimal Spectrum Partitioning and Licensing in Tiered Access Under Stochastic Market Models

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***Abstract*— We consider the problem of partitioning a spectrum band into M channels of equal bandwidth, and then further assigning these M channels into P licensed channels and $M-P$ unlicensed channels. Licensed channels can be accessed both for licensed and opportunistic use following a tiered structure that has a higher priority for licensed use. Unlicensed channels can be accessed only for opportunistic use. We address the following question in this paper. Given a market setup, what values of M and P maximize the net spectrum utilization of the spectrum band? While this problem is fundamental, it is highly relevant practically, e.g., in the context of partitioning the recently proposed Citizens Broadband Radio Service band. If M is too high or too low, it may decrease spectrum utilization due to limited channel capacity or due to wastage of channel capacity, respectively. If P is too high (low), it will not incentivize the wireless operators who are primarily interested in unlicensed channels (licensed channels) to join the market. These tradeoffs are captured in our optimization problem which manifests itself as a two-stage Stackelberg game. We design an algorithm to solve the Stackelberg game and hence find the optimal M and P. The algorithm design also involves an efficient Monte Carlo integrator to evaluate the expected value of the involved random variables like spectrum utilization and operators' revenue. We also benchmark our algorithms using numerical simulations.**

***Index Terms*— Spectrum auction, opportunistic spectrum access, CBRS band, Stackelberg game, iterated removal of strictly dominated strategies, Monte Carlo integration, optimization.**

I. INTRODUCTION

TO SUPPORT the ever-growing wireless data traffic, the Federal Communication Commission (FCC) released the underutilized Citizens Broadband Radio Service (CBRS) band for shared use in 2015 [2]. CBRS band is a 150 MHz federal spectrum band from 3.55 GHz to 3.7 GHz. The 150 MHz band is divided into 15 channels of 10 MHz each. The shared use of the CBRS band follows an order of priority. Federal users have the highest priority access to the channels. Out of the 15 channels, 7 are Priority Access Licenses (PALs). PAL licenses are sold through auctions and the lease duration of a PAL license may range between $1-10$ years [2]–[4]. A PAL license holder can use their channel only if federal users are not using it. The remaining 8 out of the 15 channels are reserved only for opportunistic use by General Authorized Access (GAA) users. Opportunistic channel allocation to GAA users can happen at a time scale of minutes to weeks. GAA users can use these 8 channels if federal users are not using the channels. GAA users can also use the 7 PAL channels provided that neither federal users nor PAL license holders are using it.

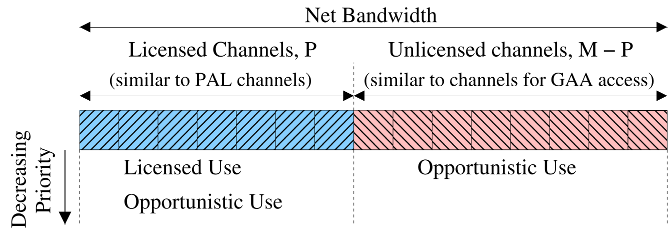

Fig. 1. Pictorial representation of the tiered spectrum model under consideration in this paper.

As mentioned in the previous paragraph, the CBRS band is divided into $M=15$ channels out of which there are $P=7$ PAL licenses. But does $M=15$ and $P=7$ maximize the utilization of the CBRS band? In this paper, we are interested in the following abstraction of this question. A net bandwidth is partitioned into M channels of equal bandwidth. These M channels are further divided into P licensed channels (similar to PAL channels) and $M-P$ unlicensed channels (similar to channels reserved for GAA users). In this paper, the process of dividing the net bandwidth into M channels is called *spectrum partitioning* and the process of allocating these M channels as licensed and unlicensed channels is called *spectrum licensing*. Licensed channels are used for both licensed use and opportunistic use with the former having higher priority. Unlicensed channels are reserved for opportunistic use only. This spectrum access model is shown in Figure 1. The wireless operators earn revenue by serving customer demands. A wireless operator is incentivized to join the market if the revenue which it can earn is above a desired threshold. For the given setup, what value of M and P maximizes spectrum utilization where spectrum utilization is defined as the net amount of customer demand served by the entire bandwidth? The application of this question is not just limited to CBRS but other spectrum sharing architectures like licensed shared access [5], high priority channels in TV White Spaces [6] etc. which have certain resemblance with CBRS.

There are various factors that decide the optimal values of M and P. Some of these factors are as follows. If the number of channels, M, increases, the bandwidth, and hence capacity, of each channel decreases. The capacity of each channel should be large enough to accommodate a good portion of the customer demand of a wireless operator but not so large that most of the capacity of the channel is not utilized for the majority of the time. This suggests that M should not be

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too small or too large. If the number of licensed channels P is too high, there is a small number of unlicensed channels. Therefore, those operators who primarily rely on unlicensed channels to serve customer demands will not be able to generate enough revenue and hence will not be incentivized to join the market. Similarly, if the P is too low, wireless operators who primarily rely on licensed channels to serve customer demands will not be incentivized to join the market. P should be set such that enough operators join the market to ensure that the customer demands served over the entire bandwidth is as high as possible. There may be other qualitative factors governing optimal M and P. Therefore, in this paper, we design an algorithm to jointly optimize M and P such that spectrum utilization is maximized.

A. Related Work

Variations of the spectrum partitioning and spectrum licensing problems considered in this paper have been studied separately, but not jointly, in the spectrum sharing and related fields. There are a few works that have addressed problems similar to partitioning a fixed bandwidth into an optimal number of channels. In [7], the authors derive an analytical expression for the optimal number of channels such that the spatial density of transmission is maximized subject to a fixed link transmission rate and packet error rate. Partitioning of bandwidth in the presence of guard bands has been considered in [8] where the authors used Stackelberg game formulation to analyze how a spectrum holder should partition its bandwidth in order to maximize its revenue in spectrum auctions.

The second problem studied in this paper deals with spectrum licensing. This has been widely studied in the literature from various perspectives. Some works concentrated on minimizing the amount of bandwidth allocated to backup channels (unlicensed channels in our case) while providing a certain level of guarantee to secondary users against channel preemption [9]. There has also been research on overlay D2D and cellular devices that studied optimal partitioning of orthogonal in-band spectrum to maximize the average throughput rates of cellular and D2D devices [10]. In [11], the authors investigated whether to allocate an additional spectrum band for licensed or unlicensed use and concluded that the licensed use is more favorable for maximizing the social surplus. A similar result has been shown in [12] which studied the effect of adding an unlicensed spectrum band in a market consisting of wireless operators with licensed channels. The authors showed that if the amount of unlicensed spectrum band is below a certain limit, the overall social welfare may decrease with the increase in unlicensed spectrum band. The authors in [13], [14] studied the CBRS band for a market setup that consists of Environmental Sensing Capability operators (ESCs) whose job is to monitor and report spectrum occupancy to the wireless operators. The authors analyzed how the ratio of the licensed and unlicensed bands affects the market competition between the ESC operators, the wireless operators, and the end users of the CBRS band. There is a line of work that studies spectrum partitioning for topics similar to licensed and unlicensed use using Stackelberg games; macro cells and small cells [15], [16], long-term leasing and short-term rental market [17], and 4G cellular and Super Wifi services [18].

Such a diverse body of work just on spectrum partitioning and licensing is justified because individual problem setups have their own salient features and hence require their own analysis. Our problem setup considers *jointly* optimizing spectrum partitioning and spectrum licensing, which has not been considered in the existing literature. This problem is novel because of the combination of the following two reasons. *First,* our spectrum access model, like CBRS, is a combination of (a) Unlicensed spectrum access model. This is because $M-P$ unlicensed channels are reserved specifically for opportunistic use. (b) Primary-secondary spectrum access model. This is because P licensed channels can be used for opportunistic access following the priority hierarchy. Prior works like [12]–[14], which solved the spectrum licensing problem, did not simultaneously consider both of the spectrum access models. *Second,* we consider a very generalized system model in terms of the number of operators, their types, and their heterogeneity. Such a setup leads to a scenario where the regulator has to decide M and P such that the right set of wireless operators are incentivized to join the market. The authors have addressed the problem of joint spectrum partitioning and licensing in [1]. But this paper uses a more realistic bidding model for licensed channels and a generalized opportunistic spectrum access (OSA) strategy. Finally, the stackelberg game formulation used in this paper is a generalized version of [1].

B. Contribution and Paper Organization

We now present an overall outline of the paper and, in the process, discuss its main contributions. In Section II, we present a system model which can mathematically capture the effect of the number of channels, M, and the number of licensed channels, P, on the spectrum utilization. The proposed system model captures spectrum auctions using a simple stochastic model without going into complex game-theoretic formulations. Based on our reading of [2]–[4] and other literature on CBRS band, it is not clear if there is a consensus in the literature/policy about whether PAL license holders are also allowed to use channels opportunistically. So it is possible that PAL license holders may or may not be allowed to use channels opportunistically. Our model is general enough to capture both of these cases. Our model can capture both overlay and interweave OSA strategies [19].

It is possible that a choice of values of M and P that incentivizes one group of wireless operators may not incentivize another group. Therefore, a M and P which incentivizes all the wireless operators may not exist. This argument can be exemplified by referring to [2]–[4] which shows a lot of debate between the wireless operators concerning the parameters of the CBRS model. Even if it is possible to satisfy all the operators, it may not be optimal to do so in terms of maximizing the spectrum utilization. We capture this idea by formulating our problem as a two-stage Stackelberg game in Section III-A which forms the second contribution of the paper. The Stackelberg game consists of the regulator (leader) and the wireless operators (followers). In Stage 1, the regulator sets M and P to maximize spectrum utilization. In Stage 2, the wireless operators decide whether or not to join the market based on the M and P set by the regulator in Stage 1.

In Section III-B, we design an algorithm to solve the Stackelberg game and hence the optimal M and P which maximize spectrum utilization. We approach this in steps. Few properties associated with the expected revenue of an operator are discussed first. We show that when these properties hold, we can design a polynomial-time algorithm to solve Stage 2 of the Stackelberg game. We finally solve Stage 1 of the

Stackelberg game using a grid search approach to find the optimal M and P which maximizes spectrum utilization. *To the best of our knowledge, joint optimization of partitioning and tiered licensing have not been considered in the existing related literature.* Designing an algorithm for joint optimization of M and P is the fourth contribution of the paper.

The solve the Stackelberg game, we have to calculate the expected revenue of an operator and expected spectrum utilization. The complex nature of the problem does not allow simple analytical formulas of these expected values. Even if such analytical formulas are possible, adapting them to changes in system model can be time consuming. Therefore, we develop a Monte Carlo integrator to evaluate these expected values in Section IV. Our choice of using a Monte Carlo integrator over deterministic numerical integration techniques is because our setup involves evaluation of high-dimensional integrals. Unlike deterministic numerical integration techniques, the computation time of Monte Carlo integration does not scale with dimension. One of the main bottlenecks of Monte Carlo integration is random sampling. While designing our Monte Carlo integrator, we reduced random sampling as much as possible to make it more time efficient. Designing an efficient Monte Carlo integrator which can easily adapt to few changes in the system model is the third contribution of the paper.

Finally, we use the algorithms designed in Sections III-B and IV to obtain important numerical results in Section V which show the importance of joint optimization of M and P, and how the how optimal values of M and P vary with market parameters. This is the final contribution of the paper.

II. System Model

In this section, we discuss individual components of our system model in Sections II-A to II-C. The list of important notations is included in Table I. Consider two sets $\mathcal{A}$ and $\mathcal{B}$. $\mathcal{A}\bigcup\mathcal{B}$ implies the union $\mathcal{A}$ and $\mathcal{B}$ while $\mathcal{A}\backslash\mathcal{B}$ is as set which consists of all those elements in $\mathcal{A}$ which are not in $\mathcal{B}$. A singleton set consisting of element a is denoted by $\{a\}$.

TABLE I

A Table of Important Notations

Notation	Description
t , γ	t^{th} time slot and epoch γ resp.
M , P	Number of channels and number of licensed channels resp.
ϕ	Indicator variable which decides if a Tier-1 operator can also use channels opportunistically. We have, $\phi \in \{0, 1\}$.
D	Capacity of the entire bandwidth for licensed access.
α_L , α_U	Interference parameter associated with opportunistic use of licensed and unlicensed channels resp.
$\mathcal{S}_L^C$, $\mathcal{S}_U^C$	Set of candidate licensed and unlicensed operators resp.
$\mathcal{S}_L$, $\mathcal{S}_U$	Set of interested licensed and unlicensed operators resp.
$\mathcal{S}$	Set of interested operators; $\mathcal{S} = \mathcal{S}_L \bigcup \mathcal{S}_U$.
$\mathcal{T}_1(\gamma)$	Set of Tier-1 operators in epoch γ, i.e. set of interested licensed operators who won licensed channels in epoch γ.
$\mathcal{T}_2(\gamma)$	Set of Tier-2 operators in epoch γ; $\mathcal{T}_2(\gamma) = \mathcal{S}\backslash\mathcal{T}_1(\gamma)$.
$x_k(t)$	Customer demand of the k^{th} operator in t^{th} time slot.
μ_k^θ , σ_k^θ	Mean and standard deviation resp. of Gaussian random variable $\theta_k(t)$ where, $x_k(t) = \max(0, \theta_k(t))$.
$\widetilde{x}_{k,i}(t)$	Amount of demand served by the k^{th} operator in t^{th} time slot if it is a Tier-i operator where $i \in \{1, 2\}$.
$\widetilde{x}_{k,a}(t)$	Amount of demand served by the k^{th} operator in t^{th} time slot when spectrum access type is a where $a \in \{lc, op\}$.
$X_{k,a}(\gamma)$	Net demand served by the k^{th} operator in epoch γ when spectrum access type is a where $a \in \{lc, op\}$.
$R_{k,a}(\gamma)$	Revenue of k^{th} operator in epoch γ using spectrum access of type a where $a \in \{lc, op\}$.
$R_{k,i}(\gamma)$	Revenue of k^{th} operator in epoch γ if it is a Tier-i operator.
$V_k(\gamma)$	Bid of the k^{th} operator, where $k \in \mathcal{S}_L$, in epoch γ.
$h_k(\cdot)$	A function associated with k^{th} operator which maps the mean of $X_{k,a}(\gamma)$ to the mean of $R_{k,a}(\gamma)$.
$\mu_{k,a}^X$, $\mu_{k,a}^R$	Mean of $X_{k,a}(\gamma)$ and $R_{k,a}(\gamma)$ resp. We have, $\mu_{k,a}^R = h_k\left(\mu_{k,a}^X\right)$.
$\sigma_{k,a}^X$, $\sigma_{k,a}^R$	Standard deviation of $X_{k,a}(\gamma)$ and $R_{k,a}(\gamma)$ resp.
ρ_k	Correlation coefficent between $X_{k,a}(\gamma)$ and $R_{k,a}(\gamma)$.
ω_k	Correlation coefficent between $V_k(\gamma)$ and $R_{k,lc}(\gamma)$.
λ_k	Minimum revenue requirement of the k^{th} operator.
ξ_k	We have, $\xi_k = \left(\mu_k^\theta,\, \sigma_k^\theta,\, h_k(\cdot),\, \sigma_{k,a}^R,\, \rho_k,\, \rho_k,\, \lambda_k\right)$.

A. Channel Model

A net bandwidth of $W\,hz$ is divided into M channels of equal bandwidth $\frac{W}{M}$. Out of the M channels, P channels are licensed channels while the remaining $M-P$ channels are unlicensed channels. In our model, time is divided into slots where $t \in \mathbb{Z}^+$ denotes the t^{th} time slot. Licensed channels are allocated for prioritized licensed use and opportunistic use while unlicensed channels are allocated only for opportunistic use. Allocation of licensed channels for licensed use happens through auctions. These auctions occur every $T \geq 1$ time slots where T is the lease duration. An entire lease duration is called an "epoch". Epoch γ is from time slot $(\gamma-1)T+1$ to γT. Allocation of licensed channels and unlicensed channels for opportunistic use occur every time slot.

Those operators who are allocated licensed channels for licensed use are called *Tier-1 operators* while those who are not are called *Tier-2 operators*. In our model, an operator can be allocated at most one licensed channel for licensed use in an epoch, i.e. spectrum cap is one. Similar assumption has been made in prior works like [20]. Spectrum cap of one ensures fairness by allocating the licensed channels to as many operators as possible. A Tier-1 operator can also use opportunistic channels to serve its customer demand in case the bandwidth of the allocated licensed channel is not sufficient. Tier-2 operators use channels opportunistically. Tier-1 operators may also use channels opportunistically. Let $\phi \in \{0, 1\}$. If $\phi = 1$, then Tier-1 operators can use channels opportunistically; otherwise they cannot.

The capacity of a channel/bandwidth is the maximum units of customer demand that can be served using that channel/bandwidth in a time slot. Let D be the capacity of the entire bandwidth of $W\,hz$ when used for licensed access. As the entire bandwidth is partitioned into M channels, each licensed channel has a capacity of $\frac{D}{M}$ when used for licensed use while the unlicensed channels have a capacity of $\frac{\alpha_U D}{M}$ where $\alpha_U \in [0, 1]$ is the interference parameter associated with unlicensed channels for opportunistic use. Licensed channels can also be used for opportunistic use following the priority hierarchy shown in Figure 1. Let the customer demand of a Tier-1 operator be d units. It will use its licensed channel to serve its customer demand. The remaining capacity of the licensed channel which can be utilized for opportunistic use is given by the function $\mathcal{C}(d, \alpha_L)$ where $\alpha_L \in [0, 1]$ is the interference parameter associated with licensed channels for opportunistic use. The expression for $\mathcal{C}(d, \alpha_L)$ depends on the OSA strategy: overlay or interweave [19]. For *overlay spectrum access*, $\mathcal{C}(d, \alpha_L) = \alpha_L\left(\frac{D}{M} - d\right)^+$, where $(x)^+ = \max(0, x)$. For *interweave spectrum access*, $\mathcal{C}(d, \alpha_L)$ is equal to 0 if $d > 0$ and equal to $\frac{\alpha_L D}{M}$ if

$d = 0$. Parameters α_L and α_U capture the lower efficiency of opportunistic use as compared to licensed use [2]. In general, we expect $\alpha_L \leq \alpha_U$. This may happen because the transmission power cap for opportunistic use may be lower for licensed channels compared to unlicensed channels in order to protect Tier-1 operators from harmful interference.

B. Operators, Their Demand and Revenue Model

The market consists of the *candidate licensed operators* denoted by $\mathcal{S}_L^C$ and the *candidate unlicensed operators* denoted by $\mathcal{S}_U^C$ where $\mathcal{S}_L^C$ and $\mathcal{S}_U^C$ are disjoint sets. A candidate licensed operator is a Tier-1 operator in those epochs in which it is allocated a licensed channel in the auction and a Tier-2 operator in those epochs in which it is not allocated a licensed channel. A candidate unlicensed operator is always a Tier-2 operator. A candidate operator has to invest in infrastructure development if it wants to join the market.

All the candidate operators have to invest in infrastructure development to join the market. In order to generate return on infrastructure cost and the cost of leasing a channel, the k^{th} candidate operator wants to earn a minimum expected revenue (MER), λ_k, in an epoch. Mathematically, $\lambda_k = C_k^I + C_k^L + \mathit{\Lambda}_k$ where, C_k^I is the infrastructure cost of the k^{th} operator, C_k^L is an estimate of the cost of leasing a channel according to the k^{th} operator,[1] and $\mathit{\Lambda}_k$ is the minimum profit the k^{th} operator wants to make in an epoch. The k^{th} candidate licensed/unlicensed operator is interested in joining the market if the value of M and P set by the regulator is such that the expected revenue of the operator in an epoch is greater than λ_k. The set of *interested licensed operators* and *interested unlicensed operators* are denoted by $\mathcal{S}_L$ and $\mathcal{S}_U$ respectively. We have $\mathcal{S}_L \subseteq \mathcal{S}_L^C$ and $\mathcal{S}_U \subseteq \mathcal{S}_U^C$. The set of operators, $\left(\mathcal{S}_L^C - \mathcal{S}_L\right) \bigcup \left(\mathcal{S}_U^C - \mathcal{S}_U\right)$, does not join the market. A candidate licensed/unlicensed operator gets to decide whether to join or not join the market only once. An operator gets to participate in auctions for licensed channels or to use channels opportunistically only if it decides join the market.

In our model, every operator has a separate pool of customers each with its own stochastic demands, i.e. we do not model price competition between operators to attract a common pool of customers. Consider the t^{th} time slot of epoch γ. The customer demand, or simply demand, of the k^{th} operator in the t^{th} time slot is $x_k(t)$. In our model, $x_k(t) = \max(0\,,\, \theta_k(t))$ where $\theta_k(t)$ are iid Gaussian random variable[2] with mean μ_k^θ and standard deviation σ_k^θ, i.e. $\theta_k(t) \sim \mathcal{N}\left(\mu_k^\theta, \left(\sigma_k^\theta\right)^2\right)$, $\forall t$. The k^{th} operator may be able to serve only a fraction of the customer demand. Let $\widetilde{x}_{k,1}(t)$ and $\widetilde{x}_{k,2}(t)$ denote the amount of customer demand served by the k^{th} operator if it is a Tier-1 and a Tier-2 operator respectively in epoch γ. We have $\widetilde{x}_{k,1}(t) \leq x_k(t)$ and $\widetilde{x}_{k,2}(t) \leq x_k(t)$. $\widetilde{x}_{k,1}(t)$ and $\widetilde{x}_{k,2}(t)$ can be expressed as follows

$$\widetilde{x}_{k,1}(t) = \widetilde{x}_{k,lc}(t) + \widetilde{x}_{k,op}(t) \tag{1}$$

$$\widetilde{x}_{k,2}(t) = \widetilde{x}_{k,op}(t) \tag{2}$$

where $\widetilde{x}_{k,lc}(t) = \min\left(x_k(t)\,,\, \frac{D}{M}\right)$. The term $\widetilde{x}_{k,lc}(t)$ in (1) is the amount of customer demand served a Tier-1 operator using the channel allocated to it for licensed use. The term $\widetilde{x}_{k,op}(t)$ in (1) and (2) is the demand served by an operator by using channels opportunistically. It will be shown in Section II-C that $\widetilde{x}_{k,op}(t)$ is a iid random variable. Also, if $\phi = 0$, then a Tier-1 operator cannot use channels opportunistically and hence $\widetilde{x}_{k,op}(t) \equiv 0$ in (1). In (1) and (2), $\widetilde{x}_{k,1}(t)$ and $\widetilde{x}_{k,2}(t)$ can be expressed as a time invariant function of iid random variables $x_k(t)$ and $\widetilde{x}_{k,op}(t)$. Therefore, $\widetilde{x}_{k,1}(t)$ and $\widetilde{x}_{k,2}(t)$ are iid random variables as well. Let $\widetilde{\mu}_{k,a}^x$ and $\widetilde{\sigma}_{k,a}^x$ denote the mean and standard deviation of $\widetilde{x}_{k,a}(t)$ respectively. We have,

$$\widetilde{\mu}_{k,lc}^x = \int_0^{\frac{D}{M}} \vartheta f_k^\theta(\vartheta)\, d\vartheta + \frac{D}{M}\int_{\frac{D}{M}}^\infty f_k^\theta(\vartheta)\, d\vartheta \tag{3}$$

$$\left(\widetilde{\sigma}_{k,lc}^x\right)^2 = \int_0^{\frac{D}{M}} \vartheta^2 f_k^\theta(\vartheta)\, d\vartheta + \left(\frac{D}{M}\right)^2 \int_{\frac{D}{M}}^\infty f_k^\theta(\vartheta)\, d\vartheta - \left(\widetilde{\mu}_{k,lc}^x\right)^2 \tag{4}$$

where $f_k^\theta(\vartheta)$ is the probability density function of $\theta_k(t)$. In general, an analytical expression for $\widetilde{\mu}_{k,op}^x$ and $\widetilde{\sigma}_{k,op}^x$ is not possible because of the complex nature of opportunistic spectrum allocation algorithm. We have designed a Monte Carlo integrator which can compute $\widetilde{\mu}_{k,op}^x$ in Section IV.

Throughout the rest of the paper we will use the subscript k, i, where $i \in \{1, 2\}$, to denote variables associated with k^{th} operator when its is a Tier-i operator. Also, we will use the subscript k, a, where $a \in \{lc, op\}$, to denote variables associated with k^{th} operator when access type is licensed ($a = lc$) or opportunistic ($a = op$). Let $X_{k,a}(\gamma)$ denote the net demand served by the k^{th} operator in epoch γ when access type is a. Mathematically,

$$X_{k,a}(\gamma) = \sum_{t=(\gamma-1)T+1}^{\gamma T} \widetilde{x}_{k,a}(t)\,; \quad a \in \{lc, op\} \tag{5}$$

Since $\widetilde{x}_{k,a}(t)$ is iid random variable and the lease duration T is quite large in practice, $X_{k,a}(\gamma)$ can be approximated as a Gaussian random variable using *Central Limit Theorem* [21, Chapter 8]. The mean, $\mu_{k,a}^X$, and standard deviation, $\sigma_{k,a}^X$, of $X_{k,a}(\gamma)$ are given by

$$\mu_{k,a}^X = \widetilde{\mu}_{k,a}^x T; \qquad \sigma_{k,a}^X = \widetilde{\sigma}_{k,a}^x \sqrt{T} \tag{6}$$

To this end we have, $X_{k,a}(\gamma) \sim \mathcal{N}\left(\mu_{k,a}^X, \left(\sigma_{k,a}^X\right)^2\right)$, $\forall \gamma$.

Remark 1: Gaussian Nature of $X_{k,a}(\gamma)$. $X_{k,a}(\gamma)$ is always a positive quantity because net demand served is always positive. But, we approximated $X_{k,a}(\gamma)$ as a Gaussian random variable and hence the approximated $X_{k,a}(\gamma)$ can be negative. However, the probability of $X_{k,a}(\gamma)$ being negative is

$$P\left[X_{k,a}(\gamma) < 0\right] = \frac{1}{2}\left(1 + \text{erf}\left(-\frac{\widetilde{\mu}_{k,a}^x \sqrt{T}}{\sqrt{2}\widetilde{\sigma}_{k,a}^x}\right)\right)$$

where erf $(\cdot)$ is the error function. For all practical setups, T is large enough that $P\left[X_{k,a}(\gamma) < 0\right]$ is very small. The use of Gaussian model for non-negative random variables has been used in prior works like [22].

An operator generates revenue by serving customer demand. Let $R_{k,a}(\gamma)$ denote the revenue earned by the k^{th} operator in

[1] The cost of leasing a channel for a given operator depends on its own bid, the bid of other operators, and also the auction mechanism. As discussed in the next section, in our model, the bid of an operator is a random variable. Hence, cost of leasing a channel for the k^{th} operator is also a random variable. C_k^L is only a point estimate of this random variable according to the k^{th} operator. The estimation strategy of C_k^L may vary among operators.

[2] All the iid random variables used throughout the paper are identical with respect to time slot, t, or epoch, γ, and *not* with respect to operator index k.

epoch γ when the access type is a. We model $R_{k,a}(\gamma)$ as a random variable that follows the stochastic model

$$\begin{bmatrix} X_{k,a}(\gamma) \\ R_{k,a}(\gamma) \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu^X_{k,a} \\ \mu^R_{k,a} \end{bmatrix}, \begin{bmatrix} \left(\sigma^X_{k,a}\right)^2 & \rho_k \sigma^X_{k,a}\sigma^R_{k,a} \\ \rho_k \sigma^X_{k,a}\sigma^R_{k,a} & \left(\sigma^R_{k,a}\right)^2 \end{bmatrix} \right) \quad (7)$$

for all γ where $\mu^R_{k,a} = h_k\left(\mu^X_{k,a}\right)$. According to (7), the net demand served and the net revenue earned in epoch γ are jointly Gaussian. The mean of $R_{k,a}(\gamma)$ is $\mu^R_{k,a} = h_k\left(\mu^X_{k,a}\right)$ where $h_k\left(\mu^X_{k,a}\right)$ is a monotonic increasing function of the mean demand served by the k^{th} operator in an epoch, $\mu^X_{k,a}$. The standard deviation of $R_{k,a}(\gamma)$ is $\sigma^R_{k,a}$ which can be used to capture the effect of exogenous stochastic processes like market dynamics on $R_{k,a}(\gamma)$. The relative change between $R_{k,a}(\gamma)$ and $X_{k,a}(\gamma)$ is captured with correlation coefficent $\rho_k \in [0,1)$. It captures how much a deviation of $X_{k,a}(\gamma)$ around its mean $\mu^X_{k,a}$ will affect the deviation of $R_{k,a}(\gamma)$ around its mean $h_k\left(\mu^X_{k,a}\right)$. A monotonic increasing function, $h_k(\cdot)$, and a positive correlation coefficient, ρ_k, are intuitive because from a statistical standpoint it implies that an operator who serves more customer demand generates higher revenue.

Let $R_{k,i}(\gamma)$ denote the revenue earned by the k^{th} operator if it is a Tier-i operator in epoch γ. Tier-1 serves customer demand using both licensed and opportunistic access while Tier-2 operator serves its customer demand using opportunistic access only. Hence,

$$R_{k,1}(\gamma) = R_{k,lc}(\gamma) + R_{k,op}(\gamma) \quad (8)$$

$$R_{k,2}(\gamma) = R_{k,op}(\gamma) \quad (9)$$

Notice that since $R_{k,lc}(\gamma)$ and $R_{k,op}(\gamma)$ are Gaussian, $R_{k,1}(\gamma)$ and $R_{k,2}(\gamma)$ are Gaussian as well.

C. Spectrum Allocation Model

Licensed channels are allocated to the set of interested licensed operators, $\mathcal{S}_L$, through spectrum auctions. The auction for epoch γ happens at time slot $(\gamma-1)T+1$. The set of interested licensed operators bids for licensed channels. Let $V_k(\gamma)$ be the bid of the k^{th} operator in epoch γ. Our model assumes truthful spectrum auctions. For such auctions, the operators always bid their true valuations of a licensed channel. The true value of a licensed channel to the k^{th} operator is $R_{k,lc}(\gamma)$, the revenue it can generate using the licensed channel in an epoch. But the k^{th} operator does not know the revenue it will earn in epoch γ when it is bidding for a licensed channel at the beginning of epoch γ. It only has an estimate of $R_{k,lc}(\gamma)$. We capture the relation between $R_{k,lc}(\gamma)$ and $V_k(\gamma)$ using the stochastic model

$$\begin{bmatrix} V_k(\gamma) \\ R_{k,lc}(\gamma) \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu^R_{k,lc} \\ \mu^R_{k,lc} \end{bmatrix}, \begin{bmatrix} \left(\sigma^R_{k,lc}\right)^2 & \omega_k\left(\sigma^R_{k,lc}\right)^2 \\ \omega_k\left(\sigma^R_{k,lc}\right)^2 & \left(\sigma^R_{k,lc}\right)^2 \end{bmatrix} \right) \quad (10)$$

for all γ where $\omega_k \in [0,1)$ is the correlation coefficient between $V_k(\gamma)$ and $R_{k,lc}(\gamma)$. Bid correlation coefficient ω_k captures how good the estimate is; higher ω_k implies a better estimate. Using a stochastic model like (10) to capture the relation between $V_k(\gamma)$ and $R_{k,lc}(\gamma)$ leads to a generalized system model because we can abstract away from the exact bid estimation strategy of the operators which may rely on the auction mechanism and other market externalities.

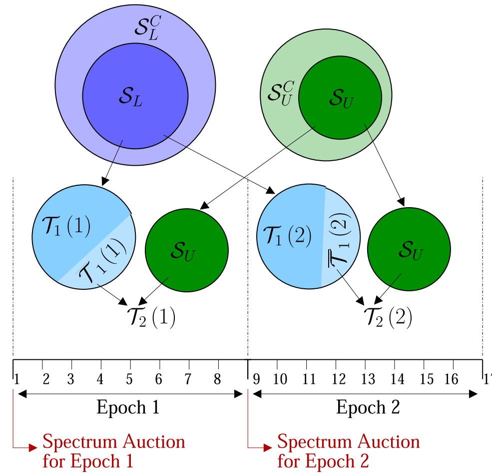

Fig. 2. Pictorial representation of the set of candiate licensed operators $\mathcal{S}_L^C$, candiate unlicensed operators $\mathcal{S}_U^C$, interested licensed operators $\mathcal{S}_L$, interested unlicensed operators $\mathcal{S}_U$, the set of interested licensed operators who are allocated (not allocated) licensed channels in an epoch $\mathcal{T}_1(\gamma)$ ($\overline{\mathcal{T}}_1(\gamma)$) and the set of Tier-2 operators in an epoch $\mathcal{T}_2(\gamma)$. Note that $\mathcal{T}_1(\gamma)$, $\overline{\mathcal{T}}_1(\gamma)$ and $\mathcal{T}_2(\gamma)$ are not the same for epochs 1 and 2.

Algorithm 1: Waterfilling Algorithm for Opportunistic Channel Allocation

Input: $D_O(t)$, $\{\overline{x}_k(t)\}_{k\in\mathcal{S}}$
Output: $\{\widetilde{x}_{k,op}(t)\}_{k\in\mathcal{S}}$

```
Sort the list {x̄_k(t)}_{k∈S} in ascending order of x̄_k(t). Let
  κ(j) denote the operator index corresponding to the j^th
  position of the sorted list.
Set unused opportunistic channel capacity C = D_O(t)
  and the remaining number of interested operators to
  allocate channel capacity N_S = |S|.
for j ← 1 to |S| do
    Set x̃_{κ(j),op}(t) = min( x̄_{κ(j)}(t) , C/N_S ).
    Set C = C − x̃_{κ(j),op}(t) and N_S = N_S − 1.
```

Given that there are P licensed channels and the spectrum cap is one, the interested licensed operators with the P highest bids $V_k(\gamma)$ are allocated one licensed channel each in epoch γ. Let $\mathcal{T}_1(\gamma) \subseteq \mathcal{S}_L$ denote the set of interested licensed operators who are allocated licensed channels in epoch γ. Similarly, $\overline{\mathcal{T}}_1(\gamma) = \mathcal{S}_L \backslash \mathcal{T}_1(\gamma)$ is the set of interested licensed operators who are not allocated licensed channels in epoch γ. The operators in $\mathcal{T}_1(\gamma)$ serve their customer demand as Tier-1 operators in epoch γ. On the other hand, operators in $\overline{\mathcal{T}}_1(\gamma)$ serve their customer demand as Tier-2 operators in epoch γ. It is to be noted that $\mathcal{T}_1(\gamma)$ and $\overline{\mathcal{T}}_1(\gamma)$ are random sets as they get decided by the bids $V_k(\gamma)$ which are random variables. The set of Tier-2 operators in epoch γ is $\mathcal{T}_2(\gamma) = \overline{\mathcal{T}}_1(\gamma) \bigcup \mathcal{S}_U$, i.e., interested unlicensed operators and interested licensed operators who are not allocated a licensed channel in epoch γ. Unlike the sets $\mathcal{S}_L$ and $\mathcal{S}_U$ which are decided once, sets $\mathcal{T}_1(\gamma)$, $\overline{\mathcal{T}}_1(\gamma)$, and $\mathcal{T}_2(\gamma)$ are decided at the beginning of every epoch. A pictorial representation of all the important sets discussed till now is shown in Figure 2. Figure 2 also shows $\mathcal{T}_1(\gamma)$, $\overline{\mathcal{T}}_1(\gamma)$, and $\mathcal{T}_2(\gamma)$ varies with epoch γ.

Opportunistic spectrum allocation happens in every time slot to all the Tier-2 operators. Tier-1 operators may or may not participate in opportunistic spectrum access depending on whether ϕ is one or zero. In order to capture both these cases under a single mathematical abstraction, we modify the demand of Tier-1 and Tier-2 operators. Let $\overline{x}_k(t)$ be the modified demand of the k^{th} operator which needs to be served using OSA. For time slot t of epoch γ,

$$\overline{x}_k(t) = \begin{cases} \phi \cdot \left(x_k(t) - \dfrac{D}{M} \right)^+ ; & k \in \mathcal{T}_1(\gamma) \\ x_k(t) ; & k \in \mathcal{T}_2(\gamma) \end{cases} \quad (11)$$

According to (11), for a Tier-2 operator, its entire demand $x_k(t)$ needs to be served using OSA. For Tier-1 operators, the excess demand which could not be satisfied with licensed use is $\left(x_k(t) - \frac{D}{M} \right)^+$. If $\phi = 1$, this excess demand has to be served using OSA. If $\phi = 0$, then $\overline{x}_k(t) = 0$ for Tier-1 operators implying that they don't participate in OSA.

Opportunistic channel capacity in time slot t of epoch γ is

$$D_O(t) = \alpha_U \left(M - \widetilde{P} \right) \frac{D}{M} + \sum_{k \in \mathcal{T}_1(\gamma)} \mathcal{C}(x_k(t), \alpha_L) \quad (12)$$

where $\widetilde{P} = \min(|\mathcal{S}_L|, P)$. In (12), the first term is the net channel capacity of unlicensed channels and the second term is the net remaining channel capacity of the licensed channels. The variable $\widetilde{P}$ is used to capture edge cases where the number of licensed channels is more than the number of interested licensed operators. In such cases, the remaining $P - |\mathcal{S}_L|$ channels which are not allocated to licensed operators are used as unlicensed channels. The expression for $\mathcal{C}(x_k(t), \alpha_L)$ depends on the OSA strategy (overlay or interweave) and has been discussed in Section II-A. As our model is inspired by the CBRS band, we have to ensure that opportunistic spectrum allocation is fair [23]. One approach to ensure fair allocation and to avoid wastage of channel capacity is to use a max-min fair algorithm, like the famous Waterfilling algorithm. A detailed exposition of max-min fairness can be found in [24], [25]. In this section, we present the Waterfilling algorithm, explain its working with an example, and qualitatively justify how it ensures fairness and avoids wastage of channel capacity. Waterfilling algorithm will be used for opportunistic channel allocation throughout this paper.

Algorithm 1 is the pseudocode of Waterfilling algorithm. Let $\mathcal{S}$ denote the set of interested operators, i.e. $\mathcal{S} = \mathcal{S}_L \bigcup \mathcal{S}_C$. The union of Tier-1 and Tier-2 operators, $\mathcal{T}_1(\gamma) \bigcup \mathcal{T}_2(\gamma)$, is equal to $\mathcal{S}$. The inputs to Algorithm 1 are the opportunistic channel capacity, $D_O(t)$, and the modified demands of all the interested operators, $\{\overline{x}_k(t)\}_{k \in \mathcal{S}}$. The output of Algorithm 1 is the opportunistic channel capacity allocated to all the interested operators, $\{\widetilde{x}_{k,op}(t)\}_{k \in \mathcal{S}}$. $\widetilde{x}_{k,op}(t)$ is also equal to the demand served by the operators using OSA. We use the following example to explain Algorithm 1: the set of interested operators is $\mathcal{S} = \{1, 2, 3, 5, 7\}$, their corresponding modified demand is $\{5, 9, 3, 7, 2\}$, and the opportunistic channel capacity $D_O(t) = 17$. The example is shown in Figure 3.

Waterfilling algorithm allocates channel capacity to the set of interested operators in ascending order of their modified demand (*lines 1* and *3*). The sorted list of modified demand is $\{2, 3, 5, 7, 9\}$ and the operator index $\kappa(j)$ corresponding to position $j = 1, 2, 3, 4, 5$ of the sorted list is 7, 3, 1, 5, 2 respectively. In *line 2*, unused opportunistic channel capacity $C = 17$ and the remaining number of interested operators who need to be allocated channel capacity $N_S = 5$. Inside the for loop, the algorithm reserves an equal portion of unused opportunistic channel capacity C for the remaining

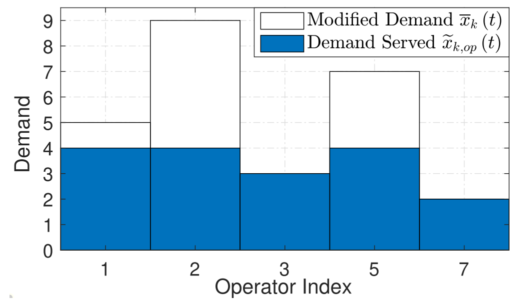

Fig. 3. Pictorial representation of the example for Waterfilling algorithm.

N_S interested operators. This is done in *line 4* where a maximum channel capacity of $\frac{C}{N_S}$ is reserved for the $\kappa(j)^{th}$ operator. This step ensures *fairness* of Waterfilling algorithm. The channel capacity allocated to the $\kappa(j)^{th}$ operator is the minimum of its modified demand (the required channel capacity) and the maximum reserved channel capacity of $\frac{C}{N_S}$. Accordingly, C and N_S are updated in *line 5*. In our example, for $j = 1$, $\widetilde{x}_{7,op}(t) = \min\left(2, \frac{17}{5}\right) = 2$ and hence the updated $C = 17 - 2 = 15$ and $N_S = 4$. For $j = 2$, $\widetilde{x}_{3,op}(t) = \min\left(3, \frac{15}{4}\right) = 3$ and hence the updated $C = 15 - 3 = 12$ and $N_S = 3$. For $j = 3$, $\widetilde{x}_{1,op}(t) = \min\left(5, \frac{12}{3}\right) = 4$ and hence the updated $C = 12 - 4 = 8$ and $N_S = 2$. Similarly, $\widetilde{x}_{5,op}(t) = \widetilde{x}_{2,op}(t) = 4$. Waterfilling algorithm *prevents wastage* of channel capacity by allocating no more than the required channel capacity in *line 4*. This ensures that the unused opportunistic channel capacity C is as high as possible for the operators with higher customer demand.

We end this section by proving that the output $\widetilde{x}_{k,op}(t)$ of Algorithm 1 are iid random variables. By referring to (12) and (11), we can conclude that $D_O(t)$ and $\overline{x}_k(t)$, which form the input to Algorithm 1, are the outputs of time-invariant functions of iid random variables $x_k(t)$ and $V_k(\gamma)$ ($V_k(\gamma)$ decides the random set $\mathcal{T}_1(\gamma)$ in (12)). This implies that $D_O(t)$ and $\overline{x}_k(t)$ are iid random variables as well. Also note that except the inputs $D_O(t)$ and $\overline{x}_k(t)$, Algorithm 1 is not dependent on time t. Therefore, Algorithm 1 can be expressed as a time-invariant function of iid random variables $D_O(t)$ and $\overline{x}_k(t)$. This directly implies that the output $\widetilde{x}_{k,op}(t)$ of Algorithm 1 are iid random variables.

Remark 2 (Generality of the OSA Model): We want to highlight that our OSA model is very general for three reasons. *First*, any opportunistic channel allocation algorithm can be used as long as $\widetilde{x}_{k,op}(t)$ are iid random variables. *Second*, the parameter ϕ helps us capture cases where Tier-1 operators can/cannot participate in OSA. *Third*, our model can capture both overlay and interweave OSA strategy.

III. Optimization Problem

We start this section by formulating the optimization problem for joint spectrum partitioning and licensing as a two-stage Stackelberg Game in Section III-A. In the process of formulating the Stackelberg Game, we introduce two functions. The first is the revenue function of an operator which captures the expected revenue of an operator in an epoch. The second is the objective function which is proportional to spectrum utilization of all the interested operators in the market. We then develop efficient algorithms to solve the two stages of the Stackelberg Game in Section III-B and hence find the optimal M and P which maximizes spectrum utilization. In this section, we assume *complete* *information* games but the overall approach can be easily extended to *incomplete information* games as discussed in Appendix A of the supplementary material.

A. Stackelberg Game Formulation

In this section, we formulate the optimal spectrum partitioning problem as a two-stage Stackelberg game. In our formulation of the Stackelberg game, the regulator is the *leader* and the wireless operators are the *followers*. The k^{th} operator can be completely characterized by seven parameters which can be represented as a tuple $\xi_k = \left(\mu_k^\theta,\, \sigma_k^\theta,\, h_k(\cdot),\, \sigma_{k,a}^R,\, \rho_k,\, \omega_k,\, \lambda_k\right)$. In sections III-A and III-B, we assume complete information games, i.e. an operator and the regulator knows ξ_k of all the operators. The player in stage-1 of the Stackelberg game is the regulator whose decision variables are M and P. The payoff of the regulator is the expected spectrum utilization over a period of $\Gamma \geq 1$ epochs which is given by

$$Q = E\left[\sum_{\gamma=1}^{\Gamma}\sum_{t=(\gamma-1)T+1}^{\gamma T}\left(Q_{lc}(\gamma,t)+Q_{op}(\gamma,t)\right)\right] = \sum_{\gamma=1}^{\Gamma}\sum_{t=(\gamma-1)T+1}^{\gamma T} E\left[Q_{lc}(\gamma,t)+Q_{op}(\gamma,t)\right] \text{ where,} \tag{13}$$

$$Q_{lc}(\gamma,t) = \sum_{k\in\mathcal{T}_1(\gamma)} \widetilde{x}_{k,lc}(t) \tag{14}$$

$$Q_{op}(\gamma,t) = \sum_{k\in\mathcal{S}} \widetilde{x}_{k,op}(t) \tag{15}$$

In (13), $Q_{lc}(\gamma,t)$ and $Q_{op}(\gamma,t)$ are the net spectrum utilization in time slot t of epoch γ by using licensed and opportunistic spectrum access respectively. The regulator wants to maximize Q. We now prove that the term $E\left[Q_{lc}(\gamma,t)+Q_{op}(\gamma,t)\right]$ of (13) is not a function of γ and t. Based on (14) and (15), $\widetilde{x}_{k,lc}(t)$, $\widetilde{x}_{k,op}(t)$, and $\mathcal{T}_1(\gamma)$ are the only random variables in the expressions of $Q_{lc}(\gamma,t)$ and $Q_{op}(\gamma,t)$. As discussed in previous sections, $\widetilde{x}_{k,lc}(t)$ and $\widetilde{x}_{k,op}(t)$ are iid random variables. $\mathcal{T}_1(\gamma)$ is a function of bids $V_k(\gamma)$ of the operators. Since, $V_k(\gamma)$ is an iid random variable, so is $\mathcal{T}_1(\gamma)$. This discussion implies that $Q_{lc}(\gamma,t)+Q_{op}(\gamma,t)$ itself is an iid random variable and hence its expectation is independent of γ and t. In fact, it is a function of M, P, $\mathcal{S}_L$ and $\mathcal{S}_U$. Let,

$$U(M,P,\mathcal{S}_L,\mathcal{S}_U) = E\left[Q_{lc}(\gamma,t)+Q_{op}(\gamma,t)\right] \tag{16}$$

Substituting (16) in (13) we get, $Q = \Gamma T U(M,P,\mathcal{S}_L,\mathcal{S}_U)$. This shows that maximizing Q is the same as maximizing $U(M,P,\mathcal{S}_L,\mathcal{S}_U)$. Therefore, we will use $U(M,P,\mathcal{S}_L,\mathcal{S}_U)$ as the payoff function of the regulator in the rest of the paper. $U(M,P,\mathcal{S}_L,\mathcal{S}_U)$ is also called the *objective function* as it is a direct measure of spectrum utilization which we are trying to maximize in this paper.

The players in stage-2 of the Stackelberg game are the candidate licensed operators, $\mathcal{S}_L^C$, and candidate unlicensed operators, $\mathcal{S}_U^C$. These candidate operators decide whether to enter the market or not in a *non-cooperative* manner, i.e. our Stackelberg game model does not consider collusion between operators. The decision variables of the Stage-2 game are the set of interested licensed operators, $\mathcal{S}_L$, and the set of interested unlicensed operators, $\mathcal{S}_U$. The k^{th} operator is interested in joining the market only if the expected revenue in an epoch is greater than λ_k. The expected revenue in an epoch of an interested licensed or unlicensed operator is given by the *revenue function*. The formula for revenue function if different for interested licensed operators and interested unlicensed operators. The revenue function of an interested licensed operator, i.e. $k \in \mathcal{S}_L$, is

$$\begin{aligned}
&\mathcal{R}_k(M,P,\mathcal{S}_L,\mathcal{S}_U) \\
&= E\left[R_{k,1}(\gamma)\mid\mathcal{E}_k^\gamma\right]P\left[\mathcal{E}_k^\gamma\right] + E\left[R_{k,2}(\gamma)\mid\overline{\mathcal{E}_k^\gamma}\right]P\left[\overline{\mathcal{E}_k^\gamma}\right] && (17)\\
&= E\left[R_{k,lc}(\gamma)\mid\mathcal{E}_k^\gamma\right]P\left[\mathcal{E}_k^\gamma\right] \\
&\quad + E\left[R_{k,op}(\gamma)\mid\mathcal{E}_k^\gamma\right]P\left[\mathcal{E}_k^\gamma\right] + E\left[R_{k,op}(\gamma)\mid\overline{\mathcal{E}_k^\gamma}\right]P\left[\overline{\mathcal{E}_k^\gamma}\right] && (18)\\
&= E\left[R_{k,lc}(\gamma)\mid\mathcal{E}_k^\gamma\right]P\left[\mathcal{E}_k^\gamma\right] + h_k\left(\mu_{k,op}^X\right) && (19)
\end{aligned}$$

where $P[Z]$ denotes the probability of event Z, $\mathcal{E}_k^\gamma$ ($\overline{\mathcal{E}_k^\gamma}$) is the event that $k\in\mathcal{T}_1(\gamma)$ ($k\in\mathcal{T}_2(\gamma)$). In (17), $E\left[R_{k,i}(\gamma)\mid\mathcal{E}_k^\gamma\right]$ is the expected revenue of the k^{th} operator if it is a Tier-i operator in epoch γ. Finally, (17) is obtained using the *law of total expectation*. Equation 18 is obtained by substituting $R_{k,1}(\gamma) = R_{k,lc}(\gamma)+R_{k,op}(\gamma)$ (refer to (8)). Equation 19 is obtained by noticing that the sum of the second and the third term of (18) is equal to $E\left[R_{k,op}(\gamma)\right]$ which in turn is equal to $h_k\left(\mu_{k,op}^X\right)$ according to (7). Similar to the objective function, the revenue function of an interested licensed operator is also not a function of epoch γ. This is because the statistical properties of the involved random variables $R_{k,1}(\gamma)$ and $R_{k,2}(\gamma)$ are independent of γ.

If the k^{th} operator is an interested unlicensed operator, i.e. $k\in\mathcal{S}_U$, it is always a Tier-2 operator. Hence, its expected revenue in an epoch is

$$\mathcal{R}_k(M,P,\mathcal{S}_L,\mathcal{S}_U) = E\left[R_{k,2}(\gamma)\right] = h_k\left(\mu_{k,op}^X\right) \tag{20}$$

where $E\left[R_{k,2}(\gamma)\right] = h_k\left(\mu_{k,op}^X\right)$ because $R_{k,2}(\gamma) = R_{k,op}(\gamma)$ (refer to (9)) and $E\left[R_{k,op}(\gamma)\right] = h_k\left(\mu_{k,op}^X\right)$.

Payoff function of an operator who is interested in joining the market either as a licensed or an unlicensed operator is

$$\pi_k(M,P,\mathcal{S}_L,\mathcal{S}_U) = \mathcal{R}_k(M,P,\mathcal{S}_L,\mathcal{S}_U) - \lambda_k \tag{21}$$

where $\mathcal{R}_k(M,P,\mathcal{S}_L,\mathcal{S}_U)$ is given by (17) if $k\in\mathcal{S}_L$ and by (20) if $k\in\mathcal{S}_U$. If an operator does not join the market, its payoff is zero. An operator decides to enter the market only if its payoff $\pi_k(M,P,\mathcal{S}_L,\mathcal{S}_U)$ is *strictly* greater than zero.

With (21) as the payoff function, Stage-2 game may have multiple Nash Equilibriums which complicates the analysis. This can be simplified if we assume that the operators are *pessimistic* in nature. By doing so, we can get an unique solution of the Stage-2 game. Pessimistic models to address the issue of multiple Nash Equilibriums have been considered in prior works like [26]–[28]. One simple approach to model pessimistic decision making strategy is to use the concept of *dominant strategy*, i.e. an operator decides to join the market if and only if joining the market is its optimal strategy irrespective of whether other operators decide to join the market. However, in this paper, we model a pessimistic operators' decision making strategy using *iterated elimination of strictly dominated strategies* (IESDS) [29]. Compared to dominant strategy, IESDS is a less pessimistic decision making strategy because more operators will join the market.

IESDS can be explained as follows. IESDS consists of iterations. Consider the first iteration which is the original Stage-2 game. We iterate through all the candidate licensed and unlicensed operators to check if either joining the market or not joining the market is a dominant strategy for any of the operators. The operators whose dominant strategy is to join (not join) the market will join (not join) the market irrespective of other operators' decisions. This reduces the size of the Stage-2 game as it effectively consists of those operators who could not decide whether to join (not join) the market in the first iteration. Such operators are called *confused operators* in this paper. These confused operators who did not have a dominant strategy in the original Stage-2 game may have a dominant strategy in the reduced Stage-2 game. Therefore, in the second iteration, we find the dominant strategy of the confused operators in the reduced Stage-2 game. Such iterations continue until convergence, which happens when the reduced Stage-2 game does not have any dominant strategy. It is possible that there are confused operators even after convergence. Those operators will not join the market because, in our model, the operators are pessimistic in nature.

B. Solution of the Stackelberg Game

In this subsection, we use backward induction [30] to solve the Stackelberg Game formulated in Section III-A. To apply backward induction, we first solve Stage-2 of the game followed by Stage-1. The following properties of the revenue function (as given by (17) and (20)) are crucial in designing an efficient algorithm to solve Stage-2 of the Stackelberg Game.

Property 1: *$\mathcal{R}_k(M,P,\mathcal{S}_L,\mathcal{S}_U)$ is monotonic decreasing in $\mathcal{S}_L$, i.e.* $\mathcal{R}_k(M,P,\mathcal{S}_L,\mathcal{S}_U) \geq \mathcal{R}_k(M,P,\mathcal{S}_L\bigcup\{a\},\mathcal{S}_U)$ *where $a \notin \mathcal{S}_L$ and $a \in \mathcal{S}_L^C$.*

Property 2: *$\mathcal{R}_k(M,P,\mathcal{S}_L,\mathcal{S}_U)$ is monotonic decreasing in $\mathcal{S}_U$, i.e.* $\mathcal{R}_k(M,P,\mathcal{S}_L,\mathcal{S}_U) \geq \mathcal{R}_k(M,P,\mathcal{S}_L,\mathcal{S}_U\bigcup\{a\})$ *where $a \notin \mathcal{S}_U$ and $a \in \mathcal{S}_U^C$.*

We have verified these properties numerically using the Monte Carlo integrator which will be described in Section IV. These properties can be justified as follows. Property 1 states that as the set of interested licensed operators, $\mathcal{S}_L$, increases, the revenue function of both the licensed and the unlicensed operators decreases. The revenue function of a licensed operator decreases with an increase in $\mathcal{S}_L$ because the operator has to compete with more operators in the spectrum auctions to get a channel. This reduces the operator's probability of winning spectrum auctions which in turn decreases its revenue function as it can effectively serve fewer customer demand. The revenue function of an unlicensed operator also decreases with an increase in $\mathcal{S}_L$. This happens because with an increase in $\mathcal{S}_L$, there is an increase in the number of operators interested in opportunistic channel access. This reduces the share of opportunistic channels for an unlicensed operator. Therefore, its revenue decreases as it can serve fewer customer demand. Property 2 states that as the set of interested unlicensed operators, $\mathcal{S}_U$, increases, the revenue function of both the licensed and the unlicensed operators decreases. This happens because with an increase in $\mathcal{S}_U$, the share of opportunistic channel decreases for a licensed or an unlicensed operator. This in turn decreases its revenue function.

The pseudocode to solve Stage-2 of the Stackelberg game is given in Algorithm 2. The inputs of Algorithm 2 are clearly described in Table I. Let $\mathcal{S}_L(M,P)$ and $\mathcal{S}_U(M,P)$ denote the set of interested licensed and unlicensed operators if the entire bandwidth is divided into M channels out of which P are licensed channels. $\mathcal{S}_L(M,P)$ and $\mathcal{S}_U(M,P)$ are the outputs of Algorithm 2. As mentioned in Section III-A, $\mathcal{S}_L(M,P)$ and $\mathcal{S}_U(M,P)$ are decided by the operators based on IESDS. Algorithm 2 uses Properties 1 and 2 to compute $\mathcal{S}_L(M,P)$ and $\mathcal{S}_U(M,P)$ in polynomial time when an operator's decision making strategy to join/not join the market is based on IESDS.

Let $\widehat{\mathcal{X}}_l$ and $\widetilde{\mathcal{X}}_l$, where $\widehat{\mathcal{X}}_l, \widetilde{\mathcal{X}}_l \in \mathcal{S}_L^C$, denote the set of licensed operators who are sure to join the market and the set of confused licensed operators respectively till the l^{th} iteration. Note that $\widehat{\mathcal{X}}_l$ and $\widetilde{\mathcal{X}}_l$ are disjoint sets and the set $\mathcal{S}_L^C\setminus\left(\widehat{\mathcal{X}}_l\bigcup\widetilde{\mathcal{X}}_l\right)$ consists of those licensed operators who are sure not to join the market till the l^{th} iteration. Similarly, $\widehat{\mathcal{Y}}_l$ and $\widetilde{\mathcal{Y}}_l$, where $\widehat{\mathcal{Y}}_l, \widetilde{\mathcal{Y}}_l \in \mathcal{S}_U^C$, denote the set of unlicensed operators who decided to join the market and the set of confused unlicensed operators till the l^{th} iteration respectively.

We will now explain the working of Algorithm 2. Algorithm 2 starts with iteration 0. Initially, none of the operators are sure whether to join the market or not; all of them are confused. Hence, in iteration 0, we initialize $\widehat{\mathcal{X}}_0=\emptyset$, $\widetilde{\mathcal{X}}_0=\mathcal{S}_L^C$, $\widehat{\mathcal{Y}}_l=\emptyset$ and $\widetilde{\mathcal{Y}}_0=\mathcal{S}_U^C$ (*line 1*). The *while loop* in *lines 3-19* finds $\widehat{\mathcal{X}}_l$, $\widetilde{\mathcal{X}}_l$, $\widehat{\mathcal{Y}}_l$ and $\widetilde{\mathcal{Y}}_l$ for the l^{th} iteration given $\widehat{\mathcal{X}}_{l-1}$, $\widetilde{\mathcal{X}}_{l-1}$, $\widehat{\mathcal{Y}}_{l-1}$ and $\widetilde{\mathcal{Y}}_{l-1}$ of the $(l-1)^{th}$ iteration. Since the operators in sets $\widehat{\mathcal{X}}_{l-1}$ and $\widehat{\mathcal{Y}}_{l-1}$ will surely join the market, we initialize $\widehat{\mathcal{X}}_l$ and $\widehat{\mathcal{Y}}_l$ to $\widehat{\mathcal{X}}_{l-1}$ and $\widehat{\mathcal{Y}}_{l-1}$ respectively at the beginning of the l^{th} iteration (*line 2*). The set of confused licensed and unlicensed operators, $\widetilde{\mathcal{X}}_l$ and $\widetilde{\mathcal{Y}}_l$, are initialized to $\widetilde{\mathcal{X}}_{l-1}$ and $\widetilde{\mathcal{Y}}_{l-1}$ respectively at the beginning of the l^{th} iteration (*line 2*). In the *for loop* in *lines 6 - 12*, we check if an licensed operator in set $\widetilde{\mathcal{X}}_{l-1}$ is sure to either join or not join the market. Similarly, in the *for loop* in *lines 13 - 19*, we check if an unlicensed operator in set $\widetilde{\mathcal{Y}}_{l-1}$ is sure to either join or not join the market.

We will now explain the working of the for loop in lines 6-12. The largest possible set of interested licensed operators in the $\widetilde{l}^{th}$ iteration, for $\widetilde{l}\geq l$ is $\widehat{\mathcal{X}}_{l-1}\bigcup\widetilde{\mathcal{X}}_{l-1}$. This is because the operators in set $\mathcal{S}_L^C\setminus\left(\widehat{\mathcal{X}}_{l-1}\bigcup\widetilde{\mathcal{X}}_{l-1}\right)$ are sure not to join the market till the $(l-1)^{th}$ iteration. Similarly, the largest possible set of interested unlicensed operators in the $\widetilde{l}^{th}$ iteration, for $\widetilde{l}\geq l$ is $\widehat{\mathcal{Y}}_{l-1}\bigcup\widetilde{\mathcal{Y}}_{l-1}$. Therefore, according to Properties 1 and 2, the minimum revenue of the k^{th} operator, where $k\in\widetilde{\mathcal{X}}_{l-1}$, in the $\widetilde{l}^{th}$ iteration, for $\widetilde{l}\geq l$ is $\mathcal{R}_k\left(M,P,\widehat{\mathcal{X}}_{l-1}\bigcup\widetilde{\mathcal{X}}_{l-1},\widehat{\mathcal{Y}}_{l-1}\bigcup\widetilde{\mathcal{Y}}_{l-1}\right)$. So if

$$\mathcal{R}_k\left(M,P,\widehat{\mathcal{X}}_{l-1}\bigcup\widetilde{\mathcal{X}}_{l-1},\widehat{\mathcal{Y}}_{l-1}\bigcup\widetilde{\mathcal{Y}}_{l-1}\right)>\lambda_k$$

then joining the market becomes the dominant strategy of the k^{th} operator in the l^{th} iteration. Therefore, in *line 8*, we remove the k^{th} operator from the set of confused licensed operators and add it to the set of licensed operators who are sure to join the market. If the k^{th} operator, where $k\in\widetilde{\mathcal{X}}_{l-1}$, joins the market, then the smallest possible set of interested licensed and unlicensed operators in the $\widetilde{l}^{th}$ iteration, for $\widetilde{l}\geq l$ are $\widehat{\mathcal{X}}_{l-1}\bigcup\{k\}$ and $\widehat{\mathcal{Y}}_{l-1}$ respectively. Therefore, according to Properties 1 and 2, the maximum revenue of the k^{th} operator, where $k\in\widetilde{\mathcal{X}}_{l-1}$, in the $\widetilde{l}^{th}$ iteration, for $\widetilde{l}\geq l$ is $\mathcal{R}_k\left(M,P,\widehat{\mathcal{X}}_{l-1}\bigcup\{k\},\widehat{\mathcal{Y}}_{l-1}\right)$. So if

$$\mathcal{R}_k\left(M,P,\widehat{\mathcal{X}}_{l-1}\bigcup\{k\},\widehat{\mathcal{Y}}_{l-1}\right)\leq\lambda_k$$

Algorithm 2: Algorithm to Solve Stage 2 of the Stackelberg Game for Joint Spectrum Partitioning

Input: M, P, T, D, α_L, α_U, $\mathcal{S}_L^C$, $\mathcal{S}_U^C$ and $\xi_k; \forall k \in \mathcal{S}_L^C \bigcup \mathcal{S}_U^C$
Output: $\mathcal{S}_L(M,P)$ and $\mathcal{S}_U(M,P)$
1 Set $\widehat{\mathcal{X}}_0 = \emptyset$, $\widetilde{\mathcal{X}}_0 = \mathcal{S}_L^C$, $\widehat{\mathcal{Y}}_0 = \emptyset$ and $\widetilde{\mathcal{Y}}_0 = \mathcal{S}_U^C$.
2 Set $converged = False$ and $l = 0$.
3 **while** $not\,(converged)$ **do**
4 Set $converged = True$ and $l = l + 1$.
5 Set $\widehat{\mathcal{X}}_l = \widehat{\mathcal{X}}_{l-1}$, $\widetilde{\mathcal{X}}_l = \widetilde{\mathcal{X}}_{l-1}$, $\widehat{\mathcal{Y}}_l = \widehat{\mathcal{Y}}_{l-1}$ and $\widetilde{\mathcal{Y}}_l = \widetilde{\mathcal{Y}}_{l-1}$.
6 **for** k ***in*** $\widetilde{\mathcal{X}}_{l-1}$ **do**
7 **if** $\mathcal{R}_k\left(M, P, \widehat{\mathcal{X}}_{l-1} \bigcup \widetilde{\mathcal{X}}_{l-1}, \widehat{\mathcal{Y}}_{l-1} \bigcup \widetilde{\mathcal{Y}}_{l-1}\right) > \lambda_k$ **then**
8 Set $\widehat{\mathcal{X}}_l = \widehat{\mathcal{X}}_l \bigcup \{k\}$ and $\widetilde{\mathcal{X}}_l = \widetilde{\mathcal{X}}_l \backslash \{k\}$.
9 Set $converged = False$.
10 **else if** $\mathcal{R}_k\left(M, P, \widehat{\mathcal{X}}_{l-1} \bigcup \{k\}, \widehat{\mathcal{Y}}_{l-1}\right) \leq \lambda_k$ **then**
11 Set $\widetilde{\mathcal{X}}_l = \widetilde{\mathcal{X}}_l \backslash \{k\}$.
12 Set $converged = False$.
13 **for** k ***in*** $\widetilde{\mathcal{Y}}_{l-1}$ **do**
14 **if** $\mathcal{R}_k\left(M, P, \widehat{\mathcal{X}}_{l-1} \bigcup \widetilde{\mathcal{X}}_{l-1}, \widehat{\mathcal{Y}}_{l-1} \bigcup \widetilde{\mathcal{Y}}_{l-1}\right) > \lambda_k$ **then**
15 Set $\widehat{\mathcal{Y}}_l = \widehat{\mathcal{Y}}_l \bigcup \{k\}$ and $\widetilde{\mathcal{Y}}_l = \widetilde{\mathcal{Y}}_l \backslash \{k\}$.
16 Set $converged = False$.
17 **else if** $\mathcal{R}_k\left(M, P, \widehat{\mathcal{X}}_{l-1}, \widehat{\mathcal{Y}}_{l-1} \bigcup \{k\}\right) \leq \lambda_k$ **then**
18 Set $\widetilde{\mathcal{Y}}_l = \widetilde{\mathcal{Y}}_l \backslash \{k\}$.
19 Set $converged = False$.
20 Set $\mathcal{S}_L(M,P) = \widehat{\mathcal{X}}_l$ and $\mathcal{S}_U(M,P) = \widehat{\mathcal{Y}}_l$

then not joining the market becomes the dominant strategy of the k^{th} operator in the l^{th} iteration. Therefore, in *line 11*, we remove the k^{th} operator from the set of confused licensed operators but we do not add it to the set of licensed operators who are sure to join the market. The for loop in lines 13-19 work in a similar way to decide if an unlicensed operator in set $\widetilde{\mathcal{Y}}_{l-1}$ is sure to either join or not join the market.

The variable $converged$ which is declared in *line 2* and updated in *lines 9, 12, 16, 19* decides when the while loop terminates. This can be explained as follows. Say that a few of the confused operators in the l^{th} iteration decide to not join the market, i.e. if statements in *lines 10* or *17* are $true$. In this case, $converged$ is set to $false$ and hence the while loop continues. Since few of the operators decide not to join the market in the l^{th} iteration, then due to Properties 1 and 2, the revenue function of the remaining confused operators in the $(l+1)^{th}$ iteration is more compared to their corresponding values in the l^{th} iteration. Therefore, it is possible that for some of these confused operators, joining the market becomes the dominant strategy in the $(l+1)^{th}$ iteration. The opposite happens when a few of the confused operators in the l^{th} iteration decide to join the market. This discussion captures the fundamental idea behind IESDS.

Say that after the end of the l_o^{th} iteration, $\widehat{\mathcal{X}}_{l_o} = \widehat{\mathcal{X}}_{l_o-1}$, $\widetilde{\mathcal{X}}_{l_o} = \widetilde{\mathcal{X}}_{l_o-1}$, $\widehat{\mathcal{Y}}_{l_o} = \widehat{\mathcal{Y}}_{l_o-1}$ and $\widetilde{\mathcal{Y}}_{l_o} = \widetilde{\mathcal{Y}}_{l_o-1}$. This happens when *if statements* in *lines 7, 10, 14, 17* are all $false$. When this happens, $converged$ is $true$ after the end of the l_o^{th} iteration and hence the while loop terminates. This is because if $\widehat{\mathcal{X}}_{l_o} = \widehat{\mathcal{X}}_{l_o-1}$, $\widetilde{\mathcal{X}}_{l_o} = \widetilde{\mathcal{X}}_{l_o-1}$, $\widehat{\mathcal{Y}}_{l_o} = \widehat{\mathcal{Y}}_{l_o-1}$ and $\widetilde{\mathcal{Y}}_{l_o} = \widetilde{\mathcal{Y}}_{l_o-1}$, then the value of the revenue function in lines 7, 10, 14, 17 in the $(l_o+1)^{th}$ iteration is the same as that in the l_o^{th} iteration. Therefore, the if statements in lines 7, 10, 14, and 17 will be $false$ in the $(l_o+1)^{th}$ iteration just like the l_o^{th} iteration. This argument suggests that $\widehat{\mathcal{X}}_l = \widehat{\mathcal{X}}_{l_o}$, $\widetilde{\mathcal{X}}_l = \widetilde{\mathcal{X}}_{l_o}$, $\widehat{\mathcal{Y}}_l = \widehat{\mathcal{Y}}_{l_o}$, and $\widetilde{\mathcal{Y}}_l = \widetilde{\mathcal{Y}}_{l_o}$ for all $l \geq l_o$ and hence convergence in $\widehat{\mathcal{X}}_l$, $\widetilde{\mathcal{X}}_l$, $\widehat{\mathcal{Y}}_l$ and $\widetilde{\mathcal{Y}}_l$ have been achieved. After convergence is achieved, there are three kinds of operators. *First*, the operators in sets $\widehat{\mathcal{X}}_{l_o}$ and $\widehat{\mathcal{Y}}_{l_o}$ who are sure that they should join the market. *Second*, the operators in sets $\widehat{\mathcal{X}}_{l_o} \backslash \widetilde{\mathcal{X}}_{l_o}$ and $\widehat{\mathcal{Y}}_{l_o} \backslash \widetilde{\mathcal{Y}}_{l_o}$ who are sure that they should not join the market. *Third*, the 'confused' operators in sets $\widetilde{\mathcal{X}}_{l_o}$ and $\widetilde{\mathcal{Y}}_{l_o}$. Since our model assumes that the operators are pessimistic, confused operators will not join the market. Hence, the set of interested licensed and unlicensed operators are $\widetilde{\mathcal{X}}_{l_o}$ and $\widehat{\mathcal{Y}}_{l_o}$ respectively where l_o is the last iteration of Algorithm 2 (*line 20*).

Proposition 1: Time complexity of Algorithm 2 is $\mathcal{O}\left(N^2\right)$ *where* $N = \left|\mathcal{S}_L^C\right| + \left|\mathcal{S}_U^C\right|$.

Proof: The while loop continues until none of the confused operators of an iteration have a dominant strategy. Such a condition is possible at most N times because there are only N candidate operators. Hence, the while loop is executed at most N times. For a given iteration of the while loop, the inner for loop in lines 6-12 is executed at most $\left|\mathcal{S}_L^C\right|$ times and that in lines 13-19 is executed at most $\left|\mathcal{S}_U^C\right|$ times. Therefore, the inner for loops runs at most $\left|\mathcal{S}_L^C\right| + \left|\mathcal{S}_U^C\right| = N$ times. This shows that the time complexity of Algorithm 2 is $\mathcal{O}\left(N^2\right)$. This completes the proof. ■

Remark 3 (Efficiency of Algorithm): 2. Algorithm 2 uses Properties 1 and 2 to decide whether a confused operator will join the market or not by computing its revenue function for the largest/smallest set of interested operators. Without these properties, we have to compute the revenue function for an exponential number of set of interested operators to decide whether a confused operator will join the market or not.

Remark 4 Comparison With Dominant Strategy): Only the 1^{st} iteration of Algorithm 2 is required to find the dominant strategies of the operators. It is for this reason that the set of interested operators, $\mathcal{S}_L(M,P)$ and $\mathcal{S}_U(M,P)$, will always be larger if operators' decision making strategy is based on IESDS rather than dominant strategy.

The objective function in (16) can be re-written as

$$\widetilde{U}(M,P) = U\left(M, P, \mathcal{S}_L(M,P), \mathcal{S}_U(M,P)\right) \tag{22}$$

where $\mathcal{S}_L(M,P)$ and $\mathcal{S}_U(M,P)$ are the solutions of the Stage-2 game. In Stage-1, the regulator chooses M and P to maximize $\widetilde{U}(M,P)$. Let the optimal solution be M^* and P^*, the optimal value of the objective function be U^*, where U^*=$\widetilde{U}(M^*, P^*)$, and the optimal set of interested licensed and unlicensed operators be $\mathcal{S}_L^*$ and $\mathcal{S}_U^*$, where $\mathcal{S}_L^* = \mathcal{S}_L(M^*, P^*)$ and $\mathcal{S}_U^* = \mathcal{S}_U(M^*, P^*)$. M^* and P^* are found by performing a grid search. The grid search is detailed in Algorithm 3. As shown in *lines 2 and 3* of Algorithm 3, the grid search is performed from $M = 1$ to a certain M_{max} and from $P = 0$ to $\min\left(\left|\mathcal{S}_L^C\right|, M\right)$. Note that since spectrum cap is one, the number of licensed channels should be lesser than the number of candidate licensed operators, $\left|\mathcal{S}_L^C\right|$. The time complexity of Algorithm 3 is $\mathcal{O}\left(M_{max}\left|\mathcal{S}_L^C\right|\right)$.

Algorithm 3: Algorithm to Solve Stage 1 of the Stackelberg Game for Joint Spectrum Partitioning

Input: T, D, α_L, α_U, $\mathcal{S}_L^C$, $\mathcal{S}_U^C$, and $\xi_k; \forall k \in \mathcal{S}_L^C \bigcup \mathcal{S}_U^C$
Output: M^*, P^*, $\mathcal{S}_L^*$, $\mathcal{S}_U^*$, and U^*

```
Set U* = −∞.
for M ← 1 to M_max do
    for P ← 0 to min(|S_L^C|, M) do
        Call Algorithm 2 to get the set of interested
         licensed and unlicensed operators, S_L(M,P) and
         S_U(M,P) respectively, for current M and P.
        Set Ũ = U(M, P, S_L(M,P), S_U(M,P)).
        if Ũ > U* then
            Set M* = M, P* = M, S_L* = S_L(M,P),
             S_U* = S_U(M,P), and U* = Ũ.
```

IV. Monte Carlo Integrator Design

Algorithms 2 and 3 rely on the computation of the objective function, $U(M, P, \mathcal{S}_L, \mathcal{S}_U)$, and the revenue function, $\mathcal{R}_k(M, P, \mathcal{S}_L, \mathcal{S}_U)$. In this section, we design an efficient Monte Carlo integrator to compute these two functions. These functions are the mean of certain random variables. Monte Carlo integrator estimates the mean of a random variable by calculating the *sample mean* of the random variable. Consider a random variable $Z \sim F_Z$, where F_Z is the probability distribution of Z. Let the mean and the standard deviation of Z be μ_Z and σ_Z respectively. The following recursive formula can be used to compute the sample mean of Z,

$$\widehat{z}^r = \frac{(r-1)\,\widehat{z}^{r-1} + z_r}{r} \tag{23}$$

where r is the number of samples, z^r is the r^{th} sample of Z, and $\widehat{z}^r$ is the sample mean of Z calculated over the first r samples. $\widehat{z}^r$ is an estimate of μ_Z. Note that $\widehat{z}^r$ itself is a random variable with mean μ_Z and standard deviation $\frac{\sigma_Z}{\sqrt{r}}$. According to *Chebyshev's inequality*, the probability that $\widehat{z}^r$ is within a Δ bound of μ_Z is lower bounded as follows

$$P\left[|\widehat{z}^r - \mu_Z| \le \Delta\right] \ge 1 - \frac{\sigma_Z^2}{r\Delta^2} \tag{24}$$

We want to design a Monte Carlo integrator whose maximum acceptable percentage error in $\widehat{z}^r$ is β_1 with a minimum probability of β_2. β_1 and β_2 capture the "goodness" of estimate $\widehat{z}^r$; a lower β_1 and a higher β_2 imply a better estimate. To achieve this we substitute $\Delta = \frac{\beta_1}{100}\mu_Z$ in (24) which makes the RHS of (24) equal to $1 - \frac{100^2\sigma_Z^2}{r\beta_1^2\mu_Z^2}$. So we have to recursively calculate $\widehat{z}^r$ until

$$100^2\sigma_Z^2 \le r\beta_1^2\mu_Z^2\,(1-\beta_2) \tag{25}$$

Inequality 25 can be used as one of the stopping criteria for the Monte Carlo integrator. However, we don't know μ_Z and σ_Z^2 of (25); in fact we want to calculate μ_Z. One possible heuristic would be to use the sample mean and the sample variance in place of μ_Z and σ_Z^2 respectively. Sample mean can be calculated using (23). Sample variance δz^r can be computed using the following recursive formula [31],

$$\delta z^r = \frac{(r-1)}{r}\delta z^{r-1} + (r-1)\left(\widehat{z}^r - \widehat{z}^{r-1}\right)^2 \tag{26}$$

To summarize, μ_Z is estimated by recursively calculating the sample mean using (23) until the sample mean and sample variance pair, $(\widehat{z}^r, \delta z^r)$, satisfies the following inequality,

$$100^2\delta z^r \le r\beta_1^2\left(\widehat{z}^r\right)^2(1-\beta_2) \tag{27}$$

Now, we discuss all the sample means which we have to calculate in order to estimate the objective and the revenue functions. By referring to (14), (15) and (16), we can say that the objective function is the expected value of the net demand served by all the interested operators in one time slot using either licensed or opportunistic access. Let $\widehat{U}^r$ denote the sample mean over r samples of the net demand served by all the interested operators in one time slot. Equation 19 shows that the revenue function of the k^{th} licensed operator consists of two terms. The first term in (19) is the expected value of the k^{th} licensed operator's revenue in an epoch generated using licensed access. Let $\widehat{R}_{k,lc}^r$ denote the sample mean over r samples of the k^{th} licensed operator's revenue in an epoch generated using licensed access. The second term in (19) is the expected value of the k^{th} licensed operator's revenue in an epoch generated using opportunistic access. This value is equal to $h_k\left(\mu_{k,op}^X\right)$ according to (7). But $\mu_{k,op}^X = \widetilde{\mu}_{k,op}^x T$ (refer to (6)) and hence $h_k\left(\mu_{k,op}^X\right) = h_k\left(\widetilde{\mu}_{k,op}^x T\right)$. $\widetilde{\mu}_{k,op}^x$ is the expected value of the demand served by the k^{th} operator in a time slot using opportunistic spectrum access. Let $\widehat{U}_{k,op}^r$ be the estimate of $\widetilde{\mu}_{k,op}^x$ over r samples. Finally, $\widehat{R}_{k,lc}^r + h_k\left(\widehat{U}_{k,op}^r T\right)$ is the estimate of the k^{th} licensed operator's revenue function over r samples. According to (20), the estimate of the k^{th} unlicensed operator's revenue function over r samples is $h_k\left(\widehat{U}_{k,op}^r T\right)$. To this end, we have to calculate the sample means $\widehat{U}^r$, $\widehat{U}_{k,op}^r$, and $\widehat{R}_{k,lc}^r$ to estimate the objective and the revenue function.

We now present a proposition, which helps in generating random samples efficiently for the Monte Carlo integrator.

Proposition 2: Define

$$\varphi_k = \int_0^{\frac{D}{M}} \vartheta^2 f_k^\theta(\vartheta)\,d\vartheta + \frac{D}{M}\int_{\frac{D}{M}}^{\infty} \vartheta f_k^\theta(\vartheta)\,d\vartheta - \mu_k^\theta \widetilde{\mu}_{k,lc}^x \tag{28}$$

where $f_k^\theta(\vartheta)$ is the probability density function of $\theta_k(t)$. Then, $\theta_k(t)$, $R_{k,lc}(\gamma)$ and $V_k(\gamma)$ are jointly Gaussian random variables with joint probability distribution,

$$\begin{bmatrix} \theta_k(t) \\ R_{k,lc}(\gamma) \\ V_k(\gamma) \end{bmatrix} \sim \mathcal{N}(\psi_k, \Sigma_k) \tag{29}$$

for all γ and for all $t \in [(\gamma-1)T+1, \gamma T]$ where

$$\psi_k = \left[\mu_k^\theta \;\; \mu_{k,lc}^R \;\; \mu_{k,lc}^R\right]^T \tag{30}$$

$$\Sigma_k = \begin{bmatrix} \left(\sigma_k^\theta\right)^2 & \rho_k \frac{\sigma_{k,lc}^R}{\sigma_{k,lc}^X}\varphi_k & \omega_k\rho_k \frac{\sigma_{k,lc}^R}{\sigma_{k,lc}^X}\varphi_k \\ \rho_k \frac{\sigma_{k,lc}^R}{\sigma_{k,lc}^X}\varphi_k & \left(\sigma_{k,lc}^R\right)^2 & \omega_k\left(\sigma_{k,lc}^R\right)^2 \\ \omega_k\rho_k \frac{\sigma_{k,lc}^R}{\sigma_{k,lc}^X}\varphi_k & \omega_k\left(\sigma_{k,lc}^R\right)^2 & \left(\sigma_{k,lc}^R\right)^2 \end{bmatrix}. \tag{31}$$

Proof: Please refer to Appendix B of the supplementary material for the proof. ■

Algorithm 4: Monte Carlo Integrator

Input: M, P, T, D, ϕ, α_L, α_U, $\mathcal{S}_L$, $\mathcal{S}_U$, and $\xi_k; \forall k \in \mathcal{S}_L \bigcup \mathcal{S}_U$

Output: $U\left(M, P, \mathcal{S}_L, \mathcal{S}_U\right)$ and $\mathcal{R}_k\left(M, P, \mathcal{S}_L, \mathcal{S}_U\right); \forall k \in \mathcal{S}_L \bigcup \mathcal{S}_U$

1 Set $\widehat{U}^0 = 0$, $\widehat{U}^0_{k,op} = 0; \forall k \in \mathcal{S}$, and $\widehat{R}^0_{k,lc} = 0; \forall k \in \mathcal{S}_L$.

2 Set $stop = False$ and $r = 0$.

3 **while** $not\,(stop)$ **do**

4 Set $r = r + 1$.

5 For all k in $\mathcal{S}_L$, sample θ^r_k, $R^r_{k,lc}$ and V^r_k from probability distribution (29). Set $x^r_k = \max\left(0, \theta^r_k\right)$.

6 For all k in $\mathcal{S}_U$, sample θ^r_k from the probability distribution $\mathcal{N}\left(\mu^\theta_k, \left(\sigma^\theta_k\right)^2\right)$. Set $x^r_k = \max\left(0, \theta^r_k\right)$.

7 Sort the list $\{V^r_k\}_{k \in \mathcal{S}_L}$ in descending order of V^r_k. Let $\mathcal{T}^r_1$ be the subset of operators in $\mathcal{S}_L$ with the P highest values of V^r_k. $\mathcal{T}^r_1$ are the Tier-1 operators. $\mathcal{T}^r_2 = \mathcal{S}\backslash\mathcal{T}^r_1$ are the Tier-2 operators.

8 Demand served by Tier-1 operators using licensed spectrum access are $\widetilde{x}^r_{k,lc}$=$\min\left(x^r_k, \frac{D}{M}\right); \forall$k$\in \mathcal{T}^r_1$.

9 Calculate modified demand, $\overline{x}^r_k$, and opportunistic channel capacity, D^r_O, using (11) and (12) respectively.

10 Call Algorithm 1 to get $\left\{\widetilde{x}^r_{k,op}\right\}_{k \in \mathcal{S}}$, the demand served by operators using opportunistic spectrum access. The input to Algorithm 1 are D^r_O and $\{\overline{x}^r_k\}_{k \in \mathcal{S}}$.

11 Set $\widehat{U}^r = \frac{(r-1)\widehat{U}^{r-1} + \sum\limits_{k \in \mathcal{T}^r_1} \widetilde{x}^r_{k,lc} + \sum\limits_{k \in \mathcal{S}} \widetilde{x}^r_{k,op}}{r}$.

12 Set $\widehat{R}^r_{k,lc} = \frac{(r-1)\widehat{R}^{r-1}_{k,lc} + R^r_{k,lc}}{r}$ for all k in $\mathcal{T}^r_1$.

13 Set $\widehat{R}^r_{k,lc} = \frac{(r-1)\widehat{R}^{r-1}_{k,lc} + 0}{r}$ for all k in $\mathcal{S}_L\backslash\mathcal{T}^r_1$.

14 Set $\widehat{U}^r_{k,op} = \frac{(r-1)\widehat{U}^{r-1}_{k,op} + \widetilde{x}^r_{k,op}}{r}$ for all k in $\mathcal{S}$.

15 **if** $r = 1$ **then**

16 Set $\widehat{U}^1 = 0$, $\widehat{U}^1_{k,op} = 0; \forall k \in \mathcal{S}$, and $\widehat{R}^1_{k,lc} = 0; \forall k \in \mathcal{S}_L$.

17 **else**

18 Set $\delta U^r = \frac{(r-2)\delta U^{r-1}}{(r-1)} + r\left(\widehat{U}^r - \widehat{U}^{r-1}\right)^2$

$\delta U^r_{k,op} = \frac{(r-2)\delta U^{r-1}_{k,op}}{(r-1)} + r\left(\widehat{U}^r_{k,op} - \widehat{U}^{r-1}_{k,op}\right)^2; \forall k \in \mathcal{S}$

$\delta R^r_{k,lc} = \frac{(r-2)\delta R^{r-1}_{k,lc}}{(r-1)} + r\left(\widehat{R}^r_{k,lc} - \widehat{R}^{r-1}_{k,lc}\right)^2; \forall k \in \mathcal{S}_L$

19 Set $stop =$ $Stop\left(r, \delta U^r, \delta U^r_{k,op}, \delta R^r_{k,lc}, \widehat{U}^r, \widehat{U}^r_{k,op}, \widehat{R}^r_{k,lc}\right)$.

20 Set $U\left(M, P, \mathcal{S}_L, \mathcal{S}_U\right) = \widehat{U}^r$

$\mathcal{R}_k\left(M, P, \mathcal{S}_L, \mathcal{S}_U\right) = \widehat{R}^r_{k,lc} + h_k\left(\widehat{U}^r_{k,op} T\right); \forall k \in \mathcal{S}_L$

$\mathcal{R}_k\left(M, P, \mathcal{S}_L, \mathcal{S}_U\right) = h_k\left(\widehat{U}^r_{k,op} T\right); \forall k \in \mathcal{S}_U$.

Recall that in (28), $\widetilde{\mu}^x_{k,lc}$ is given by (3). In (30), $\mu^R_{k,lc} = h_k\left(\widetilde{\mu}^x_{k,lc} T\right)$ (refer to (6) and (7)). In (31), $\sigma^X_{k,lc} = \widetilde{\sigma}^x_{k,lc}\sqrt{T}$ where $\widetilde{\sigma}^x_{k,lc}$ is given by (4).

The pseudocode for the Monte Carlo integrator is given in Algorithm 4. The sample means $\widehat{U}^r$, $\widehat{U}^r_{k,op}$, and $\widehat{R}^r_{k,lc}$ are initialized to zero for $r = 0$ (*line 1*). Inside the while loop, $\widehat{U}^r$, $\widehat{U}^r_{k,op}$, and $\widehat{R}^r_{k,lc}$ are computed recursively until stopping criteria. We discuss the stopping criteria later in this section. In *line 5*, the r^{th} sample of $\theta_k(t)$, $R_{k,lc}(\gamma)$, and $V_k(\gamma)$ are generated for all the licensed operators according to the probability distribution given by (29). We have dropped the γ and t inside the parenthesis for notational simplicity. Similarly, in *line 6*, $\theta_k(t)$ is generated for all the unlicensed operators. $\theta_k(t)$ follows the probability distribution $\mathcal{N}\left(\mu^\theta_k, \left(\sigma^\theta_k\right)^2\right)$ (refer to Section II-B). The r^{th} sample of $\theta_k(t)$, $R_{k,lc}(\gamma)$ and $V_k(\gamma)$ are denoted by θ^r_k, $R^r_{k,lc}$ and V^r_k respectively. The customer demand of the k^{th} operator for the r^{th} sample is $x^r_k = \max\left(0, \theta^r_k\right)$. Tier-1 and Tier-2 operators for the r^{th} sample are decided in *line 7*. Licensed operators with the P highest bids, V^r_k, are the Tier-1 operators for the r^{th} sample. $\mathcal{T}^r_1$ denotes the set of Tier-1 operators for the r^{th} sample. The remaining operators, $\mathcal{S}\backslash\mathcal{T}^r_1$, are the Tier-2 operators for the r^{th} sample. $\mathcal{T}^r_2$ denotes the set of Tier-2 operators for the r^{th} sample. In *lines 8-10*, demand served by the operators using licensed and opportunistic spectrum access are calculated. Demand served by operators using licensed and opportunistic spectrum access for the r^{th} sample are denoted using $\widetilde{x}^r_{k,lc}$ and $\widetilde{x}^r_{k,op}$ respectively.

The sample means $\widehat{U}^r$, $\widehat{U}^r_{k,op}$, and $\widehat{R}^r_{k,lc}$ are calculated in *lines 11-14* using recursive formulas analogous to (23). The formula to update $\widehat{U}^r$ is shown in *line 11*. The term $\sum\limits_{k \in \mathcal{T}^r_1} \widetilde{x}^r_{k,lc} + \sum\limits_{k \in \mathcal{S}} \widetilde{x}^r_{k,op}$ is the net demand served by all the operators in a time slot for the r^{th} sample. The formula to update $\widehat{R}^r_{k,lc}$ is shown in *lines 12 and 13*. If the k^{th} licensed operator is a Tier-1 operator for the r^{th} sample, then it earns a revenue of $R^r_{k,lc}$ in an epoch using licensed spectrum access (*line 12*). But if the k^{th} licensed operator is a Tier-2 operator for the r^{th} sample, then it earns a revenue of 0 using licensed spectrum access (*line 13*). $\widehat{U}^r_{k,op}$ is updated in *line 14*. The operators serve $\widetilde{x}^r_{k,op}$ customer demand using opportunistic spectrum access (*line 14*).

The sample variance corresponding to sample means $\widehat{U}^r$, $\widehat{U}^r_{k,op}$, and $\widehat{R}^r_{k,lc}$ are calculated in *lines 15-18*. These variances are initialized to zero for the 1^{st} sample (*line 16*) and updated using recursive formulas similar to (26) for $r > 1$ (*line 18*). In *line 19*, the $Stop(\cdot)$ function decides whether to stop the Monte Carlo integrator. The stopping criteria is based on (27). The $Stop(\cdot)$ function returns $True$ if and only if $r \geq r_{min}$ and all the sample mean and sample variance pairs $\left(\widehat{U}^r, \delta U^r\right)$, $\left(\widehat{U}^r_{k,op}, \delta U^r_{k,op}\right)$, and $\left(\widehat{R}^r_{k,lc}, \delta R^r_{k,lc}\right)$ satisfies (27). The condition $r \geq r_{min}$ ensures that the Monte Carlo integrator samples the mean over at least r_{min} samples. Finally, the estimated values of the objective function and the revenue function are set in *line 20* according to what we have discussed before in this section (refer to the paragraph before Proposition 2).

V. Numerical Results

In this section, we conduct numerical simulations to benchmark the algorithms developed in the previous sections. We also explore how the optimal solution M^* and P^* varies with interference parameters. Throughout this section, each time slot has a duration of one week and lease duration of licensed channels is one year. Hence, $T = 52$. In all our simulations we have: *(i)* $h_k\left(\mu^X_{k,a}\right) = a_k \mu^X_{k,a}$ where

$a_k > 0$. *(ii)* $\sigma^R_{k,a} = \eta_k h_k\left(\mu^X_{k,a}\right)$ where $\eta_k > 0$ is the coefficient of variation of $R_{k,a}(\gamma)$. *(iii)* $\lambda_k = \overline{\eta}_k \cdot a_k \mu^\theta_k T$ where $\overline{\eta}_k \in [0,1]$ and the term $a_k \mu^\theta_k T$ is the mean revenue of the k^{th} operator in an epoch if it can serve all its customer demand in every time slot. *(iv)* The maximum capacity of the entire bandwidth D is a fraction υ of the sum of μ^θ_k of all the candidate operators, i.e. $D = \upsilon \sum_{k \in \mathcal{S}^C} \mu^\theta_k$ where $\mathcal{S}^C = \mathcal{S}^C_L \bigcup \mathcal{S}^C_U$. Given our choice of $h_k\left(\mu^X_{k,a}\right)$, $\sigma^R_{k,a}$, and λ_k, the tuple ξ_k is equivalent to $\left(\mu^\theta_k, \sigma^\theta_k, a_k, \eta_k, \rho_k, \omega_k, \overline{\eta}_k\right)$ in this section. Parameters of convergence for the Monte Carlo integrator are: $r_{min} = 10000$, $\beta_1 = 1$ and $\beta_2 = 0.99$.

A. Benefit of Joint Optimization of M and P

In the first numerical simulation, we analyze the increase in spectrum utilization that one can obtain using joint optimization of M and P when compared to optimizing M while holding P fixed and vice-versa. Our numerical setup is as follows. There are four candidate licensed operators and no candidate unlicensed operator. There are 10 parameters which completely defines a market setting: μ^θ_k, σ^θ_k, a_k, η_k, ρ_k, ω_k, $\overline{\eta}_k$, υ, α_L, and α_U. We generate 1000 such market settings by randomly selecting these 10 parameters from uniform distributions each of which is associated with a certain range. The range of the parameters μ^θ_k, σ^θ_k, a_k, η_k, ρ_k, ω_k, and $\overline{\eta}_k$ for all the operators are $[0.75, 1.0]$, $[0.25, 0.75]$, $[0.9, 1.1]$, $[0.25, 0.75]$, $[0.5, 0.9]$, $[0.85, 0.95]$, and $[0.25, 1.0]$ respectively. The range of υ, α_L, and α_U are $[0.5, 1.0]$, $[0.75, 1.0]$, and $[0.75, 1.0]$ respectively. While generating α_L and α_U, we ensure that $\alpha_L \leq \alpha_U$.

The optimal value of the objective function corresponding to Algorithm 3 is U^*. We compare Algorithm 3 with a sub-optimal algorithm. Let the optimal value of the objective function corresponding to a sub-optimal algorithm be $\widehat{U}^*$. The percentage increase in the objective function is $\Delta U^* = \frac{U^* - \widehat{U}^*}{D} \cdot 100$. The reason for having D in the denominator is as follows. The objective function given by (16) is the mean demand served by all the operators in one time slot which cannot be greater than D, the maximum capacity of the entire bandwidth. Hence, $U^*, \widehat{U}^* \leq D$ which implies that $U^* - \widehat{U}^* \leq D$. We compute ΔU^* for sub-optimal algorithms and plot the cumulative distribution function (CDF) of ΔU^* in Figure 4. Recall that ϕ can be 0 or 1, and the OSA strategy can be either interweave or overlay. So, there are four possible combinations of OSA. For a given sub-optimal algorithm, we compute CDFs for all the four combinations.

We consider two sub-optimal algorithms. For the first algorithm, P is fixed and M is optimized. An intuitive choice of P is the number of candidate licensed operators. In that way, every candidate licensed operators win a licensed channel in every epoch. For the second algorithm, M is fixed and P is optimized. We set $M = \left\lfloor \frac{D}{\vartheta} \right\rfloor$ where $\lfloor \cdot \rfloor$ is the floor function and $\vartheta = \frac{1}{|\mathcal{S}^C|} \sum_{k \in \mathcal{S}^C} \mu^\theta_k$ is the sample mean of the mean of an operator's customer demand. This choice of M is to ensure that the bandwidth ϑ of a licensed channel is neither too high that most of it is wasted and neither too low that a licensed operator has to reject most of its customer demand.

In Figure 4, a lower value of CDF for a given ΔU^* implies that the difference in spectrum utilization between joint optimization and the sub-optimal algorithm is higher.

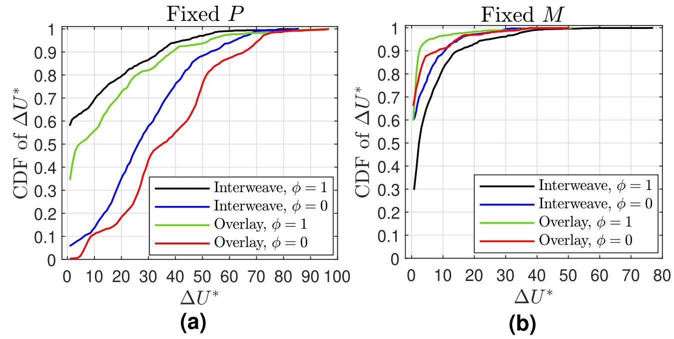

Fig. 4. Cumulative distribution function of the percentage increase in objective function, ΔU^*, for four different types of opportunistic spectrum access when the sub-optimal algorithm is: (a) optimizing M while holding P fixed. (b) optimizing P while holding M fixed.

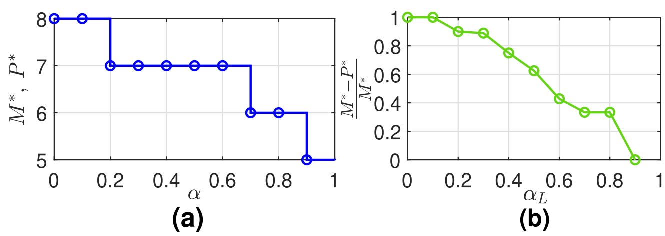

Fig. 5. (a) Plots showing the effect of interference parameter α for a market containing only candidate licensed operators on the optimal number of channel, M^*, and the optimal number of licensed channels, P^*. (b) Plots showing the effect of interference parameter of licensed channel α_L for a market containing both candidate licensed operators and unlicensed operators on the ratio of the bandwidth allocated for unlicensed channels, $\frac{M^* - P^*}{M^*}$.

By comparing Figures 4.a and 4.b we can say that joint optimization leads to more improvement in spectrum utilization when P is fixed rather than when M is fixed. Based on Figure 4.a, we can say that when P is fixed, joint optimization leads to more improvement in spectrum utilization for: (i) overlay strategy than interweave strategy when ϕ is fixed. (ii) $\phi = 0$ than $\phi = 1$. Based on Figure 4.b, we can say that when M is fixed, joint optimization leads to more improvement in spectrum utilization for interweave strategy than overlay strategy when ϕ is fixed. We don't observe any such systematic trend for ϕ when M is fixed.

B. Effect of Interference Parameters

The second numerical simulation is to study the effect of interference parameters on the optimal solution. We consider two simulation setups. The first simulation setup is as follows. For this setup, $\alpha_L = \alpha_U = \alpha$. There are 8 candidate licensed operators and no candidate unlicensed operators. We consider a homogeneous market setting. The minimum revenue requirement λ_k is set to zero for all the operators which ensure that all the operators join the market. The remaining parameters of the market are: $\mu^\theta_k = 1$, $\sigma^\theta_k = 0.5$, $a_k = 1$, $\eta_k = 0.5$, $\rho_k = 0.8$, and $\omega_k = 0.9$ for all k's. Also, $\upsilon = 0.8$. We study how M^* and P^* vary with α. The simulation result is shown in Figure 5.a. Since there are no candidate unlicensed operators, it is intuitive that there are no unlicensed channels, i.e. $M^* = P^*$. Figure 5.a shows that M^* decreases with an increase in α. This can be explained as follows. If M is low, the bandwidth, and hence the capacity of each licensed channel is high. Therefore, a licensed operator can serve more customer demand using

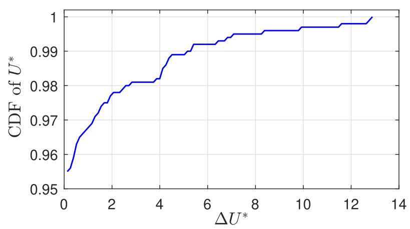

Fig. 6. Cumulative distribution function of the percentage increase in objective function, ΔU^*, when the sub-optimal algorithm is to choose the value of M and P that maximize the number of interested operators.

the allocated licensed channel thereby increasing spectrum utilization. But if M is too low, only a few of the 8 licensed operators are allocated the licensed channels in an epoch. The remaining operators uses channels opportunistically as Tier-2 operators. The efficiency of opportunistic access is decided by α. If α is low, it is better to have fewer Tier-2 operators in an epoch because opportunistic spectrum access is inefficient. This can be ensured with a higher M so that there are more Tier-1 operators in every epoch.

In our second simulation setup, we include candidate unlicensed operators. The simulation setup is similar to the first setup but differs in the following ways. *First*, out of the 8 operators, four are candidate licensed operators and four are candidate unlicensed operators. *Second*, the interference parameters α_L and α_U are not the same. We set $\alpha_U = 0.9$ and vary α_L from 0 to 0.9. We study how the ratio of the bandwidth allocated for unlicensed channels characterized by the ratio $\frac{M^*-P^*}{M^*}$ changes with α_L. This is shown in Figure 5.b. Unlike the previous simulation setup, the current simulation setup has candidate unlicensed operators. Therefore, we expect that there will be unlicensed channels dedicated for the candidate unlicensed operators. But the question is: what portion of the bandwidth should be allocated for unlicensed channels? If α_L is high, most of the bandwidth can be reserved for licensed channels because even if the Tier-1 operators are not using the licensed channels, the Tier-2 operators can use the remaining capacity of the licensed channels efficiently. But as α_L decreases, the opportunistic access of licensed channels becomes inefficient. Therefore, it is better to reserve a higher portion of the bandwidth for unlicensed channels.

C. Market Competition vs Spectrum Utilization

For most markets, an increase in competition improves social welfare. In our setup, we use the number of interested operators, $|\mathcal{S}_L| + |\mathcal{S}_U|$, as the measure of market competition and spectrum utilization as the measure of social welfare. In this numerical simulation, we show that there exist market setups where an increase in $|\mathcal{S}_L| + |\mathcal{S}_U|$ decreases spectrum utilization. The simulation setup and the definition of ΔU^* are similar to the one in section V-A but differs in the following ways. *First,* in this setup, we have three candidate licensed operators and three candidate unlicensed operators. *Second,* the sub-optimal algorithm in this setup finds M and P that maximize $|\mathcal{S}_L|+|\mathcal{S}_U|$ instead of the objective function defined in (16). If there are multiple values of M and P that maximize $|\mathcal{S}_L| + |\mathcal{S}_U|$, we choose the ones that maximize the objective function defined in (16).

The simulation result is shown in Figure 6 where we plot the CDF of ΔU^* for 1000 market setups. To establish our claim that maximizing $|\mathcal{S}_L| + |\mathcal{S}_U|$ doesn't necessarily maximize spectrum utilization, we want to find market setups where ΔU^* is strictly greater than 0. We can see that for $(1 - 0.955) \cdot 100\% = 4.5\%$ of the market setups, $\Delta U^* > 0$. This establishes our claim that there are market setups, however few, where maximizing $|\mathcal{S}_L| + |\mathcal{S}_U|$ doesn't necessarily maximize spectrum utilization. However, for these 4.5% of the market setups, ΔU^* is upper bounded by 13% implying only a marginal improvement in spectrum utilization.

VI. Conclusion

In this paper, we designed an optimization algorithm to partition a bandwidth into channels and further decide the number of licensed channels in order to maximize spectrum utilization. The access to this bandwidth is governed by a tiered spectrum access model inspired by the CBRS band. We first propose a system model which accurately captures various aspects of the tiered spectrum access model. Based on this model, we formulate our optimization problem as a two-staged Stackelberg game and then designed algorithms to solve the Stackelberg game. Finally, we get numerical results to benchmark our algorithm and to also study certain optimal trends of spectrum partitioning and licensing as a function of interference parameters.

There can be various directions for future research related to generalization of the Stackelberg Game model. *First*, is to capture collusion between operators in Stage-2 of the Stackelberg Game. *Second*, in our current model, every operator is assumed to be equally pessimistic. It would be interesting to associate each operator with a degree of pessimism.

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