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Variational fracture with boundary loads

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ABSTRACT

Variational fracture has been viewed as incompatible with boundary loads, due to a straightforward non-existence argument. We introduce a different variational formulation that includes loads, and we illustrate a method for showing existence.

1. Introduction

Variational fracture has been very successful for studying static and quasi-static fracture (see [1] and [2] for overviews), but only for problems with Dirichlet boundary conditions, possibly with traction-free conditions on part of the boundary. Considering mixed boundary conditions, with a load applied to part of the boundary, has been seen as problematic, and even impossible, as we describe below. Here we propose a different variational formulation from the most natural one, and a method for finding solutions.

To make the presentation as simple as possible, we consider only scalar valued displacements, and the simplest elastic energy, $\frac{1}{2}\int_\Omega |\nabla u|^2$, for the displacement u on $\Omega \subset \mathbb{R}^N$. We now describe the issue with combining variational Griffith fracture with boundary loads, noting that cohesive fracture is similar. First, we decompose $\partial\Omega$ into the disjoint union of measurable sets $\partial_D\Omega$ and $\partial_N\Omega$, where $\partial_D\Omega$ is the Dirichlet part of the boundary, and $\partial_N\Omega$ is the Neumann part.

The variational formulation of equilibrium Griffith fracture (based on global minimization) with specified Dirichlet data g and preexisting crack Γ is to minimize

$$E_D[\Gamma](u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \mathcal{H}^{N-1}(S_u \setminus \Gamma)$$

over $u \in SBV(\Omega)$ with $u = g$ on $\partial_D\Omega$ (see [3] for the definition and properties of SBV). We write E_D to emphasize that only Dirichlet boundary values will be imposed when minimizing this energy. A minimizer

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u will satisfy

$$\int_\Omega \nabla u \cdot \nabla \phi = 0 \tag{1.1}$$

for all $\phi \in SBV(\Omega)$ satisfying $\phi = 0$ on $\partial_D \Omega$ and $S_\phi \subset S_u \cup \Gamma$. This is the weak form of

$$\Delta u = 0 \text{ in } \Omega \setminus (S_u \cup \Gamma), \quad u = g \text{ on } \partial_D \Omega, \quad \partial_\nu u = 0 \text{ on } S_u \cup \Gamma \cup \partial_N \Omega, \tag{1.2}$$

where ν is normal to S_u and Γ, and is the outer normal to $\partial \Omega$. Minimizers of E_D will also satisfy Griffith's stability [4], that for any increment $\Delta\Gamma$ in $S_u \cup \Gamma$ and function $v \in SBV(\Omega)$ with the same boundary condition as u and $S_v = S_u \cup \Gamma \cup \Delta\Gamma$, we have

$$\frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2}\int_\Omega |\nabla v|^2 \leq \mathcal{H}^{N-1}(\Delta\Gamma). \tag{1.3}$$

At the same time, the variational formulation for solutions u of

$$\Delta u = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial_D \Omega, \quad \partial_\nu u = f \text{ on } \partial_N \Omega \tag{1.4}$$

is to minimize

$$E_N(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_{\partial_N \Omega} f u$$

over $u \in H^1(\Omega)$ with $u = g$ on $\partial_D \Omega$. Here we write E_N to emphasize that we minimize this energy to get the Neumann boundary condition on $\partial_N \Omega$. A minimizer u will satisfy

$$\int_\Omega \nabla u \cdot \nabla \phi - \int_{\partial_N \Omega} f\phi = 0 \tag{1.5}$$

for all $\phi \in H^1(\Omega)$ satisfying $\phi = 0$ on $\partial_D \Omega$, which is the weak form of (1.4).

A natural question is then how to combine (variationally) (1.4) with (1.2) and (1.3): what is a variational problem that will produce a u satisfying

$$\Delta u = 0 \text{ in } \Omega \setminus (S_u \cup \Gamma), \quad u = g \text{ on } \partial_D \Omega, \quad \partial_\nu u = f \text{ on } \partial_N \Omega, \quad \partial_\nu u = 0 \text{ on } S_u \cup \Gamma, \tag{1.6}$$

together with (1.3)?

There is an immediate answer: combine E_D and E_N, and minimize

$$E_{FN}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \mathcal{H}^{N-1}(S_u \setminus \Gamma) - \int_{\partial_N \Omega} f u \tag{1.7}$$

over $u \in SBV(\Omega)$ with $u = g$ on $\partial_D \Omega$. A minimizer u will then satisfy (1.6) and (1.3), as desired. But there is a fatal problem readily found in the literature (see, e.g., [2]): there cannot be a minimizer of (1.7) unless $f = 0$. The reason is straightforward: the crack S_u can disconnect part of $\partial_N \Omega$ from $\partial_D \Omega$, and then the infimum of (1.7) is minus infinity. In essence, the problem is that the crack and Neumann boundary term can collaborate to make the energy arbitrarily low. Our proposal here is to hide these terms from each other and prevent this collaboration.

Specifically, we propose finding u that simultaneously solves two variational problems. Instead of combining the energies E_N and E_D into one enegy, we look for u minimizing both, so that it satisfies (1.6) and (1.3), a subtle point being that its competitors for the latter (and E_D) satisfy the same Dirichlet condition on all of $\partial \Omega$. The idea is the crack energy and the Neumann boundary term do not appear in the same energy, so they cannot collaborate, and each energy can be minimized. Actually, we will see at the end, in Remark 2.10, that it is possible to consider just one energy, with a restriction on admissible competitors.

Another way of viewing this formulation is as follows. Suppose u minimizes E_D subject to $u = g$ on $\partial_D \Omega$ and $u = h$ on $\partial_N \Omega$, and happens to satisfy $\partial_\nu u = f$ on $\partial_N \Omega$. Is this not a solution to variational fracture with the boundary conditions we seek? It is solutions like this that we will be producing.

We now turn to the question of existence. We show one method, and note that there may be others that are better at avoiding material failure, defined below. For further ease of exposition, we will consider the case that the Dirichlet boundary condition g is identically zero.

2. Existence

Definition 2.1. Allowing cracks along the boundary of Ω generally involves some notational issues. Here we do something slightly different from what has been done previously, but we believe it is the simplest way of considering the issue. The problem is the meaning of u on $\partial\Omega$, when $u \in SBV(\Omega)$. The usual meaning is the trace of u, denoted Tu, which is a limit of values of u as the boundary is approached. The idea here is simply that we will consider $u : \bar{\Omega} \to \mathbb{R}$ that take values on $\partial\Omega$, but these values are not necessarily the trace, in which case such points are included in the jump set of u, S_u. So, here, for regular domains Ω, we say $u \in SBV(\bar{\Omega})$ if $u : \bar{\Omega} \to \mathbb{R}$, $u|_{\partial\Omega} \in L^1(\partial\Omega; \mathcal{H}^{N-1}\lfloor\partial\Omega)$, $u|_\Omega \in SBV(\Omega)$ (in the usual sense), and

$$S_u := S_{u|_\Omega} \cup \{x \in \partial\Omega : Tu(x) \neq u(x)\}.$$

To begin the existence argument, we first choose a minimizer u_1 of

$$E_N(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_{\partial_N\Omega} fu$$

over $u \in H^1(\Omega)$ satisfying $u = 0$ on $\partial_D\Omega$. This will be used to supply Dirichlet data on $\partial_N\Omega$ when finding v_1: minimize

$$E_D(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \mathcal{H}^{N-1}(S_u)$$

over $u \in SBV(\bar{\Omega})$ satisfying $u = 0$ on $\partial_D\Omega$ and $u = u_1$ on $\partial_N\Omega$.

We repeat this process recursively to find u_n and v_n as follows: First, $\Gamma_1 := S_{v_1}$ and for $n > 2$, $\Gamma_{n-1} := \Gamma_{n-2} \cup S_{v_{n-1}}$. Then choose u_n that minimizes

$$E_N(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_{\partial_N\Omega} fu$$

over $u \in SBV(\bar{\Omega})$ satisfying $S_u \subset \Gamma_{n-1}$ and $u = 0$ on $\partial_D\Omega$ (there is a potential issue with the existence of such a minimizer; we address this shortly in a remark below). Now choose v_n that minimizes

$$E_D[\Gamma_{n-1}](u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \mathcal{H}^{N-1}(S_u \setminus \Gamma_{n-1})$$

over $u \in SBV(\bar{\Omega})$ satisfying $u = 0$ on $\partial_D\Omega$ and $u = u_n$ on $\partial_N\Omega$. Note again that in this minimization we include in u's discontinuity set, S_u, the set $\{x \in \partial_D\Omega : Tu(x) \neq 0\}$ and $\{x \in \partial_N\Omega : Tu(x) \neq u_n(x)\}$.

The hope is that u_n weakly converges to a u_∞ that minimizes both E_N and $E_D[\Gamma_\infty]$ over the appropriate classes of competitors, where $\Gamma_\infty := \cup_n\Gamma_n$.

Remark 2.2. There is necessarily a possibility of material failure when there are loads and fracture. When we minimize, the second term in E_N, the boundary term, wants to go to minus infinity, but is held back by the combination of the Dirichlet data on $\partial_D\Omega$ and the stored elastic energy, which normally would increase as the boundary term in the energy decreases. However, if the crack at stage n, Γ_{n-1}, disconnects $\partial_D\Omega$ from part of $\partial_N\Omega$, then the infimum of the energy is minus infinity, and there is no minimizer.

Note that this lack of solution is not caused by the variational formulation—in this case, there is in fact no solution to the Neumann problem. For this reason, we must allow for the possibility that the material fails under the boundary load. This failure can also occur in the limit, as n tends to infinity (which limit we will eventually take), and so we will consider the material to have failed if at some stage n a minimizer u_n does not exist, or $\{\|u_n\|_\infty\}$ is not bounded, where $\|u\|_\infty := \max\{\|u\|_{L^\infty(\Omega)}, \|u\|_{L^\infty(\partial\Omega)}\}$.

A similar failure occurs if $\Gamma_\infty = \cup_n\Gamma_n$ partly lies along $\partial_N\Omega$, resulting in the Neumann problem not being solvable for the limit. For simplicity, we strengthen this condition, and say that the material fails if $\mathrm{dist}(\Gamma_\infty, \partial_N\Omega) = 0$ (in the natural measure theoretic sense). This is all encapsulated in the definition below.

At this point, a natural question, or even objection, might be, how is the above failure different from the non-existence problem described earlier? The key difference is that, with the latter, failure is guaranteed due to direct interaction between the crack and boundary term. Here, it is only a possibility, as the crack might happen to grow to meet $\partial_N\Omega$, or in a way that disconnects $\partial_D\Omega$ from part of $\partial_N\Omega$, oblivious to the existence of a boundary energy term.

Finally, we note that it seems possible that in some situations, there does exist a u that minimizes both E_N and E_D, but when we do the above recursive approach, the crack grows in a way that results in failure. For this reason, it would be interesting to explore other approaches to existence.

Definition 2.3. We say the material does not fail under the boundary load f if the following hold:

(1) Each u_n exists, and $\{\|u_n\|_\infty\}$ is bounded
(2) $\mathrm{dist}(\Gamma_\infty, \partial_N\Omega) > 0$.

We will see below that condition (1) implies $\{E_N(u_n)\}$ and $\{E_D(u_n)\}$ are bounded, and $\mathcal{H}^{N-1}(\Gamma_\infty) < \infty$.

Theorem 2.4. *If the material does not fail under the boundary load $f \in L^1(\partial_N\Omega)$, then there exists $u_\infty \in SBV(\bar{\Omega})$ such that, up to a subsequence,*

$$u_n \rightharpoonup u_\infty \text{ in } SBV(\Omega),$$

u_∞ minimizes E_N over functions in $\{u \in SBV(\bar{\Omega}) : S_u \subset \Gamma_\infty, u = 0 \text{ on } \partial_D\Omega\}$, and it minimizes $E_D[\Gamma_\infty]$ over $\{u \in SBV(\bar{\Omega})\colon u = u_\infty \text{ on } \partial\Omega\}$.

The proof will be done in two parts. From here on, we will assume that Definition 2.3 holds. First, we have

Lemma 2.5. *There exists $u_\infty \in SBV(\bar{\Omega})$ such that, up to a subsequence, $u_n \rightharpoonup u_\infty$, and u_∞ minimizes*

$$E_D[\Gamma_\infty](u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \mathcal{H}^{N-1}(S_u \setminus \Gamma_\infty)$$

over $u \in SBV(\bar{\Omega})$, $u = u_\infty$ on $\partial\Omega$. Furthermore, for all $\phi \in SBV(\bar{\Omega})$ satisfying $\phi = 0$ on $\partial\Omega$ and $S_\phi \subset \Gamma_\infty$, we have

$$\int_\Omega \nabla u_n \cdot \nabla\phi \to 0.$$

Proof. The existence of a weak limit u_∞ (of a subsequence) will follow from the boundedness of $\{\|u_n\|_\infty\}$, $\{E_N(u_n)\}$, $\{\mathcal{H}^{N-1}(\Gamma_n)\}$, and SBV compactness (see [3]), with it remaining to define u_∞ on $\partial\Omega$. We first note some monotonicity properties. For all $n \geq 2$, it is immediate from the minimality of u_n that

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 - \int_{\partial_N\Omega} fu_n \leq \frac{1}{2}\int_\Omega |\nabla v_{n-1}|^2 - \int_{\partial_N\Omega} fv_{n-1}, \tag{2.1}$$

and from the minimality of v_{n-1}, that, since $u_{n-1} = v_{n-1}$ on $\partial_N\Omega$,

$$\frac{1}{2}\int_\Omega |\nabla v_{n-1}|^2 - \int_{\partial_N\Omega} fv_{n-1} \leq \frac{1}{2}\int_\Omega |\nabla u_{n-1}|^2 - \int_{\partial_N\Omega} fu_{n-1} - \mathcal{H}^{N-1}(\Gamma_{n-1} \setminus \Gamma_{n-2}). \tag{2.2}$$

Furthermore, since u_n is an admissible variation for its minimality, as in (1.5) we get

$$\int_\Omega \nabla u_n \cdot \nabla u_n = \int_{\partial_N\Omega} fu_n.$$

Below we will refer to the right hand side as the Neumann energy of u_n.

We now have

$$\frac{1}{2}\int_{\Omega}|\nabla u_n|^2-\int_{\partial_N\Omega}fu_n=-\frac{1}{2}\int_{\Omega}|\nabla u_n|^2=-\frac{1}{2}\int_{\partial_N\Omega}fu_n. \tag{2.3}$$

Combining this with (2.1) and (2.2), we get

$$\frac{1}{2}\int_{\Omega}|\nabla u_n|^2\geq\frac{1}{2}\int_{\Omega}|\nabla u_{n-1}|^2 \text{ and } \int_{\partial_N\Omega}fu_n\geq\int_{\partial_N\Omega}fu_{n-1}. \tag{2.4}$$

By the boundedness of $\|u_n\|_\infty$ and by (2.3), we have boundedness of the elastic and Neumann energies, and by the monotonicity (2.4), the sequences of elastic and Neumann energies converge. Let C_L and $2C_L$ be their respective limits. Then $E_N(u_n)$ converges to $-C_L$, by (2.3).

Next, we look at $\mathcal{H}^{N-1}(\Gamma_n)$. Combining (2.1) and (2.2) we get

$$\mathcal{H}^{N-1}(\Gamma_{n-1}\setminus\Gamma_{n-2})+E_N(u_n)\leq E_N(u_{n-1}).$$

For $m>n$ and iterating the above, we get

$$\mathcal{H}^{N-1}(\Gamma_{m-1}\setminus\Gamma_{n-1})+E_N(u_m)\leq E_N(u_n). \tag{2.5}$$

Since $\Gamma_i\subset\Gamma_{i+1}$ $\forall i\in\mathbb{N}$, it follows that $\mathcal{H}^{N-1}(\Gamma_{m-1}\setminus\Gamma_{n-1})\to\mathcal{H}^{N-1}(\Gamma_\infty\setminus\Gamma_{n-1})$ as $m\to\infty$. So taking the limit in (2.5) as $m\to\infty$ gives

$$\mathcal{H}^{N-1}(\Gamma_\infty\setminus\Gamma_{n-1})-C_L\leq E_N(u_n),$$

so that $\mathcal{H}^{N-1}(\Gamma_\infty)<\infty$. We then also have $\mathcal{H}^{N-1}(\Gamma_\infty\setminus\Gamma_n)\to 0$, and by SBV compactness, a subsequence of $\{u_n\}$ weakly converges to a $u_\infty\in SBV(\bar{\Omega})$ with $S_{u_\infty}\subset\Gamma_\infty$, where

$$u_\infty|_{\partial_D\Omega}:=0,\quad u_\infty|_{\partial_N\Omega}:=Tu_\infty.$$

We now want to establish that the elastic energies of v_n also converge to C_L. Since by assumption the u_n are bounded in L^∞, so are the v_n, by the maximum principle. From (2.2), the fact that v_n are bounded in $L^\infty(\Omega)$, and the fact that

$$\mathcal{H}^{N-1}(\Gamma_{n-1}\setminus\Gamma_{n-2})\to 0,$$

we have

$$\limsup\frac{1}{2}\int_{\Omega}|\nabla v_n|^2\leq C_L.$$

Similarly, from (2.1), we have

$$\liminf\frac{1}{2}\int_{\Omega}|\nabla v_n|^2\geq C_L, \text{ and so } \frac{1}{2}\int_{\Omega}|\nabla v_n|^2\to C_L.$$

Now suppose there exists $\phi\in SBV(\bar{\Omega})$ with $\phi=0$ on $\partial\Omega$ and

$$\frac{1}{2}\int_{\Omega}|\nabla(u_\infty+\phi)|^2+\mathcal{H}^{N-1}(S_\phi\cup\Gamma_\infty)<\frac{1}{2}\int_{\Omega}|\nabla u_\infty|^2+\mathcal{H}^{N-1}(\Gamma_\infty).$$

This is equivalent to the energy difference being negative:

$$\int_{\Omega}\nabla u_\infty\cdot\nabla\phi+\frac{1}{2}\int_{\Omega}|\nabla\phi|^2+\mathcal{H}^{N-1}(S_\phi\setminus\Gamma_\infty)=:\eta<0. \tag{2.6}$$

But the left hand side of (2.6) is equal to the limit of

$$\int_{\Omega}\nabla u_n\cdot\nabla\phi+\frac{1}{2}\int_{\Omega}|\nabla\phi|^2+\mathcal{H}^{N-1}(S_\phi\setminus\Gamma_n),$$

since $\mathcal{H}^{N-1}(\Gamma_\infty \setminus \Gamma_n) \to 0$, which means that, for n large enough, since

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 - \frac{1}{2}\int_\Omega |\nabla v_n|^2 \to 0,$$

we have

$$\frac{1}{2}\int_\Omega |\nabla(u_n+\phi)|^2 + \mathcal{H}^{N-1}(S_\phi \cup \Gamma_n) < \frac{1}{2}\int_\Omega |\nabla u_n|^2 + \mathcal{H}^{N-1}(\Gamma_n) + \frac{\eta}{2} < \frac{1}{2}\int_\Omega |\nabla v_n|^2 + \mathcal{H}^{N-1}(\Gamma_n),$$

contradicting the minimality of v_n.

Similarly, if $\phi \in SBV(\bar{\Omega})$ with $\phi = 0$ on $\partial\Omega$ and $S_\phi \subset \Gamma_\infty$, and

$$\int_\Omega \nabla u_n \cdot \nabla\phi \to \eta \neq 0$$

then for n large enough

$$\frac{1}{2}\int_\Omega |\nabla(u_n+\lambda\phi)|^2 + \mathcal{H}^{N-1}(S_\phi \cup \Gamma_n) < \frac{1}{2}\int_\Omega |\nabla v_n|^2 + \mathcal{H}^{N-1}(\Gamma_n),$$

for λ small enough, and with the correct sign, since the difference between the energy of $u_n + \lambda\phi$ and u_n is

$$\lambda\int_\Omega \nabla u_n \cdot \nabla\phi + \lambda^2\frac{1}{2}\int_\Omega |\nabla\phi|^2 + \mathcal{H}^{N-1}(S_\phi \setminus \Gamma_n),$$

and the last term goes to zero since $S_\phi \subset \Gamma_\infty$. Again, this contradicts the minimality of v_n for n large enough. $\square$

We then address minimizing E_N.

Lemma 2.6. *u_∞ minimizes*

$$E_N(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_{\partial_N\Omega} fu$$

over $u \in SBV(\bar{\Omega})$ with $u = 0$ on $\partial_D\Omega$ and $S_u \subset \Gamma_\infty$.

Proof. Let $\psi \in SBV(\bar{\Omega})$ with $\psi = 0$ on $\partial_D\Omega$, $S_\psi \subset \Gamma_\infty$, and $E_N(\psi) < \infty$. Choose $\phi \in H^1(\Omega)$ such that $\phi = 0$ on $\partial_D\Omega$ and $\phi = \psi$ on $\partial_N\Omega$, which is possible because of (2) in the non-failure condition, $\operatorname{dist}(\Gamma_\infty, \partial_N\Omega) > 0$ – because of this condition, ψ is an H^1 function in a neighborhood of $\partial_N\Omega$, and so the extension to ϕ is straightforward. We also consider $\phi \in SBV(\bar{\Omega})$, extending to $\partial\Omega$ using its trace.

Suppose $E_N(u_\infty + \psi) < E_N(u_\infty)$, which means

$$\int_\Omega \nabla u_\infty \cdot \nabla\psi = \gamma + \int_{\partial_N\Omega} f\psi$$

for some $\gamma < 0$. From the minimality of u_n, the convergence of u_n to u_∞, and the fact that ϕ is an admissible variation of u_n for E_N, we have

$$0 = \int_\Omega \nabla u_n \cdot \nabla\phi - \int_{\partial_N\Omega} f\phi \to \int_\Omega \nabla u_\infty \cdot \nabla\phi - \int_{\partial_N\Omega} f\phi.$$

Hence, since $\psi - \phi \in SBV(\bar{\Omega})$ with $(\psi - \phi) = 0$ on $\partial\Omega$ and $S_{\psi-\phi} \subset \Gamma_\infty$, by Lemma 2.5 we have

$$0 \leftarrow \int_\Omega \nabla u_n \cdot \nabla(\psi - \phi) \to \gamma + \int_{\partial_N\Omega} f\psi - \int_{\partial_N\Omega} f\phi = \gamma,$$

a contradiction. $\square$

This completes the proof of Theorem 2.4.

Remark 2.7. A natural question, since we show existence only when there is not failure, is whether failure is common, or even certain. We note that, as we mentioned in the introduction, every solution u to a pure Dirichlet problem is also a solution to a mixed problem, as we can designate part of the boundary that is away from the crack as $\partial_N \Omega$, and set $f := \partial_\nu u$ on $\partial_N \Omega$. Then, u is a solution to variational fracture with boundary load f on $\partial_N \Omega$. This shows that the formulation here is not vacuous. Furthermore, studying conditions on Ω and f guaranteeing existence (or non failure) seems to be an interesting direction to explore.

In addition, we note that part 2 of our failure definition is, for simplicity, a bit unnecessarily strong. It would be interesting to show that it is sufficient if $\mathcal{H}^{N-1}(\Gamma_\infty \cap \partial_N \Omega) = 0$.

We conclude with some small extensions and further remarks.

Note first that if we replace Γ_∞ with S_{u_∞} in the previous lemmas, there is no effect on the energy of u_∞, but there is an increase in the energy of competitors, or there is a reduction in the class of competitors, so we have

Lemma 2.8. *u_∞ minimizes*

$$E_D[S_{u_\infty}](u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \mathcal{H}^{N-1}(S_u \setminus S_{u_\infty})$$

over $u \in SBV(\bar{\Omega})$, $u = 0$ on $\partial_D \Omega$, $u = u_\infty$ on $\partial_N \Omega$.

Lemma 2.9. *u_∞ minimizes*

$$E_N(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_{\partial_N \Omega} fu$$

over $u \in SBV(\bar{\Omega})$ with $u = 0$ on $\partial_D \Omega$ and $S_u \subset S_{u_\infty}$.

Remark 2.10. We can now claim that u_∞ actually does minimize

$$E_{FN}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \mathcal{H}^{N-1}(S_u \setminus \Gamma_\infty) - \int_{\partial_N \Omega} fu,$$

if the class of competitors is restricted to

$$\{u \in SBV(\bar{\Omega}) : u = u_\infty \text{ on } \partial\Omega\} \bigcup \{u \in SBV(\bar{\Omega}) : S_u \subset \Gamma_\infty, u = 0 \text{ on } \partial_D \Omega\}.$$

That is, competitors are not allowed to simultaneously vary both their boundary data on $\partial_N \Omega$ and the crack. But this is consistent with Griffith's idea that cracks compete with elastic energy, not boundary loads.

Note also that in the above minimality, Γ_∞ can be replaced with S_{u_∞}, as before.

Finally, we make the following remark.

Remark 2.11. It might seem natural to instead minimize

$$\left\{ \frac{1}{2}\int_\Omega |\nabla u|^2 + \mathcal{H}^{N-1}(S_u) : u \text{ minimizes} \right.$$
$$\left. v \mapsto \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_{\partial_N \Omega} fv, \text{ over } v \text{ in } SBV(\bar{\Omega}),\ v = 0 \text{ on } \partial_D \Omega,\ S_v \subset S_u \right\}.$$

This would be incorrect, however, since as S_u grows, the class of minimizers for E_N grows, so E_N decreases, which means the elastic energy *increases*, by (2.3). The solution to this minimization problem will therefore necessarily be $S_u = \varnothing$.

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