SCHWARZIAN DERIVATIVES, PROJECTIVE STRUCTURES, AND THE WEIL–PETERSSON GRADIENT FLOW FOR RENORMALIZED VOLUME

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Abstract

To a complex projective structure Σ on a surface, Thurston associates a locally convex pleated surface. We derive bounds on the geometry of both in terms of the norms $\|\phi_\Sigma\|_\infty$ and $\|\phi_\Sigma\|_2$ of the quadratic differential ϕ_Σ of Σ given by the Schwarzian derivative of the associated locally univalent map. We show that these give a unifying approach that generalizes a number of important, well-known results for convex cocompact hyperbolic structures on 3-manifolds, including bounds on the Lipschitz constant for the nearest-point retraction and the length of the bending lamination. We then use these bounds to begin a study of the Weil–Petersson gradient flow of renormalized volume on the space $\mathrm{CC}(N)$ of convex cocompact hyperbolic structures on a compact manifold N with incompressible boundary, leading to a proof of the conjecture that the renormalized volume has infimum given by one half the simplicial volume of DN, the double of N.

1. Introduction

Throughout the work of Bers, Sullivan, and Thurston, the precise relation between the conformal boundary of a hyperbolic 3-manifold and its internal geometry has been a key subtlety. For example, the classical *Bers inequality* bounds the lengths of geodesics in the 3-manifold in terms of their lengths in the hyperbolic metric on the conformal boundary, and a related theorem of Sullivan gives uniform bounds on the Teichmüller distance between the conformal boundary and the boundary of the convex core for 3-manifolds with incompressible boundary. There is a long history of results of this type, obtained by Canary (see [10]), Bishop (see [2]), Epstein, Marden, and Markovic (see [13]), and Bridgeman and Canary (see [4], [5]), that have made important advances through delicate arguments.

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This paper provides a unifying perspective to these considerations via the *Schwarzian derivative*, which naturally associates a holomorphic quadratic differential to each component of the conformal boundary of a hyperbolic 3-manifold. Remarkably, in addition to shining new light on a number of important results in the literature, the "Schwarzian" is key to proving a conjectured lower bound on the *renormalized volume* of hyperbolic 3-manifolds, a notion whose import we elucidate here.

To begin, the following initial result illustrates these explicit connections.

THEOREM 1.1
Let M be a hyperbolic 3-manifold, $\partial_c M$ its conformal boundary, and $C(M)$ its convex core. Let ϕ_M be the holomorphic quadratic differential obtained from the Schwarzian derivative of the map comparing $\partial_c M$ to its Fuchsian uniformization. Then

1. *the retract map $\partial_c M \to \partial C(M)$ is $\sqrt{1+2\|\phi_M\|_\infty}$-Lipschitz, and*
2. *$L(\lambda_M) \le 4\pi|\chi(\partial_c M)|\|\phi_M\|_\infty$, where $L(\lambda_M)$ is the length of the bending lamination λ_M of $\partial C(M)$.*

Indeed, Theorem 1.1 follows almost directly from a theorem of G. Anderson, bounding Thurston's *projective metric* in terms of the hyperbolic metric where the bound depends on the Schwarzian derivative. Taking Anderson's result together with the classical Nehari bound on the Schwarzian, we obtain many well-known results, such as the Lipschitz bounds of Epstein, Marden, and Markovic (see [13]) and Bridgeman and Canary (see [5]), and the length bounds of Bridgeman and Canary (see [4]), as immediate corollaries.

Working a bit harder, we obtain bounds on $L(\lambda_M)$ in terms of the L^2-norm of the Schwarzian, which we employ to study the powerful notion of *renormalized volume*. Motivated by considerations from theoretical physics, the notion of renormalized volume was first introduced by Graham and Witten (see [14]) in the general setting of conformally compact Einstein manifolds. In the setting of infinite-volume, convex cocompact hyperbolic 3-manifolds, renormalized volume has been seen to be of particular interest as a more analytically natural proxy for convex core volume (see, e.g., [26], [27]). The approach here follows the work of Krasnov and Schlenker (see [17]) and Schlenker (see [24]). Our L^2-bounds give the following tight relationship between the convex core volume $V_C(M)$ and the renormalized volume $V_R(M)$ of M.

THEOREM 1.2
There is a function $G(t) \sim t^{1/5}$ such that if M is a convex cocompact hyperbolic 3-manifold with incompressible boundary, then

$$V_C(M) - |\chi(\partial M)|G(\|\phi_M\|_2) \le V_R(M) \le V_C(M),$$

and $V_R(M) = V_C(M)$ if and only if the convex core of M has totally geodesic boundary.

The result reveals the close connection of the renormalized volume to the volume of the convex core, but the renormalized volume carries the advantage that if we fix a hyperbolizable 3-manifold N, then V_R is a smooth function on the space $\mathrm{CC}(N)$ of all convex cocompact hyperbolic 3-manifolds homeomorphic to N. A formula for the derivative was established by Taktajan and Teo (see [26]) and Zograf and Taktajan (see [27]), and was re-proved by Krasnov and Schlenker (see [17]) using different methods more germane to the present considerations (see Theorem 3.9 for a precise statement). It is natural to conjecture that the infimum $\mathcal{V}_R(N)$ of V_R is the purely topologically defined *simplicial volume* of N. By applying the variational formula of Krasnov and Schlenker and Theorem 1.2 to study the Weil–Petersson gradient flow of V_R, we establish the conjectured lower bound.

COROLLARY 1.3
Let N be a compact hyperbolizable 3-manifold with nonempty incompressible boundary and without torus boundary components. Then $\mathcal{V}_R(N) = \frac{1}{2}V_S(DN)$, where DN is the double of N and $V_S(DN)$ is the simplicial volume. The infimum is realized if and only if N is acylindrical or has the homotopy type of a closed surface.

Corollary 1.3 is an analogue of a well-known result of Storm on the convex core volume (see [25]).

Partial results in this direction were established prior to our work. It follows immediately from the Krasnov–Schlenker variational formula that all critical points of V_R occur at $M \in \mathrm{CC}(N)$, where the convex core of M has totally geodesic boundary. Note that this can only occur when N is acylindrical, in which case there is a unique such structure in $\mathrm{CC}(N)$, or N is homotopy equivalent to a surface and there is a half-dimensional subspace of $\mathrm{CC}(N)$ of Fuchsian structures where the renormalized volume is zero. In the acylindrical setting, Moroianu (see [20]) and Pallete (see [22]) have independently shown that this critical point is a local minimum of V_R. When N is a homotopy equivalent to a closed surface—the "surface group" case—our result implies that $\mathcal{V}_R(N) = 0$. Previously, Krasnov and Schlenker (see [17]) had proved that the renormalized volume has zero infimum when taken over quasi-Fuchsian manifolds with finitely bent convex core boundary. In the special case of *almost-Fuchsian* structures, this was proved by Ciobotaru and Moroianu (see [11]). Finally, when N is acylindrical, Corollary 1.3 was proved by Pallete [23] using very different methods. In fact, combining our methods with those of Pallete gives a new and technically simpler proof of the Storm theorem on convex core volume for acylindrical manifolds.

Note that, prior to the work here, it was not even known that the renormalized volume was positive.

1.1. Core volume, renormalized volume, and Weil–Petersson distance

In a sequel (see [3]), we study the Weil–Petersson gradient flow further, supplying a direct proof of renormalized volume lower bounds in terms of Weil–Petersson distance.

THEOREM 1.4
Given $\epsilon > 0$, there exists $c = c(\epsilon, S) > 0$, so that if $d_{\mathrm{WP}}(X, Y) \geq \epsilon$, then we have

$$V_R\big(Q(X,Y)\big) \geq c \cdot d_{\mathrm{WP}}(X,Y).$$

Here, $Q(X,Y)$ denotes the *Bers simultaneous uniformization* of X and Y in $\mathrm{Teich}(S)$, and $d_{\mathrm{WP}}(X,Y)$ is their *Weil–Petersson distance*. Together with the comparison of Theorem 1.2, we obtain direct proofs of the lower bounds on convex core volume in [7] and [8]. Previously, these results had been obtained by building a combinatorial model for the Weil–Petersson metric (the *pants graph*), and showing that these combinatorics also give volume estimates for the relevant convex cores. The model relies on delicate combinatorial arguments involving the complex of curves and its hierarchical structure developed in [18] and [19], while the renormalized volume flow produces a natural analytic link between Weil–Petersson distance and volume.

1.2. Outline

We begin with a discussion of locally univalent maps and complex projective structures. On a projective structure there are two natural metrics: the hyperbolic metric, which depends only on the underlying conformal structure, and Thurston's *projective metric*. By comparing a projective structure to its Fuchsian uniformization, one also obtains a holomorphic quadratic differential via the Schwarzian derivative. The main technical tool of the paper is an unpublished theorem of G. Anderson (Theorem 2.1), bounding the projective metric in terms of the hyperbolic metric and a function of the L^∞-norm of the Schwarzian quadratic differential. Section 2.1 is devoted to a short, new proof of this theorem. As with the original, the proof is based on a construction of Epstein which associates a surface in $\mathbb{H}^3$ to a conformal metric on the unit disk Δ and a locally univalent map $f : \Delta \to \widehat{\mathbb{C}}$.

In Sections 2.2 and 2.3, we review Thurston's parameterization of locally univalent maps and of projective structures in terms of measured laminations. In particular, Thurston parameterizes projective structures on a surface by locally convex pleated surfaces. There is a natural "retract" map from the projective structure to the pleated surface that is 1-Lipschitz from the projective metric to the path metric on the pleated surface. Using the Schwarzian bound on the projective metric, we obtain a bound on the Lipschitz constant for the retract map when we take the hyperbolic metric on the domain (Corollary 2.7). The length of the bending lamination is also controlled by the Schwarzian, as it is a linear function of the area of the projective metric (Theorem 2.10).

In Section 2.4, we review the classical bounds of Nehari on the L^∞-norm of the Schwarzian derivative of univalent maps and use the Nehari bounds to bound the Schwarzian when the locally univalent map is a covering map for a domain in Ω. In Sections 2.5 and 2.6, we combine the Nehari bounds to derive Lipschitz bounds on retract maps from domains in $\widehat{\mathbb{C}}$ to convex hulls in $\mathbb{H}^3$ (Theorem 2.14) and from the conformal boundary of a hyperbolic 3-manifold to the boundary of the convex core (Theorem 2.15). We also obtain bounds on the length of the bending lamination of the convex core (Theorems 2.16 and 2.17).

All of these bounds are based on the L^∞-norm of the Schwarzian. In Section 2.7, we bound the length of the bending lamination in terms of the L^2-norm of the Schwarzian. This will be used in our study of renormalized volume.

In Section 3, the last part of the paper, we begin our study of the renormalized volume of a convex cocompact hyperbolic 3-manifold. After reviewing definitions, we improve on bounds, originally due the Schlenker, comparing the renormalized volume to the volume of the convex core. In particular, we show that the difference of the two volumes is bounded by a function of the L^2-norm of the Schwarzian of the projective boundary (Theorem 1.2).

We use these bounds to study the Weil–Petersson gradient flow of $-V_R$. Along flow lines, the L^2-norm of the Schwarzian of the projective boundary will decay to zero. It will follow that the infimum of renormalized volume will agree with the infimum of convex core volume (Theorem 3.11).

We highlight one other novelty of our approach: a new definition of the *W-volume*. The usual definition of W-volume involves the integral of the mean curvature over the boundary of the manifold. We will see that it can be reinterpreted as a function of the volume of the submanifold, the area of the boundary, and the area of its associated metric at infinity. This reinterpretation is valid even when the boundary is not smooth and clarifies the formula for the W-volume of the convex core given in [24].

The proof of our theorem on the lower bound for renormalized volume is actually quite short. The reader who is solely interested in this result can skip much of the paper, as it only depends on the bound on the projective metric (Theorem 2.8), the bound on the length of the bending lamination in terms of the L^2-norm of the Schwarzian (Section 2.7), and Section 3.

2. Epstein surfaces and projective structures

Let $f : \Delta \to \widehat{\mathbb{C}}$ be a locally univalent map. Thurston defined a natural metric on Δ associated to f, the *Thurston* or *projective metric*. Here is the definition: let $D \subset \Delta$ be an open topological disk and define ρ_D to be the hyperbolic metric on D. Then D is *round with respect to* f if $f(D)$ is round in $\widehat{\mathbb{C}}$. We then define

$$\rho_f(z) = \inf_D \rho_D(z),$$

where D ranges over all round disks containing z. By the Schwarz lemma, if ρ_Δ is the hyperbolic metric on Δ, then $\rho_\Delta \leq \rho_D$ for all disks D contained in Δ with equality if and only if $D = \Delta$. Therefore, $\rho_\Delta \leq \rho_f$ with equality if and only if f is the restriction of an element of $\mathrm{PSL}(2, \mathbb{C})$. In particular, $\rho_f > 0$. Upper bounds for ρ_f are more subtle. The following theorem of Anderson will be a key tool for what follows.

THEOREM 2.1 (Anderson, [1, Theorem 4.2])
$\rho_f(z) \leq \rho_\Delta(z)\sqrt{1 + 2\|Sf\|_\infty}$.

Here Sf is the *Schwarzian derivative* quadratic differential on Δ given by

$$Sf = \left(\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2\right) dz^2.$$

Then $\|Sf(z)\| = |Sf(z)|/\rho_\Delta^2(z)$ is a function on Δ. In particular, for any conformal automorphism γ of Δ, we have $\|S(f \circ \gamma)(z)\| = \|(Sf)(\gamma z)\|$. Furthermore, the sup norm is given by

$$\|Sf\|_\infty = \sup_{z \in \Delta} \|Sf(z)\|.$$

2.1. Epstein surfaces

Using that $\widehat{\mathbb{C}}$ can be naturally identified as the boundary of $\mathbb{H}^3$, we describe a construction of Epstein that associates a surface in $\mathbb{H}^3$ to a locally univalent map $f : \Delta \to \widehat{\mathbb{C}}$ and a conformal metric ρ on Δ.

Given a point $x \in \mathbb{H}^3$, let ρ_x be the *visual metric* on $\widehat{\mathbb{C}}$ centered at x. There are several ways to define ρ_x, and we will choose one that fits our needs for later. For $z \in \widehat{\mathbb{C}}$, let r be the geodesic ray based at x that limits to z at infinity. Then there will be a unique totally geodesic copy of $\mathbb{H}^2 \subset \mathbb{H}^3$ that contains x and is orthogonal to r. The hyperbolic plane will limit to a round circle in $\widehat{\mathbb{C}}$. Let D be the disk bounded by this circle that contains z and ρ_D, its hyperbolic metric. We then define $\rho_x(z) = \rho_D(z)$. Note that ρ_x is invariant under any isometry of $\mathbb{H}^3$ that fixes x. In fact, up to a normalization, this last property also determines ρ_x.

Given a conformal metric ρ on a domain in $\widehat{\mathbb{C}}$ containing a point z, we observe that the set $\mathfrak{h}_{\rho,z} = \{x \in \mathbb{H}^3 | \rho_x(z) = \rho(z)\}$ is a horosphere. We will be interested in the horospheres associated to the pushforward metric $f_*\rho$. Unfortunately, as f is only locally univalent, this pushforward is in general not well defined. To get around this, we define $f_*\rho(z)$ by restricting f to a neighborhood of z, where f is injective, pushing the metric forward on this neighborhood, and then evaluating at $f(z)$.

Let $T^1\mathbb{H}^3$ be the unit tangent bundle of $\mathbb{H}^3$, and let $\pi : T^1\mathbb{H}^3 \to \mathbb{H}^3$ be the projection to $\mathbb{H}^3$. If ρ is smooth, Epstein shows that there is a unique smooth immersion $\widetilde{\mathrm{Ep}}_\rho : \Delta \to T^1\mathbb{H}^3$ such that $\widetilde{\mathrm{Ep}}_\rho(z)$ is an inward pointing normal to the horosphere $\mathfrak{h}_{f_*\rho,z}$, and when $\mathrm{Ep}_\rho = \pi \circ \widetilde{\mathrm{Ep}}_\rho$ is an immersion at z, the surface will be tangent to $\mathfrak{h}_{f_*\rho,z}$. We emphasize that if ρ is smooth, then while $\widetilde{\mathrm{Ep}}_\rho$ will always be an immersion, Ep_ρ may not be. For example, if ρ_x is the visual metric for a point $x \in \mathbb{H}^3$, then $\widetilde{\mathrm{Ep}}_{\rho_x}$ is a diffeomorphism onto $T^1_x\mathbb{H}^3$, but Ep_{ρ_x} will be be the constant map to x.

The maps $\widetilde{\mathrm{Ep}}_\rho$, Ep_ρ have some nice properties.

PROPOSITION 2.2 (Epstein [12, Theorem 2.1 and Equation 3.10])
Let $g_t : T^1\mathbb{H}^3 \to T^1\mathbb{H}^3$ be the geodesic flow. Then $g_t \circ \widetilde{\mathrm{Ep}}_\rho = \widetilde{\mathrm{Ep}}_{e^t\rho}$. Furthermore, if ρ is smooth, then there are functions $\kappa_t^0, \kappa_t^1 : \Delta \to (\mathbb{R}\backslash\{-1\}) \cup \infty$ satisfying

$$\kappa_t^i(z) = \frac{\kappa_0^i(z)\cosh t + \sinh t}{\kappa_0^i(z)\sinh t + \cosh t},$$

such that if neither $\kappa_t^0(z)$, $\kappa_t^1(z)$ are infinite, then Ep_t is an immersion at z and $\kappa_t^0(z)$, $\kappa_t^1(z)$ are the principal curvatures. In particular, if $t \geq \log \sqrt{|1+\kappa_0^i(z)|/|1-\kappa_0^i(z)|}$ for $i = 0, 1$, then $\mathrm{Ep}_{e^t\rho}$ is an immersion and locally convex at z.

The map $\mathrm{Ep}_\rho : \Delta \to \mathbb{H}^3$ is the (parameterized) *Epstein surface* of ρ associated to the locally univalent map f. We will be particularly interested in the Epstein surface $\mathrm{Ep}_{\rho_\Delta}$ associated to the hyperbolic metric ρ_Δ in Δ. The importance of the Schwarzian derivative in studying the Epstein surface for the hyperbolic metric is evident in the following theorem.

THEOREM 2.3 (Epstein [12])
The principal curvatures of the Epstein surface $\mathrm{Ep}_{\rho_\Delta}$ at the image of $z \in \Delta$ are $\frac{-\|Sf(z)\|}{\|Sf(z)\| \pm 1}$.

Theorem 2.1 will follow from the following proposition.

PROPOSITION 2.4
If ρ is a smooth conformal metric and Ep_ρ is locally convex, then $\rho_f \leq \rho$.

Proof
Define a map $F\colon \Delta \times [0,\infty] \to \mathbb{H}^3$ by $F(z,t) = \mathrm{Ep}_{e^t\rho}(z)$ if $t \in [0,\infty)$ and $F(z,\infty) = f(z)$. By Proposition 2.2, F restricted to $\Delta \times [0,\infty)$ will be an immersion to $\mathbb{H}^3$ and will extend continuously on $\Delta \times [0,\infty]$ to a map to $\mathbb{H}^3 \cup \widehat{\mathbb{C}}$. Since F is an immersion, F pulls back a hyperbolic structure on $\Delta \times [0,\infty)$ that is foliated by the Epstein surfaces. By convexity, a hyperbolic plane tangent to any Epstein surface in $\Delta \times [0,\infty)$ will be embedded and extend to a round disk on $\Delta = \Delta \times \{\infty\}$ with respect to f. For a point $z \in \Delta$, let D be the round disk bounded by the boundary of the hyperbolic plane tangent to the Epstein surface at $(z,0)$. By definition, $\rho_f \le \rho_D$. On the other hand, $\rho_D = \rho$ from the definition of the Epstein surface (and our normalization of the visual metric), and therefore $\rho_f(z) \le \rho(z)$ for all $z \in \Delta$. □

Proof of Theorem 2.1
By Theorem 2.3, the principal curvatures of $\mathrm{Ep}_{\rho_\Delta}$ at $\mathrm{Ep}_{\rho_\Delta}(z)$ are $\frac{-\|Sf(z)\|}{\|Sf(z)\|\pm 1}$. By the curvature equations in Proposition 2.2, if $t > \log(\sqrt{1+2\|Sf(z)\|})$, then the principal curvatures of $\mathrm{Ep}_{e^t\rho_\Delta}$ at $\mathrm{Ep}_{e^t\rho_\Delta}(z)$ are positive. So if $t > \log(\sqrt{1+2\|Sf\|_\infty})$, then $\mathrm{Ep}_{e^t\rho_\Delta}$ is locally convex. The theorem then follows from Proposition 2.4. □

If Sf has small norm on a large neighborhood of $z \in \Delta$, then we can get stronger bounds on the Thurston metric.

COROLLARY 2.5
If $\|Sf(z)\| \le K$ for all $z \in B(z_0,r)$, then

$$\rho_f(z_0) \le \rho_\Delta(z_0)\sqrt{1+2K}\coth(r/2).$$

Proof
Let $B = B(z_0,r)$. By the Schwarz lemma $\frac{|Sf(z)|}{\rho_B(z)^2} \le \frac{|Sf(z)|}{\rho_\Delta(z)^2}$, and therefore by Theorem 2.1,

$$\rho_{f|_B}(z_0) \le \rho_B(z_0)\sqrt{1+2K},$$

where $\rho_{f|_B}$ is the projective metric for f restricted to B. By the definition of the Thurston metric, $\rho_f(z_0) \le \rho_{f|_B}(z_0)$, and an explicit calculation shows that $\rho_B(z_0) = \rho_\Delta(z_0)\coth(r/2)$. This gives the desired inequality. □

2.2. *The Thurston parameterization*

Let $P(\Delta)$ be locally univalent maps $f\colon \Delta \to \widehat{\mathbb{C}}$ with the equivalence $f \sim g$ if $f = \phi \circ g$ for some $\phi \in \mathrm{PSL}(2,\mathbb{C})$. Thurston described a natural parameterization of $P(\Delta)$ by $\mathcal{ML}(\mathbb{H}^2)$, the space of measure geodesic laminations on $\mathbb{H}^2$. We briefly review this construction.

A round disk $D \subset \widehat{\mathbb{C}}$ shares a boundary with a hyperbolic plane $\mathbb{H}^2_D \subset \mathbb{H}^3$. Let $r_D \colon D \to \mathbb{H}^3$ be the nearest point projection to $\mathbb{H}^2_D$, and let $\tilde{r}_D \colon D \to T^1\mathbb{H}^3$ be the normal vector to $\mathbb{H}^2_D$ at $r_D(z)$ pointing towards D. We can use these maps to define a version of the Epstein map for ρ_f. In particular, define $\widetilde{\mathrm{Ep}}_{\rho_f} \colon \Delta \to T^1\mathbb{H}^3$ by $\widetilde{\mathrm{Ep}}_{\rho_f}(z) = \tilde{r}_{f(D)}(f(z))$, where D is the unique round disk with respect to f such that $\rho_D(z) = \rho_f(z)$, and let $\mathrm{Ep}_{\rho_f}(z) = \pi \circ \widetilde{\mathrm{Ep}}_{\rho_f}(z) = r_{f(D)}(f(z))$. (For the existence of this disk, see [15, Theorem 1.2.7].) We also define $\widetilde{\mathrm{Ep}}_{e^t\rho_f} = g_t \circ \widetilde{\mathrm{Ep}}_{\rho_f}$ and $\mathrm{Ep}_{e^t\rho_f} = \pi \circ \widetilde{\mathrm{Ep}}_{e^t\rho_f}$.

The image of Ep_{ρ_f} is a *locally convex pleated plane*. More precisely, let $\mathcal{ML}(\mathbb{H}^2)$ be measured geodesic laminations on $\mathbb{H}^2$, and let $\mathcal{ML}_0(\mathbb{H}^2) \subset \mathcal{ML}(\mathbb{H}^2)$ be the subspace of laminations with finite support. That is, $\lambda \in \mathcal{ML}_0(\mathbb{H}^2)$ if it is the union of a finite collection of disjoint geodesics ℓ_i with positive weights θ_i. Then λ determines a continuous map $p_\lambda \colon \mathbb{H}^2 \to \mathbb{H}^3$, unique up to postcomposition with isometries of $\mathbb{H}^3$, that is, an isometry on the complement of the support of λ and is "bent" with angle θ_i at ℓ_i. By continuity, we can extend this construction to a general $\lambda \in \mathcal{ML}(\mathbb{H}^2)$. An exposition of the following theorem of Thurston can be found in [15].

THEOREM 2.6
Given $f \in P(\Delta)$, there exist maps $c_f \colon \Delta \to \mathbb{H}^2$ and $p_f \colon \mathbb{H}^2 \to \mathbb{H}^3$ and a lamination λ_f such that p_f is a locally convex pleated surface pleated along λ_f, $\mathrm{Ep}_{\rho_f} = p_f \circ c_f$, and the map $f \mapsto \lambda_f$ is a homeomorphism from $P(\Delta) \to \mathcal{ML}(\mathbb{H}^2)$. Furthermore, the maps $c_f \colon (\Delta, \rho_f) \to \mathbb{H}^2$ and $\mathrm{Ep}_{\rho_f} \colon (\Delta, \rho_f) \to \mathbb{H}^3$ are 1-Lipschitz.

Combined with Theorem 2.1, we have an immediate corollary.

COROLLARY 2.7
Given $f \in P(\Delta)$, the Epstein map $\mathrm{Ep}_{\rho_f} \colon (\Delta, \rho_\Delta) \to \mathbb{H}^3$ is $\sqrt{1 + 2\|Sf\|_\infty}$-Lipschitz.

2.3. *Projective structures*

A projective structure Σ on a surface S is an atlas of charts to $\widehat{\mathbb{C}}$ with transition maps the restriction of Möbius transformations. We let $P(S)$ be the space of projective structures on S. One way to construct a projective structure on S is to take an $f \in P(\Delta)$ such that there exists a Fuchsian group Γ with $S = \Delta/\Gamma$ and a representation $\sigma \colon \Gamma \to \mathrm{PSL}(2, \mathbb{C})$ with $f \circ \gamma = \sigma(\gamma) \circ f$ for all $\gamma \in \Gamma$. In fact, every projective structure on S arises in this way.

This description of projective structures allow us to associate a number of objects to a given projective structure. First, we observe that a projective structure determines a complex structure X on S, and we let $P(X) \subset P(S)$ be projective structures on S with underlying complex structure X. Given $\Sigma \in P(X)$, the Schwarzian derivative Sf

of f will descend to a holomorphic quadratic differential ϕ_Σ on X. The lamination λ_f will be Γ-equivariant and descend to measured lamination λ_Σ on S. The hyperbolic metric ρ_Δ and the projective metric ρ_f on Δ will also descend to conformal metrics ρ_X and ρ_Σ on X.

In the equivariant setting, Corollary 2.7 becomes the following theorem.

THEOREM 2.8
Given a projective structure $\Sigma \in P(X)$, we have

$$\rho_\Sigma(z) \le \rho_X(z)\sqrt{1 + 2\|\phi_\Sigma\|_\infty}.$$

If $\|\phi_\Sigma(z)\| \le K$ for all $z \in B(z_0, r)$, then

$$\rho_\Sigma(z_0) \le \rho_X(z_0)\sqrt{1 + 2K}\coth(r/2).$$

If the measured lamination λ_Σ has support of a finite collection of closed geodesics $\gamma_1, \dots, \gamma_n$ with weights $\theta_1, \dots, \theta_n$, then the length of λ_Σ is $L(\lambda_\Sigma) = \sum \theta_i \ell(\gamma_i)$, where $\ell(\gamma_i)$ is the hyperbolic length of γ_i. This length extends continuously to general measure laminations on S.

We have the following useful relationship between the area of the projective metric and the length of the bending lamination.

LEMMA 2.9
Given a projective structure $\Sigma \in P(S)$ with bending lamination $\lambda_\Sigma \in \mathcal{ML}(S)$, we have $\mathbf{area}(\rho_\Sigma) = L(\lambda_\Sigma) + 2\pi|\chi(S)|$.

Proof
Both the area of the projective metric and the length of the bending lamination vary continuously in $P(S)$. The set of projective structures whose bending laminations is supported on finitely many geodesics is dense in $P(S)$ and the formula $\mathbf{area}(\rho_\Sigma) = L(\lambda_\Sigma) + 2\pi|\chi(S)|$ holds on such laminations by direct computation. The lemma follows. □

This immediately leads to bounds on the length.

THEOREM 2.10
If λ_Σ is the bending lamination for a projective structure Σ, then

$$L(\lambda_\Sigma) \le 4\pi|\chi(\Sigma)|\|\phi_\Sigma\|_\infty.$$

Proof
Squaring the inequality from Theorem 2.8, we get a bound on the area of the projective metric in terms of the area of the hyperbolic metric:

$$\mathbf{area}(\rho_\Sigma) \leq (1 + 2\|\phi_\Sigma\|_\infty)\mathbf{area}(\rho_X).$$

Subtracting $\mathbf{area}(\rho_X) = 2\pi|\chi(\Sigma)|$ from both sides and applying Lemma 2.9, we have

$$L(\lambda_\Sigma) \leq 4\pi|\chi(\Sigma)|\|\phi_\Sigma\|_\infty,$$

as claimed. □

2.4. Schwarzian bounds
We recall the classical Nehari bound on the Schwarzian derivative. (The upper bound was proved independently by Kraus.)

THEOREM 2.11 (Nehari [21, Theorem I])
We have the following:

- *If $f : \Delta \to \widehat{\mathbb{C}}$ is univalent, then $\|Sf\|_\infty \leq \frac{3}{2}$.*
- *If $\|Sf\|_\infty \leq \frac{1}{2}$, then f is univalent.*

In particular, if $\Omega \subset \widehat{\mathbb{C}}$ is a simply connected hyperbolic domain, then the above theorem bounds the Schwarzian derivative of the uniformizing map $f : \Delta \to \Omega$. If Ω is hyperbolic but not necessarily simply connected, we can still bound the Schwarzian for f (which in this case will be a covering map), but our bounds depend on the injectivity radius of the hyperbolic metric of Ω. Let $\mathrm{inj}_\Omega(z)$ be the supremum of the radii of embedded disks in Ω centered at z, and let

$$\delta_\Omega = \inf_{z \in \Omega} \mathrm{inj}_\Omega(z).$$

The following result bounding the Schwarzian in terms of inj_Ω and δ_Ω is due to Kra and Maskit.

COROLLARY 2.12 (Kra and Maksit, [16, Lemma 5.1])
Let Ω be a hyperbolic domain in the plane that is not simply connected, and let $f : \Delta \to \Omega$ be the uniformizing covering map. Then $\|Sf(z)\| \leq \frac{3}{2}\coth^2(\mathrm{inj}_\Omega(z)/2)$ and $\frac{1}{2}\coth^2(\delta_\Omega/2) \leq \|Sf\|_\infty$.

Proof
For each $z \in \Delta$, the restriction of f to the disk $B = B(z, \mathrm{inj}_\Omega(z))$ is univalent. By applying Theorem 2.11, we have that

$$\frac{|Sf(z)|}{\rho_B^2(z)} \leq \frac{3}{2},$$

where ρ_B is the hyperbolic metric on B. We also have $\rho_\Delta(z) = \tanh(\mathrm{inj}_\Omega(z)/2) \times \rho_B(z)$. The upper bound follows.

Given any $\delta' > \delta_\Omega$, there exists a disk $B = B(z, \delta)$ such that $f|_B$ is not injective. Therefore, by Theorem 2.11, there exists a $z' \in B$ such that

$$\frac{|Sf(z')|}{\rho_B^2(z')} \geq \frac{1}{2}.$$

A calculation shows that

$$\frac{\rho_B(z')}{\rho_\Delta(z')} \geq \frac{\rho_B(z)}{\rho_\Delta(z)} = \coth(\delta'/2),$$

so $\|Sf(z')\| \geq \frac{1}{2}\coth^2(\delta'/2)$. As this holds for all $\delta' > \delta_\Omega$, the lower bounds follows. $\square$

We will only use the upper bound in what follows.

2.5. *Lipschitz maps and hyperbolic domes*

Let $\Omega \subset \widehat{\mathbb{C}}$ be a hyperbolic domain, and let $\Lambda = \widehat{\mathbb{C}} \backslash \Omega$. Then the *convex hull*, $H(\Lambda) \subset \mathbb{H}^3$, is the smallest closed convex subset of $\mathbb{H}^3$ whose closure in $\widehat{\mathbb{C}}$ is Λ. The boundary of $H(\Lambda)$ is the *dome* of Ω, which we denote as $\mathbf{dome}(\Omega)$. With its intrinsic path metric, $\mathbf{dome}(\Omega)$ is a hyperbolic surface. The nearest point retraction of $\mathbb{H}^3$ to $H(\Lambda)$ extends to a continuous map $r\colon \Omega \to \mathbf{dome}(\Omega)$. We are interested in comparing the hyperbolic metric on Ω with the intrinsic path metric on $\mathbf{dome}(\Omega)$.

We would like to relate the retract r to an Epstein map. Let $f\colon \Delta \to \Omega$ be the uniformizing map. Then f is a covering map, and for any conformal metric ρ on Δ that is invariant with respect to the covering, the Epstein map for ρ will descend to a map with domain Ω which (in abuse of notation) we will continue to denote $\mathrm{Ep}_\rho\colon \Omega \to \mathbb{H}^3$. We then have the following.

PROPOSITION 2.13
If $f\colon \Delta \to \Omega$ is the uniformizing map, then $r \circ f = \mathrm{Ep}_{\rho_f}$.

Proof
Given $z \in \Omega$, there is a unique horosphere $\mathfrak{h}$ based at z that intersects $H(\Lambda)$ at exactly one point with this point being the projection $r(z)$.

The hyperbolic plane tangent to $\mathfrak{h}$ at $r(z)$ is a support plane for $H(\Lambda)$ and its boundary bounds a round disk $D_z \subset \Omega$ which contains z. If $\rho_{D_z}(z) = \rho_f(z)$, then $\mathrm{Ep}_{\rho_f}(z) = r \circ f$ from the construction of the Epstein map for the projective metric.

By the definition of the projective metric, $\rho_{D_z}(z) \geq \rho_f(z)$, so we just need to show that $\rho_{D_z}(z) \leq \rho_f(z)$.

If $\rho_{D_z}(z) > \rho_f(z)$, then there exists a round disk $D \subset \Omega$ with $\rho_{D_z}(z) > \rho_D(z)$. Let $\mathfrak{h}'$ be the horosphere of points whose visual metrics agree with ρ_D at z. Since $\rho_{D_z}(z) > \rho_D(z)$, the horosphere $\mathfrak{h}'$ bounds a horoball whose interior contains $\mathfrak{h}$. The open hyperbolic half-space bounded by D will contain the interior of this horoball and hence $\mathfrak{h}$. Since $\mathfrak{h}$ intersects $H(\Lambda)$, this open half-space will intersect $H(\Lambda)$, a contradiction. $\square$

Combining this proposition with Theorem 2.11 and Corollaries 2.7 and 2.12, we have the following.

THEOREM 2.14
*If $f\colon \Delta \to \Omega \subset \widehat{\mathbb{C}}$ is a conformal homeomorphism, then the retract $r\colon \Omega \to$ **dome**(Ω) is a $\sqrt{1+2\|Sf\|_\infty}$-Lipschitz map from the hyperbolic metric on Ω to the path metric on* **dome**(Ω). *In particular, if Ω is simply connected, then r is 2-Lipschitz, and if Ω is not simply connected with $\delta_\Omega > 0$, then r is $\sqrt{1+3\coth^2(\delta_\Omega/2)}$-Lipschitz.*

When Ω is simply connected, Epstein, Marden, and Markovic proved that the retract map was 2-Lipschitz (see [13, Theorem 3.1]). When Ω is not simply connected, Bridgeman and Canary showed that r was $(A + \frac{B}{\delta_\Omega})$-Lipschitz for universal constants $A, B > 0$ (see [5, Corollary 1.8]). Our bounds are better both when δ_Ω is small and large. The simplicity of the proof here indicates one strength of our methods.

2.6. *Hyperbolic 3-manifolds*

The above result in turn can also be interpreted in terms of hyperbolic 3-manifolds. Let Γ be a discrete, torsion-free subgroup of $\mathrm{PSL}(2,\mathbb{C})$. Let Ω be a component the domain of discontinuity of Γ, and let $\Gamma_\Omega \subset \Gamma$ be the subgroup that stabilizes Ω. Then the projective structure $\Sigma = \Omega/\Gamma_\Omega$ is a component of the conformal boundary of the hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$, and $X =$ **dome**$(\Omega)/\Gamma_\Omega$ is the component of the boundary of the convex core, $C(M)$, of M that faces Σ. The nearest point retraction $M \to C(M)$ extends continuously to a map $r\colon \Sigma \to X$. Note that Σ is incompressible in M if and only if Ω is simply connected. If Σ is compressible in M, then compressible curves in Σ lift to homotopically nontrivial closed curves in Ω. In particular, the length of the shortest compressible geodesic in Σ will be twice the injectivity radius of Ω. In this setting, Theorem 2.14 becomes the following.

THEOREM 2.15
Let M be a complete hyperbolic 3-manifold, Σ the complex projective structure on a component of the conformal boundary of M, and X the component of the boundary of the convex core of M facing Σ. Then the retraction $r: \Sigma \to X$ is a $\sqrt{1+2\|Sf\|_\infty}$-Lipschitz map from the hyperbolic metric on Σ to the path metric on X. In particular, if X is incompressible in M, then r is 2-Lipschitz, and if the length of every compressible curve on X has length $\geq \delta > 0$, then r is $\sqrt{1+3\coth^2(\delta/4)}$-Lipschitz.

We can also apply the Schwarzian bounds to obtain bounds on the length of the bending lamination. In particular, Theorem 2.10 becomes the following.

THEOREM 2.16
Let Σ be a component of the projective boundary of a hyperbolic 3-manifold M with bending lamination λ_Σ. Then $L(\lambda_\Sigma) \leq 4\pi|\chi(\Sigma)|\|\phi_\Sigma\|_\infty$. In particular, we have the following:

- *If Σ is incompressible, then $L(\lambda_\Sigma) \leq 6\pi|\chi(\Sigma)|$.*
- *If Σ is compressible and the length of the shortest compressible curve is $\delta > 0$, then $L(\lambda_\Sigma) \leq 6\pi|\chi(\Sigma)|\coth^2(\delta/4)$.*

The bound in the incompressible case was first obtained by Bridgeman and Canary in [6]. In the compressible case, the bound in [4] is $(\frac{A}{\delta}+B)|\chi(\Sigma)|$, which is stronger than the bound here. With more work, our methods can obtain similar bounds as in [4]. The proof is technical and this result will not be used in the rest of the paper.

THEOREM 2.17
If Σ is a compressible component of the boundary of a hyperbolic 3-manifold with bending lamination λ_Σ and the length shortest compressible curve is $\delta > 0$, then $L(\lambda_\Sigma) \leq (\frac{A}{\delta}+B)|\chi(\Sigma)|$ for universal constants $A, B > 0$.

Proof
The central idea is that that the ratio between the projective metric and the hyperbolic metric can only be large in the (compressible) thin part of the surface.

The complex projective structure Σ is the quotient of a domain $\Omega \subset \widehat{\mathbb{C}}$. Let X be the conformal structure on Σ with hyperbolic metric ρ_X. Then Ω is a covering space of X, and the hyperbolic metric ρ_Ω is the lift of ρ_X. Similarly, the Schwarzian ϕ_Ω on Ω is the lift of the Schwarzian ϕ_Σ on Σ, and $\rho_{\tilde{\Sigma}}$ is the lift of the projective metric ρ_Σ. We would like to bound above the ratio $\rho_\Sigma(z)/\rho_X(z)$. To do this, we will use Corollary 2.7, which will require us to bound the Schwarzian $\|\phi_\Sigma(z')\|$ for all z' in the disk $B(z,1)$. (The choice of radius 1 is essentially arbitrary.) To bound ϕ_Σ, we will use Corollary 2.12 to bound ϕ_Ω, and then use ϕ_Ω as the lift of ϕ_Σ.

For $z \in \Omega$, let

$$\operatorname{inj}^1_\Omega(z) = \inf_{z' \in B(z,1)} \operatorname{inj}_\Omega(z').$$

A simple estimate shows that there exists a constant $A_0 > 0$ such that $\operatorname{inj}^1_\Omega(z) \geq \operatorname{inj}_\Omega(z)/A_0$ (which holds for any complete hyperbolic surface).

Let $\epsilon > 0$ be the two-dimensional Margulis constant, and let $C \subset \Omega$ be a component of ϵ-thin part $\Omega^{<\epsilon} = \{z \in \Omega \mid \operatorname{inj}_\Omega(z) < \epsilon\}$. There is also a constant $A_1 > 0$ such that $\frac{3}{2}\coth^2(x/2) \leq \frac{A_1}{x^2}$ for $x \leq \epsilon$. Then by Corollary 2.12, for all $z' \in B(z,1)$ with $z \in C$, we have

$$\|\phi_\Omega(z')\| \leq \frac{A_1}{\operatorname{inj}^1_\Omega(z)^2} \leq \frac{A_0^2 A_1}{\operatorname{inj}_\Omega(z)^2}.$$

After applying Corollary 2.5, we see that, for $z \in C$,

$$\rho^2_{\tilde{\Sigma}}(z) \leq \rho^2_\Omega(z)\Big(1 + \frac{2A_0^2 A_1}{\operatorname{inj}_\Omega(z)^2}\Big)\coth^2(1/2).$$

We want to bound the area of C in the projective metric. We let ℓ be the length of the core geodesic of C in the hyperbolic metric. We give C coordinates $S^1 \times (-w(\ell), w(\ell))$, where $S^1 \times \{0\}$ is the geodesic and is parameterized by arc length, and each $\{\theta\} \times (-w(\ell), w(\ell))$ is a geodesic segment orthogonal to the core geodesic. The area form for the hyperbolic metric is then $\cosh t \, d\theta \, dt$. The constant $w(\ell)$ is chosen such that $\operatorname{inj}_\Omega(\theta, \pm w(\ell)) = \epsilon$. Another basic estimate in hyperbolic geometry gives that there exists $A_2 > 1$ such that

$$\ell e^{|t|}/A_2 \leq \operatorname{inj}_\Omega(\theta, t).$$

Here, it is important that $z = (\theta, t) \in C$ is in the ϵ-thin part.

We now calculate the area of C in the projective metric:

$$\begin{aligned}\mathbf{area}(\rho_{\tilde{\Sigma}}|_C) &= \int_{-w}^{w}\int_0^\ell \cosh t \rho^2_{\tilde{\Sigma}}/\rho^2_\Omega \, d\theta \, dt \\ &\leq \coth^2(1/2)\Big(\mathbf{area}(\rho_{\Omega|_C}) + \int_{-w}^{w}\int_0^\ell \frac{2A_0^2A_1}{\operatorname{inj}_\Omega(\theta,t)^2}\cosh t \, d\theta \, dt\Big).\end{aligned}$$

Then we use the lower bound on the injectivity radius to bound the remaining integral:

$$\begin{aligned}\int_{-w}^{w}\int_0^\ell \frac{2A_0^2A_1}{\operatorname{inj}_\Omega(\theta,t)^2}\cosh t \, d\theta \, dt &\leq 2\int_0^w\int_0^\ell \frac{2A_0^2A_1A_2^2}{\ell^2 e^{2t}}\cosh t \, d\theta \, dt \\ &= 2\int_0^w \frac{2\ell A_0^2A_1A_2^2}{\ell^2 e^{2t}}\cosh t \, d\theta \, dt\end{aligned}$$

$$\begin{aligned} &\le A_3 \int_0^w \frac{e^t}{\ell e^{2t}}\,dt \\ &= \frac{A_3}{\ell}(1-e^{-w}) \le \frac{2A_3}{\ell}. \end{aligned}$$

Since $\ell \ge \delta$, this becomes

$$\mathbf{area}(\rho_{\tilde{\Sigma}}|_C) \le \coth^2(1/2)\Big(\mathbf{area}(\rho_\Omega|_C) + \frac{A_3}{\delta}\Big).$$

Given a point $z \in \Sigma$, let $\tilde{z} \in \Omega$ be a point in the preimage of z. We then define $\tilde{\mathrm{inj}}_X(z) = \mathrm{inj}_\Omega(\tilde{z})$ and observe that this definition is independent of our choice of $\tilde{z}$. Injectivity radius can only increase in a cover so $\mathrm{inj}_X(z) \le \tilde{\mathrm{inj}}_X(z)$. The *compressible ϵ-thin part* is the set of points

$$X_c^{<\epsilon} = \big\{z \in X \,\big|\, \tilde{\mathrm{inj}}_X(z) < \epsilon\big\}.$$

If C is a component of the compressible ϵ-thin part, then each component of the preimage of C in Ω will be contained in a component $\tilde{C}$, as well as the ϵ-thin part of Ω, and we will have $\mathbf{area}(\rho_\Sigma|_C) \le \mathbf{area}(\rho_{\tilde{\Sigma}}|_C)$. Furthermore, each C will contain a simple closed geodesic, so there can be at most $3g-3 = \frac{3}{2}|\chi(\Sigma)|$ components of the compressible ϵ-thin part, and, therefore,

$$\mathbf{area}(\rho_\Sigma|_{X_c^{<\epsilon}}) \le \coth^2(1/2)\Big(\mathbf{area}(\rho_X|_{X_c^{<\epsilon}}) + \frac{3A_3}{2\delta}|\chi(\Sigma)|\Big).$$

On the other hand, if $\tilde{\mathrm{inj}}_X(z) \ge \epsilon$, then for $z' \in B(z,1)$, we have as above that $\tilde{\mathrm{inj}}_X(z') \ge \epsilon/A_0$. Therefore, by Corollary 2.5,

$$\rho_\Sigma^2(z)/\rho_X^2(z) = \rho_{\tilde{\Sigma}}^2(\tilde{z})/\rho_\Omega^2(\tilde{z}) \le \Big(1 + 3\coth^2\Big(\frac{\epsilon}{2A_0}\Big)\Big)\coth^2(1/2) = A_4.$$

Therefore, we have that

$$\mathbf{area}(\rho_\Sigma|_{X_c^{\ge\epsilon}}) \le A_4\mathbf{area}(\rho_X|_{X_c^{\ge\epsilon}}),$$

where $X_c^{\ge\epsilon}$ is the compressible ϵ-thick part of X.

By letting $A = \frac{3}{2}A_3\coth^2(1/2)$ and $B = 2\pi A_4$ and combining our two area bounds, we have

$$L(\lambda_\Sigma) \le \mathbf{area}(\rho_\Sigma) = \mathbf{area}(\rho_\Sigma|_{X_x^{<\epsilon}}) + \mathbf{area}(\rho_\Sigma|_{X_c^{\ge\epsilon}}) \le |\chi(\Sigma)|\Big(\frac{A}{\delta} + B\Big). \qquad \square$$

2.7. L^2-Bounds for the bending lamination

Given a quadratic differential ϕ on hyperbolic surface X with metric ρ_X, the ratio $|\phi|/\rho_X^2$ is a function on X. We define the L^2-norm of ϕ to be the L^2-norm of this function with respect to the hyperbolic metric. In order to prove our main theorem about renormalized volume, we will need a bound on $L(\lambda_\Sigma)$ in terms of the L^2-norm of the quadratic differential ϕ_Σ. We begin with the following lemma.

LEMMA 2.18
Let ϕ be a holomorphic quadratic differential on a hyperbolic surface X.
Then

$$\|\phi\|_2 \geq 2\sqrt{\frac{\pi}{3}}\tanh^2\bigl(\mathrm{inj}_X(z)/2\bigr)\|\phi(z)\|.$$

Proof
Let $B = B(z,r)$ be the disk centered at z of radius $r = \mathrm{inj}_X(z)$. Let $\|\phi\|_{X,2}$ be the L^2-norm of ϕ on X, and let $\|\phi\|_{B,2}$ be the L^2-norm of ϕ on B. Then $\|\phi\|_{X,2} \geq \|\phi\|_{B,2}$ by the Schwarz lemma. By [9, Lemma 5.1], we have $\|\phi\|_{B,2} \geq 2\sqrt{\pi/3}\|\phi(z)\|_B$, where $\|\phi(z)\|_B$ is the norm of ϕ on B. Comparing the complete hyperbolic metric on B to that on X, we see $\|\phi(z)\|_B = \tanh^2(r/2)\|\phi(z)\|_X$. □

We now combine the above with the prior results to obtain comparisons of the Thurston metric and Poincaré metric for quadratic differentials with small L^2-norm on the thick part of the surface. For $\epsilon > 0$, we define the ϵ thick-thin decomposition to be $X^{\geq\epsilon} = \{z \in X \mid \mathrm{inj}_X(z) \geq \epsilon\}$ and $X^{<\epsilon} = \{z \in X \mid \mathrm{inj}_X(z) < \epsilon\}$.

LEMMA 2.19
Let $\Sigma \in P(X)$ be a projective structure such that $\|\phi_\Sigma\|_2 \leq \epsilon^5$. Then for $z \in X^{\geq\epsilon}$,

$$\rho_\Sigma(z) \leq \bigl(1 + F(\epsilon)\bigr)\rho_X(z),$$

where $F(t) \simeq (2 + 4\sqrt{3/\pi})t$ as $t \to 0$.

Proof
We can assume that $\epsilon < 1$ and then define $r > 0$ such that $\epsilon = e^{-r}$. Let $z \in X^{\geq\epsilon}$. For $w \in B(z,r)$ then, a simple calculation shows that $\mathrm{inj}_X(w) \geq \mathrm{inj}_X(z)e^{-r} \geq \epsilon^2$. This follows from the fact that for C, a hyperbolic annulus with core geodesic of length ℓ is then

$$\sinh\bigl(\mathrm{inj}_C(x)\bigr) = \sinh(\ell/2)\cosh\bigl(d(x)\bigr),$$

where $d(x)$ is the distance from x to the geodesic. By comparing two points x, y with $\mathrm{inj}_C(x) > \mathrm{inj}_C(y)$, one obtains

$$\frac{\mathrm{inj}_C(x)}{\mathrm{inj}_C(y)} \leq \frac{\sinh(\mathrm{inj}_C(x))}{\sinh(\mathrm{inj}_C(y))} \leq \frac{\cosh(d(x))}{\cosh(d(y))} \leq e^{d(x)-d(y)}.$$

Therefore, for $w \in B(z,r)$ by Lemma 2.18,

$$\|\phi_\Sigma(w)\| \leq \sqrt{\frac{3}{4\pi}}\Big(\frac{\|\phi_\Sigma\|_2}{\tanh^2(\epsilon^2/2)}\Big) \leq \sqrt{\frac{3}{4\pi}}\Big(\frac{\epsilon^5}{\tanh^2(\epsilon^2/2)}\Big).$$

Therefore, by the local bound in Theorem 2.8, we have

$$\begin{aligned}\frac{\rho_\Sigma(z)}{\rho_X(z)} &\leq \sqrt{1+\sqrt{\frac{3}{\pi}}\Big(\frac{\epsilon^5}{\tanh^2(\epsilon^2/2)}\Big)\coth(r/2)} \\ &= \sqrt{1+\sqrt{\frac{3}{\pi}}\Big(\frac{\epsilon^5}{\tanh^2(\epsilon^2/2)}\Big)\Big(\frac{1+\epsilon}{1-\epsilon}\Big)} = 1+F(\epsilon).\end{aligned}$$

Computing the first two terms of the Taylor series shows that, as $t \to 0$,

$$F(t) \simeq \Big(2+4\sqrt{\frac{3}{\pi}}\Big)t. \qquad \square$$

We now use the above to get prove the L^2-bound on the length of the bending lamination.

THEOREM 2.20

Let $\Sigma \in P(X)$ be a projective structure with Schwarzian quadratic differential ϕ_Σ with $\|\phi_\Sigma\|_\infty \leq K$. Then

$$L(\lambda_\Sigma) \leq 2\pi|\chi(X)|G_K(\|\phi_\Sigma\|_2),$$

where $G_K(t) \sim t^{1/5}$ as $t \to 0$.

Proof

We let $\epsilon = \|\phi\|_2^{1/5}$. As $\|\phi_\Sigma\|_\infty \leq K$, by Theorem 2.8 we have $\rho_\Sigma(z) \leq \sqrt{1+2K} \times \rho_X(z)$ for all z. We decompose X into the thick-thin pieces

$$\mathbf{area}(\rho_\Sigma) = \int_{X^{\geq\epsilon}} \rho_\Sigma^2 + \int_{X^{<\epsilon}} \rho_\Sigma^2 \leq \int_{X^{\geq\epsilon}} \big(1+F(\epsilon)\big)^2\rho_X^2 + \int_{X^{<\epsilon}} (1+2K)\rho_X^2.$$

Therefore,

$$\mathbf{area}(\rho_\Sigma) \leq \big(1+F(\epsilon)\big)^2\mathbf{area}(\rho_{X^{\geq\epsilon}}) + (1+2K)\mathbf{area}(\rho_{X^{<\epsilon}}).$$

Since $\mathbf{area}(\rho_{X^{\geq\epsilon}}) \leq \mathbf{area}(\rho_X) = 2\pi|\chi(S)|$ and for the genus g surface S there are at most $(3g-3)$ ϵ-thin parts each with area bounded by 2ϵ, we have

$$2\pi|\chi(S)| + L(\lambda_\Sigma) \leq \big(1+F(\epsilon)\big)^2 2\pi|\chi(S)| + (1+2K)(3g-3)2\epsilon.$$

Since $|\chi(S)| = 2g - 2$ when we apply Lemma 2.9, we have

$$\begin{aligned} L(\lambda_\Sigma) &\le 2\pi|\chi(S)|\Big(\big(1 + F(\epsilon)\big)^2 + \frac{3\epsilon}{2\pi}(1 + K) - 1\Big) \\ &= 2\pi|\chi(S)|\Big(2F(\epsilon) + F(\epsilon)^2 + \frac{3\epsilon}{2\pi}(1 + 2K)\Big). \end{aligned} \qquad \square$$

3. Renormalized volume

We now describe the renormalized volume for a convex cocompact hyperbolic 3-manifold M. We also review many of its fundamental properties as developed by Krasnov and Schlenker. While it will take some setup before we state the definition, we will see that renormalized volume has many nice properties that make its definition natural.

3.1. The W-volume

Throughout this subsection and the next, we fix a convex cocompact hyperbolic 3-manifold M and let $\partial_c M$ be its conformal boundary, Σ be its projective boundary, λ_M be the bending lamination of the convex core, and ϕ_M be the Schwarzian derivative of Σ. We also let ρ_M be the hyperbolic metric on $\partial_c M$ and ρ_Σ be the projective metric determined by Σ.

Let $N \subset M$ be a smooth, compact convex submanifold of M with $C^{1,1}$ boundary. Here, and in what follows, N is convex if every geodesic segment with endpoints in N is contained in N. Then the *W-volume* of N is

$$W(N) = \mathrm{vol}(N) - \frac{1}{2}\int_{\partial N} H\, da,$$

where H is the mean curvature function on ∂N. That is, H is the average of the principal curvatures or, equivalently, one half the trace of the shape operator. The $C^{1,1}$ condition (the normal vector field is defined everywhere and is Lipschitz) implies that H is defined almost everywhere and that the integral

$$2\int_{\partial N} H\, da$$

is the variation of area of ∂N under the normal flow.

We let N_t be the t-neighborhood of N in M. Then there is a very simple formula for the W-volume of N_t in terms of N.

PROPOSITION 3.1 (Krasnov and Schlenker [17])
Let M be a convex cocompact hyperbolic 3-manifold, and let N be a convex submanifold with $C^{1,1}$ boundary. Then

$$W(N_t) = W(N) - t\pi\chi(\partial_c M).$$

(The desired statement follows directly from [17, Lemma 4.2].)

As defined, the W-volume is a function on the space of convex submanifolds of M with $C^{1,1}$ boundary. We would like to reinterpret it as a function on smooth, conformal metrics on $\partial_c M$. We need the following lemma.

LEMMA 3.2
Let H be a closed convex submanifold of $\mathbb{H}^3$, and let $\Lambda = \overline{H} \cap \widehat{\mathbb{C}}$ and $\Omega = \widehat{\mathbb{C}} \backslash \Lambda$. Then there exists a conformal metric $\rho = \rho_H$ on Ω such that Ep_ρ is the nearest point retraction $r \colon \Omega \to \partial H$. If $\gamma \in \mathrm{PSL}(2, \mathbb{C})$ with $\gamma(H) = H$, then $\gamma^* \rho = \rho$.

In particular, if $N \subset M$ is a convex submanifold of a convex cocompact hyperbolic manifold M, then there exists a smooth metric $\rho = \rho_N$ on $\partial_c M$ such that $\mathrm{Ep}_\rho = r$, where $r \colon \partial_c M \to \partial N$ is the nearest point retraction.

Finally, if N_t is the t-neighborhood of N, then $\rho_{N_t} = e^t \rho_N$.

Proof
For each $z \in \Omega$, there is a unique horosphere $\mathfrak{h}_z$ based at z that intersects H at exactly one point and $r(z)$ is the point of intersection. We then define $\rho(z) = \nu_{r(z)}$, where $\nu_{r(z)}$ is the visual metric. Then r satisfies all the properties of the Epstein map for ρ, and since the Epstein map is unique, we have $r = \mathrm{Ep}_\rho$. The construction is clearly equivariant. Equivariance implies the second paragraph, and the last statement then follows from Proposition 2.2. □

We then have a nice formula for the integral of the mean curvature in terms of the of the area of ρ_N and ∂N.

LEMMA 3.3
Let N be a smooth convex submanifold of a convex cocompact hyperbolic 3-manifold M. Then

$$\int_{\partial N} H\, da = \frac{1}{2}\mathbf{area}(\rho_N) - \mathbf{area}(\partial N) - \pi \chi(\partial M).$$

Furthermore, if $\rho_N = \rho_M$, then

$$\int_{\partial N} H\, da = \|\phi_M\|_2^2.$$

Proof
Let $B \colon T(\partial N) \to T(\partial N)$ be the shape operator given by $B(v) = -\nabla_v n$, where n is the normal vector field to ∂N. In particular, the eigenvalues of B are the principal curvatures of ∂N. Then

$$H = \frac{1}{2}\operatorname{tr}(B) = \frac{1}{4}\big(\det(I+B) - \det(I-B)\big),$$

where $I : T(\partial N) \to T(\partial N)$ is the identity operator. An elementary calculation shows that the pullback via the retraction $r : \partial_c M \to \partial N$ of the 2-form $\det(I+B)\,da$ is the area form for the metric ρ_N on $\partial_c M$ (see [17, Equation 30]). Therefore,

$$\mathbf{area}(\rho_N) = \int_{\partial N} \det(I+B)\,da.$$

On the other hand,

$$\det(I+B) + \det(I-B) = 4 + 2K,$$

where $K = \det(B) - 1$ is the Gaussian curvature of ∂N. Therefore,

$$\int_{\partial N} \big(\det(I+B) + \det(I-B)\big)\,da = 4\mathbf{area}(\partial N) + 4\pi\chi(\partial N).$$

Rearranging the terms proves the first statement in the lemma.

For $\rho_N = \rho_M$ the hyperbolic metric, by Theorem 2.3 the principal curvatures at $r(z)$ are $\frac{-\|\phi_M(z)\|}{\|\phi_M(z)\|\pm 1}$. Therefore, if da^* is the area form for ρ_M, then

$$\begin{aligned}\mathbf{area}(\partial N) &= \int_{\partial N} da = \int_{\partial_c M} \frac{1}{\det(I+B)}\,da^* = \int_{\partial_c M} \big(1 - \|\phi_M(z)\|^2\big)\,da^* \\ &= \mathbf{area}(\rho_M) - \|\phi_M\|_2^2.\end{aligned}$$

Therefore as $\mathbf{area}(\rho_M) = 2\pi|\chi(\partial M)|$, the result follows. ☐

This gives us an alternate way of defining the W-volume by setting

$$W(N) = \operatorname{vol}(N) - \frac{1}{4}\mathbf{area}(\rho_N) + \frac{1}{2}\mathbf{area}(\partial N) + \frac{1}{2}\pi\chi(\partial N).$$

Note that the definition makes sense even if the boundary N is not $C^{1,1}$. Also, regardless of the regularity of N, the t-neighborhood N_t will always have $C^{1,1}$ boundary. In particular, the scaling property (Proposition 3.1) still holds for this alternative definition of the W-volume, even when the boundary of N is not $C^{1,1}$. One advantage of this definition is that we can use it to see that the W-volume varies continuously.

PROPOSITION 3.4

The W-volume is continuous on the space of compact convex submanifolds of M with the Hausdorff topology. The map from compact, convex submanifolds to metrics on $\partial_c M$ is continuous in the L^∞-topology.

Proof
Fix a convex submanifold N, and let V_i be convex submanifolds such that the distance between N and V_i in the Hausdorff metric is less then $1/i$. We can assume that $N \subset V_i$, and if not, we can replace V_i with its $1/i$-neighborhood. By Proposition 3.1, the W-volume of the V_i will converge to $W(N)$ if and only if the W-volume of the $1/i$-neighborhoods also converge to $W(N)$.

To see that the W-volume converges, we first observe that volume is continuous in the Hausdorff topology on the space of convex submanifolds. Next we note that the nearest point retraction of ∂V_i to ∂N is 1-Lipschitz, so $\mathbf{area}(\partial V_i) \geq \mathbf{area}(\partial N)$. Since $V_i \subset N_{1/i}$, we similarly have that $\mathbf{area}(\partial N_{1/i}) \geq \mathbf{area}(\partial V_i)$. We also have $\mathbf{area}(\partial N_{1/i}) \to \mathbf{area}(\partial N)$, and therefore $\mathbf{area}(\partial V_i) \to \mathbf{area}(\partial N)$.

To compare the metrics ρ_N and ρ_{V_i}, fix a point $z \in \partial_c M$ and let $\mathfrak{h}_z$ and $\mathfrak{h}_z^i$ be the horospheres based at z that meet N and V_i, respectively, in a single point. Then $\mathfrak{h}_z^i$ will be disjoint from N, but its $1/i$-neighborhood will intersect N. This implies that $1 \leq \rho_{V_i}(z)/\rho_N(z) \leq e^{1/i}$. It follows that the map from convex submanifolds with the Hausdorff topology to the space of conformal metrics with the L^∞-topology is continuous. Therefore, $\mathbf{area}(\rho_N)$ varies continuously in N and this, along with the previous paragraph, implies that the W-volume varies continuously. □

Let $\mathcal{M}(\partial_c M)$ be continuous conformal metrics on $\partial_c M$ with the L^∞-topology, and let $\mathcal{M}_C(\partial_c M)$ be the subspace of metrics ρ such that there exists a convex submanifold N with $\rho_N = e^t \rho$ for some $t \in \mathbb{R}$. We can then define the W-volume as a function on $\mathcal{M}_C(\partial_c M)$ by setting

$$W(\rho) = W(N) + t\pi\chi(\partial_c M).$$

Note that $\mathcal{M}_C(\partial_c M)$ will not be all continuous metrics. For example, a metric that locally has the form $\frac{|dz|}{1+|z|}$ will not be in $\mathcal{M}_C(\partial_c M)$. However, we have the following proposition.

PROPOSITION 3.5 (Krasnov and Schenker [17, Theorem 5.8])
Let ρ be a smooth metric on $\partial_c M$. Then for t sufficiently large there exists a convex submanifold $N \subset M$ such that $e^t\rho = \rho_N$. In particular, $\rho \in \mathcal{M}_C(\partial_c M)$.

We are now finally in position to define the renormalized volume. We let ρ_M be the hyperbolic metric on $\partial_c M$, and define

$$V_R(M) = W(\rho_M).$$

We have, by Lemma 3.3, that if N is the submanifold corresponding to ρ_M, then

$$V_R(M) = \mathrm{vol}(N) - \|\phi_M\|_2^2.$$

THEOREM 3.6 (Schlenker [24, Proposition 3.11, Corollary 3.8])
Let M be convex cocompact. Then we have the following.

- *(Maximality): If $\rho \in \mathcal{M}_C(\partial_c M)$ with* $\mathbf{area}(\rho) = \mathbf{area}(\rho_M)$*, then $W(\rho) \leq V_R(M)$ with equality if and only if $\rho = \rho_M$.*
- *(Monotonicity): If $\rho_0, \rho_1 \in \mathcal{M}_C(\partial_c M)$ have nonpositive curvature on $\partial_c M$ and $\rho_0 \leq \rho_1$, then*

$$W(\rho_0) \leq W(\rho_1).$$

3.2. Bounds on renormalized volume

For quasi-Fuchsian manifolds, Schlenker used the W-volume of the convex core to get upper and lower bounds on the renormalized volume (see [24, Theorem 1.1]). This is generalized easily to convex cocompact 3-manifolds with incompressible boundary (see [6, Theorem 1.1]).

Schlenker's approach was to use the monotonicity property and maximality property of W-volume. If N is the convex core of M, then by Proposition 2.13, the metric at infinity $\rho_N = \rho_\Sigma$ is the projective metric. As the projective metric is nonpositively curved and is greater than the hyperbolic metric, the monotonicity property of W-volume (see Theorem 3.6) gives an upper bound on the renormalized volume. By rescaling the metric so that it has the same area as the hyperbolic metric, the maximality property (again, see Theorem 3.6) gives a lower bound on renormalized volume.

In order to obtain L^2-bounds, we will use the same strategy as Schlenker. We first need the following theorem. The upper bound is due to Schlenker (see [24]), and the lower bound is a simple application of the monotonicity and maximality properties of renormalized volume.

THEOREM 3.7
Let M be a convex cocompact hyperbolic 3*-manifold. Then*

$$V_C(M) - \frac{1}{2}L(\lambda_M) \leq V_R(M) \leq V_C(M) - \frac{1}{4}L(\lambda_M),$$

and $V_C(M) = V_R(M)$ if and only if the convex core of M has totally geodesic boundary.

Proof
As noted above, the metric at infinity ρ_N for the convex core N is the projective metric ρ_Σ. Using our formula for the W-volume in terms of area, we compute that

$$W(\rho_\Sigma) = W(N) = V_C(M) - \frac{1}{4}\mathbf{area}(\rho_\Sigma) + \frac{1}{2}\mathbf{area}(\partial N) + \frac{1}{2}\pi\chi(\partial N).$$

By Lemma 2.9, $\mathbf{area}(\rho_\Sigma) = L(\lambda_M) + 2\pi|\chi(S)|$. Since the boundary of the convex core is a hyperbolic surface, we have $\mathbf{area}(\partial N) = 2\pi|\chi(S)|$. Therefore,

$$W(\rho_\Sigma) = V_C(M) - \frac{1}{4}L(\lambda_M).$$

Since $\rho_\Sigma \geq \rho_M$, by the monotonicity property we have $W(\rho_\Sigma) \geq V_R(M)$, so

$$V_C(M) \geq V_C(M) - \frac{1}{4}L(\lambda_M) \geq V_R(M),$$

with $V_C(M) = V_R(M)$ if and only if $L(\lambda_M) = 0$. This proves the upper bound.

For the lower bound, let

$$\hat{\rho}_\Sigma = \sqrt{\frac{\mathbf{area}(\rho_\Sigma)}{\mathbf{area}(\rho_M)}}\rho_\Sigma.$$

Then $\mathbf{area}(\hat{\rho}_\Sigma) = \mathbf{area}(\rho_M)$, so by the maximality property (see Theorem 3.6), $W(\hat{\rho}_\Sigma) \leq V_R(M)$. Similarly, by the scaling property (see Theorem 3.6) and the formula for $\mathbf{area}(\rho_\Sigma)$, we have

$$W(\hat{\rho}_\Sigma) = W(\rho_\Sigma) - \frac{\pi}{2}\log\Big(1 + \frac{L(\lambda_M)}{\mathbf{area}(\rho_M)}\Big)|\chi(\partial M)|.$$

As $\log(1+x) \leq x$ and $\mathbf{area}(\rho_M) = 2\pi|\chi(\partial M)|$, we have

$$V_R(M) \geq W(\hat{\rho}_\Sigma) \geq W(\rho_\Sigma) - \frac{1}{4}L(\lambda_M) = V_C(M) - \frac{1}{2}L(\lambda_M).$$

Thus it follows that

$$V_C(M) - \frac{1}{2}L(\lambda_M) \leq V_R(M) \leq V_C(M) - \frac{1}{4}L(\lambda_M).$$

We therefore have $V_C(M) = V_R(M)$ if and only if $L(\lambda_M) = 0$. Thus, $V_C(M) = V_R(M)$ if and only if M has totally geodesic boundary. $\square$

Combining the L^2-bound for length in Theorem 2.20 with the above theorem, we obtain the following.

THEOREM 3.8 (see Theorem 1.2)
There is a function $G(t) \sim t^{1/5}$ such that if M is a convex cocompact hyperbolic 3-manifold with incompressible boundary, then

$$V_C(M) - |\chi(\partial M)|G(\|\phi_M\|_2) \leq V_R(M) \leq V_C(M),$$

and $V_R(M) = V_C(M)$ if and only if the convex core of M has totally geodesic boundary.

Proof
As M has incompressible boundary, then we have the Nehari bound $\|\phi_M\| \leq \frac{3}{2}$. By Theorem 2.20, we have

$$L(\lambda_M) \leq 2\pi|\chi(\partial_c M)|G_{\frac{3}{2}}(\|\phi_M\|_2),$$

where $G_{\frac{3}{2}}(t) \sim t^{1/5}$ and the result follows with $G = \pi \cdot G_{\frac{3}{2}}$. □

The results here should be compared to earlier work of Bridgeman and Canary. For manifolds with incompressible boundary, they prove a lower bound where the function $G(t)$ in Theorem 1.2 is replaced by a universal constant (see [6, Theorem 1.1]). For manifolds with compressible boundary, they give upper and lower bounds on the $V_R(M)$ that depend on the length of the shortest compressible curve in the boundary (see [6, Theorem 1.3]). In particular, one can produce a sequence of Schottky manifolds (convex cocompact hyperbolic structures on a handlebody) of fixed genus whose convex core volume is bounded above but the length of the shortest compressible curve approaches zero. Then the Bridgeman–Canary bounds imply that the renormalized volume of this sequence limits to $-\infty$.

3.3. *The gradient flow of V_R*

Let N be a compact, hyperbolizable 3-manifold with incompressible boundary, and let $\mathrm{CC}(N)$ be the space of convex cocompact hyperbolic structures on N. Then for each $M \in \mathrm{CC}(N)$, the map $M \mapsto \partial_c M$ defines an isomorphism from $\mathrm{CC}(N)$ to $\mathrm{Teich}(\partial N)$. The renormalized volume is then a function on $\mathrm{CC}(N)$ and, via the above identification, a function on $\mathrm{Teich}(\partial N)$. By [17, Corollary 8.6], V_R is a smooth function and we let V be the gradient flow of $-V_R$ with respect to the Weil–Petersson metric on $\mathrm{Teich}(\partial N)$.

Recall that a tangent vector to Teichmüller space is given by a Beltrami differential and a cotangent vector is a holomorphic quadratic differential. The following variational formula appears in [26, Theorem 6.2].

THEOREM 3.9
Given $M \in \mathrm{CC}(N)$ and $\mu \in T_{\partial_c M}\mathrm{Teich}(\partial N)$, we have

$$dV_R(\mu) = \mathrm{Re}\int_{\partial_c M} \mu\phi_M.$$

Using this variational formula, we get an explicit description of the gradient flow.

PROPOSITION 3.10
The flow for V is defined for all time and for each $M \in \mathrm{CC}(N)$, $V(M) = -\frac{\bar{\phi}_M}{\rho_M^2}$.

Proof
The isomorphism $T_X^* \operatorname{Teich}(S) \to T_X \operatorname{Teich}(S)$ determined by the Weil–Petersson metric is given by $\phi \mapsto \frac{\bar{\phi}}{\rho_X^2}$. Therefore, the second statement follows by Theorem 3.9.

To see that V is defined for all time, we observe that in the Teichmüller metric, the norm of V is bounded by $3/2$ from the Kraus–Nehari bound (Theorem 2.11). Since the Teichmüller metric is complete, a bounded vector field has a flow for all time. □

Recall that $\mathcal{V}_R(N)$ is the infimum of the renormalized volume of $M \in \operatorname{CC}(N)$. We define $\mathcal{V}_C(N)$ to be the same quantity with renormalized volume replaced by convex core volume. While $\mathcal{V}_C(N)$ is trivially nonnegative, this is not clear for $\mathcal{V}_R(N)$. As noted above, if N is a handlebody, then work of Bridgeman and Canary (see [6]) implies that $\mathcal{V}_R(N) = -\infty$. However, if N has incompressible boundary, we prove the following.

THEOREM 3.11
Let N be compact hyperbolizable 3-manifold with nonempty incompressible boundary and without torus boundary components. Then $\mathcal{V}_R(N) = \mathcal{V}_C(N)$. If there exists an $M \in \operatorname{CC}(N)$ with $V_R(M) = \mathcal{V}_R(N)$, then either N is acylindrical and M is the unique manifold in $\operatorname{CC}(N)$ whose convex core boundary is totally geodesic, or N has the homotopy of a closed surface and $\mathcal{V}_R(N) = M$ if and only if M is a Fuchsian manifold.

Proof
We first observe that, by the upper bound on renormalized volume from Theorem 1.2, $\mathcal{V}_R(N) \le \mathcal{V}_C(N)$. If we have $M \in \operatorname{CC}(N)$ with $V_R(M) = \mathcal{V}_R(N)$, then M is critical point of V_R and therefore by the variational formula (Theorem 3.9), $\phi_M = 0$. This occurs exactly when the convex core of M has totally geodesic boundary which implies that either N is acylindrical or M is Fuchsian. In the acylindrical case, there is a unique $M \in \operatorname{CC}(N)$ whose convex core boundary is totally geodesic.

Now choose $M \in \operatorname{CC}(N)$, let M_t be the flow of V with $M = M_0$, and let ϕ_t be the Schwarzian derivative of the projective boundary M_t. We have

$$V_R(M_T) = V_R(M) - \int_0^T \|\phi_t\|_2^2 \, dt.$$

Since V_R is bounded below on $\operatorname{CC}(N)$, the integral $\int_0^\infty \|\phi_t\|_2^2 \, dt$ converges. Therefore, there is a increasing sequence $\{t_i\}$ such that $t_i \to \infty$ and $\|\phi_{t_i}\|_2 \to 0$ as $i \to \infty$. We also have that $V_R(M_t)$ is a decreasing function of t that is bounded below, and hence $V_R(M_t)$ is convergent as $t \to \infty$. Together with Theorem 1.2, these two facts imply that

$$\lim_{i\to\infty} V_R(M_{t_i}) = \lim_{i\to\infty} V_C(M_{t_i}).$$

Since $V_R(M_{t_i})$ is a decreasing sequence, we have

$$V_R(M) \geq \lim_{i\to\infty} V_R(M_{t_i}).$$

By definition, $V_C(M_t) \geq \mathcal{V}_C(N)$, so

$$\lim_{i\to\infty} V_C(M_{t_i}) \geq \mathcal{V}_C(N).$$

Therefore, $V_R(M) \geq \mathcal{V}_C(N)$. Since M is arbitrary, we have $\mathcal{V}_R(N) \geq \mathcal{V}_C(N)$ completing the proof. □

By a theorem of Storm (see [25, Theorem 5.9]), the infimum of the volume of the convex core is half the simplicial volume of the double of the manifold with the infimum realized if and only N is acylindrical or N has the homotopy type of a closed surface. As an immediate corollary of our result and Storm's theorem, we have the following.

COROLLARY 3.12 (see Corollary 1.3)
Let N be a compact hyperbolizable 3-manifold with nonempty incompressible boundary and without torus boundary components. Then $\mathcal{V}_R(N) = \frac{1}{2}V_S(DN)$, where DN is the double of N and $V_S(DN)$ is the simplicial volume. The infimum is realized if and only if N is acylindrical or has the homotopy type of a closed surface.

The manifold DN is hyperbolic if and only if N is acylindrical; then $V_S(DN)$ is twice the volume of the convex core of the unique $M \in \mathrm{CC}(N)$ with totally geodesic boundary. As noted in the introduction, Pallete has proved Corollary 1.3 if N is acylindrical. Pallete's proof does not appeal to Storm's result, so combining Theorem 3.11 together with Pallete's work gives a new proof of the Storm theorem in the acylindrical case. In fact, by studying the limit of the M_t as $t \to \infty$, one could directly prove Storm's theorem without appealing to [23]. This will be discussed further in [3].

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