SYMMETRY IN STATIONARY AND UNIFORMLY ROTATING SOLUTIONS OF ACTIVE SCALAR EQUATIONS

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Abstract

We study the radial symmetry properties of stationary and uniformly rotating solutions of the 2D Euler and gSQG equations, both in the smooth setting and the patch setting. For the 2D Euler equation, we show that any smooth stationary solution with compactly supported and nonnegative vorticity must be radial, without any assumptions on the connectedness of the support or the level sets. For the 2D Euler equation in the patch setting, we show that every uniformly rotating patch D with angular velocity $\Omega \le 0$ or $\Omega \ge \frac{1}{2}$ must be radial, where both bounds are sharp. For the gSQG equation, we obtain a similar symmetry result for $\Omega \le 0$ or $\Omega \ge \Omega_\alpha$ (with the bounds being sharp), under the additional assumption that the patch is simply connected. These results settle several open questions posed by Hmidi, de la Hoz, Hassainia, and Mateu on uniformly rotating patches. Along the way, we close a question by Choksi, Neumayer, and Topaloglu on overdetermined problems for the fractional Laplacian, which may be of independent interest. The main new ideas come from a calculus-of-variations point of view.

1. Introduction

Let us start by considering the initial value problem for the 2-dimensional incompressible Euler equation in vorticity form. Here the evolution of the vorticity ω is given by

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 & \text{in } \mathbb{R}^2 \times \mathbb{R}_+, \\ u(\cdot,t) = -\nabla^\perp (-\Delta)^{-1} \omega(\cdot,t) & \text{in } \mathbb{R}^2, \\ \omega(\cdot,0) = \omega_0 & \text{in } \mathbb{R}^2, \end{cases} \tag{1.1}$$

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where $\nabla^\perp := (-\partial_{x_2}, \partial_{x_1})$. Note that we can express u as $u(\cdot,t) = \nabla^\perp(\omega(\cdot,t) * \mathcal{N})$, where $\mathcal{N}(x) := \frac{1}{2\pi}\ln|x|$ is the Newtonian potential in two dimensions. More generally, the 2D Euler equation belongs to the following family of active scalar equations indexed by a parameter α, $(0 \le \alpha < 2)$, known as the *generalized surface quasi-geostrophic* (gSQG) equations:

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 & \text{in } \mathbb{R}^2 \times \mathbb{R}_+, \\ u(\cdot,t) = -\nabla^\perp(-\Delta)^{-1+\frac{\alpha}{2}}\omega(\cdot,t) & \text{in } \mathbb{R}^2, \\ \omega(\cdot,0) = \omega_0 & \text{in } \mathbb{R}^2. \end{cases} \tag{1.2}$$

Here we can also express the Biot–Savart law as

$$u(\cdot,t) = \nabla^\perp\big(\omega(\cdot,t) * K_\alpha\big), \tag{1.3}$$

where K_α is the fundamental solution for $-(-\Delta)^{-1+\frac{\alpha}{2}}$; that is,

$$K_\alpha(x) = \begin{cases} \frac{1}{2\pi}\ln|x| & \text{for } \alpha = 0, \\ -C_\alpha |x|^{-\alpha} & \text{for } \alpha \in (0,2), \end{cases} \tag{1.4}$$

where $C_\alpha = \frac{1}{2\pi}\frac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})}$ is a positive constant only depending on α.

We will focus here on establishing radial symmetry properties for stationary and uniformly rotating solutions to equations (1.1) and (1.2). We either work with the *patch* setting, where $\omega(\cdot,t) = 1_{D(t)}$ is an indicator function of a bounded set that moves with the fluid, or the *smooth* setting, where $\omega(\cdot,t)$ is smooth and compactly supported in x. (For well-posedness results for patch solutions, see the global well-posedness results in [7] and [20] for (1.1), and the local well-posedness results in [26], [35], [60], and [77] for (1.2) with $\alpha \in (0,2)$.)

Let us begin with the definition of a stationary/uniformly rotating solution in the patch setting. For a bounded open set $D \subset \mathbb{R}^2$, we say that $\omega = 1_D$ is a *stationary patch* solution to (1.2) for some $\alpha \in [0,2)$ if $u(x)\cdot\vec{n}(x) = 0$ on ∂D, with u given by (1.3). This leads to the integral equation

$$1_D * \mathcal{K}_\alpha \equiv C_i \quad \text{on } \partial D, \tag{1.5}$$

where the constant C_i can differ on different connected components of ∂D. And if $\omega(x,t) = 1_D(R_{\Omega t}x)$ is a *uniformly rotating patch* solution with angular velocity Ω (where $R_{\Omega t}x$ rotates a vector $x \in \mathbb{R}^2$ counterclockwise by angle Ωt about the origin), then 1_D becomes stationary in the rotating frame with angular velocity Ω; that is, $(\nabla^\perp(1_D * \mathcal{K}_\alpha) - \Omega x^\perp)\cdot\vec{n}(x) = 0$ on ∂D. This is equivalent to $\nabla^\perp(1_D * \mathcal{K}_\alpha - \frac{\Omega}{2}|x|^2)\cdot\vec{n}(x) = 0$ on ∂D, and as a result we have

$$1_D * \mathcal{K}_\alpha - \frac{\Omega}{2}|x|^2 \equiv C_i \quad \text{on } \partial D, \tag{1.6}$$

where C_i again can take different values along different connected components of ∂D. Note that a stationary patch D also satisfies (1.6) with $\Omega = 0$, and it can be considered as a special case of uniformly rotating patch with zero angular velocity.

Likewise, in the smooth setting, if $\omega(x,t) = \omega_0(R_{\Omega t}x)$ is a uniformly rotating solution of (1.2) with angular velocity Ω (which becomes a stationary solution in the $\Omega = 0$ case), then we have $(\nabla^\perp(\omega_0 * K_\alpha) - \Omega x^\perp)\cdot\nabla\omega_0 = 0$. As a result, ω_0 satisfies

$$\omega_0 * \mathcal{K}_\alpha - \frac{\Omega}{2}|x|^2 \equiv C_i \quad \text{on each connected component of a regular level set of } \omega_0, \tag{1.7}$$

where C_i can be different if a regular level set $\{\omega_0 = c\}$ has multiple connected components.

Clearly, every radially symmetric patch/smooth function automatically satisfies (1.6)/(1.7) for all $\Omega \in \mathbb{R}$. The goal of this article is to address the complementary question, which can be roughly stated as follows.

QUESTION 1
In the patch or smooth setting, under what condition must a stationary/uniformly rotating solution be radially symmetric?

Below we summarize the previous literature related to this question, and state our main results. We will first discuss the 2D Euler equation in the patch and smooth settings, respectively. Then we will discuss the gSQG equation with $\alpha \in (0,2)$.

1.1. 2D Euler in the patch setting

Let us deal with the patch setting first. So far, affirmative answers to Question 1 have only been obtained for simply connected patches, for angular velocities $\Omega = 0$, $\Omega < 0$ (under some additional convexity assumptions), and $\Omega = \frac{1}{2}$. For stationary patches ($\Omega = 0$), Fraenkel [33, Chapter 4] proved that if D satisfies (1.6) (where $K_\alpha = \mathcal{N}$) with the *same* constant C on the whole ∂D, then D must be a disk. The idea is that in this case the stream function $\psi = 1_D * \mathcal{N}$ solves a semilinear elliptic equation $\Delta\psi = g(\psi)$ in $\mathbb{R}^2$ with $g(\psi) = 1_{\{\psi < C\}}$, where the monotonicity of the discontinuous function g allows one to apply the moving-plane method developed in [40] and [80] to obtain the symmetry of ψ. As a direct consequence, every simply connected stationary patch must be a disk. But if D is not simply connected, (1.6) gives that $\psi = C_i$ on different connected components of ∂D, and thus ψ might not solve a single semilinear elliptic equation in $\mathbb{R}^2$. Even if ψ satisfies $\Delta\psi = g(\psi)$, g might not have the right monotonicity. For these reasons, whether a nonsimply connected stationary patch must be radial still remains an open question.

For $\Omega < 0$, Hmidi [49] used the moving-plane method to show that if a simply connected, uniformly rotating patch D satisfies some additional convexity assumption (which is stronger than star-shapedness but weaker than convexity), then D must be a disk. In the special case $\Omega = \frac{1}{2}$, Hmidi [49] also showed that a simply connected, uniformly rotating patch D must be a disk, using the fact that $1_D * \mathcal{N} - \frac{\Omega}{2}|x|^2$ becomes a harmonic function in D when $\Omega = \frac{1}{2}$.

On the other hand, it is known that there can be nonradial uniformly rotating patches for $\Omega \in (0, \frac{1}{2})$. The first example dates back to the Kirchhoff ellipse in [59], where it was shown that any ellipse D with semiaxes a, b is a uniformly rotating patch with angular velocity $\frac{ab}{(a+b)^2}$. Deem and Zabusky [31] numerically found families of patch solutions of (1.1) with m-fold symmetry by bifurcating from a disk at explicit angular velocities $\Omega_m^0 = \frac{m-1}{2m}$, and they coined the term *V-states*. Further numerics were done in [32], [69], [78], and [90]. Burbea [9] gave the first rigorous proof of their existence by using (local) bifurcation theory arguments close to the disk. There have been many recent developments in a series of works by Hmidi, Mateu, Verdera, and de la Hoz (see [30], [52], [53]) in different settings and directions (regularity of the boundary, different topologies, and so forth). In particular, [30] showed the existence of m-fold doubly connected nonradial patches bifurcating at any angular velocity $\Omega \in (0, \frac{1}{2})$ from some annulus of radii $b \in (0, 1)$ and 1.

There are many other interesting perspectives of the V-states, which we briefly review below, although they are not directly related to Question 1. Hassainia, Masmoudi, and Wheeler [48] were able to perform global bifurcation arguments and study the whole branch of V-states. Other scenarios such as the bifurcation from ellipses instead of disks have also been studied: first numerically by Kamm [56] and later theoretically by Castro, Córdoba, and Gómez-Serrano [17] and by Hmidi and Mateu [50]. See also the work of Carrillo, Mateu, Mora, Rondi, Scardia, and Verdera [15] for variational techniques applied to other anisotropic problems related to vortex patches. Love [65] established linear stability for ellipses of aspect ratio bigger than $\frac{1}{3}$ and linear instability for ellipses of aspect ratio smaller than $\frac{1}{3}$. Most of these efforts have been devoted to establishing nonlinear stability and instability in the range predicted by the linear part. Wan [87] and Tang [84] proved the nonlinear stable case, whereas Guo, Hallstrom, and Spirn [43] settled the nonlinear unstable one (see also [25]). In [86], Turkington considered N vortex patches rotating around the origin in the variational setting, yielding solutions of the problem which are close to point vortices.

Our first main result is summarized below in Theorem A, which gives a complete answer to Question 1 for 2D Euler in the patch setting. Note that D is allowed to be disconnected, and each connected component can be nonsimply connected. Figure 1 illustrates a comparison of our result (in red color) with the previous results (in black color).

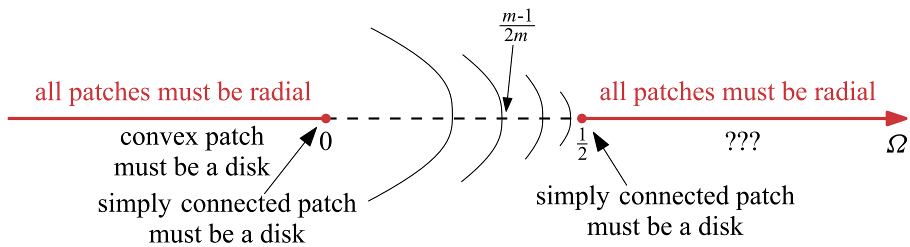

Figure 1. (Color online) For 2D Euler in the patch setting, previous results on Question 1 are summarized in black color. Our results in Theorem A are colored in red.

THEOREM A (= Corollary 2.10, Theorems 2.12 and 2.14)
Let $D \subset \mathbb{R}^2$ be a bounded open set with $C^{1,\gamma}$ boundary. Assume that D is a stationary/uniformly rotating patch of (1.1), in the sense that D satisfies (1.6) (with $K_\alpha = \mathcal{N}$) for some $\Omega \in \mathbb{R}$. Then D must be radially symmetric if $\Omega \in (-\infty, 0) \cup [\frac{1}{2}, \infty)$, and radially symmetric up to a translation if $\Omega = 0$.

Remark 1.1
The $C^{1,\gamma}$ boundary regularity is not optimal and can be pushed down to Lipschitz. We have outlined the necessary modifications to make Theorem A work in Remarks 2.4 and 2.7, which also apply to Corollary 2.10 and Theorems 2.12 and 2.14.

1.2. 2D Euler in the smooth setting
One of the main motivations of this paper is to find sufficient rigidity conditions in terms of the vorticity, such that the only stationary/uniformly rotating solutions are radial ones. Heuristically speaking, this belongs to the broader class of the "Liouville theorem" type of results, which shows that solutions satisfying certain conditions must have a simpler geometric structure, such as being constant (in one direction, or all directions) or being radial. We were unable to find any conditions on 2D Euler in the literature that lead to radial symmetry, although several other Liouville-type results have been established for 2D fluid equations. For 2D Euler, Hamel and Nadirashvili [44], [45] proved that any stationary solution without a stagnation point must be a shear flow. (But note that this result does not apply to our setting (1.7), since the velocity field u associated with any compactly supported ω_0 must have a stagnation point. See also the Liouville theorem by Koch, Nadirashvili, Seregin, and Šverák [61] for the 2D Navier–Stokes equations.)

Let us briefly review some results on the characterization of stationary solutions to 2D Euler, although they are not directly related to Question 1. Nadirashvili [74] studied the geometry and the stability of stationary solutions, following the works of Arnold [3]–[5]. Izosimov and Khesin [54] characterized stationary solutions of 2D Euler on surfaces. Choffrut and Šverák [21] showed that locally near each stationary smooth solution there exists a manifold of stationary smooth solutions transversal to the foliation, and Choffrut and Székelyhidi [22] showed that there is an abundant set of stationary weak (L^∞) solutions near a smooth stationary one. Luo–Shvydkoy in [67] and [68] classified the set of stationary smooth solutions of the form $v = \nabla^\perp(r^\gamma f(\omega))$, where (r,ω) are polar coordinates. In a different direction, Turkington [85] used variational methods to construct stationary vortex patches of a prescribed area in a bounded domain, imposing that the patch is a characteristic function of the set $\{\Psi > 0\}$, and also studied the asymptotic limit of the patches tending to point vortices. Long, Wang, and Zeng [64] studied their stability, as well as the regularity in the smooth setting (see also [12]). For other variational constructions close to point vortices, we refer to the work of Cao, Liu, and Wei [10], Cao, Peng, and Yan [11], and Smets and van Schaftingen [82]. Musso, Pacard, and Wei [73] constructed nonradial smooth stationary solutions without compact support in ω. The (nonlinear L^1) stability of circular patches was proved by Wan and Pulvirenti [88] (a shorter proof was later given by Sideris and Vega [81]). See also Beichman and Denisov [6] for similar results on the strip.

Recently, Gavrilov in [37] and [38] provided a remarkable construction of nontrivial stationary solutions of 3D Euler with compactly supported velocity. See also Constantin, La, and Vicol [24] for a simplified proof with extensions to other fluid equations.

Regarding uniformly rotating smooth solutions ($\Omega \neq 0$) for 2D Euler, Castro, Córdoba, and Gómez-Serrano [18] were able to desingularize a vortex patch to produce a smooth m-fold V-state with $\Omega \sim \frac{m-1}{2m} > 0$ for $m \geq 2$. Recently, García, Hmidi, and Soler [36] studied the construction of V-states bifurcating from other radial profiles (Gaussians and piecewise quadratic functions).

Our second main result is the following theorem, which gives radial symmetry of compactly supported stationary/uniformly rotating solutions in the smooth setting for $\Omega \leq 0$, under the additional assumption $\omega_0 \geq 0$.

THEOREM B (= Theorem 3.5 and Corollary 3.6)
Let $\omega_0 \geq 0$ be smooth[1] *and compactly supported. Assume that $\omega(x,t) = \omega_0(R_{\Omega t}x)$ is a stationary/uniformly rotating solution of (1.1) with $\Omega \leq 0$, in the sense that it satisfies (1.7) with $K_\alpha = \mathcal{N}$. Then ω_0 must be radially symmetric if $\Omega < 0$, and radially symmetric up to a translation if $\Omega = 0$.*

[1]It is enough to assume that $\omega_0 \in C^2(\mathbb{R}^2)$; see the footnote under Theorem 3.5 for more discussions.

Note that there exist nonradial nonnegative smooth uniformly rotating solutions for *every* $\Omega > 0$: taking the nonradial smooth V-state $\omega_0 \geq 0$ constructed in [18] (with $\Omega > 0$) and multiplying it by any $\lambda > 0$, one obtains a new V-state $\lambda\omega_0$ with angular velocity $\lambda\Omega > 0$.

Although the extra assumption $\omega_0 \geq 0$ might seem unnatural at first glance, in a forthcoming work [42] we will show that it is indeed necessary: if we allow ω_0 to change sign, then by applying bifurcation arguments to sign-changing radial patches, we are able to show that there exists a compactly supported, sign-changing smooth stationary vorticity ω_0 that is nonradial.

1.3. The gSQG case ($0 < \alpha < 2$)

Recall that in the patch setting, a stationary/uniformly rotating patch satisfies (1.6) with K_α given in (1.4). Even though the kernels K_α are qualitatively similar for all $\alpha \in [0, 2)$, there is a key difference on the symmetry versus nonsymmetry results between the cases $\alpha = 0$ and $\alpha > 0$. For the 2D Euler equation ($\alpha = 0$), we proved in Theorem A that any rotating patch D with $\Omega \leq 0$ must be radial, even if D is not simply connected. However, this result is not true for any $\alpha \in (0, 2)$: de la Hoz, Hassainia, Hmidi, and Mateu [28] showed that there exist nonradial patches bifurcating from annuli at $\Omega < 0$ and Gómez-Serrano [41] constructed nonradial, doubly connected stationary patches ($\Omega = 0$). Therefore, we cannot expect a nonsimply connected rotating patch D with $\Omega \leq 0$ to be radial for $\alpha \in (0, 2)$.

However, if D is a simply connected stationary patch, then radial symmetry results were obtained in a series of works for $\alpha \in [0, \frac{5}{3})$, which we review below. These works consider (1.6) in a more general context not limited to dimension 2. Let $K_{\alpha,d}$ be the fundamental solution of $-(-\Delta)^{-1+\frac{\alpha}{2}}$ in $\mathbb{R}^d$ for $d \geq 2$, given by

$$K_{\alpha,d} := -C_{\alpha,d}|x|^{-d+2-\alpha} \tag{1.8}$$

for some $C_{\alpha,d} > 0$; except that in the special case $-d + 2 - \alpha = 0$ it becomes $K_{\alpha,d} = C_d \ln|x|$ for some $C_d > 0$. Note that $K_{\alpha,d} \in L^1_{\text{loc}}(\mathbb{R}^d)$ for all $\alpha < 2$. Consider the following question.

QUESTION 2

Let $\alpha \in [0, 2)$. *Assume that* $D \subset \mathbb{R}^d$ *is a bounded open set such that*

$$K_{\alpha,d} * 1_D - \frac{\Omega}{2}|x|^2 = \text{const} \quad \text{on } \partial D \tag{1.9}$$

for some $\Omega \leq 0$, *where the constant is the same along all connected components of* ∂D. *Must* D *be a ball in* $\mathbb{R}^d$?

Positive answers to Question 2 were obtained in the $\Omega = 0$ case for $\alpha < \frac{5}{3}$ in the following works. As we discussed before, Fraenkel [33] proved that D must be a ball for $\alpha = 0$. Also using the moving-plane method, Reichel [75, Theorem 2], Lu and Zhu [66], and Han, Lu, and Zhu [46] generalized this result to $\alpha \in [0, 1)$. Here [66] also covered generic radially increasing potentials not too singular at the origin (which include all Riesz potentials $K_{\alpha,d}$ with $\alpha \in [0, 1)$). Recently, Choksi, Neumayer, and Topaloglu [23, Theorem 1.3] further pushed the range to $\alpha \in [0, \frac{5}{3})$, leaving the range $\alpha \in [\frac{5}{3}, 2)$ an open problem. We point out that in all these results for $\alpha > 0$, ∂D was assumed to be at least C^1. All the above results were obtained using the moving-plane method.

In our third main result, we use a completely different approach to give an affirmative answer to Question 2 for all $\Omega \le 0$ and $\alpha \in [0, 2)$, under a weaker assumption on the regularity of ∂D.

THEOREM C (= Theorem 4.2)
Let D be a bounded open set in $\mathbb{R}^d$ with Lipschitz boundary (and if $d = 2$, then we only require ∂D to be rectifiable). If D satisfies (1.9) for some $\Omega \le 0$ and $\alpha \in [0, 2)$, then it must be a ball in $\mathbb{R}^d$.

As a direct consequence, Theorem C implies that for the gSQG equation with $\alpha \in [0, 2)$, any simply connected rotating patch with $\Omega \le 0$ must be a disk (see Theorem 4.4). In addition, in the smooth setting (1.7), we prove a similar result in Corollary 4.7 for uniformly rotating solutions with $\Omega \le 0$ for all $\alpha \in [0, 2)$: if the superlevel sets $\{\omega_0 > h\}$ are all simply connected for all $h > 0$, then ω_0 must be radially decreasing.

Next we review the previous literature on uniformly rotating solutions for the gSQG equation. Note that the case of $\alpha \in (0, 2)$ is more challenging than the 2D Euler case, since the velocity is more singular and this produces obstructions to the bifurcation theory when it comes to the choice of spaces and the regularity of the functionals involved in the construction. Hassainia and Hmidi [47] showed the existence of V-states with C^k boundary regularity in the case $0 < \alpha < 1$, and in [16], Castro, Córdoba, and Gómez-Serrano upgraded the result to show existence and C^∞ boundary regularity in the remaining open cases: $\alpha \in [1, 2)$ for the existence, $\alpha \in (0, 2)$ for the regularity. In that case, the solutions bifurcate at angular velocities given by $\Omega_m^\alpha := 2^{\alpha-1} \frac{\Gamma(1-\alpha)}{\Gamma(1-\frac{\alpha}{2})^2} \left(\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})} - \frac{\Gamma(m+\frac{\alpha}{2})}{\Gamma(m+1-\frac{\alpha}{2})} \right)$. This boundary regularity was subsequently improved to analytic in [17]. (See also [51] for another family of rotating solutions, [28] and [76] for the doubly connected case, and [19] for a construction in the smooth setting.)

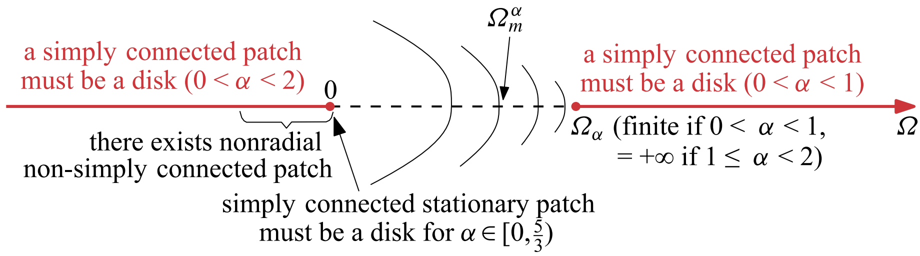

Figure 2. (Color online) For gSQG in the patch setting, previous results on Question 1 are summarized in black color, with our results in Theorems C and D colored in red.

One can check that Ω_m^α are increasing functions of m for any α, whose limit is a finite number $\Omega^\alpha := 2^{\alpha-1} \frac{\Gamma(1-\alpha)}{\Gamma(1-\frac{\alpha}{2})^2} \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})}$ for $\alpha \in [0, 1)$, and $+\infty$ if $\alpha \ge 1$. It is then a natural question to ask whether there exist V-states (with area π) that rotate with angular velocity faster than Ω_α for $\alpha \in (0, 1)$. Our fourth main theorem answers this question among all simply connected patches.

THEOREM D ($=$ Theorem 5.1)
For $\alpha \in (0, 1)$, let 1_D be a simply connected V-state of area π, and let its angular velocity be $\Omega \ge \Omega^\alpha$. Then D must be the unit disk.

Finally, we illustrate a comparison of our results in Theorems C and D (in red color) with the previous results (in black color) in Figure 2.

1.4. *Structure of the proofs*

While all the previous symmetry results on Questions 1 and 2 (see, e.g., [23], [33], [46], [49], [66], [75]) are done by moving-plane methods, our approaches are completely different, with more of a variational flavor.

Theorem A is based on computing the first variation of the energy functional

$$\mathcal{E}[1_D] = -\frac{1}{2} \int_{\mathbb{R}^2} 1_D(x)(1_D * \mathcal{N})(x) - \frac{\Omega}{2}|x|^2 1_D(x)\, dx$$

in two different ways, as we deform D along a carefully chosen vector field that is divergence-free in D. On the one hand, we show that the first variation should be zero if D is a stationary/rotating patch with angular velocity Ω; on the other hand, we show that the first variation must be nonzero if $\Omega \le 0$ or $\Omega \ge \frac{1}{2}$, leading to a contradiction. If D is simply connected, we give a very short proof in Section 2.1, where a rearrangement inequality due to Talenti [83] is crucial to get a sign condition. For a nonsimply connected patch D, the choice of the right vector field is more involved.

Since $1_D * \mathcal{N} - \frac{\Omega}{2}|x|^2$ now takes different constant values on different connected components of ∂D, in order to keep the first variation at zero it is necessary to modify our perturbation vector field such that it also preserves the area of each hole. We then prove a new version of a rearrangement inequality for this modified vector field in a similar spirit as Talenti's result, leading to a nonzero first variation if D is nonradial and $\Omega \leq 0$ or $\Omega \geq \frac{1}{2}$.

The smooth setting in Theorem B is based on a similar idea, but it is technically more difficult. The point of view is to approximate a smooth function by step functions and consider the above perturbation in each set where the step function is constant. To do this, we need to obtain some quantitative (stability) estimates on our version of Talenti's rearrangement inequality, particularly in terms of the Fraenkel asymmetry of the domain in the spirit of Fusco, Maggi, and Pratelli [34].

Theorem C is also based on a variational approach, but we need a different perturbation from the vector field in Theorem A, which heavily relies on the Newtonian potential and also fails for general Riesz potential K_α. The key ingredient to prove Theorem C is to perturb D using the continuous Steiner symmetrization in [8], which has been successfully applied in other contexts by Carrillo, Hittmeir, Volzone, and Yao [13] (nonlinear aggregation models) or by Morgan [72] (minimizers of the gravitational energy). This method is much more flexible and allows the treatment of more singular kernels than is possible using moving-plane methods. Due to the low regularity of the kernels, instead of computing the derivative of the energy under the perturbation, we work with finite differences instead.

Theorem D uses maximum principles and monotonicity formulas for nonlocal equations. The idea is to find the smallest disk $B(0, R)$ containing D (which intersects ∂D at some x_0), then use two different ways to compute $\nabla(1_{B(0,R)\setminus D} * K_\alpha)$ at x_0, and obtain a contradiction if $\Omega \geq \Omega_\alpha$ and D is not a disk. The proof works for the full range of $\alpha \in [0, 2)$, thus closing the problem raised by Hmidi [49] and by de la Hoz, Hassainia, Hmidi, and Mateu [29] among all simply connected patches.

1.5. Organization

Our work here is split into sections according to the cases $\alpha = 0$ (Euler) and $\alpha \neq 0$ (gSQG). Sections 2 and 3 are devoted to proving the symmetry results for $\alpha = 0$ in the patch setting (Section 2) and in the smooth setting (Section 3). Sections 4 and 5 deal with the gSQG equations with $0 < \alpha < 2$. Section 4 is concerned with the case $\Omega \leq 0$, whereas Section 5 handles the case $\Omega \geq \Omega_c$.

1.6. Notation

In Sections 2 and 3 of this paper, we use the following notation.

For a simple closed curve Γ, denote by $\mathrm{int}(\Gamma)$ its interior, which is the bounded connected component of $\mathbb{R}^2$ separated by the curve Γ. Note that the Jordan–Schoenflies theorem guarantees that $\mathrm{int}(\Gamma)$ is open and simply connected.

We say that two disjoint simple closed curves Γ_1 and Γ_2 are nested if $\Gamma_1 \subset \mathrm{int}(\Gamma_2)$ or vice versa. We say that two domains D_1, D_2 are nested if one is contained in a hole of the other one.

For a bounded domain $D \subset \mathbb{R}^2$, we denote by $\partial_{\mathrm{out}} D$ its outer boundary. And if D is doubly connected, then we denote by $\partial_{\mathrm{in}} D$ its inner boundary,

For a set D, we use $1_D(x)$ to denote its indicator function. And for a statement S, we let

$$\mathbb{1}_S = \begin{cases} 1 & \text{if } S \text{ is true,} \\ 0 & \text{if } S \text{ is false.} \end{cases}$$

(e.g., $\mathbb{1}_{\pi<3} = 0$).

For an open set $U \subset \mathbb{R}^2$, in the boundary integral $\int_{\partial U} \vec{n} \cdot \vec{f}\, d\sigma$, the vector $\vec{n}$ is taken as the outer normal of the open set U in that integral.

For a countable number of disjoint sets $U_i \subset \mathbb{R}^2$, we denote their union by $\dot{\bigcup}_i U_i$ to emphasize the disjointedness.

2. Radial symmetry of steady/rotating patches for the 2D Euler equation

Throughout this section, we work with the 2D Euler equation (1.1) in the patch setting. For a stationary or uniformly rotating patch D with angular velocity $\Omega \in \mathbb{R}$, let

$$f_\Omega(x) := (1_D * \mathcal{N})(x) - \frac{\Omega}{2}|x|^2.$$

Recall that in (1.6) we have shown that $f_\Omega \equiv C_i$ on each connected component of ∂D, where the constants can be different on different connected components.

Our goal in this section is to prove Theorem A, which completely answers Question 1 for 2D Euler patches. As we described in the Introduction, our proof has a variational flavor, which is done by perturbing D by a carefully chosen vector field, and computing the first variation of an associated energy functional in two different ways. In Section 2.1, we will define the energy functional and the perturbation vector field, and give a one-page proof in Theorem 2.2 that answers Question 1 among simply connected patches. (Note that even among simply connected patches, it is an open question whether every rotating patch with $\Omega > \frac{1}{2}$ or $\Omega < 0$ must be a disk.) In the following subsections, we further develop this method, and modify our perturbation vector field to cover nonsimply connected patches.

2.1. Warm-up: Radial symmetry of simply connected rotating patches

We begin by providing a sketch and some motivations of our approach, and then give a rigorous proof afterwards in Theorem 2.2. Suppose that D is a $C^{1,\gamma}$ simply connected rotating patch with angular velocity Ω that is *not* a disk. We perturb D in "time" (here the "time" t is just a name for our perturbation parameter, and is irrelevant with the actual time in the Euler equation) with a velocity field $\vec{v}(x) \in C^1(D) \cap C(\overline{D})$ that is divergence-free in D, which we will fix later. That is, consider the transport equation

$$\rho_t + \nabla \cdot (\rho \vec{v}) = 0$$

with $\rho(\cdot, 0) = 1_D$. We then investigate how the "energy functional"

$$\mathcal{E}[\rho] := -\int_{\mathbb{R}^2} \frac{1}{2}\rho(x)(\rho * \mathcal{N})(x) - \frac{\Omega}{2}|x|^2 \rho(x)\, dx$$

changes in time under the perturbation. Formally, we have

$$\begin{aligned} \frac{d}{dt}\mathcal{E}[\rho]|_{t=0} &= -\int_{\mathbb{R}^2} \rho_t(x,0)\Big(\big(\rho(\cdot,0) * \mathcal{N}\big)(x) - \frac{\Omega}{2}|x|^2\Big)\, dx \\ &= -\int_D \vec{v}(x) \cdot \nabla\Big((1_D * \mathcal{N})(x) - \frac{\Omega}{2}|x|^2\Big)\, dx. \end{aligned} \tag{2.1}$$

The above transport equation and the energy functional only serve as our motivation, and will not appear in the proof. In the actual proof, we only focus on the right-hand side of (2.1), which is an integral that is well defined by itself:

$$\mathcal{I} := -\int_D \vec{v}(x) \cdot \nabla\Big((1_D * \mathcal{N})(x) - \frac{\Omega}{2}|x|^2\Big)\, dx = -\int_D \vec{v} \cdot \nabla f_\Omega\, dx. \tag{2.2}$$

We will use two different ways to compute $\mathcal{I}$, and show that if D is not a disk, then the two ways lead to a contradiction for $\Omega \le 0$ or $\Omega \ge \frac{1}{2}$.

On the one hand, since f_Ω is a constant on ∂D (denote it by c), the divergence theorem yields the following for *every* $\vec{v} \in C^1(D) \cap C(\overline{D})$ that is divergence-free in D:

$$\begin{aligned} \mathcal{I} &= -c\int_{\partial D} \vec{n} \cdot \vec{v}\, d\sigma + \int_D (\nabla \cdot \vec{v}) f_\Omega\, dx \\ &= -c\int_D \nabla \cdot \vec{v}\, dx + \int_D (\nabla \cdot \vec{v}) f_\Omega\, dx = 0. \end{aligned} \tag{2.3}$$

On the other hand, we fix $\vec{v}$ as follows, which is at the heart of our proof. Let $\vec{v}(x) := -\nabla\varphi(x)$ in D, where

$$\varphi(x) := \frac{|x|^2}{2} + p(x) \quad \text{in } D, \tag{2.4}$$

with $p(x)$ being the solution to Poisson's equation

$$\begin{cases} \Delta p(x) = -2 & \text{in } D, \\ p(x) = 0 & \text{on } \partial D. \end{cases} \tag{2.5}$$

Note that φ is harmonic in D, thus $\vec{v}$ is indeed divergence-free in D. This definition of $\vec{v}$ is motivated by the fact that among all divergence-free vector fields in D, such $\vec{v}$ is the closest one to $-\vec{x}$ in the $L^2(D)$ distance. (In fact, such $\vec{v}$ is connected to the gradient flow of $\int_D \frac{|x|^2}{2}\,dx$ in the metric space endowed by 2-Wasserstein distance, under the constraint that $|D(t)|$ must remain constant; see [2], [70], [71].) Formally, one expects that D becomes "more symmetric" as we perturb it by $\vec{v}$, which inspires us to consider the first variation of $\mathcal{E}$ under such perturbation.

In the proof we will show that with such choice of $\vec{v}$, we can compute $\mathcal{I}$ in another way and obtain that $\mathcal{I} > 0$ for $\Omega \le 0$ and $\mathcal{I} < 0$ for $\Omega \ge \frac{1}{2}$. Therefore, in both cases, we obtain a contradiction with $\mathcal{I} = 0$ in (2.3).

Our proof makes use of a rearrangement inequality for solutions to elliptic equations, which is due to Talenti [83]. Below is the form that we will use; the original theorem works for a more general class of elliptic equations.

PROPOSITION 2.1 ([83, Theorem 1])
Let $D \subset \mathbb{R}^2$ be a bounded open set, and let p be defined as in (2.5). Let B be an open disk centered at the origin with $|B| = |D|$, and let p_B solve (2.5) in B. Then we have $p^* \le p_B$ pointwise in B, where p^* is the radial decreasing rearrangement of p. This leads to

$$\int_D p(x)\,dx \le \int_B p_B(x)\,dx.$$

Using that $p_B(x) = \frac{1}{2}(r_B^2 - |x|^2)$, where $\pi r_B^2 = |B| = |D|$ and $\int_B p_B(x) = \frac{\pi}{4} r_B^4$ it follows that

$$\int_D p(x)\,dx \le \frac{1}{4\pi}|D|^2,$$

where the equality is achieved if and only if D is an open disk (cf. [58, Theorem 3.1, Remark 3.1]).

Now we are ready to prove the following theorem, saying that any simply connected stationary/rotating patch with $\Omega \le 0$ or $\Omega \ge \frac{1}{2}$ must be a disk. Interestingly, the same proof can treat the two disjoint intervals $\Omega \le 0$ and $\Omega \ge \frac{1}{2}$ all at once.

THEOREM 2.2
Let D be a simply connected bounded domain with $C^{1,\gamma}$ boundary. If D is a rotating

patch solution with angular velocity Ω, where $\Omega \le 0$ or $\Omega \ge \frac{1}{2}$, then D must be a disk, and it must be centered at the origin unless $\Omega = 0$.

Proof
Let D be a rotating patch with $\Omega \in (-\infty, 0] \cup [\frac{1}{2}, \infty)$. As we described above, in this theorem we will use two different ways to compute the integral $\mathcal{I}$ defined in (2.2), where we fix $\vec{v}(x) := -\nabla\varphi(x)$, with φ and p defined as in (2.4) and (2.5).

On the one hand, we have that $\vec{v}$ is divergence-free in D, and elliptic regularity theory immediately yields that $\vec{v} \in C^1(D) \cap C(\overline{D})$. Using the assumption that D is a rotating patch, we know f_Ω is a constant on ∂D. (Note that ∂D is a connected closed curve since we assume that D is simply connected). Thus the computation in (2.3) directly gives that $\mathcal{I} = 0$.

On the other hand, we compute $\mathcal{I}$ as follows:

$$\begin{aligned} \mathcal{I} &= -\int_D \vec{v} \cdot \nabla f_\Omega \, dx = \int_D \nabla\varphi \cdot \nabla f_\Omega \, dx \\ &= \underbrace{\int_D x \cdot \nabla f_\Omega \, dx}_{=:\mathcal{I}_1} + \underbrace{\int_D \nabla p \cdot \nabla f_\Omega \, dx}_{=:\mathcal{I}_2} . \end{aligned} \tag{2.6}$$

For $\mathcal{I}_1$, we have

$$\begin{aligned} \mathcal{I}_1 &= \int_D x \cdot \nabla(1_D * \mathcal{N}) \, dx - \int_D x \cdot \Omega x \, dx \\ &= \frac{1}{2\pi} \int_D \int_D \frac{x \cdot (x-y)}{|x-y|^2} \, dy \, dx - \Omega \int_D |x|^2 \, dx \\ &= \frac{1}{4\pi} \int_D \int_D \frac{x \cdot (x-y) - y \cdot (x-y)}{|x-y|^2} \, dy \, dx - \Omega \int_D |x|^2 \, dx \\ &= \frac{1}{4\pi} |D|^2 - \Omega \int_D |x|^2 \, dx, \end{aligned} \tag{2.7}$$

where the third equality is obtained by exchanging x with y in the first integral, then taking the average with the original integral. To compute $\mathcal{I}_2$, using the divergence theorem (and the fact that $p = 0$ on ∂D), we have

$$\mathcal{I}_2 = -\int_D p \Delta f_\Omega \, dx = (2\Omega - 1) \int_D p \, dx. \tag{2.8}$$

Plugging (2.7) and (2.8) into (2.6) gives

$$\mathcal{I} = \frac{1}{4\pi} |D|^2 - \Omega \int_D |x|^2 \, dx + (2\Omega - 1) \int_D p \, dx. \tag{2.9}$$

When $\Omega = 0$, Proposition 2.1 directly gives that $\mathcal{I} > 0$ if D is not a disk, contradicting $\mathcal{I} = 0$.

When $\Omega \in (-\infty, 0) \cup [\frac{1}{2}, \infty)$, let B be a disk centered at the origin with the same area as D. Towards a contradiction, assume that $D \neq B$. Among all sets with the same area as D, the disk B is the unique one that minimizes the second moment. To see this, denoting by r_B the radius of B, we have

$$\begin{aligned}\int_D |x|^2\,dx - \int_B |x|^2\,dx &= \int_{D\setminus B} |x|^2\,dx - \int_{B\setminus D} |x|^2\,dx \\ &\geq \int_{D\setminus B} r_B^2\,dx - \int_{B\setminus D} r_B^2\,dx = 0,\end{aligned}$$

where the last equality follows from $|D \setminus B| = |B \setminus D|$, which is due to $|D| = |B|$; and the inequality is strict whenever $D \neq B$. Thus, if $D \neq B$, then we have

$$\int_D |x|^2\,dx > \int_B |x|^2\,dx = \frac{1}{2\pi}|D|^2,$$

where the last step follows from an elementary computation. Plugging this into (2.9) gives the following inequality for $\Omega \in [\frac{1}{2}, \infty)$:

$$\mathcal{I} < \frac{1}{4\pi}|D|^2 - \frac{\Omega}{2\pi}|D|^2 + (2\Omega - 1)\int_D p\,dx = (1 - 2\Omega)\Big(\frac{1}{4\pi}|D|^2 - \int_D p\,dx\Big) \leq 0.$$

On the other hand, for $\Omega \in (-\infty, 0)$, we have

$$\mathcal{I} > \frac{1}{4\pi}|D|^2 - \frac{\Omega}{2\pi}|D|^2 + (2\Omega - 1)\int_D p\,dx = (1 - 2\Omega)\Big(\frac{1}{4\pi}|D|^2 - \int_D p\,dx\Big) > 0,$$

and we obtain a contradiction to $\mathcal{I} = 0$ in all the cases, thus the proof is finished. □

Remark 2.3

In fact, one can easily check that the proof of Theorem 2.2 applies to a bounded disconnected patch $D = \dot{\bigcup}_{i=1}^N D_i$ with $C^{1,\gamma}$ boundary, as long as each connected component D_i is simply connected, which yields that such a patch cannot be a rotating solution. Here the proof remains the same, except for one small change. On the one hand, since now we have $f_\Omega = c_i$ on ∂D_i, (2.3) should be replaced by

$$\mathcal{I} = -\sum_{i=1}^N \Big(c_i \int_{\partial D_i} \vec{n} \cdot \vec{v}\,d\sigma + \int_{D_i} (\nabla \cdot \vec{v}) f_\Omega\,dx\Big) = 0.$$

On the other hand, the same argument as in the proof of Theorem 2.2 shows that $\mathcal{I} \neq 0$ whenever D contains more than one connected component.

Remark 2.4
When the domain D is Lipschitz, we no longer have $\vec{v} = -x - \nabla p \in C^1(\overline{D})$. Nevertheless, the divergence theorem in the proof of Theorem 2.2 can still be justified, thanks to the fact that $\nabla p \in L^2(\partial D)$ for a Lipschitz domain D, where $\nabla p|_{\partial D}$ is taken in the nontangential limit sense, and this follows from [55, Theorem 5.6].

Even in the regime $\Omega \in (0, \frac{1}{2})$, where nonradial rotating patches are known to exist (recall that there exist patches bifurcating from a disk at $\Omega_m = \frac{m-1}{2m}$ for all $m \geq 2$), our approach still allows us to obtain the following quantitative estimate, saying that if a simply connected patch D rotates with angular velocity $\Omega \in (0, \frac{1}{2})$ that is very close to $\frac{1}{2}$, then D must be very close to a disk, in the sense that their symmetric difference must be small.

COROLLARY 2.5
Let D be a simply connected bounded domain with $C^{1,\gamma}$ boundary. Assume that D is a rotating patch solution with angular velocity Ω, where $\Omega \in (\frac{1}{4}, \frac{1}{2})$. Let $\delta := \frac{1}{2} - \Omega$. Then we have

$$|D \triangle B| \leq 2\sqrt{2\delta}|D|,$$

where B is the disk centered at the origin with the same area as D.

Proof
In the proof of Theorem 2.2, combining the equation $\mathcal{I} = 0$ and (2.9) together, we have that

$$\frac{1}{4\pi}|D|^2 - \Omega \int_D |x|^2 \, dx - (1 - 2\Omega) \int_D p \, dx = 0.$$

Dividing both sides by Ω and rearranging the terms, we obtain

$$\int_D |x|^2 \, dx - \frac{1}{2\pi}|D|^2 = \frac{1 - 2\Omega}{\Omega}\Big(\frac{1}{4\pi}|D|^2 - \int_D p \, dx\Big) \leq \frac{2\delta|D|^2}{\pi},$$

where in the inequality we used that $2\delta := 1 - 2\Omega$, $\Omega > \frac{1}{4}$, and $\int_D p \, dx \geq 0$.

Since $\int_B |x|^2 \, dx = \frac{1}{2\pi}|D|^2$, the above inequality implies that

$$\int_{D \setminus B} |x|^2 \, dx - \int_{B \setminus D} |x|^2 \, dx \leq \frac{2\delta|D|^2}{\pi}. \tag{2.10}$$

Since D and B have the same area, let us denote $\beta := |D \setminus B| = |B \setminus D|$. Among all sets $U \subset B^c$ with area β, $\int_U |x|^2 \, dx$ is minimized when U is an annulus with area β and inner circle coinciding with ∂B. To see this, let U_0 be such an annulus, and denote by r_{in} and r_{out} its inner and outer radius (note that r_{in} is also the radius of B). Then we have

$$\begin{aligned}\int_U |x|^2\,dx - \int_{U_0} |x|^2\,dx &= \int_{U\setminus U_0} |x|^2\,dx - \int_{U_0\setminus U} |x|^2\,dx \\ &\geq \int_{U\setminus U_0} r_{\text{out}}^2\,dx - \int_{U_0\setminus U} r_{\text{out}}^2\,dx = 0,\end{aligned}$$

where the inequality follows from $U \setminus U_0 \subset B^c \setminus U_0 \subset B(0, r_{\text{out}})^c$ (recall that $U \subset B^c$), and the last equality follows from $|U \setminus U_0| = |U_0 \setminus U|$, which is due to $|U| = |U_0|$.

Thus an elementary computation gives

$$\int_{D\setminus B} |x|^2\,dx \geq \inf_{U\subset B^c, |U|=\beta} \int_U |x|^2\,dx = \frac{\beta(2|B|+\beta)}{2\pi}.$$

Likewise, among all sets $V \subset B$ with area β, $\int_V |x|^2\,dx$ is maximized when V is an annulus with area β and outer circle coinciding with ∂B, thus

$$\int_{B\setminus D} |x|^2\,dx \leq \sup_{V\subset B, |V|=\beta} \int_V |x|^2\,dx = \frac{\beta(2|B|-\beta)}{2\pi}.$$

Subtracting these two inequalities yields

$$\int_{D\setminus B} |x|^2\,dx - \int_{B\setminus D} |x|^2\,dx \geq \frac{\beta^2}{\pi},$$

and combining this with (2.10) immediately gives

$$\beta^2 \leq 2\delta |D|^2,$$

thus $|D \triangle B| = 2\beta \leq 2\sqrt{2\delta}|D|$. □

2.2. Radial symmetry of nonsimply connected stationary patches

In this subsection, we aim to prove the radial symmetry of a connected rotating patch D with $\Omega \leq 0$, where D is allowed to be nonsimply connected. Let $D \subset \mathbb{R}^2$ be a bounded domain with $C^{1,\gamma}$ boundary. Assume that D has n holes with $n \geq 0$, and then let $h_1, \ldots, h_n \subset \mathbb{R}^2$ denote the n holes of D (each h_i is a bounded open set). Note that ∂D has $n+1$ connected components: they include the outer boundary of D, which we denote by ∂D_0, and the inner boundaries ∂h_i for $i = 1, \ldots, n$. We orient ∂h_i the opposite way as ∂D_0.

To begin with, we point out that even for the steady patch case $\Omega = 0$, the proof of Theorem 2.2 cannot be directly adapted to the nonsimply connected patch. If we define $\vec{v}$ in the same way, then the second way to compute $\mathcal{I}$ still goes through (since Proposition 2.1 still holds for nonsimply connected D), and leads to $\mathcal{I} > 0$ if D is not a disk. But the first way to compute $\mathcal{I}$ no longer gives $\mathcal{I} = 0$: if D is stationary and not simply connected, then $f(x) := (1_D * \mathcal{N})(x)$ may take different constant values on different connected components of ∂D, thus the identity (2.3) no longer holds.

In order to fix this issue, we still define $\vec{v} = -\nabla\varphi = -\nabla(\frac{|x|^2}{2} + p)$, but modify the definition of p in the following lemma. Compared to the previous definition (2.5), the difference is that p now takes different values $0, c_1, \ldots, c_n$ on each connected component of ∂D. The lemma shows that there exist values of $\{c_i\}_{i=1}^n$ such that $\int_{\partial h_i} \nabla p \cdot \vec{n}\, d\sigma = -2|h_i|$ along the boundary of each hole. As we will see later, this leads to $\int_{\partial h_i} \vec{v} \cdot \vec{n}\, d\sigma = 0$ for $i = 1, \ldots, n$, which ensures that $\mathcal{I} = 0$. (Of course, with p defined in the new way, the second way of computing $\mathcal{I}$ no longer follows from Proposition 2.1, and we will take care of this later in Proposition 2.8.)

LEMMA 2.6

Let D, h_i, and ∂D_0 be given as in the first paragraph of Section 2.2. Then there exist positive constants $\{c_i\}_{i=1}^n$ such that the solution $p : \overline{D} \to \mathbb{R}$ to the Poisson equation

$$\begin{cases} \Delta p = -2 & \text{in } D, \\ p = c_i & \text{on } \partial h_i \text{ for } i = 1, \ldots, n, \\ p = 0 & \text{on } \partial D_0, \end{cases} \tag{2.11}$$

satisfies

$$\int_{\partial h_i} \nabla p \cdot \vec{n}\, d\sigma = -2|h_i| \quad \text{for } i = 1, \ldots, n. \tag{2.12}$$

Here $|h_i|$ is the area of the domain $h_i \subset \mathbb{R}^2$.

Proof

Let u satisfy that

$$\begin{cases} \Delta u = -2 & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases}$$

Furthermore, let the function v_j for $j = 1, \ldots, n$ be the solution to

$$\begin{cases} \Delta v_j = 0 & \text{in } D, \\ v_j = 0 & \text{on } \partial D \setminus \partial h_j, \\ v_j = 1 & \text{on } \partial h_j, \end{cases}$$

where $v_j \in C^2(D) \cap C^1(\overline{D})$ by elliptic regularity. Now we consider the following linear equation,

$$Ax = b, \tag{2.13}$$

where $A_{i,j} = \int_{\partial h_i} \nabla v_j \cdot \vec{n}\, d\sigma$ and $b_i = -2|h_i| - \int_{\partial h_i} \nabla u \cdot \vec{n}\, d\sigma$. We argue that (2.13) has a unique solution. Thanks to the divergence theorem, we have

$$0 = \int_D \Delta v_j \, dx = \int_{\partial D_0} \nabla v_j \cdot \vec{n}\, d\sigma - \sum_{i=1}^{n} \int_{\partial h_i} \nabla v_j \cdot \vec{n}\, d\sigma.$$

Therefore,

$$\sum_{i=1}^{n} A_{i,j} = \int_{\partial D_0} \nabla v_j \cdot \vec{n}\, d\sigma < 0, \tag{2.14}$$

where the last inequality follows from the Hopf lemma (see [57]) since v_j attains its minimum value 0 on ∂D_0, and $v_j \not\equiv 0$ on ∂D. A similar argument gives that $A_{i,j} > 0$ for $i \neq j$ and $A_{j,j} < 0$. Thus A is invertible by the Gershgorin circle theorem (see [39]), leading to a unique solution of (2.13). Let us denote the solution by $x = (c_1, \ldots, c_n)^t$. Then the function p defined by

$$p := u + \sum_{i=1}^{n} c_i v_i$$

satisfies the desired properties (2.12).

Now we prove that $c_i > 0$ for $i \geq 1$. Suppose that $c_{i*} := \min_i c_i \leq 0$. Then by the minimum principle, p attains its minimum on ∂h_{i*}. Therefore,

$$0 \leq \int_{\partial h_{i*}} \nabla p \cdot \vec{n}\, d\sigma = -2|h_{i*}| < 0,$$

which is a contradiction. □

Remark 2.7

In order to push the regularity of the domain down to Lipschitz, to prove (2.14), instead of the Hopf Lemma, we may use the following observation in two dimensions (see [1, pp. 935–936]. Let $D \subset \mathbb{R}^2$ be a Lipschitz domain, and let $\Sigma \subset \partial D$ be relatively open with respect to ∂D. Let v_j be a harmonic function in D and continuous in $\overline{D}$. If v_j vanishes on Σ and its normal derivative vanishes in a subset of Σ with positive surface measure, then $v_j \equiv 0$ in D. In addition, the use of the divergence theorem is justified by the same argument as in Remark 2.4.

Next we prove a parallel version of Talenti's theorem for the function p constructed in Lemma 2.6. We will use this result throughout Sections 2 and 3.

PROPOSITION 2.8
Let $D \subset \mathbb{R}^2$ be a bounded domain with $C^{1,\gamma}$ boundary. Assume that D has n holes with $n \geq 0$, and denote by $h_1, \dots, h_n \subset \mathbb{R}^2$ the holes of D (each h_i is a bounded open set). Let $p : \overline{D} \to \mathbb{R}$ be the function constructed in Lemma 2.6. Then the following two estimates hold:

$$\sup_{\overline{D}} p \leq \frac{|D|}{2\pi} \tag{2.15}$$

and

$$\int_D p(x)\,dx \leq \frac{|D|^2}{4\pi}. \tag{2.16}$$

Furthermore, for each of the two inequalities above, the equality is achieved if and only if D is either a disk or an annulus.

Proof
The proof is divided into two parts. In step 1 we prove the two inequalities (2.15) and (2.16), and in step 2 we show that equality can be achieved if and only if D is a disk or an annulus.

Step 1. When D is simply connected, (2.15) and (2.16) directly follow from Talenti's theorem Proposition 2.1. Next we consider a nonsimply connected domain D, and prove that these inequalities also hold when $p : \overline{D} \to \mathbb{R}$ is defined as in Lemma 2.6.

For $k \in \mathbb{R}^+$, let us denote $D_k := \{x \in D : \; p(x) > k\}$, $g(k) := |D_k|$ and $\tilde{D}_k := D_k \dot{\cup} (\dot{\bigcup}_{\{i:c_i>k\}} \overline{h_i})$. Elliptic regularity theory gives that $p \in C^\infty(D)$, thus by Sard's theorem, k is a regular value for almost every $k \in (0, \sup_D p)$, that is, $|\nabla p(x)| > 0$ on $\{x \in D : \; p(x) = k\}$. Thus $\{x \in D : \; p(x) = k\}$ is a union of smooth simple closed curves and equal to $\partial \tilde{D}_k$ for almost every $k \in (0, \sup_D p)$.

Since $\partial D_k = \partial \tilde{D}_k \dot{\cup} (\dot{\bigcup}_{\{i:c_i>k\}} \partial h_i)$ for $k \notin \{c_1, \dots, c_n\}$, we compute

$$\begin{aligned}
g(k) &= -\frac{1}{2}\int_{D_k} \Delta p(x)\,dx = -\frac{1}{2}\int_{\partial D_k} \nabla p \cdot \vec{n}\,d\sigma \\
&= -\frac{1}{2}\int_{\partial \tilde{D}_k} \nabla p \cdot \vec{n}\,d\sigma + \frac{1}{2}\sum_{\{i:c_i>k\}} \int_{\partial h_i} \nabla p \cdot \vec{n}\,d\sigma \\
&= -\frac{1}{2}\int_{\partial \tilde{D}_k} \nabla p \cdot \vec{n}\,d\sigma - \sum_{\{i:c_i>k\}} |h_i|,
\end{aligned}$$

where the last identity is due to (2.12). Therefore, it follows that

$$g(k) + \sum_{\{i:c_i>k\}} |h_i| = -\frac{1}{2}\int_{\partial \tilde{D}_k} \nabla p \cdot \vec{n}\, d\sigma = \frac{1}{2}\int_{\partial \tilde{D}_k} |\nabla p|\, d\sigma, \tag{2.17}$$

where the last equality follows from the fact that ∇p is perpendicular to the tangent vector on the level set.

On the other hand, the coarea formula yields that

$$g(k) = \int_{\mathbb{R}} \int_{\partial \tilde{D}_s} 1_{D_k} \frac{1}{|\nabla p|}\, d\sigma ds = \int_k^\infty \int_{\partial \tilde{D}_s} \frac{1}{|\nabla p|}\, d\sigma\, ds.$$

Therefore, it follows that for almost every $k \in (0, \sup_D p)$,

$$g'(k) = -\int_{\partial \tilde{D}_k} \frac{1}{|\nabla p|}\, d\sigma. \tag{2.18}$$

Thus it follows from (2.17) and (2.18) that

$$\begin{aligned} g'(k)\Big(g(k) + \sum_{\{i:c_i>k\}} |h_i|\Big) &= -\frac{1}{2}\Big(\int_{\partial \tilde{D}_k} |\nabla p|\, d\sigma\Big)\Big(\int_{\partial \tilde{D}_k} \frac{1}{|\nabla p|}\, d\sigma\Big) \\ &\le -\frac{1}{2} P(\tilde{D}_k)^2, \end{aligned} \tag{2.19}$$

where $P(E)$ denotes the perimeter of a rectifiable curve ∂E. Note that the last inequality becomes equality if and only if $|\nabla p|$ is a constant on $\partial \tilde{D}_k$. Also, the isoperimetric inequality gives that

$$P(\tilde{D}_k)^2 \ge 4\pi |\tilde{D}_k|, \tag{2.20}$$

where equality holds if and only if $\tilde{D}_k$ is a disk. This yields that

$$g'(k)\Big(g(k) + \sum_{\{i:c_i>k\}} |h_i|\Big) \le -2\pi |\tilde{D}_k| = -2\pi\Big(g(k) + \sum_{\{i:c_i>k\}} |h_i|\Big). \tag{2.21}$$

Therefore, $g'(k) \le -2\pi$ for almost every $k \in (0, \sup_D p)$. Combining it with the fact that $g(0) = |D|$, we have

$$g(k) \le \big(g(0) - 2\pi k\big)_+ = \big(|D| - 2\pi k\big)_+ \quad \text{for almost every } k \ge 0.$$

This proves that $\sup_{\overline{D}} p \le \frac{|D|}{2\pi}$. It follows that

$$\begin{aligned} \int_D p(x)\, dx &= \int_D \int_0^{\frac{|D|}{2\pi}} 1_{\{k<p(x)\}}\, dk\, dx = \int_0^{\frac{|D|}{2\pi}} g(k)\, dk \\ &\le \int_0^{\frac{|D|}{2\pi}} \big(|D| - 2\pi k\big)_+\, dx = \frac{|D|^2}{4\pi}. \end{aligned}$$

Step 2. Now we show that for the two inequalities (2.15) and (2.16), the equality is achieved if and only if D is either a disk or an annulus. First, if D is either a disk or an annulus centered at some $x_0 \in \mathbb{R}^2$, then uniqueness of solution to Poisson's equation gives that p is radially symmetric about x_0. Since we have $\Delta p = -2$ in D and $p = 0$ on the outer boundary of D, this gives an explicit formula $p(x) = -\frac{|x-x_0|^2}{2} + \frac{R^2}{2}$ for $x \in D$, where R is the outer radius of D. For either a disk or an annulus, one can explicitly compute $\sup_D p$ and $\int_D p\,dx$ to check that equalities in (2.15) and (2.16) are achieved.

To prove the converse, assume that either (2.15) or (2.16) achieves equality, and we aim to show that D is either a disk or an annulus. In order for either equality to be achieved, (2.21) needs to achieve equality at almost every $k \in (0, \sup_D p)$. In addition, $g(k)$ needs to be continuous in k since $g(k)$ is decreasing. Since (2.21) follows from a combination of the Cauchy-Schwarz inequality in (2.19) and the isoperimetric inequality in (2.20), we need to have all the three conditions below in order for either (2.15) or (2.16) to achieve equality:

(1) $|\nabla p|$ is a constant on each level set $\partial \tilde{D}_k$ for almost every $k \in (0, \sup_D p)$;

(2) $\tilde{D}_k$ is a disk for almost every $k \in (0, \sup_D p)$;

(3) $g(k) = |D_k|$ is continuous in k; as a result, $|\tilde{D}_k|$ is continuous in k at all $k \neq c_i$, with $c_i > 0$ defined as in (2.11).

Next we will show that if these three conditions are satisfied, then D must be an annulus or disk. First, note that by sending $k \searrow 0$ in condition (2), and combining it with the continuity of $|\tilde{D}_k|$ as $k \searrow 0$, it already gives that the outer boundary of D must be a circle. Therefore, if D is simply connected, then it must be a disk.

If D is nonsimply connected, using condition (2) and (3), we claim that D can have only one hole, which must be a disk, and p must achieve its maximum value in $\overline{D}$ on the boundary of the hole. To see this, let h_i be any hole of D, and recall that $p|_{\partial h_i} = c_i$. As we consider the set limit of $\tilde{D}_k$ as k approaches c_i from below and above, by definition of $\tilde{D}_k$ we have

$$\lim_{k \nearrow c_i} \tilde{D}_k = \lim_{k \searrow c_i} \tilde{D}_k \,\dot{\cup} \Big(\dot{\bigcup_{\{j : c_j = c_i\}}} \overline{h_j} \Big).$$

By (2) and (3), the left-hand side $\lim_{k \nearrow c_i} \tilde{D}_k$ is a disk, and the set $\lim_{k \searrow c_i} \tilde{D}_k$ on the right-hand side is also a disk (if the limit is nonempty). But after taking union with the holes $\{h_j : c_j = c_i\}$ (each is a simply connected set), the right-hand side will be a disk if and only if $\lim_{k \searrow c_i} \tilde{D}_k$ is empty, $\dot{\bigcup}_{\{j : c_j = c_i\}} \overline{h_j} = \overline{h_i}$, and h_i is a disk. This implies that $c_i = \sup_D p$ and $c_j < c_i$ for all $j \neq i$. But since h_i is chosen to be any hole of D, we know that D can have only one hole (call it h), which is a disk, and $\sup_D p = p|_{\partial h}$. Finally, note that condition (1) gives that all the disks $\{\tilde{D}_k\}$ are concentric, and as a result we have that D is an annulus, finishing the proof. $\square$

Finally, we are ready to show that every connected stationary patch D with $C^{1,\gamma}$ boundary must be either a disk or an annulus.

THEOREM 2.9
Let $D \subset \mathbb{R}^2$ be a bounded domain with $C^{1,\gamma}$ boundary. Suppose that $\omega(x) := 1_D(x)$ is a stationary patch solution to the 2D Euler equation in the sense of (1.5). Then D is either a disk or an annulus.

Proof
If D has n holes (where $n \geq 0$), denote them by $h_1, \dots, h_n$. By (1.5), the function $f := 1_D * \mathcal{N}$ is constant on each connected component of ∂D, and let us denote

$$f(x) = \begin{cases} a_i & \text{on } \partial h_i, \\ a_0 & \text{on } \partial D_0. \end{cases} \tag{2.22}$$

Let $p : \overline{D} \to \mathbb{R}$ be defined as in Lemma 2.6, and let $\varphi := \frac{|x|^2}{2} + p$. Similar to the proof of Theorem 2.2, we calculate $\mathcal{I} := \int_D \nabla\varphi \cdot \nabla f \, dx$ in two different ways. Note that $\nabla f = \nabla(f - a_0)$ in D. Applying the divergence theorem to $\mathcal{I}$ and using (2.22) and $\Delta\varphi = 0$ in D, it follows that

$$\begin{aligned} \mathcal{I} &= \int_{\partial D} (\nabla\varphi \cdot \vec{n})(f - a_0)\, d\sigma - \int_D \Delta\varphi (f - a_0)\, dx \\ &= \sum_{i=1}^n (a_i - a_0) \int_{\partial h_i} \nabla\varphi \cdot \vec{n}\, d\sigma. \end{aligned} \tag{2.23}$$

By definition of φ, and combining it with the property of p in (2.12), we have

$$\begin{aligned} \int_{\partial h_i} \nabla\varphi \cdot \vec{n}\, d\sigma &= \int_{\partial h_i} \nabla\Big(\frac{|x|^2}{2}\Big) \cdot \vec{n}\, d\sigma + \int_{\partial h_i} \nabla p \cdot \vec{n}\, d\sigma \\ &= \int_{h_i} 2\, dx + \int_{\partial h_i} \nabla p \cdot \vec{n}\, d\sigma = 0. \end{aligned} \tag{2.24}$$

Plugging this into (2.23) gives $\mathcal{I} = 0$. On the other hand, we also have

$$\mathcal{I} = \int_D x \cdot \nabla f\, dx + \int_D \nabla p \cdot \nabla f\, dx =: E_1 + E_2.$$

We compute

$$E_1 = \int_D x \cdot (1_D * \nabla\mathcal{N})\, dx = \int_D \int_D \frac{1}{2\pi} \frac{x \cdot (x - y)}{|x - y|^2}\, dy\, dx = \frac{|D|^2}{4\pi}, \tag{2.25}$$

where the last equality is obtained by exchanging x with y and taking the average with the original integral. For E_2, the divergence theorem yields that

$$E_2 = \int_{\partial D} p \nabla f \cdot \vec{n}\, d\sigma - \int_D p \Delta f\, dx = \int_{\partial D} p \nabla f \cdot \vec{n}\, d\sigma - \int_D p\, dx.$$

Using the property of p in (2.11) and the fact that $\Delta f = 0$ in h_i, the divergence theorem yields that

$$\int_{\partial D} p \nabla f \cdot \vec{n}\, d\sigma = -\sum_{i=1}^{n} \int_{\partial h_i} p \nabla f \cdot \vec{n}\, d\sigma = -\sum_{i=1}^{n} c_i \int_{h_i} \Delta f\, dx = 0. \tag{2.26}$$

As a result, we have $E_2 = -\int_D p\, dx$. If D is neither a disk nor an annulus, then Proposition 2.8 gives

$$\mathcal{I} = E_1 + E_2 = \frac{|D|^2}{4\pi} - \int_D p\, dx > 0,$$

contradicting $\mathcal{I} = 0$. $\square$

In the next corollary, we generalize the above result to a nonnegative stationary patch with multiple (disjoint) patches.

COROLLARY 2.10
Let $\omega(x) := \sum_{i=1}^{n} \alpha_i 1_{D_i}$, where $\alpha_i > 0$, each D_i is a bounded domain with $C^{1,\gamma}$ boundary, and $D_i \cap D_j = \emptyset$ if $i \neq j$. Assume that ω is a stationary patch solution, that is, the function $f(x) := \omega * \mathcal{N}$ satisfies $\nabla^{\perp} f \cdot \vec{n} = 0$ on ∂D_i for all $i = 1, \dots, n$. Then ω is radially symmetric up to a translation.

Proof
Following similar notation as the beginning of Section 2.2, we denote the outer boundary of D_i by ∂D_{i0}, and the holes of each D_i (if any) by h_{ik} for $k = 1, \dots, N_i$. Let $p_i : \overline{D_i} \to \mathbb{R}$ be defined as in Lemma 2.6, that is, p_i satisfies

$$\begin{cases} \Delta p_i = -2 & \text{in } D_i, \\ p_i = c_{ik} & \text{on } \partial h_{ik}, \\ p_i = 0 & \text{on } \partial D_{i0}, \end{cases}$$

where c_{ik} is chosen such that $\int_{\partial h_{ik}} \nabla p_i \cdot \vec{n}\, d\sigma = -2|h_{ik}|$. We then define $\varphi : \bigcup_{i=1}^{n} \overline{D_i} \to \mathbb{R}$, such that in each $\overline{D_i}$ we have $\varphi = \varphi_i := \frac{|x|^2}{2} + p_i$.

Similar to Theorem 2.9, we compute

$$\mathcal{I} := \int_{\mathbb{R}^2} \omega \nabla\varphi \cdot \nabla f \, dx = \sum_{i=1}^{n} \int_{D_i} \alpha_i \nabla\varphi_i \cdot \nabla f \, dx$$

in two different ways. On the one hand, since $f = \omega * \mathcal{N}$ is a constant on each connected component of ∂D_i, the same computation of Theorem 2.9 yields that $\int_{D_i} \nabla\varphi_i \cdot \nabla f \, dx = 0$, therefore $\mathcal{I} = 0$. On the other hand, since $\nabla\varphi = x + \nabla p_i$ in each D_i, we break $\mathcal{I}$ into

$$\begin{aligned}\mathcal{I} &= \sum_{i,j=1}^{n} \alpha_i\alpha_j \int_{D_i} x \cdot \nabla(1_{D_j} * \mathcal{N}) \, dx + \sum_{i,j=1}^{n} \alpha_i\alpha_j \int_{D_i} \nabla p_i \cdot \nabla(1_{D_j} * \mathcal{N}) \, dx \\ &=: \mathcal{I}_1 + \mathcal{I}_2.\end{aligned}$$

For $\mathcal{I}_1$, we compute

$$\begin{aligned}\mathcal{I}_1 &= \sum_{i,j=1}^{n} \frac{\alpha_i\alpha_j}{2} \Big(\int_{D_i} x \cdot \nabla(1_{D_j} * \mathcal{N}) \, dx + \int_{D_j} x \cdot \nabla(1_{D_i} * \mathcal{N}) \, dx \Big) \\ &= \sum_{i,j=1}^{n} \frac{\alpha_i\alpha_j}{2} \Big(\int_{D_i} \int_{D_j} \frac{x \cdot (x-y)}{2\pi|x-y|^2} \, dy \, dx + \int_{D_j} \int_{D_i} \frac{x \cdot (x-y)}{2\pi|x-y|^2} \, dy \, dx \Big) \\ &= \sum_{i,j=1}^{n} \frac{\alpha_i\alpha_j}{4\pi} |D_i||D_j|, \end{aligned} \tag{2.27}$$

where we exchanged i with j to get the first equality. For $\mathcal{I}_2$, we have

$$\begin{aligned}\mathcal{I}_2 &= \sum_{i=1}^{n} \alpha_i^2 \int_{D_i} \nabla p_i \cdot \nabla(1_{D_i} * \mathcal{N}) \, dx + \sum_{i \neq j} \alpha_i\alpha_j \int_{D_i} \nabla p_i \cdot \nabla(1_{D_j} * \mathcal{N}) \, dx \\ &=: I_{21} + I_{22}.\end{aligned}$$

By the same computation for E_2 in the proof of Theorem 2.9, we have

$$I_{21} = -\sum_{i=1}^{n} \alpha_i^2 \int_{D_i} p_i \, dx. \tag{2.28}$$

For $i \neq j$, we denote $j \prec i$ if D_j is contained in a hole of D_i. (And if D_j is not contained in any hole of D_i, we say $j \not\prec i$.) Using this notation, the divergence theorem directly yields that

$$\begin{aligned}\int_{\partial D_i} p_i \nabla(1_{D_j} * \mathcal{N}) \cdot \vec{n} \, d\sigma &= -\sum_{k=1}^{N_i} \int_{\partial h_{ik}} p_i \nabla(1_{D_j} * \mathcal{N}) \cdot \vec{n} \, d\sigma \\ &= 0 \quad \text{if } j \not\prec i. \end{aligned} \tag{2.29}$$

And if $j \prec i$, then the divergence theorem and (2.15) in Proposition 2.8 yield

$$\int_{\partial D_i} p_i \nabla(1_{D_j} * \mathcal{N}) \cdot \vec{n}\, d\sigma \geq -\sup_{\partial D_i} p_i |D_j| \geq -\frac{1}{2\pi}|D_i||D_j| \quad \text{if } j \prec i. \tag{2.30}$$

Hence it follows that

$$I_{22} \geq -\sum_{i \neq j} \mathbb{1}_{j \prec i} \frac{\alpha_i \alpha_j}{2\pi}|D_i||D_j| = -\sum_{i \neq j}(\mathbb{1}_{j \prec i} + \mathbb{1}_{i \prec j})\frac{\alpha_i \alpha_j}{4\pi}|D_i||D_j|, \tag{2.31}$$

where the last step is obtained by exchanging i, j and taking the average with the original sum. Note that we have $\mathbb{1}_{j \prec i} + \mathbb{1}_{i \prec j} \leq 1$ for any $i \neq j$. From (2.27), (2.28), and (2.31), we obtain

$$\mathcal{I} \geq \sum_{i=1}^{n} \alpha_i^2 \Big(\frac{|D_i|^2}{4\pi} - \int_{D_i} p_i\, dx\Big) + \sum_{\substack{j \not\prec i \text{ and } i \not\prec j \\ i \neq j}} \alpha_i \alpha_j \frac{|D_i||D_j|}{4\pi}. \tag{2.32}$$

Since we already know that $\mathcal{I} = 0$ and all the summands in (2.32) are nonnegative, it follows that

$$\frac{|D_i|^2}{4\pi} = \int_{D_i} p_i\, dx \quad \text{for all } i = 1, \dots, n \text{ and } \big\{(i, j) : i \neq j, i \not\prec j \text{ and } j \not\prec i\big\} = \emptyset.$$

Therefore, every D_i is either a disk or an annulus by Proposition 2.8 and they are nested. By relabeling the indices, we can assume that $i \prec i+1$ for $i = 1, \dots, n-1$.

Next we prove that all D_i's are concentric by induction. For $k \geq 1$, suppose that $D_1, \dots, D_k$ are known to be concentric about some $o \in \mathbb{R}^2$. To show that D_{k+1} is also centered at o, we break f into

$$f = \sum_{i=1}^{k} (\alpha_i 1_{D_i}) * \mathcal{N} + \sum_{i=k+1}^{n} (\alpha_i 1_{D_i}) * \mathcal{N}.$$

In the first sum, each D_i is centered at o for $i \leq k$; thus Lemma 2.11(a) (which we prove right after this theorem) yields that $\sum_{i=1}^{k} (\alpha_i 1_{D_i}) * \mathcal{N} = \frac{C}{2\pi} \ln|x - o|$ on $\partial_{\text{in}} D_{k+1}$, where $C = \sum_{i=1}^{k} \alpha_i |D_i| > 0$. In the second sum, for each $i \geq k+1$, since each D_i is an annulus with $\partial_{\text{in}} D_{k+1}$ in its hole, Lemma 2.11(b) gives that $1_{D_i} * \mathcal{N} \equiv$ const on $\partial_{\text{in}} D_{k+1}$ for all $i \geq k+1$. Thus, overall, we have $f = \frac{C}{2\pi} \ln|x - o| + C_2$ on $\partial_{\text{in}} D_{k+1}$ for $C > 0$. Combining it with the assumption that f is a constant on $\partial_{\text{in}} D_{k+1}$, we know that D_{k+1} must also be centered at o, finishing the induction step. $\square$

Now we state and prove the lemma used in the proof of Corollary 2.10, which follows from standard properties of the Newtonian potential.

LEMMA 2.11
Assume that $g \in L^\infty(\mathbb{R}^2)$ is radially symmetric about some $o \in \mathbb{R}^2$, and is compactly supported in $B(o,R)$. Then $\eta := g * \mathcal{N}$ satisfies the following:
(a) $\eta(x) = \frac{\int_{\mathbb{R}^2} g\,dx}{2\pi} \ln|x-o|$ *for all $x \in B(0,R)^c$;*
(b) *if in addition we have $g \equiv 0$ in $B(o,r)$ for some $r \in (0,R)$, then $\eta = \text{const}$ in $B(o,r)$.*

Proof
To show (a), we take any $x \in B(o,R)^c$ and consider the circle $\Gamma \ni x$ centered at o. By radial symmetry of η about o and the divergence theorem, we have

$$\nabla\eta \cdot \frac{x}{|x|} = \frac{1}{|\Gamma|}\int_\Gamma \nabla\eta\cdot\vec{n}\,d\sigma = \frac{1}{|\Gamma|}\int_{\mathrm{int}(\Gamma)} \Delta\eta\,dx = \frac{\int_{\mathbb{R}^2} g(x)\,dx}{2\pi|x-o|},$$

which implies $\eta(x) = \frac{\int g\,dx}{2\pi}\ln|x-o| + C$. To show that $C=0$, for $|x|$ sufficiently large we have

$$\begin{aligned}|C| &= \Big|\int_{B(o,R)} g(x)\big(\mathcal{N}(x-y)-\mathcal{N}(x-o)\big)\,dy\Big|\\ &\le \|g\|_{L^\infty(\mathbb{R}^2)} \sup_{y\in B(o,R)} \big|\mathcal{N}(x-o)-\mathcal{N}(x-y)\big|,\end{aligned}$$

and by sending $|x|\to\infty$ we have $C=0$, which gives (a). To show (b), it suffices to prove that $\nabla\eta \equiv 0$ in $B(o,r)$. Take any $x \in B(o,r)$, and consider the circle $\Gamma_2 \ni x$ centered at o. Again, symmetry and the divergence theorem yield that

$$\big|\nabla\eta(x)\big| = \frac{1}{|\Gamma_2|}\int_{\Gamma_2}\nabla\eta\cdot\vec{n}\,d\sigma = \frac{1}{|\Gamma_2|}\int_{\mathrm{int}(\Gamma_2)}\Delta\eta\,dx = \frac{\int_{\mathrm{int}(\Gamma)} g(x)\,dx}{|\Gamma|} = 0,$$

finishing the proof of (b). ☐

2.3. Radial symmetry of nonsimply connected rotating patches with $\Omega < 0$
In this subsection, we show that a nonnegative uniformly rotating patch solution (with multiple disjoint patches) must be radially symmetric if the angular velocity $\Omega < 0$.

THEOREM 2.12
For $i = 1,\dots,n$, let D_i be a bounded domain with $C^{1,\gamma}$ boundary, and assume that $D_i \cap D_j = \emptyset$ for $i \neq j$. If $\omega = \sum_{i=1}^n \alpha_i 1_{D_i}$ is a nonnegative rotating patch solution with $\alpha_i > 0$ and angular velocity $\Omega < 0$, then ω must be radially symmetric.

Proof
In this proof, let

$$f_\Omega(x) := \omega * \mathcal{N} - \frac{\Omega}{2}|x|^2.$$

In each D_i, let us define p_i as in Lemma 2.6. Let $\varphi_i := \frac{|x|^2}{2} + p_i$ in each D_i. As in Theorem 2.12, we compute $\mathcal{I} := \sum_{i=1}^n \alpha_i \int_{D_i} \nabla\varphi_i \cdot \nabla f_\Omega \, dx$ in two different ways. Since f_Ω is a constant on each connected component of ∂D_i and $\nabla\varphi_i$ is divergence-free in D_i, we still have $\mathcal{I} = 0$ as in the proof of Theorem 2.9.

On the other hand, we have

$$\begin{aligned}\mathcal{I} &= \sum_{i=1}^n \alpha_i \int_{D_i} (x + \nabla p_i) \cdot \nabla(\omega * \mathcal{N}) \, dx + \underbrace{(-\Omega)}_{\geq 0} \sum_{i=1}^n \alpha_i \int_{D_i} (x + \nabla p_i) \cdot x \, dx \\ &=: \mathcal{I}_1 + (-\Omega)\mathcal{I}_2.\end{aligned}$$

As in the proof of Corollary 2.10, we have

$$\mathcal{I}_1 = \sum_{i=1}^n \alpha_i^2 \Big(\frac{|D_i|^2}{4\pi} - \int_{D_i} p_i \, dx \Big) + \sum_{\substack{j \not\prec i \text{ and } i \not\prec j \\ i \neq j}} \alpha_i \alpha_j \frac{|D_i||D_j|}{4\pi} \geq 0. \tag{2.33}$$

Note that $\mathcal{I}_1 = 0$ as long as all D_i's are nested annuli/disks, even if they are not concentric. For $\mathcal{I}_2$, using the Cauchy–Schwarz inequality in the second step, and Lemma 2.13 in the third step (which we will prove right after this theorem), we have

$$\begin{aligned}\mathcal{I}_2 &= \sum_{i=1}^n \alpha_i \Big(\int_{D_i} |x|^2 \, dx + \int_{D_i} \nabla p_i \cdot x \, dx \Big) \\ &\geq \sum_{i=1}^n \alpha_i \Big(\int_{D_i} |x|^2 \, dx - \Big(\int_{D_i} |\nabla p_i|^2 \, dx \Big)^{1/2} \Big(\int_{D_i} |x|^2 \, dx \Big)^{1/2} \Big) \geq 0. \end{aligned} \tag{2.34}$$

Combining (2.33) and (2.34) gives us $\mathcal{I} \geq 0$. If there is any D_i that is not a disk or annulus centered at the origin, Lemma 2.13 would give a strict inequality in the last step of (2.34), which leads to $\mathcal{I} > 0$ and thus contradicts with $\mathcal{I} = 0$. $\square$

Now we state and prove the lemma that is used in the proof of Theorem 2.12.

LEMMA 2.13
Let D be a bounded domain with $C^{1,\gamma}$ boundary, and let p be as in Lemma 2.6. Then we have

$$-\int_D \nabla p \cdot x \, dx = \int_D |\nabla p|^2 \, dx \leq \int_D |x|^2 \, dx. \tag{2.35}$$

Furthermore, in the inequality, "$=$" is achieved if and only if D is a disk or annulus centered at the origin.

Proof
We compute

$$\int_D |\nabla p|^2\,dx = \int_{\partial D} p\nabla p\cdot\vec{n}\,d\sigma + \int_D 2p\,dx$$
$$= -\int_{\partial D} px\cdot\vec{n}\,d\sigma + \int_D 2p\,dx,$$

where in the last equality we use that p is constant along each ∂h_i, as well as the following identity due to (2.12) and the divergence theorem (here $\vec{n}$ is the outer normal of h_i):

$$\int_{\partial h_i} \nabla p\cdot\vec{n}\,d\sigma = -2|h_i| = -\int_{h_i} \Delta\frac{|x|^2}{2}\,dx = -\int_{\partial h_i} x\cdot\vec{n}\,d\sigma.$$

On the other hand, the divergence theorem yields

$$-\int_D \nabla p\cdot x\,dx = -\int_{\partial D} px\cdot\vec{n}\,d\sigma + \int_D 2p\,dx.$$

Therefore, using Young's inequality $-\nabla p\cdot x \le \frac{1}{2}|\nabla p|^2 + \frac{1}{2}|x|^2$ (where the equality is achieved if and only if $-\nabla p = x$), we have

$$\int_D |\nabla p|^2\,dx = -\int_D \nabla p\cdot x\,dx \le \frac{1}{2}\int_D |\nabla p|^2\,dx + \frac{1}{2}\int_D |x|^2\,dx,$$

which proves (2.35). Here the equality is achieved if and only if $-\nabla p = x$ in D, which is equivalent with $p + \frac{|x|^2}{2}$ being a constant in D, and it can be extended to $\overline{D}$ due to continuity of p. By our construction of p in Lemma 2.6, p is already a constant on each connected component of ∂D, implying that $\frac{|x|^2}{2}$ is constant on each piece of ∂D; hence ∂D must be a family of circles centered at the origin. By the assumption that D is connected, it must be either a disk or annulus centered at the origin. □

2.4. Radial symmetry of nonsimply connected rotating patches with $\Omega \ge \frac{1}{2}$
In this final subsection for patches, we consider a bounded open set D with $C^{1,\gamma}$ boundary. The set D can have multiple connected components, and each connected component can be nonsimply connected. If 1_D is a rotating patch solution to the Euler equation with angular velocity $\Omega \ge \frac{1}{2}$, then we will show that D must be radially symmetric and centered at the origin.

To do this, one might be tempted to proceed as in Theorem 2.2 and replace $p : D \to \mathbb{R}$ by the function defined in Lemma 2.6. Here the first way of computing $\mathcal{I} = \int_D (x + \nabla p)\cdot\nabla f_\Omega\,dx$ still yields $\mathcal{I} = 0$, but the second way gives some undesired terms caused by the holes h_i:

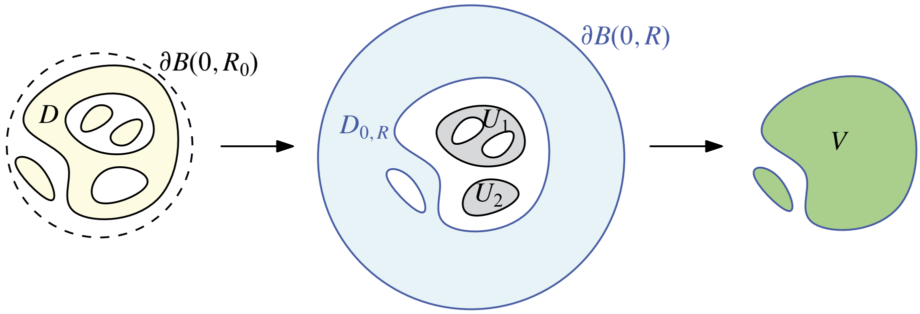

Figure 3. (Color online) For a set $D \subset B(0, R_0)$ (the whole yellow region on the left), the middle figure illustrates the definition of $D_{0,R}$ (the blue region), $\{U_i\}$ (the gray regions), and the right figure illustrates $V = B_R \setminus D_{0,R}$ (the green region).

$$\mathcal{I} = \frac{1}{4\pi}|D|^2 - \Omega \int_D |x|^2 \, dx + (2\Omega - 1) \int_D p \, dx + 2\Omega \sum_{i=1}^{n} p|_{\partial h_i} |h_i|.$$

Due to the last term on the right-hand side, we are unable to show that $\mathcal{I} \leq 0$ when $\Omega \geq \frac{1}{2}$ as we did before in Theorem 2.2. For this reason, we take a different approach in the next theorem. Instead of defining p as a function in D and $\mathcal{I}$ as an integral in D, we want to define them in D^c. But since D^c is unbounded, we define p^R and $\mathcal{I}_R$ in a truncated set $B(0, R) \setminus D^c$, and then use two different ways to compute $\mathcal{I}_R$. By sending $R \to \infty$, we will show that the two ways give a contradiction unless D is radially symmetric.

THEOREM 2.14
For a bounded open set D with $C^{1,\gamma}$ boundary, assume that 1_D is a rotating patch solution to the Euler equation with angular velocity $\Omega \geq \frac{1}{2}$. Then D is radially symmetric and centered at the origin.

Proof
Since D is bounded, let us choose $R_0 > 0$ such that $B_{R_0} \supset D$. For any $R > R_0$, consider the open set $B_R \setminus \overline{D}$, which may have multiple connected components. We denote the component touching ∂B_R by $D_{0,R}$, and name the other connected components by $U_1, \ldots, U_n$. Throughout this proof, we assume that $n \geq 1$: if not, then each connected component of D is simply connected, which has already been treated in Theorem 2.2 and Remark 2.3. We also define $V := B_R \setminus D_{0,R}$, which is the union of D and all its holes. Note that V may have multiple connected components, but each must be simply connected. (See Figure 3 for an illustration of $D_{0,R}$, $\{U_i\}_{i=1}^n$ and V.)

To prove the theorem, the key idea is to define p^R and $\mathcal{I}_R$ in $B_R \setminus D$, instead of in D. Let $p_{0,R}$ and p_i be defined as in Lemma 2.6 in $D_{0,R}$ and U_i, respectively, then set $\varphi_{0,R} := p_{0,R} + \frac{|x|^2}{2}$ in $D_{0,R}$, and $\varphi_i := p_i + \frac{|x|^2}{2}$ in U_i for $i = 1, \dots, n$. Finally, define p^R and $\varphi^R : \mathbb{R}^2 \to \mathbb{R}$ as

$$p^R := p_{0,R} 1_{D_{0,R}} + \sum_{i=1}^n p_i 1_{U_i}, \qquad \varphi^R := \varphi_{0,R} 1_{D_{0,R}} + \sum_{i=1}^n \varphi_i 1_{U_i}.$$

Since 1_D rotates with angular velocity $\Omega \geq \frac{1}{2}$, we know that $f_\Omega := 1_D * \mathcal{N} - \frac{\Omega}{2}|x|^2$ is constant on each connected component of ∂D. Next we will compute

$$\mathcal{I}_R := \int_{B_R \setminus \overline{D}} \nabla f_\Omega \cdot \nabla \varphi^R \, dx \tag{2.36}$$

in two different ways. If some connected component of ∂D is not a circle, then we will derive a contradiction by sending $R \to \infty$. We point out that as we increase R, the domain $D_{0,R}$ will change, but the sets $\{U_i\}_{i=1}^n$ and V all remain unchanged.

On the one hand, we break $\mathcal{I}_R$ into

$$\mathcal{I}_R = \int_{D_{0,R}} \nabla f_\Omega \cdot \nabla \varphi_{0,R} \, dx + \sum_{i=1}^n \int_{U_i} \nabla f_\Omega \cdot \nabla \varphi_i \, dx =: \mathcal{I}_R^1 + \mathcal{I}_R^2.$$

Since f_Ω is constant on each connected component of ∂U_i, the same computation as (2.23)–(2.24) gives $\mathcal{I}_R^2 = 0$. For $\mathcal{I}_R^1$, note that although f_Ω is a constant along the boundary of each hole of $D_{0,R}$, it is *not* a constant along $\partial_{\text{out}} D_{0,R} = \partial B_R$. Thus similar computations as (2.23)–(2.24) now give

$$\begin{aligned} \mathcal{I}_R^1 &= \int_{\partial B_R} \Big(1_D * \mathcal{N} - \frac{\Omega}{2} R^2\Big) \nabla \varphi_{0,R} \cdot \vec{n} \, d\sigma \\ &= \int_{\partial B_R} \big((1_D * \mathcal{N})(x) - |D|\mathcal{N}(x)\big) \nabla \varphi_{0,R} \cdot \vec{n} \, d\sigma(x), \end{aligned} \tag{2.37}$$

where in the second equality we used $\int_{\partial B_R} \nabla \varphi_{0,R} \cdot \vec{n} \, d\sigma = 0$ and the fact that $\mathcal{N}(x)$ is constant on ∂B_R. For any $x \in \partial B_R$, since $D \subset B_{R_0}$ and $R > R_0$, we can control $(1_D * \mathcal{N})(x) - |D|\mathcal{N}(x)$ as

$$\begin{aligned} \big|(1_D * \mathcal{N})(x) - \big|D\big|\mathcal{N}(x)\big| &\leq \frac{1}{2\pi} \int_D \big|\log|x-y| - \log|x|\big| \, dy \\ &\leq \frac{|D|}{2\pi} \Big|\log\Big(1 - \frac{R_0}{R}\Big)\Big| \quad \text{on } \partial B_R. \end{aligned} \tag{2.38}$$

We introduce the following lemma to control $|\nabla \varphi_{0,R} \cdot \vec{n}|$ on ∂B_R. The proof is postponed to the end of this subsection.

LEMMA 2.15
Let $D \subset B_{R_0}$ be an open set with $C^{1,\gamma}$ boundary. For any $R > R_0$, let $D_{0,R}$, V, $p_{0,R}$, and $\varphi_{0,R}$ be defined as in the proof of Theorem 2.14. Then we have

$$|\nabla\varphi_{0,R} \cdot \vec{n}| \le \frac{NR_0^2}{2R\log(R/R_0)} \quad \text{on } \partial B_R, \tag{2.39}$$

where $N > 0$ is the number of connected components of V (and is independent of R).

Once we have this lemma, plugging (2.39) and (2.38) into (2.37) yields

$$|\mathcal{I}_R^1| \le \frac{N|D|R_0^2}{2}\Big|\log\Big(1-\frac{R_0}{R}\Big)\Big|\big(\log(R/R_0)\big)^{-1} \le |D|\frac{C(D,R_0)}{R\log R} \to 0 \quad \text{as } R \to \infty.$$

Combining this with $\mathcal{I}_R^2 = 0$ gives

$$\lim_{R\to\infty} \mathcal{I}_R = 0. \tag{2.40}$$

Next we compute $\mathcal{I}_R$ in another way. Note that $1_{B_R} * \mathcal{N} - \frac{|x|^2}{4}$ is a radial harmonic function in B_R, and thus is equal to some constant C_R in B_R. Using this fact, we can rewrite f_Ω as

$$f_\Omega = 1_D * \mathcal{N} - \frac{\Omega}{2}|x|^2 = (1_D - 1_{B_R}) * \mathcal{N} - \Big(\frac{\Omega}{2} - \frac{1}{4}\Big)|x|^2 + C_R.$$

As a result, $\mathcal{I}_R$ can be rewritten as

$$\begin{aligned}\mathcal{I}_R &= -\int_{B_R\setminus\overline{D}} \nabla(1_{B_R\setminus\overline{D}} * \mathcal{N}) \cdot \nabla\varphi^R\,dx - \frac{(2\Omega-1)}{2}\int_{B_R\setminus\overline{D}} x \cdot \nabla\varphi^R\,dx \\ &=: -\mathcal{J}_R^1 - \frac{(2\Omega-1)}{2}\mathcal{J}_R^2. \end{aligned} \tag{2.41}$$

Next we will show that $\mathcal{J}_R^1, \mathcal{J}_R^2 \ge 0$, leading to $\mathcal{I}_R \le 0$. Let us start with $\mathcal{J}_R^2$. Applying Lemma 2.13 to each of $D_{0,R}$ and $\{U_i\}_{i=1}^n$ immediately gives

$$\begin{aligned}\mathcal{J}_R^2 &\ge \int_{D_{0,R}} |x|^2 + \nabla p_{0,R} \cdot x\,dx + \sum_{i=1}^n \int_{U_i} |x|^2 + \nabla p_i \cdot x\,dx \\ &=: T_{0,R} + \sum_{i=1}^n T_i \ge 0. \end{aligned} \tag{2.42}$$

Note that the T_i's are independent of R for $i = 1, \dots, n$, and we know that $T_i \ge 0$ with equality is achieved if and only if U_i is an annulus or a disk centered at the origin. This will be used later to show that all $\{U_i\}_{i=1}^n$ are centered at the origin in the

$\Omega > \frac{1}{2}$ case. (When $\Omega = \frac{1}{2}$, the coefficient of $\mathcal{J}_R^2$ becomes 0 in (2.41), thus a different argument is needed in this case.)

We now move on to $\mathcal{J}_R^1$. We first break it into

$$\mathcal{J}_R^1 = \int_{B_R\setminus\overline{D}} \nabla(1_{B_R\setminus\overline{D}} * \mathcal{N})\cdot x\,dx + \int_{B_R\setminus\overline{D}} \nabla(1_{B_R\setminus\overline{D}} * \mathcal{N})\cdot \nabla p^R\,dx =: J_{11} + J_{12}.$$

An identical computation as (2.25) gives $J_{11} = \frac{1}{4\pi}(|D_{0,R}| + \sum_{i=1}^n |U_i|)^2$. For J_{12}, the same computation as (2.28)–(2.30) gives the following (where we used that each U_i lies in a hole of $D_{0,R}$ for $i = 1,\dots,n$):

$$J_{12} \ge -\int_{D_{0,R}} p_{0,R}\,dx - \sum_{i=1}^n \int_{U_i} p_i\,dx - \sum_{1\le i\le n} |U_i| \sup_{D_{0,R}} p_{0,R} - \sum_{\substack{1\le i,j\le n\\ j\prec i}} \frac{|U_i||U_j|}{2\pi}.$$

Adding up the estimates for J_{11} and J_{12}, we obtain

$$\begin{aligned}\mathcal{J}_R^1 \ge \Big(\frac{1}{4\pi}|D_{0,R}|^2 - \int_{D_{0,R}} p_{0,R}\,dx\Big) + \Big(\sum_{1\le i\le n}|U_i|\Big)\Big(\frac{1}{2\pi}|D_{0,R}| - \sup_{D_{0,R}} p_{0,R}\Big)\\ + \sum_{i=1}^n\Big(\frac{1}{4\pi}|U_i|^2 - \int_{U_i} p_i\,dx\Big) + \sum_{\substack{j\not\prec i \text{ and } i\not\prec j\\ i\neq j}} \frac{1}{4\pi}|U_i||U_j|.\end{aligned} \tag{2.43}$$

By Proposition 2.8, all terms on the right-hand side are nonnegative. But note that only the two terms in the second line are independent of R. Plugging (2.43) and (2.42) into (2.41) gives the following (where we only keep the terms independent of R on the right-hand side):

$$\begin{aligned}\liminf_{R\to\infty}(-\mathcal{I}_R) \ge \sum_{i=1}^n\Big(\frac{1}{4\pi}|U_i|^2 - \int_{U_i} p_i\,dx\Big)\\ + \sum_{\substack{j\not\prec i \text{ and } i\not\prec j\\ i\neq j}} \frac{1}{4\pi}|U_i||U_j| + \frac{2\Omega-1}{2}\sum_{i=1}^n T_i \ge 0.\end{aligned}$$

Combining this with the previous limit (2.40), we know that U_i must be an annulus or a disk for $i = 1,\dots,n$, and they must be nested in each other. In addition, if $\Omega > \frac{1}{2}$, then we have $T_i = 0$ for $i = 1,\dots,n$, implying that each U_i is centered at the origin.

The radial symmetry of $D_{0,R}$ is more difficult to obtain. Although the first two terms on the right-hand side of (2.43) are both strictly positive if $D_{0,R}$ is not an annulus, we need some uniform-in-R lower bound to get a contradiction in the $R\to\infty$ limit. Between these two terms, it turns out the second term is easier to control. This is done in the next lemma, whose proof we postpone to the end of this subsection.

LEMMA 2.16

Let $D \subset B_{R_0}$ be an open set with $C^{1,\gamma}$ boundary. For any $R > R_0$, let $D_{0,R}$, V, and $p_{0,R}$ be given as in the proof of Theorem 2.14. If V is not a single disk, then there exists some constant $C(V) > 0$ depending only on V such that

$$\liminf_{R\to\infty}\Big(\frac{1}{2\pi}|D_{0,R}| - \sup_{D_{0,R}} p_{0,R}\Big) \geq C(V) > 0.$$

If V is not a disk, then Lemma 2.16 gives $\liminf_{R\to\infty} \mathcal{J}_R^1 > (\sum_{1\leq i\leq n}|U_i|) \times C(V) > 0$. (Recall that in the beginning of this proof we assume that $\sum_{1\leq i\leq n}|U_i| > 0$, and it is independent of R.) This implies that $\liminf_{R\to\infty}(-\mathcal{J}_R) \geq C(V) > 0$, contradicting (2.40).

So far we have shown that ∂D is a union of nested circles, and it remains to show that they are all centered at 0. For the $\Omega > \frac{1}{2}$ case, we already showed that all $\{U_i\}_{i=1}^n$ are centered at 0, so it suffices to show that the outermost circle ∂V (denoted by $B(\tilde{o},\tilde{r})$) is also centered at 0. By definition of $\{U_i\}_{i=1}^n$, we have $D = B(\tilde{o},\tilde{r}) \setminus (\dot{\bigcup}_{i=1}^n U_i)$. Note that $1_{B(\tilde{o},\tilde{r})} * \mathcal{N} = \frac{|x-\tilde{o}|^2}{4} + C$ for some constant C, and $1_{\dot{\bigcup}_{i=1}^n U_i} * \mathcal{N}$ is radially increasing. Therefore, f_Ω can be written as

$$f_\Omega = 1_{B(\tilde{o},\tilde{r})} * \mathcal{N} - 1_{\dot{\bigcup}_{i=1}^n U_i} * \mathcal{N} - \frac{\Omega}{2}|x|^2 = \frac{|x-\tilde{o}|^2}{4} - g(x),$$

where g is radially symmetric, and strictly increasing in the radial variable. Since both f_Ω and $\frac{|x-\tilde{o}|^2}{4}$ are known to take constant values on $\partial B(\tilde{o},\tilde{r})$, it implies that g must be constant on $\partial B(\tilde{o},\tilde{r})$ too, and the fact that g is a radially increasing function gives that $\tilde{o} = 0$. This finishes the proof for $\Omega > \frac{1}{2}$.

For $\Omega = \frac{1}{2}$, we do not yet know whether $\{U_i\}_{i=1}^n$ are centered at 0. Denote by U_1 the innermost one. Then we have

$$f_\Omega(x) = \frac{|x-\tilde{o}|^2}{4} - 1_{\dot{\bigcup}_{i=1}^n U_i} * \mathcal{N} - \frac{|x|^2}{4} = \frac{\tilde{o}\cdot x}{2} + \text{const} \quad \text{for } x \in \partial_{\text{out}} U_1, \quad (2.44)$$

where the second equality follows from Lemma 2.11(b), where we used that $1 \prec j$ for all $2 \leq j \leq n$. Combining (2.44) with the fact that $f_\Omega = \text{const}$ on $\partial_{\text{out}} U_1$ gives $\tilde{o} = 0$, that is, the outermost circle must be centered at 0. This leads to $f_\Omega = -\sum_{i=1}^n 1_{U_i} * \mathcal{N}$. Since $f_\Omega = \text{const}$ on each connected component of ∂U_i, we can apply the last part in the proof of Corollary 2.10 to show that the $\{U_i\}_{i=1}^n$ are all concentric. Denoting their center by o_1, we can show that $o_1 = 0$: Lemma 2.11(a) gives $f_\Omega(x) = C \ln|x - o_1|$ for some $C < 0$ on $\partial B(\tilde{o},\tilde{r})$, and since we have $f = \text{const}$ on $\partial B(\tilde{o},\tilde{r})$ and $\tilde{o} = 0$, it implies that $o_1 = 0$, finishing the proof. $\square$

Proof of Lemma 2.15

For notational simplicity, we shorten $p_{0,R}$, $D_{0,R}$, and $\varphi_{0,R}$ into p_R, D_R, and φ_R

throughout this proof. Recall that $\partial D_R = \partial B_R \cup \partial V$. Clearly we have $\varphi_R = \frac{R^2}{2}$ on ∂B_R, due to $p_R = 0$ on $\partial_{\text{out}} D_R = \partial B_R$. We claim that

$$-\frac{NR_0^2}{2} \leq \varphi_R - \frac{R^2}{2} \leq \frac{R_0^2}{2} \quad \text{on } \partial V, \tag{2.45}$$

where $N \geq 1$ is the number of connected components of V. Once it is proved, we apply the comparison principle to the functions $\varphi_R - \frac{R^2}{2}$ and $\pm g$, where

$$g(x) := \frac{NR_0^2}{2\log(R/R_0)} \log \frac{R}{|x|}.$$

Note that $g \equiv 0$ on ∂B_R, and $g \geq \frac{NR_0^2}{2}$ on ∂V since $\partial V \subset B_{R_0}$. If $0 \notin \overline{D_R}$, then the functions $\varphi_R - \frac{R^2}{2}$ and $\pm g$ are all harmonic in D_R, their values on ∂B_R are all 0, and their boundary values on ∂V are ordered due to (2.45). The comparison principle in D_R then yields

$$-g(x) \leq \varphi_R(x) - \frac{R^2}{2} \leq g(x) \quad \text{in } D_R. \tag{2.46}$$

Since $\varphi_R - \frac{R^2}{2} \equiv g \equiv 0$ on ∂B_R, (2.46) gives $|\nabla \varphi_R \cdot \vec{n}| \leq |\nabla g \cdot \vec{n}| = \frac{NR_0^2}{2R\log(R/R_0)}$ on ∂B_R, which is the desired estimate (2.39). And if $0 \in \overline{D_R}$, then (2.46) still holds in $D_R \setminus B_\epsilon$ for all sufficiently small $\epsilon > 0$ by applying the comparison principle in this set, and (2.39) again follows as a consequence.

In the rest of the proof we will show (2.45). Its second inequality is straightforward:

$$\varphi_R - \frac{R^2}{2} \leq \frac{R_0^2}{2} + \sup_{\overline{D_R}} p_R - \frac{R^2}{2} \leq \frac{R_0^2}{2} \quad \text{on } \partial V.$$

Here the first inequality follows from the definition of φ_R and the fact that $V \subset B_{R_0}$, and the second inequality is due to $\sup_{\overline{D_R}} p_R \leq \frac{|D_R|}{2\pi} \leq \frac{R^2}{2}$ in Proposition 2.8.

It remains to prove the first inequality of (2.45). Let us fix any $R > R_0$. Denote the N connected components of ∂V by $\{\Gamma_i\}_{i=1}^N$, and let $\Gamma_0 := \partial B_R$. This notation leads to $\partial D_R = \bigcup_{i=0}^N \Gamma_i$. For $i = 0, \ldots, N$, let $L_i \subset \mathbb{R}$ be the range of $\varphi_R - \frac{R^2}{2}$ on Γ_i. By continuity of φ_R, each L_i is a closed bounded interval, which can be a single point. Clearly, $L_0 = \{0\}$ due to $\varphi_R|_{\partial B_R} = \frac{R^2}{2}$. Towards a contradiction, suppose that

$$v_{\min} := \min_{1 \leq i \leq N} \inf L_i = \inf_{\partial V}\Big(\varphi_R - \frac{R^2}{2}\Big) =: -\frac{N|R_0|^2}{2} - \delta \quad \text{for some } \delta > 0. \tag{2.47}$$

As for the maximum value, since $L_0 = \{0\}$, we have

$$v_{\max} := \max_{0\le i\le N} \sup L_i \ge 0. \tag{2.48}$$

For $i = 1, \dots, N$, using $p_R|_{\Gamma_i} = \text{const}$, $\varphi_R = p_R + \frac{|x|^2}{2}$, and $\Gamma_i \subset B_{R_0}$, the length of each interval L_i satisfies

$$|L_i| = \operatorname{osc}_{\Gamma_i} \frac{|x|^2}{2} \le \frac{R_0^2}{2} \quad \text{for } i = 1, \dots, N. \tag{2.49}$$

Comparing (2.49) with (2.47)–(2.48), we know that the union of $\{L_i\}_{i=0}^N$ cannot fully cover the interval $[v_{\min}, v_{\max}]$, thus they can be separated in the following sense: there exists a nonempty proper subset $S \subset \{0, \dots, N\}$ such that the range of L_i for indices in S and $S^c := \{0, \dots, N\} \setminus S$ are strictly separated by at least δ, that is, $\min_{i\in S} \inf L_i \ge \max_{i\in S^c} \sup L_i + \delta$. In terms of φ_R, we have

$$\min_{i\in S} \inf_{\Gamma_i} \varphi_R \ge \max_{i\in S^c} \max_{\Gamma_i} \varphi_R + \delta. \tag{2.50}$$

Since φ_R is harmonic in D_R, whose boundary is $\bigcup_{i=0}^N \Gamma_i$, it is a standard comparison principle exercise to show that (2.50) implies

$$\sum_{i\in S} \int_{\Gamma_i} \nabla\varphi_R \cdot \vec{n}\, d\sigma > 0, \tag{2.51}$$

where $\vec{n}$ denotes the outer normal of D_R. But on the other hand, we have

$$\int_{\Gamma_i} \nabla\varphi_R \cdot \vec{n}\, d\sigma = 0 \quad \text{for } i = 0, \dots, N. \tag{2.52}$$

To see this, the cases $i = 1, \dots, N$ can be done by an identical computation as (2.24), and the $i = 0$ case follows from $\int_{\partial D_R} \varphi_R \cdot \vec{n}\, d\sigma = \int_{D_R} \Delta\varphi_R\, dx = 0$ and the fact that $\partial D_R = \bigcup_{i=0}^N \Gamma_i$. Thus we have obtained a contradiction between (2.51) and (2.52), completing the proof. □

Proof of Lemma 2.16

Assume that V has N connected components $\{V_j\}_{j=1}^N$ for $N \ge 1$. For notational simplicity, we shorten $D_{0,R}$, $p_{0,R}$, and $\varphi_{0,R}$ into D_R, p_R, and φ_R in this proof. Let $\epsilon_R := \frac{1}{2\pi}|D_R| - \sup_{D_R} p_R$, which is nonnegative by Proposition 2.8. Towards a contradiction, assume that there exists a diverging subsequence $\{R_i\}_{i=1}^\infty$ such that $\lim_{i\to 0} \epsilon_{R_i} = 0$.

Define $\tilde{\varphi}_{R_i} := \varphi_{R_i} - \frac{R_i^2}{2}$. We claim that $\{\tilde{\varphi}_{R_i}\}_{i=1}^\infty$ has a subsequence that converges locally uniformly to some bounded harmonic function φ_∞ in $\mathbb{R}^2 \setminus V$.

To show this, we will first obtain a uniform bound of $\{\tilde{\varphi}_{R_i}\}_{i=1}^\infty$. Note that (2.45) gives that $\sup_{\partial V} |\tilde{\varphi}_{R_i}| \le \frac{NR_0^2}{2}$ for all $i \in \mathbb{N}^+$. Since $\tilde{\varphi}_{R_i} \equiv 0$ on ∂B_{R_i} for all $i \in \mathbb{N}^+$, the maximum principle for harmonic functions gives $\sup_{D_{R_i}} |\tilde\varphi_{R_i}| \le \frac{NR_0^2}{2}$ for all $i \in \mathbb{N}^+$.

For any $R > 2R_0$, we will obtain a uniform gradient estimate for $\{\tilde\varphi_{R_i}\}$ in D_R for all $R_i > 2R$. First note that since ∂B_R is in the interior of D_{R_i} (due to $R_i > 2R$), the interior estimate for harmonic functions (together with the above uniform bound) gives that $\|\tilde\varphi_{R_i}\|_{C^2(\partial B_R)} \le C(N, R_0)$. On the other boundary ∂V, recall that $\tilde\varphi_{R_i}|_{\partial V_j} = \frac{|x|^2}{2} + c_{i,j}$, with $|c_{i,j}| \le \frac{(N+1)R_0^2}{2}$. Thus $\|\tilde\varphi_{R_i}\|_{C^2(\partial D_R)} \le C(N, R_0)$ for all $R_i > 2R$, and the standard elliptic regularity theory gives the uniform gradient estimate $\sup_{D_R} |\nabla\tilde\varphi_{R_i}| \le C(V)$. This allows us to take a further subsequence (which we still denote by $\{\tilde\varphi_{R_i}\}$) that converges uniformly in $\overline{D_R}$ to some harmonic function $\tilde\varphi_\infty \in C(\overline{D_R})$. Since $R > 2R_0$ is arbitrary, we can repeat this procedure (for countably many times) to obtain a subsequence that converges locally uniformly to a harmonic function $\tilde\varphi_\infty$ in $\mathbb{R}^2 \setminus V$, where $\tilde\varphi_\infty|_{\partial V_j} = \frac{|x|^2}{2} + c_j$ with $|c_j| \le \frac{(N+1)R_0^2}{2}$. This finishes the proof of the claim.

Now define

$$\tilde p_{R_i} := p_{R_i} - \frac{R_i^2}{2} = \tilde\varphi_{R_i} - \frac{|x|^2}{2},$$

which is known to converge locally uniformly to $\tilde p_\infty := \tilde\varphi_\infty - \frac{|x|^2}{2}$ in $\mathbb{R}^2 \setminus V$. Note that $\tilde p_\infty$ is *not* radially symmetric up to any translation. To see this, recall that $\tilde p_\infty|_{\partial V_j} \equiv c_j$. If $\tilde p_\infty$ is radial about some x_0, then it must be of the form $-\frac{|x-x_0|^2}{2} + c$ due to $\Delta \tilde p_\infty = -2$. As a result, the level sets of $\tilde p_\infty$ are all nested circles, thus V must be a single disk (where we used that each connected component of V is simply connected).

Next we will show that $\lim_{i\to 0} \epsilon_{R_i} = 0$ implies that $\tilde p_\infty$ is radial up to a translation, leading to a contradiction. For $k \in \mathbb{R}$, let $g_i(k) := |\{x \in D_{R_i} : p_{R_i}(x) > k\}|$. In the proof of Proposition 2.8, we have shown that $g_i(0) = |D_{R_i}|$, g_i is decreasing in k, with $g_i'(k) \le -2\pi$ for almost every $k \in (0, \sup_{D_{R_i}} p_{R_i})$. Since $\sup_{D_{R_i}} p_{R_i} = \frac{1}{2\pi}|D_{R_i}| - \epsilon_{R_i}$, we can control $g_i(k)$ from below and above as follows:

$$\big(|D_{R_i}| - 2\pi k - 2\pi\epsilon_{R_i}\big)_+ \le g_i(k) \le \big(|D_{R_i}| - 2\pi k\big)_+ \quad \text{for all } k \ge 0. \tag{2.53}$$

Likewise, define $\tilde g_i(k) := |\{x \in D_{R_i} : \tilde p_{R_i}(x) > k\}|$, and $\tilde g_\infty(k) := |\{x \in D_{R_i} : \tilde p_\infty(x) > k\}|$. Since $\tilde p_{R_i} = p_{R_i} - \frac{R_i^2}{2}$, we have $\tilde g_i(k) = g_i(k + \frac{R_i^2}{2})$ for all $k \ge -\frac{R_i^2}{2}$, thus (2.53) is equivalent to

$$\big(-|V| - 2\pi k - 2\pi\epsilon_{R_i}\big)_+ \le \tilde g_i(k) \le \big(-|V| - 2\pi k\big)_+ \quad \text{for all } k \ge -\frac{R_i^2}{2}.$$

The locally uniform convergence of p_{R_i} gives $\lim_{i\to\infty} \tilde g_i = \tilde g_0$, and since we assume that $\lim_{i\to\infty} \epsilon_{R_i} = 0$, we take the $i \to \infty$ limit in the above inequality and obtain

$$\tilde{g}_\infty(k) = \left(-2\pi k - |V|\right)_+ \quad \text{for all } k \in \mathbb{R},$$

which implies that

$$\tilde{g}'_\infty(k) = -2\pi \quad \text{for all } k \in \Big(-\infty, \sup_{\mathbb{R}^2\setminus V} \tilde{p}_\infty\Big). \tag{2.54}$$

Applying the proof of Proposition 2.8 to $\tilde{p}_\infty$ (note that the proof still goes through even though $\tilde{p}_\infty$ takes negative values, and is defined in an unbounded domain), we have that (2.54) can happen only if $\tilde{D}_k := \{\tilde{p}_\infty > k\} \dot{\cup} (\dot{\bigcup}_{\{j:c_j>k\}} \overline{V_j})$ is a disk for almost every $k \in (-\infty, \sup_{\mathbb{R}^2\setminus V} \tilde{p}_\infty)$, and $|\nabla \tilde{p}_\infty|$ is a constant on almost every $\partial \tilde{D}_k$. These two conditions imply that all regular sets of $\tilde{p}_\infty$ are concentric circles, thus $\tilde{p}_\infty$ is radial up to a translation, and we have obtained a contradiction. $\square$

3. Radial symmetry of nonnegative smooth stationary/rotating solutions for 2D Euler with $\Omega \leq 0$

Let $\omega(x,t) = \omega_0(R_{\Omega t} x)$ be a nonnegative compactly supported stationary/rotating solution of 2D Euler with angular velocity $\Omega \leq 0$. Recall that by (1.7), $f_\Omega := \omega_0 * \mathcal{N} - \frac{\Omega}{2}|x|^2$ is a constant along each connected component of a regular level set of ω_0. In this section, we prove that ω_0 is radial up to a translation for $\Omega = 0$, and radial for $\Omega < 0$. As we discussed in the Introduction, the $\omega \geq 0$ condition is necessary: in a forthcoming work [42] we will show that there exists a compactly supported, sign-changing smooth stationary vorticity ω_0 that is nonradial, and the construction also works for $\Omega < 0$ that is close to 0.

Most of this section is devoted to the proof of Theorem 3.5 in the $\Omega = 0$ case (the $\Omega < 0$ case is done in Corollary 3.6 as a simple extension). In the proof, the two key steps are to show that every connected component of a regular level set of ω is a circle, and that these circles are concentric. These are done by approximating ω by a step function $\omega_n = \sum_{i=1}^{M_n} \alpha_i 1_{D_i}$ such that the sets $\{D_i\}$ are disjoint, and $\|\omega - \omega_n\|_{L^\infty} = O(1/n)$. We then define $\varphi^n = \frac{|x|^2}{2} + \sum_{i=1}^{M_n} 1_{D_i} p_i$ corresponding to this step function ω_n, and compute $\mathcal{I}_n = \int \omega_n \nabla\varphi^n \cdot \nabla f dx$ in two ways.

Due to the $O(1/n)$ error in the approximation, the qualitative statement in Proposition 2.8 that "the equality is achieved if and only if D is a disk or an annulus" is no longer good enough for us. We need to obtain various quantitative versions of (2.15) for doubly connected domains, and three such versions are stated below.

In Lemma 3.2, the quantitative constant $c_0 > 0$ depends on the Fraenkel asymmetry of the outer boundary defined in Definition 3.1.

Definition 3.1 (cf. [34, Section 1.2])
For a bounded open set $E \subseteq \mathbb{R}^2$, we define the *Fraenkel asymmetry* $\mathcal{A}(E) \in [0,1)$ as

$$\mathcal{A}(E) := \inf_{x_0}\Big\{\frac{|E\Delta(x_0+rB)|}{|E|} : x_0 \in \mathbb{R}^2, \pi r^2 = |E|\Big\},$$

where B is a unit disk in $\mathbb{R}^2$.

LEMMA 3.2

Let D be a doubly connected set. Let us denote the hole of D by an open set h, and let $\tilde{D} := D \cup \overline{h}$. We define p in D as in Lemma 2.6. Then if $\mathcal{A}(\tilde{D}) > 0$, there is a constant $c_0 > 0$ that depends only on $\mathcal{A}(\tilde{D})$ such that

$$p|_{\partial h} \le \frac{|D|}{2\pi}(1-c_0).$$

Lemma 3.2 will be used in the proof of Theorem 3.5 to show that all level sets of ω are circles. To obtain radial symmetry of ω, we also need to show that all these level sets are concentric. To do this, we need to obtain some quantitative lemmas for a region between two nonconcentric disks. In Lemma 3.3 we consider a thin tubular region between two nonconcentric disks whose radii are close to each other, and obtain a quantitative version of (2.15) for such a domain.

LEMMA 3.3

For $\epsilon > 0$, consider two open disks $B_1 := B(o_1, 1)$ and $B_2 = B(o_2, 1+\epsilon)$ such that $B_1 \subset B_2$. Suppose that $|o_1 - o_2| = a\epsilon$ with $a \in (0,1)$, and let p be defined as in Lemma 2.6 in $D := \overline{B_2} \setminus B_1$. Then if ϵ and a satisfy that $0 < \epsilon < \frac{a^2}{64}$, we have

$$p|_{\partial B_1} \le \frac{|D|}{2\pi}\Big(1 - \frac{a^2}{16}\Big). \tag{3.1}$$

In Lemma 3.4 we consider a region between two nonconcentric disks (that is not necessarily a thin tubular region), and obtain a quantitative version of (2.15) for such a domain.

LEMMA 3.4

Consider two open disks $B_r := B(o_1, r)$ and $B_R = B(o_2, R)$ such that $B_r \subset B_R$. Let p be defined as in (2.6) in $D := B_R \setminus \overline{B_r}$. Suppose that $l := |o_1 - o_2| > 0$ and there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $\delta_1 < r < R < \delta_2$. Then there exists a constant c_0 that depends only on δ_1, δ_2, and l such that

$$p|_{\partial B_r} \le \frac{|D|}{2\pi}(1-c_0).$$

The proofs of the above quantitative lemmas will be postponed to Section 3.1. Now we are ready to prove the main theorem.

THEOREM 3.5
Let ω be a smooth[2] *compactly supported nonnegative stationary solution to the 2D Euler equation. Then ω is radially symmetric up to a translation.*

Proof
Note that as mentioned in step 1 of Proposition 2.8, by Sard's theorem, we have that for almost every $k \in (0, \|\omega\|_{L^\infty})$, $\omega^{-1}(\{k\})$ is a smooth 1-manifold. Furthermore, since ω is compactly supported, each such level set is a disjoint union of a finite number of simple closed curves. For any such closed curve, we call it a *level set component* in this proof.

We split the proof into several steps. Throughout steps 1, 2, and 3, we prove that all level set components of ω must be circles. In step 4, we will prove that any two level set components are nested, that is, one is contained in the other. Lastly, we present the proof that all level set components are concentric in steps 5 and 6.

Step 1. Towards a contradiction, suppose that there is $k > 0$ that is a regular value of ω, and suppose that $\omega^{-1}(\{k\})$ has a connected component Γ that differs from a circle. Recall that $\operatorname{int}(\Gamma)$ denotes the interior of Γ, which is open and simply connected. Since Γ is not a circle, we have $\mathcal{A}(\operatorname{int}(\Gamma)) > 0$, with $\mathcal{A}$ as in Definition 3.1.

In this step, we investigate level set components near Γ. Since k is a regular value, we can find an open neighborhood U of Γ and a constant $\eta > 0$ such that $|\nabla\omega| > \eta$ in U. For any $x \in \Gamma$, consider the flow map $\Phi_t(x)$ originating from x, given by

$$\frac{d}{dt}\Phi_t(x) = \frac{\nabla\omega(\Phi_t(x))}{|\nabla\omega(\Phi_t(x))|^2}$$

with initial condition $\Phi_0(x) = x$. Since $\frac{\nabla\omega}{|\nabla\omega|^2}$ is smooth and bounded in U, we can choose $\delta_1 > 0$ so that $\Phi_t(\Gamma) := \{\Phi_t(x) : x \in \Gamma\}$ lies in U for any $t \in (-\delta_1, \delta_1)$. Note that the Φ_t's are diffeomorphisms, thus $\Phi_t(\Gamma)$ is also a smooth simple closed curve for $t \in (-\delta_1, \delta_1)$. Then we observe that

$$\frac{d}{dt}\omega\big(\Phi_t(x)\big) = \nabla\omega\big(\Phi_t(x)\big) \cdot \frac{\nabla\omega(\Phi_t(x))}{|\nabla\omega(\Phi_t(x))|^2} = 1 \quad \text{for } (t,x) \in (-\delta_1, \delta_1) \times \Gamma. \tag{3.2}$$

Hence for each $t \in (-\delta_1, \delta_1)$, $\Phi_t(\Gamma)$ is a level set component and

$$\omega\big(\Phi_{t_1}(\Gamma)\big) \neq \omega\big(\Phi_{t_2}(\Gamma)\big) \quad \text{if } t_1 \neq t_2. \tag{3.3}$$

By continuity of the map $(t,x) \mapsto \Phi_t(x)$, we can find $\delta_2 \in (0, \delta_1)$ such that

[2]In fact, it is sufficient to assume that $\omega_0 \in C^2(\mathbb{R}^2)$. In the proof, the application of Sard's theorem in step 1 is the step that requires the highest regularity and it is applicable for functions in $C^2(\mathbb{R}^2)$. (See [79, Theorem 6.1].) Therefore, $\omega_0 \in C^2(\mathbb{R}^2)$ ensures that $\omega^{-1}(\{k\})$ is a C^2 1-manifold for almost every $k \in (0, \|\omega\|_{L^\infty})$, whose regularity is sufficient for all the following steps.

$$\mathcal{A}\big(\mathrm{int}\big(\Phi_t(\Gamma)\big)\big) > \frac{1}{2}\mathcal{A}\big(\mathrm{int}(\Gamma)\big) \quad \text{for any } t \in (-\delta_2, \delta_2). \tag{3.4}$$

Since two different level sets cannot intersect, we can assume without loss of generality that $\mathrm{int}(\Phi_{-\delta_2}(\Gamma)) \subset \mathrm{int}(\Phi_{\delta_2}(\Gamma))$. Then it follows from the intermediate value theorem and (3.2) that

$$\mathrm{int}\big(\Phi_{-\delta_2}(\Gamma)\big) \subset \Phi_t(\Gamma) \subset \mathrm{int}\big(\Phi_{\delta_2}(\Gamma)\big) \quad \text{for any } t \in (-\delta_2, \delta_2). \tag{3.5}$$

Lastly, we denote $V := \mathrm{int}(\Phi_{\delta_2}(\Gamma)) \backslash \overline{\mathrm{int}(\Phi_{-\delta_2}(\Gamma))}$ which is a nonempty open doubly connected set; therefore $|V| > 0$.

Step 2. For any integer $n > 1$, we claim that we can approximate ω by a step function ω_n of the form $\omega_n(x) = \sum_{i=1}^{M_n} \alpha_i 1_{D_i}(x)$, which satisfies all the following properties:

(a) each D_i is a domain with smooth boundary and possibly has a finite number of holes;

(b) each connected component of ∂D_i is a level set component of ω;

(c) $D_i \cap D_j = \emptyset$ if $i \neq j$;

(d) $\|\omega_n - \omega\|_{L^\infty(\mathbb{R}^2)} \leq \frac{2}{n}\|\omega\|_{L^\infty(\mathbb{R}^n)}$.

To construct such ω_n for a fixed $n > 1$, let $r_0 = 0$ and $r_{n+1} = \|\omega\|_{L^\infty}$. We pick $r_1, \dots, r_n$ to be regular values of ω such that $0 < r_1 < \cdots < r_n < \|\omega\|_{L^\infty}$, and $r_{i+1} - r_i < \frac{2}{n}\|\omega\|_{L^\infty}$ for $i = 0, \dots, n$. We denote $D_i := \{x \in \mathbb{R}^2 : r_i < \omega(x) < r_{i+1}\}$ for $i = 1, \dots, n-1$, and let $D_n := \{x \in \mathbb{R}^2 : \omega(x) > r_n\}$. Thus for each $i = 1, \dots, n$, D_i is a bounded open set with smooth boundary. We can then write it as $D_i = \dot{\bigcup}_{l=1}^{m_i} D_i^l$ for some $m_i \in \mathbb{N}$, where the D_i^l's are connected components of D_i. Then let $\omega_n(x) := \sum_{i=1}^n r_i \sum_{l=1}^{m_i} 1_{D_i^l}$. By relabeling the indices, we rewrite $\omega_n(x) = \sum_{i=1}^{M_n} \alpha_i 1_{D_i}$, where $M_n = \sum_{i=1}^n m_i$, and each $\alpha_i \in \{r_1, \dots, r_n\}$. One can easily check that such ω_n satisfies properties (a)–(d).

Of course, there are many ways to choose the values $r_1, \dots, r_n$, with each choice leading to a different ω_n. From now on, for any $n > 1$, we fix $\omega_n(x) := \sum_{i=1}^{M_n} \alpha_i 1_{D_i}(x)$ as any function constructed in the above way. (Note that α_i and D_i all depend on n, but we omit their n dependence for notational simplicity.)

Finally, let us point out that for $\omega_n(x)$ constructed above, if $D_i \subset V$ for some i, then D_i must be doubly connected, since step 1 shows that all level set components in V are nested curves. We will use this in steps 3 and 5.

Step 3. For any $n > 1$, let $\omega_n(x) = \sum_{i=1}^{M_n} \alpha_i 1_{D_i}(x)$ be constructed as in step 2. For each D_i, we define p_i^n in D_i as in Lemma 2.6. We set

$$\begin{cases} p^n := \sum_{i=1}^{M_n} p_i^n 1_{D_i}, \\ \varphi_i^n := p_i + \frac{|x|^2}{2} \quad \text{in } D_i, \\ \varphi^n := \sum_{i=1}^{M_n} \varphi_i^n 1_{D_i}. \end{cases} \tag{3.6}$$

As in Theorem 2.9, let $f := \omega * \mathcal{N}$, and we will compute

$$\mathcal{I}^n := \int_{\mathbb{R}^2} \omega_n(x)\nabla\varphi^n(x)\cdot\nabla f(x)\,dx = \sum_{i=1}^{M_n}\alpha_i \int_{D_i} \nabla\varphi^n(x)\cdot\nabla f(x)\,dx \tag{3.7}$$

in two different ways and derive a contradiction by taking the $n\to\infty$ limit.

On the one hand, the same computation as in (2.23)–(2.24) yields that

$$\mathcal{I}^n = \sum_{i=1}^{M_n}\alpha_i\Big(\int_{\partial D_i} f(x)\nabla\varphi_i^n(x)\cdot\vec{n}\,d\sigma - \int_{D_i} f(x)\Delta\varphi_i^n(x)\,dx\Big) = 0. \tag{3.8}$$

On the other hand,

$$\begin{aligned}\mathcal{I}^n &= \int_{\mathbb{R}^2}\omega_n(x)x\cdot\nabla f(x)\,dx + \int_{\mathbb{R}^2}\omega_n(x)\nabla p^n(x)\cdot\nabla f(x)\,dx\\ &=: \mathcal{I}_1^n + \mathcal{I}_2^n.\end{aligned}$$

Since ω_n satisfies property (d) in step 2, it follows that

$$\lim_{n\to\infty}\mathcal{I}_1^n = \int_{\mathbb{R}^2}\omega(x)x\cdot\nabla f(x)\,dx.$$

A similar computation as in (2.25) yields that

$$\begin{aligned}\int_{\mathbb{R}^2}\omega(x)x\cdot\nabla f(x)\,dx &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\omega(x)\omega(y)\frac{x\cdot(x-y)}{|x-y|^2}\,dx\,dy\\ &= \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\omega(x)\omega(y)\,dx\,dy\\ &= \frac{1}{4\pi}\Big(\int_{\mathbb{R}^2}\omega(x)\,dx\Big)^2,\end{aligned} \tag{3.9}$$

where we used the symmetry of the integration domain to get the second equality.

Now we estimate the limit of $\mathcal{I}_2^n$. By Lemma 2.13, we have $\int_{D_i}|\nabla p_i^n|^2\,dx \le \int_{D_i}|x|^2\,dx$, hence $\|\omega_n\nabla p\|_{L^2(\mathbb{R}^2)}$ is uniformly bounded. Since $\omega_n\to\omega$ in L^∞, the bounded convergence theorem yields that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\omega_n\nabla p^n\cdot\nabla\big((\omega_n-\omega)*\mathcal{N}\big)(x)\,dx = 0.$$

Therefore,

$$\liminf_{n\to\infty}\mathcal{I}_2^n = \liminf_{n\to\infty}\int_{\mathbb{R}^2}\omega_n(x)\nabla p^n(x)\cdot\nabla(\omega_n*\mathcal{N})\,dx.$$

From now on, we will omit the n dependence in p_i^n for notational simplicity. Let us break the integral on the right-hand side into

$$\begin{aligned}
&\int_{\mathbb{R}^2} \omega_n(x)\nabla p^n(x)\cdot\nabla(\omega_n * \mathcal{N})\,dx \\
&\quad= \sum_{i,j=1}^{M_n} \alpha_i\alpha_j \int_{D_i} \nabla p_i \cdot \nabla(1_{D_j} * \mathcal{N})\,dx \\
&\quad= \sum_{i=1}^{M_n}\alpha_i^2 \int_{D_i} \nabla p_i\cdot\nabla(1_{D_i} * \mathcal{N})\,dx + \sum_{i\neq j}\alpha_i\alpha_j \int_{D_i}\nabla p_i\cdot\nabla(1_{D_j} * \mathcal{N})\,dx \\
&\quad=: F_1 + F_2. \qquad (3.10)
\end{aligned}$$

For F_1, the divergence theorem yields

$$\begin{aligned}
F_1 &= \sum_{i=1}^{M_n}\alpha_i^2\Big(\int_{\partial D_i} p_i\nabla(1_{D_i} * \mathcal{N})\cdot\vec{n}\,d\sigma - \int_{D_i} p_i\,dx\Big) \\
&= -\sum_{i=1}^{M_n}\alpha_i^2\int_{D_i} p_i\,dx, \qquad (3.11)
\end{aligned}$$

where the second equality follows from an identical computation as in (2.26). Then by Proposition 2.8, we have

$$F_1 \geq -\frac{1}{4\pi}\sum_{i=1}^{M_n}\alpha_i^2|D_i|^2. \qquad (3.12)$$

For F_2, the divergence theorem yields

$$\begin{aligned}
F_2 &= \sum_{i\neq j}\alpha_i\alpha_j\Big(\int_{\partial D_i} p_i\nabla(1_{D_j} * \mathcal{N})\cdot\vec{n}\,d\sigma - \int_{D_i} p_i 1_{D_j}\,dx\Big) \\
&= \sum_{i\neq j}\alpha_i\alpha_j\int_{\partial D_i} p_i\nabla(1_{D_j} * \mathcal{N})\cdot\vec{n}\,d\sigma,
\end{aligned}$$

where we used property (c) in step 2 to get the last equality.

For $i \neq j$, recall that as in the proof of Corollary 2.10, we denote $j \prec i$ if D_j is contained in a hole of D_i. Then the divergence theorem gives

$$\int_{\partial D_i} p_i\nabla(1_{D_j} * \mathcal{N})\cdot\vec{n}\,d\sigma \begin{cases} = 0 & \text{if } j \not\prec i, \\ \geq -\sup_{\partial D_i} p_i|D_j| & \text{if } j \prec i. \end{cases} \qquad (3.13)$$

Next we will improve this inequality for $j \prec i$ and $i \in L$, where $L := \{i : D_i \subset V\}$. (Note that L depends on ω_n, where we omit this dependence for notational simplicity.) From the discussion at the end of step 2, we know that D_i has exactly one hole for all $i \in L$. Using the divergence theorem together with this observation, (3.13) becomes

$$\int_{\partial D_i} p_i \nabla(1_{D_j} * \mathcal{N}) \cdot \vec{n}\, d\sigma \begin{cases} = 0 & \text{if } j \not\prec i, \\ \geq -\sup_{D_i} p_i |D_j| & \text{if } j \prec i \text{ and } i \notin L, \\ = -p_i|_{\partial_{\mathrm{in}} D_i} |D_j| & \text{if } j \prec i \text{ and } i \in L. \end{cases} \tag{3.14}$$

For the second case on the right-hand side, we simply use the crude bound $\sup_{D_i} p_i \leq \frac{|D_i|}{2\pi}$ from Proposition 2.8. For the third case, we can have a better bound: for any $i \in L$, by Lemma 3.2 and (3.4), there exists an $\epsilon > 0$ that depends only on $\mathcal{A}(\mathrm{int}(\Gamma))$ (and in particular is independent of n) such that $p_i|_{\partial_{\mathrm{in}} D_i} \leq (\frac{1}{2\pi} - \epsilon)|D_i|$. Thus (3.14) now becomes

$$\int_{\partial D_i} p_i \nabla(1_{D_j} * \mathcal{N}) \cdot \vec{n}\, d\sigma \begin{cases} = 0 & \text{if } j \not\prec i, \\ \geq -\frac{1}{2\pi}|D_i||D_j| & \text{if } j \prec i \text{ and } i \notin L, \\ \geq -(\frac{1}{2\pi} - \epsilon)|D_i||D_j| & \text{if } j \prec i \text{ and } i \in L. \end{cases} \tag{3.15}$$

Now we are ready to estimate F_2. Let us break it into

$$\begin{aligned} F_2 = & \sum_{\substack{j \prec i \\ (i,j) \notin L \times L}} \alpha_i \alpha_j \int_{\partial D_i} p_i \nabla(1_{D_j} * \mathcal{N}) \cdot \vec{n}\, d\sigma \\ & + \sum_{\substack{j \prec i \\ (i,j) \in L \times L}} \alpha_i \alpha_j \int_{\partial D_i} p_i \nabla(1_{D_j} * \mathcal{N}) \cdot \vec{n}\, d\sigma \\ \geq & - \sum_{\substack{j \prec i \\ (i,j) \notin L \times L}} \alpha_i \alpha_j \frac{1}{2\pi} |D_i||D_j| - \sum_{\substack{j \prec i \\ (i,j) \in L \times L}} \alpha_i \alpha_j \Big(\frac{1}{2\pi} - \epsilon\Big) |D_i||D_j|, \end{aligned}$$

where the first equality follows from case (1) of (3.15), and the second inequality follows from cases (2) and (3) of (3.15). Finally, by exchanging i with j and taking the average with the original inequality, we have

$$\begin{aligned} F_2 \geq & -\frac{1}{4\pi} \sum_{\substack{i \neq j \\ (i,j) \notin L \times L}} (\mathbb{1}_{i \prec j} + \mathbb{1}_{j \prec i}) \alpha_i \alpha_j |D_i||D_j| \\ & - \frac{1}{2} \sum_{\substack{i \neq j \\ (i,j) \in L \times L}} (\mathbb{1}_{i \prec j} + \mathbb{1}_{j \prec i}) \alpha_i \alpha_j \Big(\frac{1}{2\pi} - \epsilon\Big) |D_i||D_j| \end{aligned}$$

$$\geq -\frac{1}{4\pi}\sum_{\substack{i\neq j\\(i,j)\notin L\times L}}\alpha_i\alpha_j|D_i||D_j| - \frac{1}{2}\sum_{\substack{i\neq j\\(i,j)\in L\times L}}\alpha_i\alpha_j\Big(\frac{1}{2\pi}-\epsilon\Big)|D_i||D_j|$$

$$= -\frac{1}{4\pi}\sum_{i\neq j}\alpha_i\alpha_j|D_i||D_j| + \frac{\epsilon}{2}\sum_{\substack{i\neq j\\(i,j)\in L\times L}}\alpha_i\alpha_j|D_i||D_j|, \tag{3.16}$$

where the second inequality is due to the fact that for any $i\neq j$, at most one of $i\prec j$ and $j\prec i$ can be true, thus we always have $\mathbb{1}_{i\prec j}+\mathbb{1}_{j\prec i}\leq 1$.

Therefore, from (3.12) and (3.16) it follows that

$$\begin{aligned} F_1+F_2 &\geq -\frac{1}{4\pi}\sum_{i,j=1}^{M_n}\alpha_i\alpha_j|D_i||D_j| \\ &\quad + \frac{\epsilon}{2}\sum_{(i,j)\in L\times L}\alpha_i\alpha_j|D_i||D_j| - \frac{\epsilon}{2}\sum_{i\in L}\alpha_i^2|D_i|^2 \\ &= -\frac{1}{4\pi}\Big(\sum_{i=1}^{M_n}\alpha_i|D_i|\Big)^2 + \frac{\epsilon}{2}\Big(\sum_{i\in L}\alpha_i|D_i|\Big)^2 - \frac{\epsilon}{2}\sum_{i\in L}\alpha_i^2|D_i|^2. \end{aligned} \tag{3.17}$$

Since we will send $n\to\infty$, in the rest of step 3 we will denote L by L^n to emphasize that L depends on ω_n. (In fact, α_i and D_i depend on n as well, and we omit the n dependence for them to avoid overcomplicating the notation.)

Note that $\sum_{i\in L^n}\alpha_i 1_{D_i}$ converges to $\omega 1_V$ in $L^1(\mathbb{R}^2)$. Also if $i\in L^n$, then the nondegeneracy of $|\nabla\omega|$ on V yields that $\lim_{n\to\infty}\sup_{i\in L^n}|D_i|=0$; consequently

$$\lim_{n\to\infty}\sum_{i\in L^n}\alpha_i^2|D_i|^2 \leq \|\omega\|_{L^\infty}\lim_{n\to\infty}\sup_{i\in L^n}|D_i|\int_{\mathbb{R}^2}\omega\,dx = 0.$$

Therefore, it follows that

$$\begin{aligned} \liminf_{n\to\infty}\mathcal{I}_2^n &= \liminf_{n\to\infty}(F_1+F_2) \\ &\geq -\lim_{n\to\infty}\frac{1}{4\pi}\Big(\sum_{i=1}^{M_n}\alpha_i|D_i|\Big)^2 + \lim_{n\to\infty}\frac{\epsilon}{2}\Big(\sum_{i\in L^n}\alpha_i|D_i|\Big)^2 \\ &= -\frac{1}{4\pi}\Big(\int_{\mathbb{R}^2}\omega(x)\,dx\Big)^2 + \frac{\epsilon}{2}\Big(\int_V\omega(x)\,dx\Big)^2. \end{aligned} \tag{3.18}$$

Note that ω is strictly positive in V, due to $|\nabla\omega|>0$ in V and $\omega\geq 0$ in $\mathbb{R}^2$. Thus from (3.8), (3.9), and (3.18), it follows that

$$0 = \lim_{n\to\infty}\mathcal{I}^n \geq \lim_{n\to\infty}\mathcal{I}_1^n + \liminf_{n\to\infty}\mathcal{I}_2^n \geq \frac{\epsilon}{2}\Big(\int_V\omega(x)\,dx\Big)^2 > 0, \tag{3.19}$$

which is a contradiction, and we have proved that any level set component is a circle.

Step 4. In this step, we show that every pair of level set components are nested. Towards a contradiction, assume that there exist two level set components Γ_1 and Γ_2 that are not nested.

From step 3, we know that Γ_1 and Γ_2 are circles. Then the disks $\mathrm{int}(\Gamma_1)$ and $\mathrm{int}(\Gamma_2)$ are disjoint, and they must be separated by a positive distance since Γ_1 and Γ_2 are level sets of regular values of ω. As in step 1, using the flow map Φ_t originating from the two circles, we can find disjoint open annuli V_1 and V_2 such that $\Gamma_i \subset V_i$ for $i = 1, 2$, and both $\partial_{\mathrm{out}} V_i$ and $\partial_{\mathrm{in}} V_i$ are level set components of ω.

For any $n > 1$, let $\omega_n(x) = \sum_{i=1}^{M_n} \alpha_i 1_{D_i}(x)$ be constructed as in step 2, and let

$$L_1^n := \{i : D_i \subset V_1\} \quad \text{and} \quad L_2^n := \{i : D_i \subset V_2\}.$$

Let p_i be defined as in (3.6) of step 3, and let $\mathcal{I}^n$ be defined as in (3.7). Then on the one hand, the same computations as in step 3 give

$$\lim_{n\to\infty} \mathcal{I}^n = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathcal{I}_1^n = \frac{1}{4\pi}\Big(\int_{\mathbb{R}^2} \omega(x)\,dx\Big)^2. \tag{3.20}$$

Let F_1 and F_2 be given by (3.10). For F_1, the estimate (3.12) still holds. For F_2, using (3.13) and Proposition 2.8, we have

$$F_2 \ge -\frac{1}{4\pi} \sum_{i \prec j \text{ or } j \prec i} \alpha_i \alpha_j |D_i||D_j|.$$

Since V_1 and V_2 are assumed to be not nested, if $(i, j) \in L_1^n \times L_2^n$, then neither $i \prec j$ nor $j \prec i$. Therefore, it follows that

$$\begin{aligned} F_2 \ge &-\frac{1}{4\pi} \sum_{i \ne j} \alpha_i \alpha_j |D_i||D_j| + \frac{1}{4\pi} \sum_{(i,j)\in L_1 \times L_2} \alpha_i \alpha_j |D_i||D_j| \\ &+ \frac{1}{4\pi} \sum_{(j,i)\in L_1 \times L_2} \alpha_i \alpha_j |D_i||D_j|. \end{aligned}$$

Combining the estimates for F_1 and F_2 yields

$$F_1 + F_2 \ge -\frac{1}{4\pi} \sum_{i,j=1}^{M_n} \alpha_i \alpha_j |D_i||D_i| + \frac{1}{2\pi}\Big(\sum_{i\in L_1^n} \alpha_i |D_i|\Big)\Big(\sum_{i\in L_2^n} \alpha_i |D_i|\Big).$$

As $n \to \infty$, since $\sum_{i\in L_1^n} \alpha_i 1_{D_i}$ and $\sum_{i\in L_2^n} \alpha_i 1_{D_i}$ converge to $\omega 1_{V_1}$ and $\omega 1_{V_2}$, respectively, in $L^1(\mathbb{R}^2)$, we have

$$\liminf_{n\to\infty} \mathcal{I}_2^n \geq \lim_{n\to\infty}(F_1+F_2) = -\frac{1}{4\pi}\Big(\int_{\mathbb{R}^2}\omega(x)\,dx\Big)^2 + \frac{1}{2\pi}\Big(\int_{V_1}\omega(x)\,dx\Big)\Big(\int_{V_2}\omega(x)\,dx\Big). \tag{3.21}$$

Combining (3.20) and (3.21) gives us a similar contradiction as in (3.19), except that $\frac{\epsilon}{2}(\int_V \omega(x)\,dx)^2$ is now replaced by $\frac{1}{2\pi}(\int_{V_1}\omega(x)\,dx)(\int_{V_2}\omega(x)\,dx)$. Thus we complete the proof that level sets are nested.

Step 5. In this step, we aim to show that all level set components are concentric within the *same connected component* of $\{\omega > 0\} := \{x \in \mathbb{R}^2 : \omega > 0\}$. This immediately implies that each connected component of $\{\omega > 0\}$ is an annulus or a disk, and ω is radially symmetric about its center.

Towards a contradiction, suppose that there are two level set components Γ_{in} and Γ_{out} in the same connected component of $\{\omega > 0\}$, such that they are nested circles, but their centers O_{in} and O_{out} do not coincide. We denote their radii by r_{in} and r_{out}, and define

$$U := \overline{\text{int}(\Gamma_{\text{out}})} \setminus \text{int}(\Gamma_{\text{in}}).$$

For an illustration of Γ_{in} and Γ_{out} and U, see Figure 4(a).

We claim that ω is uniformly positive in U. Recall that all level set components of ω are nested by step 4. Thus if ω achieves zero in U, then the zero level set must also be nested between Γ_{in} and Γ_{out}, since it can be taken as a limit of level set components whose value approaches 0; but this contradicts with the assumption that Γ_{in} and Γ_{out} lie in the same connected component of $\{\omega > 0\}$. As a result, we have $\omega_{\min} := \inf_U \omega > 0$.

For a sufficiently large n, let $\omega_n = \sum_{i=1}^{M_n} \alpha_i 1_{D_i}(x)$ be given as in step 2, where we further require that both Γ_{in} and Γ_{out} coincide with some boundary of D_i. (This is allowed in our construction of ω_n in step 2, since ω is regular along both Γ_{in} and Γ_{out}.) Let us denote

$$B_n := \{1 \leq i \leq M_n : D_i \subset U\},$$

and note that $U := \bigcup_{i\in B_n} \overline{D_i}$. See Figure 4(b) for an illustration of $\{D_i\}_{i\in B_n}$.

As before, we denote $i \prec j$ if D_i is nested in D_j. For the integral $\mathcal{I}^n$ in (3.7), on the one hand, we have $\mathcal{I}^n = 0$ for all $n > 1$ by (3.8). On the other hand, following the same argument as in step 3 up to (3.13) (where we also use that each D_i is already known to be doubly connected, thus $\int_{\partial D_i} p_i \nabla(1_{D_j} * \mathcal{N}) \cdot \vec{n}\,d\sigma = -p_i|_{\partial_{\text{in}} D_i}|D_j|$ if $j \prec i$), we have

$$\liminf_{n\to\infty} \mathcal{I}^n = \liminf_{n\to\infty}\Big(\frac{1}{4\pi}\Big(\sum_{i=1}^{M_n}\alpha_i|D_i|\Big)^2$$

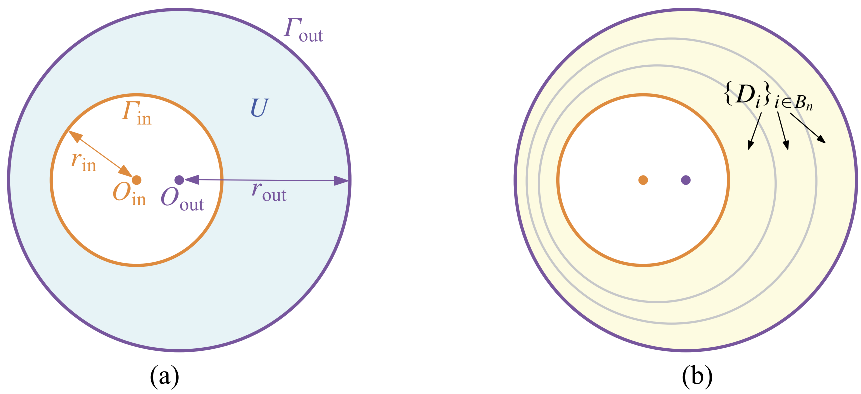

Figure 4. (Color online) (a) Illustration of the circles Γ_{in} and Γ_{out}, whose centers are O_{in} and O_{out}. The set U is colored in blue. (b) For a fixed n, each open set $\{D_i\}_{i\in B_n}$ is colored in yellow. Note that their union gives exactly the set U.

$$
\begin{aligned}
&-\sum_{i=1}^{M_n}\alpha_i^2\int_{D_i} p_i\,dx-\sum_{1\le i,j\le M_n,\,j\prec i}\alpha_i\alpha_j\,p_i|_{\partial_{\text{in}}D_i}|D_j|\Big)\\
&\ge \liminf_{n\to\infty}\underbrace{\Big(\sum_{1\le i,j\le M_n,\,j\prec i}\alpha_i\alpha_j\Big(\frac{1}{2\pi}|D_i|-p_i|_{\partial_{\text{in}}D_i}\Big)|D_j|\Big)}_{=:T_n},
\end{aligned}
$$

where in the last step we used Proposition 2.8.

Note that Proposition 2.8 gives $T_n>0$, where we have strict positivity, since $O_{\text{in}}\ne O_{\text{out}}$ implies that some $\{D_i\}_{i\in B_n}$ must be nonradial. But since the area of these D_i's may approach 0 as $n\to\infty$, in order to derive a contradiction after taking $\liminf_{n\to\infty}$, we need to obtain a quantitative estimate for Proposition 2.8 for a thin tubular region D_i between two circles, which is done in Lemma 3.3.

Next we show that the sets $\{D_i\}_{i\in B_n}$ that are "nonradial to some extent" must occupy a certain portion of U. For $i\in B_n$, denote by o^i_{in} and r^i_{in} the center and radius of $\partial_{\text{in}}D_i$, and likewise o^i_{out} and r^i_{out} the center and radius of $\partial_{\text{out}}D_i$. Note that if D_i is the innermost set in $\{D_i\}_{i\in B_n}$, then we have $o^i_{\text{in}}=O_{\text{in}}$, and the outermost D_i satisfies $o^i_{\text{out}}=O_{\text{out}}$. In addition, if $\partial_{\text{out}}D_i=\partial_{\text{in}}D_j$ for some $i,j\in B_n$, then $o^i_{\text{out}}=o^j_{\text{in}}$. Thus the triangle inequality gives

$$\sum_{i\in B_n}|o^i_{\text{in}}-o^i_{\text{out}}|\ge|O_{\text{in}}-O_{\text{out}}|=:c_0>0. \tag{3.22}$$

In order to apply Lemma 3.3 (which requires the region to have inner radius 1), for each $i\in B_n$, consider the scaling

$$\tilde{p}_i(x) := (r_{\text{in}}^i)^{-2} p_i(r_{\text{in}}^i x).$$

Then $\tilde{p}_i$ is defined in $\tilde{D}_i := (r_{\text{in}}^i)^{-1} D_i$. Due to the scaling, $\tilde{D}_i$ has inner radius 1 (denote the hole by $\tilde{h}_i$), and outer radius $1+\epsilon_i$, where $\epsilon_i := \frac{r_{\text{out}}^i - r_{\text{in}}^i}{r_{\text{in}}^i} > 0$. In addition, the distance between the centers of $\partial_{\text{in}} \tilde{D}_i$ and $\partial_{\text{out}} \tilde{D}_i$ is $a_i \epsilon_i$, where

$$a_i := \frac{|o_{\text{in}}^i - o_{\text{out}}^i|}{|r_{\text{in}}^i - r_{\text{out}}^i|}.$$

One can also easily check that $\tilde{p}_i$ satisfies $\Delta \tilde{p}_i = -2$ in $\tilde{D}_i$, and $\int_{\partial \tilde{h}_i} \nabla \tilde{p} \cdot \vec{n}\, d\sigma = -2|\tilde{h}_i| = -2\pi$. By Lemma 3.3, if $0 < \epsilon_i < \frac{(a_i)^2}{64}$, then $\tilde{p}_i|_{\partial \tilde{h}_i} \le \frac{|\tilde{D}_i|}{2\pi}(1 - \frac{a_i^2}{16})$. Thus in terms of p_i, we have

$$p_i|_{\partial_{\text{in}} D_i} \le \frac{1}{2\pi}|D_i| - c_1 a_i^2 |D_i| \quad \text{if } i \in B_n \text{ satisfies } r_{\text{out}}^i - r_{\text{in}}^i \le c_2 a_i^2, \tag{3.23}$$

where $c_1 := \frac{1}{32\pi}$ and $c_2 := \frac{r_{\text{in}}}{64}$ are independent of n and i, due to the fact that $r_{\text{in}}^i \ge r_{\text{in}} > 0$ for all $i \in B_n$. Using the definition of a_i, (3.22) can be written as

$$\sum_{i \in B_n} a_i |r_{\text{in}}^i - r_{\text{out}}^i| \ge c_0. \tag{3.24}$$

Note that $\sum_{i \in B_n} |r_{\text{in}}^i - r_{\text{out}}^i|$ satisfies the upper bound

$$\sum_{i \in B_n} |r_{\text{in}}^i - r_{\text{out}}^i| \le \frac{|U|}{2\pi r_{\text{in}}} =: M, \tag{3.25}$$

which follows from

$$|U| = \sum_{i \in B_n} |D_i| = \pi \sum_{i \in B_n} |r_{\text{in}}^i - r_{\text{out}}^i| \underbrace{(r_{\text{in}}^i + r_{\text{out}}^i)}_{>2r_{\text{in}}}.$$

Combining (3.24) and (3.25) gives

$$\sum_{i \in B_n} \mathbb{1}_{a_i > \frac{c_0}{2M}} |r_{\text{in}}^i - r_{\text{out}}^i| \ge \sum_{i \in B_n} \left(a_i - \frac{c_0}{2M}\right) |r_{\text{in}}^i - r_{\text{out}}^i| \ge \frac{c_0}{2}, \tag{3.26}$$

where the first inequality follows from $\mathbb{1}_{a_i > \frac{c_0}{2M}} \ge a_i - \frac{c_0}{2M}$ (recall that $a_i \in (0,1)$), and the second inequality follows from subtracting $\frac{c_0}{2M}$ times (3.25) from (3.24).

Let

$$K_n := \left\{ i \in B_n : a_i > \frac{c_0}{2M} \right\}.$$

Using this definition and the fact that $|D_i| > 2\pi r_{\text{in}} |r^i_{\text{in}} - r^i_{\text{out}}|$, (3.26) can be rewritten as

$$\sum_{i\in K_n} |D_i| > 2\pi r_{\text{in}} \sum_{i\in K_n} |r^i_{\text{in}} - r^i_{\text{out}}| \geq \pi r_{\text{in}} c_0. \tag{3.27}$$

Now we take a sufficiently large n, and discuss two cases (note that different n may lead to different cases).

Case 1. Every $i \in K_n$ satisfies $r^i_{\text{out}} - r^i_{\text{in}} \leq \min\{c_2(\frac{c_0}{2M})^2, \frac{r_{\text{in}} c_0}{4 r_{\text{out}}}\}$. By definition of K_n, we have $r^i_{\text{out}} - r^i_{\text{in}} \leq c_2(\frac{c_0}{2M})^2 \leq c_2 a_i^2$ for $i \in K_n$. (The motivation of the second term in the min function will be made clear later.) Then by (3.23), we have

$$\frac{1}{2\pi}|D_i| - p_i|_{\partial_{\text{in}} D_i} \geq c_1 a_i^2 |D_i| \geq \frac{c_1 c_0^2}{4M^2}|D_i| \quad \text{for all } i \in K_n.$$

Since K_n is a subset of B_n (and recall that $\alpha_i \geq \omega_{\min} > 0$ for all $i \in B_n$), we have the following lower bound for T_n:

$$\begin{aligned} T_n &\geq \omega_{\min}^2 \sum_{i,j\in K_n, j\prec i} \Big(\frac{1}{2\pi}|D_i| - p_i|_{\partial_{\text{in}} D_i}\Big)|D_j| \\ &\geq \omega_{\min}^2 \sum_{i,j\in K_n, j\prec i} \frac{c_1 c_0^2}{4M^2}|D_i||D_j| = \omega_{\min}^2 \sum_{i,j\in K_n, i\neq j} \frac{c_1 c_0^2}{8M^2}|D_i||D_j| \\ &= \omega_{\min}^2 \frac{c_1 c_0^2}{16M^2}\Big(\Big(\sum_{i\in K_n}|D_i|\Big)^2 - \sum_{i\in K_n}|D_i|^2\Big). \end{aligned} \tag{3.28}$$

Note that the second term in the min function in the assumption gives

$$\max_{i\in K_n} |D_i| < 2\pi r_{\text{out}}(r^i_{\text{out}} - r^i_{\text{in}}) \leq \frac{\pi r_{\text{in}} c_0}{2} \leq \frac{1}{2}\sum_{i\in K_n}|D_i|,$$

where we used (3.27) in the last inequality. Applying this to the right-hand side of (3.28) gives

$$T_n \geq \omega_{\min}^2 \frac{c_1 c_0^2}{16M^2} \cdot \frac{1}{2}\Big(\sum_{i\in K_n}|D_i|\Big)^2 \geq \omega_{\min}^2 \frac{c_1 c_0^2}{32M^2}(\pi r_{\text{in}} c_0)^2.$$

Case 2. If Case 1 is not true, then there must be some $i_0 \in K_n$ satisfying $r^{i_0}_{\text{out}} - r^{i_0}_{\text{in}} > \min\{c_2(\frac{c_0}{2M})^2, \frac{r_{\text{in}} c_0}{4 r_{\text{out}}}\} =: c_3$, which leads to

$$|o^{i_0}_{\text{in}} - o^{i_0}_{\text{out}}| = a_{i_0}(r^{i_0}_{\text{out}} - r^{i_0}_{\text{in}}) > \frac{c_0 c_3}{2M} =: l.$$

Although this set D_{i_0} is likely not thin enough for us to apply Lemma 3.3, since $|o^{i_0}_{\text{in}} - o^{i_0}_{\text{out}}|$ is bounded below by a positive constant independent of n, we can still use

Lemma 3.4 to conclude that $\frac{1}{2\pi}|D_{i_0}| - p_{i_0}|_{\partial_{\text{in}} D_{i_0}} \geq c_4$ for some $c_4 > 0$ depending only on $r_{\text{in}}, r_{\text{out}}$, and l. This leads to

$$T_n \geq \sum_{i_0 \succ j} \omega_{\min} \alpha_j c_4 |D_j| \geq \omega_{\min} c_4 \sum_{D_j \subset \text{int}(\Gamma_{\text{in}})} \alpha_j |D_j| \geq \omega_{\min} c_4 \cdot \frac{1}{2} \int_{\text{int}(\Gamma_{\text{in}})} \omega \, dx,$$

where the last inequality follows from the fact that for all sufficiently large n, the definition of ω_n gives $\sum_{D_j \subset \text{int}(\Gamma_{\text{in}})} \alpha_j |D_j| = \int_{\text{int}(\Gamma_{\text{in}})} \omega_n \, dx \geq \frac{1}{2} \int_{\text{int}(\Gamma_{\text{in}})} \omega \, dx$. Note that the last integral is positive since $\omega > 0$ on Γ_{in}, and it is clearly independent of n.

From the above discussion, for all sufficiently large n, regardless of whether we are in Case 1 or 2 for this n, we always have that T_n is bounded below by some uniformly positive constant independent of n. Therefore, taking the $n \to \infty$ limit gives

$$\liminf_{n \to \infty} \mathcal{I}^n \geq \liminf_{n \to \infty} T_n > 0.$$

This contradicts $\mathcal{I}^n = 0$, therefore finishing the proof of step 5.

Step 6. It remains to show that all connected components of $\{\omega > 0\}$ are concentric. If $\{\omega > 0\}$ has finitely many connected components, then we could proceed similarly as the end of the proof of Corollary 2.10. But since $\{\omega > 0\}$ may have countably many connected components, we need to use a different argument.

Let us denote the connected components of $\{\omega > 0\}$ by $\{U_i\}_{i \in I}$, where I may have countably many elements. Denote their centers by $\{o_i\}_{i \in I}$, their radii by $\{R_i\}_{i \in I}$, and their outer boundaries by $\{\partial_{\text{out}} U_i\}_{i \in I}$. Without loss of generality, suppose that the x-coordinates of their centers $\{o_i^1\}_{i \in I}$ are not all identical.

Among the (possibly infinitely many) collection of circles $\{\partial_{\text{out}} U_i\}_{i \in I}$, let Γ_r be the "circle with rightmost center" among them, in the following sense:

- If there exists some $i_0 \in I$ such that $o_{i_0}^1 = \sup_{i \in I} o_i^1$, then we define $\Gamma_r := \partial_{\text{out}} U_{i_0}$. (If the supremum is achieved at more than one index, then we set i_0 to be any of them.)
- Otherwise, take any subsequence $\{i_k\}_{k \in \mathbb{N}} \subset I$ such that $\lim_{k \to \infty} o_{i_k}^1 = \sup_{i \in I} o_i^1$. Since ω has compact support, we can extract a further subsequence (which we still denote by $\{i_k\}_{k \in \mathbb{N}}$) such that both o_{i_k} and r_{i_k} converge as $k \to \infty$, and denote their limit by $O_r \in \mathbb{R}^2$ and $R_r \in \mathbb{R}$. Finally, let $\Gamma_r := \partial B(O_r, R_r)$.

With the above definition, we point out that $f := \omega * \mathcal{N} = \text{const}$ on Γ_r. Note that in both cases above, we can find a sequence of level set components of ω that converges to Γ_r, in the sense that the Hausdorff distance between the two sets goes to 0. Since $f = \text{const}$ on each level set component of ω, continuity of f gives that $f = \text{const}$ on Γ_r.

Let $f_i(x) := (\omega 1_{U_i}) * \mathcal{N}$ for $i \in I$; note that by definition we have $f = \sum_{i \in I} f_i$. Lemma 2.11 gives the following:

(a) for all $x \in (\mathrm{int}(\partial_{\mathrm{out}} U_i))^c$, we have $f_i(x) = \frac{1}{2\pi}(\int_{U_i} \omega\, dx) \ln |x - o_i|$;

(b) If U_i is doubly connected, then $f_i = \mathrm{const}$ in $\mathrm{int}(\partial_{\mathrm{in}} U_i)$, where the constants are different for different i.

Note that for any $i \in I$, U_i must be either nested inside Γ_r, or have Γ_r nested in its hole. (By a slight abuse of notation, we use $i \prec \Gamma_r$ and $i \succ \Gamma_r$ to denote these two relations.) Let $\Gamma_r^R := (O_r^1 + R_r, O_r^2)$ and $\Gamma_r^L := (O_r^1 - R_r, O_r^2)$ be the rightmost/leftmost points of the circle Γ_r. Note that (b) implies that $f_i(\Gamma_r^R) = f_i(\Gamma_r^L)$ for all $i \succ \Gamma_r$, whereas (a) gives the following for all $i \prec \Gamma_r$:

$$f_i(\Gamma_r^R) = \frac{\int_{U_i} \omega\, dx}{2\pi} \ln |\Gamma_r^R - o_i| \geq \frac{\int_{U_i} \omega\, dx}{2\pi} \ln |\Gamma_r^L - o_i| = f_i(\Gamma_r^L),$$

where the inequality follows from the fact that $|O_r^1 + R_r - o_i^1| \geq |O_r^1 - R_r - o_i^1|$, which is a consequence of $o_i^1 \leq O_r^1$ due to our choice of O_r. (Also note that Γ_r^R and Γ_r^L have the same y-coordinate.)

As a result, summing over all $i \in I$ gives $f(\Gamma_r^R) \geq f(\Gamma_r^L)$, where the equality is achieved if and only if $o_i^1 = O_r$ for all $i \prec \Gamma_r$. Now we discuss two cases:

Case 1. There is some $i \prec \Gamma_r$ with $o_i^1 < O_r$. In this case, the above discussion gives $f(\Gamma_r^R) > f(\Gamma_r^L)$, which leads directly to the contradiction $f = \mathrm{const}$ on Γ_r.

Case 2. If case 1 does not hold, then let us define Γ_l as a "circle with leftmost center" among $\{\partial_{\mathrm{out}} U_i\}_{i \in I}$ in the same way as Γ_r. Then we must have $O_l^1 < O_r^1$, and since case 1 does not hold (i.e., all $i \prec \Gamma_r$ satisfy that $o_i^1 = O_r$), we must have $\Gamma_l \succ \Gamma_r$. By definition of Γ_r, there exists some U_{i_0} whose outer boundary is sufficiently close to Γ_r and whose center is sufficiently close to O_r. As a result, $i_0 \prec \Gamma_l$ and $o_{i_0}^1 > O_l^1$.

Let Γ_l^L and Γ_l^R be the leftmost/rightmost points of Γ_l. A parallel argument as above then gives that $f_i(\Gamma_l^L) \geq f_i(\Gamma_l^R)$ for all $i \in I$. Since we have found an $i_0 \prec \Gamma_l$ with $o_{i_0}^1 > O_l^1$, we have $f_{i_0}(\Gamma_l^L) > f_{i_0}(\Gamma_l^R)$; thus summing over all $i \in I$ gives the strict inequality $f(\Gamma_l^L) > f(\Gamma_l^R)$, contradicting with $f = \mathrm{const}$ on Γ_l.

In both cases above we have obtained a contradiction, thus $\{o_i\}_{i \in I}$ must have the same x-coordinate. An identical argument shows that their y-coordinates must also be identical, thus the $\{U_i\}_{i \in I}$ are concentric. Since ω is known to be radial within each U_i (about its own center) in steps 1–5, the proof is now finished. □

In the next corollary, we show that the above proof for stationary smooth solutions can be extended (with some modifications) to show radial symmetry of nonnegative rotating smooth solutions with $\Omega < 0$.

COROLLARY 3.6

Let $\omega(x,t) = \omega_0(R_{\Omega t} x)$ be a smooth, nonnegative compactly supported uniformly *rotating solution of 2D Euler with angular velocity* $\Omega < 0$. *Then* ω_0 *is radially symmetric about the origin.*

Proof

The proof is very similar to the proof of Theorem 3.5, and we only highlight the differences. Let $\{\omega_n\}$ be the same approximation for ω_0 as in step 2 of Theorem 3.5. We consider the same setting as in (3.6) and (3.7), except with $f(x)$ replaced by $f_\Omega(x) := \omega_0 * \mathcal{N} - \frac{\Omega}{2}|x|^2$. From the assumption on ω_0, we have that f_Ω is a constant on each level set component of ω_0. Thus the same computations in (3.8) give $\mathcal{I}^n = 0$ for all $n > 1$.

On the other hand, we have

$$\begin{aligned} \mathcal{I}^n &= \int_{\mathbb{R}^2} \omega_n \nabla\varphi^n \cdot \nabla(\omega_0 * \mathcal{N})\,dx + (-\Omega)\int_{\mathbb{R}^2} \omega_n \nabla\varphi^n \cdot x\,dx \\ &=: \mathcal{I}_1^n + \underbrace{(-\Omega)}_{\ge 0}\,\mathcal{I}_2^n. \end{aligned} \tag{3.29}$$

The same argument as in (2.34) of Theorem 2.12 gives that $\mathcal{I}_2^n \ge 0$. As for $\mathcal{I}_1^n$, in steps 3–5 of the proof of Theorem 3.5, we have already shown that $\liminf_{n\to\infty} \mathcal{I}_1^n \ge 0$, and the equality is achieved if and only if each connected component of $\{\omega_0 > 0\}$ is radially symmetric up to a translation, and they are all nested.

Let us decompose $\operatorname{supp}\omega_0$ into (possibly infinitely many) connected components $\bigcup_{i\in I} U_i$, with centers $\{o_i\}_{i\in I}$. Our goal is to show $o_i \equiv (0,0)$ for $i \in I$. Note that it suffices to show that their x-coordinates satisfy $\sup_{i\in I} o_i^1 \le 0$. Once we prove this, a parallel argument gives $\inf_{i\in I} o_i^1 \ge 0$, which implies $o_i^1 \equiv 0$ for $i \in I$, and the same can be done for the y-coordinate.

Towards a contradiction, suppose that $\sup_{i\in I} o_i^1 > 0$. We can then define Γ_r in the same way as step 6 of the proof of Theorem 3.5, that is, it is the “circle with rightmost center” among $\{\partial_{\text{out}} U_i\}_{i\in I}$, and its center O_r satisfies $O_r^1 = \sup_{i\in I} o_i^1 > 0$. Since $f_\Omega = \text{const}$ along each level set component of ω_0, we again have that $f_\Omega = \text{const}$ on Γ_r. Let Γ_r^R and Γ_r^L be the rightmost/leftmost points on Γ_r. Note that their distances to the origin satisfy $|\Gamma_r^R| > |\Gamma_r^L|$, where the strict inequality is due to the assumption $O_r^1 > 0$.

Let us define $f_i(x) = (\omega_0 1_{U_i}) * \mathcal{N}$ for $i \in I$, and note that $f_\Omega = (\sum_{i\in I} f_i) - \Omega|x|^2$. Properties (a) and (b) in step 6 of Theorem 3.5 still hold for f_i, thus we have $f_i(\Gamma_r^R) \ge f_i(\Gamma_r^L)$ for all $i \in I$. This leads to

$$f_\Omega(\Gamma_r^R) = \Big(\sum_{i\in I} f_i(\Gamma_r^R)\Big) + \underbrace{(-\Omega)}_{>0}|\Gamma_r^R|^2 > \Big(\sum_{i\in I} f_i(\Gamma_r^L)\Big) + (-\Omega)|\Gamma_r^L|^2 = f_\Omega(\Gamma_r^L),$$

contradicting the fact that $f_\Omega \equiv \text{const}$ on Γ_r. $\square$

3.1. *Proofs of the quantitative lemmas*

Before the proof of Lemma 3.2, let us state two lemmas that we will use in the proof. The first one is a quantitative version of the isoperimetric inequality obtained by Fusco, Maggi, and Pratelli [34].

LEMMA 3.7 (cf. [34, Section 1.2])
Let $E \subseteq \mathbb{R}^2$ be a bounded open set. Then there is some constant $c \in (0,1)$ such that

$$P(E) \geq 2\sqrt{\pi}\sqrt{|E|}\big(1 + c\mathcal{A}(E)^2\big),$$

where $P(E) = \mathcal{H}^1(\partial E)$ denotes the perimeter of E.

The second lemma is a simple result relating the Fraenkel asymmetry of a set E with its subsets U.

LEMMA 3.8 (cf. [27, Lemma 4.4])
Let $E \subseteq \mathbb{R}^2$ be a bounded open set. For all $U \subseteq E$ satisfying $|U| \geq |E|(1 - \frac{\mathcal{A}(E)}{4})$, we have

$$\mathcal{A}(U) \geq \frac{\mathcal{A}(E)}{4}.$$

Proof of Lemma 3.2
The proof of the Lemma 3.2 is similar to [27, Proposition 4.5] obtained by Craig, Kim, and the last author. For the sake of completeness, we give a proof below. Let $g(k)$, D_k, and $\tilde{D}_k$ be defined as in Proposition 2.8, let $\tilde{D} = D \cup \overline{h}$, and define $p_h := p|_{\partial h}$. We start by following the proof of Proposition 2.8, except that after obtaining (2.19), instead of using the isoperimetric inequality, we use the stability version in Lemma 3.7 to control $P(\tilde{D}_k)$. This gives

$$\begin{aligned} g'(k)\big(g(k) + |h|\mathbb{1}_{p_h>k}\big) &\leq -\frac{1}{2}P(\tilde{D}_k)^2 \\ &\leq -2\pi|\tilde{D}_k|\big(1 + c\mathcal{A}(\tilde{D}_k)^2\big)^2 \\ &\leq -2\pi\big(g(k) + |h|\mathbb{1}_{p_h>k}\big)\big(1 + c\mathcal{A}(\tilde{D}_k)^2\big). \end{aligned}$$

Hence it follows from Lemma 3.8 that

$$g'(k) \leq -2\pi\Big(1 + c\frac{\mathcal{A}(\tilde{D})^2}{16}\Big) \quad \text{for all } k \text{ such that } |\tilde{D}_k| \geq |\tilde{D}|\Big(1 - \frac{\mathcal{A}(\tilde{D})}{4}\Big). \tag{3.30}$$

We claim that

$$g(k) \leq |D| - 2\pi\Big(1 + c\frac{\mathcal{A}(\tilde{D})^2}{16}\Big)k \quad \text{for } k \leq \min\Big\{p_h, \frac{\mathcal{A}(\tilde{D})|\tilde{D}|}{16\pi}\Big\}. \tag{3.31}$$

Towards a contradiction, suppose that there is $k_0 \le \min\{p_h, \frac{\mathcal{A}(\tilde{D})|\tilde{D}|}{16\pi}\}$ such that (3.31) is violated. Since $1 + c\frac{\mathcal{A}(\tilde{D})^2}{16} \le 2$, we have

$$g(k_0) > |D| - 4\pi k_0 \ge |D| - \frac{\mathcal{A}(\tilde{D})|\tilde{D}|}{4}.$$

Therefore,

$$\begin{aligned}|\tilde{D}_{k_0}| &= g(k_0) + |h| \\ &> |D| - \frac{\mathcal{A}(\tilde{D})|\tilde{D}|}{4} + |h| \\ &= |\tilde{D}| - \frac{\mathcal{A}(\tilde{D})|\tilde{D}|}{4} = |\tilde{D}|\Big(1 - \frac{\mathcal{A}(\tilde{D})}{4}\Big).\end{aligned}$$

Hence for all $k \in (0, k_0]$, $g'(k)$ satisfies the inequality (3.30). Thus we have

$$\begin{aligned}g(k_0) &\le \int_0^{k_0} -2\pi\Big(1 + c\frac{\mathcal{A}(\tilde{D})^2}{16}\Big)\,dk + |D| \\ &= |D| - 2\pi\Big(1 + c\frac{\mathcal{A}(\tilde{D})^2}{16}\Big)k_0,\end{aligned}$$

contradicting our assumption on k_0.

Finally, to control p_h, we discuss two cases below, depending on which one in the minimum function in (3.31) is smaller. For simplicity, we denote $A := \frac{\mathcal{A}(\tilde{D})|\tilde{D}|}{16\pi}$ and $B := c\frac{\mathcal{A}(\tilde{D})^2}{16}$.

Case 1: $p_h \le A$. In this case (3.31) holds for all $k \le p_h$. Thus

$$0 \le g(p_h) \le |D| - 2\pi(1 + B)p_h,$$

implying that

$$p_h \le \frac{|D|}{2\pi(1 + B)} \le \frac{|D|}{2\pi}(1 - c_0)$$

for some constant c_0 which depends only on $\mathcal{A}(\tilde{D})$.

Case 2: $p_h > A$. In this case (3.31) gives $g(A) \le |D| - 2\pi(1 + B)A$ and we use a crude bound for $k \ge A$, that is, $g'(k) \le -2\pi$. Therefore, for $k > A$,

$$\begin{aligned}g(k) &= \int_A^k g'(k)\,dk + g(A) \le -2\pi(k - A) + |D| - 2\pi(1 + B)A \\ &= |D| - 2\pi k - 2\pi AB \\ &\le |D|\Big(1 - \frac{\mathcal{A}(\tilde{D})}{8}B\Big) - 2\pi k,\end{aligned}$$

where the last inequality follows from $A > \frac{\mathcal{A}(\tilde{D})|D|}{16\pi}$. Plugging in $k = p_h$ gives

$$0 \le g(p_h) \le |D|\Big(1 - \frac{\mathcal{A}(\tilde{D})}{8} B\Big) - 2\pi p_h,$$

leading to

$$p_h \le \frac{|D|}{2\pi}(1 - c_0),$$

where again c_0 depends only on $\mathcal{A}(\tilde{D})$. □

Next we prove Lemma 3.3.

Proof of Lemma 3.3
Without loss of generality, we can assume that $o_1 = (0,0)$ and $o_2 = (a\epsilon, 0)$. To estimate $p|_{\partial B_1}$, we decompose p into

$$p = p|_{\partial B_1} g + u,$$

where g satisfies

$$\begin{cases} \Delta g = 0 & \text{in } D, \\ g = 1 & \text{on } \partial B_1, \\ g = 0 & \text{on } \partial B_2, \end{cases} \tag{3.32}$$

and u satisfies

$$\begin{cases} \Delta u = -2 & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases} \tag{3.33}$$

Using this decomposition as well as the definition of p, we have

$$-2|B_1| = \int_{\partial B_1} \nabla p \cdot \vec{n}\, d\sigma = p|_{\partial B_1} \int_{\partial B_1} \nabla g \cdot \vec{n}\, d\sigma + \int_{\partial B_1} \nabla u \cdot \vec{n}\, d\sigma,$$

where $\vec{n}$ is the outer normal of B_1 throughout this proof. Thus

$$p|_{\partial B_1} = \frac{1}{\int_{\partial B_1} \nabla g \cdot \vec{n}\, d\sigma}\Big(-2\pi - \int_{\partial B_1} \nabla u \cdot \vec{n}\, d\sigma\Big). \tag{3.34}$$

To estimate $p|_{\partial B_1}$, it remains to estimate the two integrals in (3.34).

The function g can be explicitly constructed using the conformal mapping from D to a perfect annulus centered at 0. Consider the Möbius map $h : \mathbb{C} \to \mathbb{C}$ given by

$$h(z) := \frac{z+b}{1+bz},$$

where $b \in \mathbb{R}$ will be fixed soon. Note that the unit circle and the real line are both invariant under h, and ∂B_2 is mapped to some circle centered on the real line. In order to make $h(\partial B_2)$ centered at 0, since the left/right endpoints of ∂B_2 are $\pm(1+\epsilon)+a\epsilon$, we look for $b \in \mathbb{R}$ that solves

$$h(1+\epsilon+a\epsilon) = -h(-1-\epsilon+a\epsilon). \tag{3.35}$$

Plugging the definition of h into the above equation, we know that b is a root of the quadratic polynomial

$$f(b) := b^2 - \frac{2+(1-a^2)\epsilon}{a} b + 1.$$

Clearly, for $0 < a < 1$, f has two positive roots whose product is 1, thus one is in $(0,1)$ and the other in $(1,+\infty)$. We define b to be the root in $(0,1)$. One can easily check that $f(a) < 0$, and $f(\frac{a}{2}) > 0$ if $a^2 > 2(1-a^2)\epsilon$, which is true due to our assumption $a^2 > 64\epsilon$. Thus for all $\epsilon \in (0, \frac{a^2}{64})$, we have

$$0 < \frac{a}{2} < b < a < 1.$$

Note that h is holomorphic in $\mathbb{C}$ except at the two singularity points $-b$ and $-\frac{1}{b}$. We have already shown that $-b \in B_1$, thus it is outside of D. Next we will show that $-\frac{1}{b} \in B_2^c$, which is thus also outside of D. To see this, note that

$$\begin{aligned}\frac{-1-\epsilon+a\epsilon+b}{1+b(-1-\epsilon+a\epsilon)} &= h(-1-\epsilon+a\epsilon)\\ &= -h(1+\epsilon+a\epsilon) = -\frac{1+\epsilon+a\epsilon+b}{1+b(1+\epsilon+a\epsilon)} < 0,\end{aligned}$$

where the inequality follows from the fact that $a, b, \epsilon > 0$. Since the numerator of the left-hand side is already known to be negative due to $a, b \in (0,1)$, its denominator must be positive, leading to $-\frac{1}{b} < -1-\epsilon+a\epsilon$, that is, $-\frac{1}{b} \in B_2^c$.

Now we define $g : \mathbb{R}^2 \setminus \{(-b,0) \cup (-1/b,0)\} \to \mathbb{R}$ as

$$g(x) := -\frac{1}{\log|h(1+\epsilon+a\epsilon)|} \log\big|h(z)\big| + 1 \quad \text{for } z = x_1 + ix_2.$$

Let us check that g indeed satisfies (3.32). First note that g satisfies the boundary conditions in (3.32), since h maps D to a perfect annulus centered at the origin, whose inner boundary is ∂B_1. In addition, g is harmonic in $\mathbb{R}^2 \setminus \{(-b,0) \cup (-1/b,0)\}$, thus harmonic in D.

Using the explicit formula of g, we have

$$\Delta g(x) = -\frac{2\pi}{\log|h(1+\epsilon+a\epsilon)|}\big(\delta_{(-b,0)}(x) - \delta_{(-\frac{1}{b},0)}(x)\big)$$

in the distribution sense. We can then apply the divergence theorem to g in B_1, and compute the integral containing g in (3.34) explicitly as

$$\int_{\partial B_1} \nabla g \cdot \vec{n}\, d\sigma = -\frac{2\pi}{\log|h(1+\epsilon+a\epsilon)|}. \tag{3.36}$$

As for the integral containing u in (3.34), we compare u with a radial barrier function

$$w(x) := -2\big(|x|-1\big)\big(|x|-1-2\epsilon\big),$$

which satisfies $w = 0$ on ∂B_1 and $w > 0$ on ∂B_2. Note that

$$\Delta w = \Big(\partial_{rr} + \frac{1}{r}\partial_r\Big)w = -8 + \frac{4+4\epsilon}{r} \le -2 \quad \text{in } D,$$

where we used that $\epsilon \in (0, \frac{1}{2})$ and $r > 1$ in D in the last inequality. Thus $w - u$ is superharmonic in D and nonnegative on ∂D, which allows us to apply the classical maximum principle to obtain $u \le w$ in $\overline{D}$. Combining this with the fact that $u = w = 0$ on ∂B_1, we have

$$\nabla u(x) \cdot \vec{n}(x) \le \nabla w(x) \cdot \vec{n}(x) = \frac{d}{dr}w(r)|_{r=1} = 4\epsilon \quad \text{for all } x \in \partial B_1,$$

hence

$$\int_{\partial B_1} \nabla u \cdot \vec{n}\, d\sigma \le 8\pi\epsilon. \tag{3.37}$$

Plugging (3.36) and (3.37) into (3.34), we obtain

$$p|_{\partial B_1} \le \log\big(\big|h(1+\epsilon+a\epsilon)\big|\big)(1+4\epsilon).$$

Since $\log s \le s - 1$ for $s > 1$, it follows that

$$\begin{aligned}
\log\big|h(1+\epsilon+a\epsilon)\big| &\le h(1+\epsilon+a\epsilon) - 1 = \frac{1+\epsilon+a\epsilon+b}{1+b(1+\epsilon+a\epsilon)} - 1 \\
&= \epsilon\Big(1 + \frac{a-2b-ab-b\epsilon-ab\epsilon}{1+b(1+\epsilon+a\epsilon)}\Big) \\
&\le \epsilon\Big(1 - \frac{ab}{4}\Big) \le \epsilon\Big(1 - \frac{a^2}{8}\Big),
\end{aligned}$$

where we used $b > \frac{a}{2}$ to obtain the last two inequalities. Finally, using that $\epsilon < \frac{a^2}{64}$, we have

$$p|_{\partial B_1} \le \epsilon\Big(1 - \frac{a^2}{8}\Big)\Big(1 + \frac{a^2}{16}\Big) \le \epsilon\Big(1 - \frac{1}{16}a^2\Big) < \frac{|D|}{2\pi}\Big(1 - \frac{1}{16}a^2\Big),$$

where in the last step we use that $|D| = \pi(1+\epsilon)^2 - \pi > 2\pi\epsilon$. This finishes the proof of the lemma. □

Finally we give the proof of Lemma 3.4.

Proof of Lemma 3.4
Without loss of generality, we can assume that o_2 is the origin. Let $\beta := p|_{\partial B_r}$. From the proof of Proposition 2.8, we already know that $g'(k) \le -2\pi$, where $g(k) := |\{x \in D : p(x) > k\}|$. This implies that $g(k) \ge -2\pi(k - \beta)$. Therefore, we have

$$\int_D p\,dx \ge \int_0^\beta g(k)\,dk \ge \int_0^\beta -2\pi(k-\beta)\,dk = \pi\beta^2.$$

On the other hand, the same computation in the proof of Lemma 2.13 gives

$$\beta|B_r| + \int_D p\,dx = \frac{1}{2}\int_D |\nabla p|^2\,dx \le \frac{1}{2}\int_D |x|^2\,dx.$$

Since

$$\begin{aligned}\frac{1}{2}\int_D |x|^2\,dx &= \frac{1}{2}\Big(\int_{B_R} |x|^2\,dx - \int_{B_r} |x|^2\,dx\Big)\\ &= \frac{|D|^2}{4\pi} + \frac{|D||B_r|}{2\pi} + \frac{|B_r|^2}{4\pi} - \frac{1}{2}\int_{B_r} |x|^2\,dx\\ &= \frac{|D|^2}{4\pi} + \frac{|D||B_r|}{2\pi} - \frac{l^2|B_r|}{2},\end{aligned}$$

it follows that

$$\pi\beta^2 + \beta|B_r| \le \frac{|D|^2}{4\pi} + \frac{|D||B_r|}{2\pi} - \frac{l^2|B_r|}{2}. \tag{3.38}$$

By solving the quadratic inequality (3.38), we find that

$$\beta \le \frac{|D|}{2\pi}(1 - c_0)$$

for some constant c_0 which depends only on δ_1, δ_2, and l. □

4. Radial symmetry for stationary/rotating gSQG solutions with $\Omega \le 0$

In this section, we consider the family of gSQG equations with $0 < \alpha < 2$, and study the symmetry property for rotating patch/smooth solutions with angular velocity $\Omega \le 0$.

Let us deal with patch solutions first. As we have discussed in the Introduction, we cannot expect a nonsimply connected patch D with $\Omega \le 0$ to be radial, due to the nonradial examples in [28] and [41] for $\alpha \in (0,2)$. For a simply connected patch D, the constant on the right-hand side of (1.6) is the same on ∂D, which motivates us to consider Question 2 in the Introduction. The goal of this section is to prove Theorem C, which gives an affirmative answer to Question 2 for the whole range $\alpha \in [0,2)$.

Our results are not limited to the Riesz potentials $K_{\alpha,d}$ in (1.4); in fact, we only need the potential being radially increasing and not too singular at the origin. Below we state our assumption on the potential K, which covers the whole range of $K_{\alpha,d}$ with $\alpha \in [0,2)$.

(HK) Let $K \in C^1(\mathbb{R}^d \setminus \{0\})$ be radially symmetric with $K'(r) > 0$ for all $r > 0$. (Here we denote $K(x) = K(r)$ by a slight abuse of notation.) Also assume that there is some $C > 0, \delta > 0$ such that $K'(r) \le C r^{-d-1+\delta}$ for all $0 < r \le 1$.

Our proof is done by a variational approach, which relies on a continuous Steiner symmetrization argument similar to that of [13].

4.1. *Definition and properties of continuous Steiner symmetrization*

Below we define the continuous Steiner symmetrization for a bounded open set $D \subset \mathbb{R}^d$ with respect to the direction $e_1 = (1,0,\dots,0)$, which can be easily adapted to any other direction in $\mathbb{R}^d$. The definition is the same as [13, Section 2.2.1], which we briefly outline below for completeness.

For a 1-dimensional open set $U \subset \mathbb{R}$, we define its continuous Steiner symmetrization $M^\tau[U]$ as follows. If $U = (a,b)$ is an open interval, then $M^\tau[U]$ shifts the midpoint of this interval towards the origin with velocity 1, while preserving the length of interval. That is,

$$M^\tau[U] := \begin{cases} \left(a - \tau \operatorname{sgn}(\frac{a+b}{2}), b - \tau \operatorname{sgn}(\frac{a+b}{2})\right) & \text{for } 0 \le \tau < \frac{|a+b|}{2}, \\ \left(-\frac{b-a}{2}, \frac{b-a}{2}\right) & \text{for } \tau \ge \frac{|a+b|}{2}. \end{cases}$$

If $U = \bigcup_{i=1}^N U_i$ is a finite union of open intervals, then $M^\tau[U]$ is defined by $\bigcup_{i=1}^N M^\tau[U_i]$, and as soon as two intervals touch each other, we merge them into one interval as in [13, Definition 2.10(2)]. Finally, if $U = \bigcup_{i=1}^\infty U_i$ is a countable union of open intervals, we define $M^\tau[U]$ as a limit of $M^\tau[\bigcup_{i=1}^N U_i]$ as $N \to \infty$ as in [13, Definition 2.10(3)]. See [13, Figure 1] for an illustration of $M^\tau[U]$.

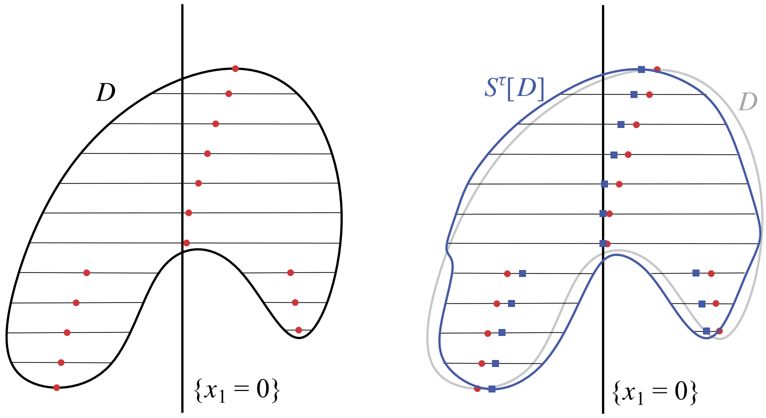

Figure 5. (Color online) Illustration of the continuous Steiner symmetrization $S^\tau[D]$ for a set $D \subset \mathbb{R}^2$. The left figure is the set D, with the midpoints of all subintervals of its 1D section highlighted in red circles. The right figure shows the set $S^\tau[D]$ for some small $\tau > 0$, with the new midpoints denoted by blue squares.

Next we move on to higher dimensions. We denote a point $x \in \mathbb{R}^d$ by (x_1, x'), where $x' = (x_2, \dots, x_d) \in \mathbb{R}^{d-1}$. For a bounded open set $D \subset \mathbb{R}^d$ and any $x' \in \mathbb{R}^{d-1}$, we define the *section* of D with respect to the direction x_1 as

$$D_{x'} := \big\{x_1 \in \mathbb{R} : (x_1, x') \in D\big\},$$

which is an open set in $\mathbb{R}$. If the section $D_{x'}$ is a single open interval centered at 0 for all $x' \in \mathbb{R}^{d-1}$, then we say that the set D is *Steiner symmetric* about the hyperplane $\{x_1 = 0\}$. Note that this definition is stronger than being symmetric about $\{x_1 = 0\}$. For example, an annulus in $\mathbb{R}^2$ is symmetric about $\{x_1 = 0\}$, but not Steiner symmetric about it.

Finally, for any $\tau > 0$, the *continuous Steiner symmetrization* of $D \subset \mathbb{R}^d$ is defined as

$$S^\tau[D] := \big\{(x_1, x') \in \mathbb{R}^d : x_1 \in M^\tau[D_{x'}]\big\},$$

with M^τ given above being the continuous Steiner symmetrization for 1-dimensional open sets. See Figure 5 for a comparison of the sets D and $S^\tau[D]$ for small $\tau > 0$.

One can easily check that $S^\tau[D]$ satisfies the following properties.

LEMMA 4.1
For any bounded open set $D \subset \mathbb{R}^d$, its continuous Steiner symmetrization $S^\tau[D]$ satisfies the following properties:

(a) $|S^\tau[D]| = |D|$ *for any* $\tau > 0$*, where* $|\cdot|$ *denotes the Lebesgue measure in* $\mathbb{R}^d$*;*

(b) $(S^\tau[D])\triangle D \subset B^\tau[D]$ *for any* $\tau > 0$*, where* $\triangle$ *is the symmetric difference between the two sets, and* $B^\tau[D]$ *is the* τ*-neighborhood of* ∂D*, given by*

$$B^\tau[D] := \{x \in \mathbb{R}^d : \operatorname{dist}(x, \partial D) \le \tau\}. \tag{4.1}$$

Proof

Property (a) is a direct consequence of the fact that $|M^\tau[U]| = |U|$ for any open set $U \subset \mathbb{R}$ and $\tau > 0$ (see [13, Lemma 2.11(b)]). To prove (b), one can start with the 1-dimensional version. For any bounded open set $U \subset \mathbb{R}$, we have $M^\tau[U]\triangle U \subset \{x \in \mathbb{R} : \operatorname{dist}(x, \partial U) \le \tau\}$, which follows from the fact that the intervals move with velocity at most 1. Thus for any bounded open set $D \subset \mathbb{R}^d$,

$$\begin{aligned} S^\tau[D]\triangle D &= \{(x_1, x') \in \mathbb{R}^d : x_1 \in M^\tau[D_{x'}]\triangle D_{x'}\} \\ &\subset \{(x_1, x') : \operatorname{dist}(x_1, \partial(D_{x'})) \le \tau\} \\ &\subset B^\tau[D], \end{aligned}$$

finishing the proof. ☐

4.2. *Simply connected patch solutions with* $\Omega \le 0$

We assume that $D \subset \mathbb{R}^d$ satisfies the following condition.

(HD) $D \subset \mathbb{R}^d$ is an open set, and there exists some $M > 0$ depending on D such that $|B^\tau[D]| \le M\tau$ for all sufficiently small $\tau > 0$, where $B^\tau[D]$ is given in (4.1).

It can be easily checked that for $d \ge 2$, any bounded open set D with Lipschitz continuous boundary satisfies condition **(HD)**. In fact, for $d = 2$, we will show that any bounded open set $D \subset \mathbb{R}^2$ with a rectifiable boundary satisfies **(HD)**, with a precise bound

$$\left|B^\tau[D]\right| \le 2|\partial D|\tau \quad \text{for all } \tau \ge 0, \tag{4.2}$$

where $|\partial D|$ is the total length of ∂D. Let us first prove that (4.2) holds for any polygon $P \subset \mathbb{R}^2$. Erect two polygons at distance τ from P with the transversal sides being bisectors of the inner angles of P (see Figure 6). It is clear that $B^\tau[P]$ is contained in the trapezoidal region, which has area no more than $2|\partial P|\tau$. Finally, this can be extended to the general case by approximating any rectifiable curve by polygons.

Below we state our main theorem of this section, which is slightly more general than Theorem C.

THEOREM 4.2

Let $D \subset \mathbb{R}^d$ *and* $K \in C^1(\mathbb{R}^d \setminus \{0\})$ *satisfy the conditions* **(HD)** and **(HK)***, respectively. Let* $g \in C^1(\mathbb{R}^d)$ *be a radial function with* $g'(r) > 0$ *for all* $r > 0$*.*

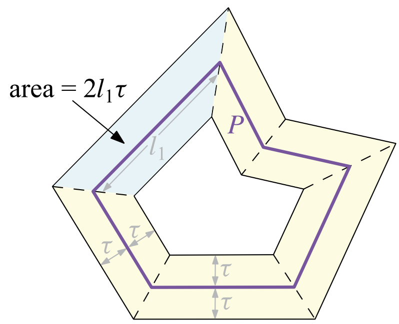

Figure 6. (Color online) Illustration of the polygon P and the underlying trapezoidal region (the whole colored region). Here the blue trapezoid has area $2l_1\tau$ (l_1 is the corresponding side length in P), and summing over all edges gives a total area of $2|\partial P|\tau$. Since the trapezoids may intersect for large τ, the whole trapezoidal region has area no more than $2|\partial P|\tau$.

If D satisfies that

$$1_D * K - \frac{\Omega}{2} g(x) = \text{const} \quad \textit{on } \partial D \tag{4.3}$$

for some $\Omega \leq 0$ (where the constant is the same on all connected components of ∂D), then D is a ball. Moreover, the ball is centered at the origin if $\Omega < 0$.

Remark 4.3

(1) Note that D does not need to be simply connected in Theorem 4.2. However, since the constant on the right-hand side of (4.3) is assumed to be the same on all connected components of ∂D, comparing with (1.6), Theorem 4.2 only implies that all simply connected patches with $\Omega \leq 0$ must be a disk.

(2) In the case $\Omega = 0$, the problem is translation-invariant, so in the proof we assume without loss of generality that the center of mass of D is at the origin.

Proof

We prove it by contradiction. Without loss of generality, we assume that K satisfies the additional assumption that $K(1) = 0$ (note that (4.3) still holds if we add any constant to K), and D is not Steiner symmetric about the hyperplane $\{x_1 = 0\}$. Let $D^\tau := S^\tau[D]$ be the continuous Steiner symmetrization of D at time $\tau > 0$. By Lemma 4.1(b), we have

$$D^\tau \triangle D \subset B^\tau[D], \tag{4.4}$$

where B^τ is defined as in (4.1). Let us consider the functional

$$\mathcal{E}[D] := \underbrace{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} 1_D(x)1_D(y)K(x-y)\,dx\,dy}_{=:\mathcal{I}[D]} + \underbrace{(-\Omega)\int_{\mathbb{R}^d} g(x)1_D(x)\,dx}_{=:\mathcal{V}[D]}.$$

We will use two different ways to compute $\frac{d^+}{d\tau}\mathcal{E}[D^\tau]|_{\tau=0}$, where $\frac{d^+}{d\tau}$ denotes the right derivative. On the one hand, using the equation (4.3) and the regularity assumptions on D, K, and g, we aim to show that

$$\frac{d^+}{d\tau}\mathcal{E}[D^\tau]|_{\tau=0} = 0. \tag{4.5}$$

Instead of directly taking the derivative, we consider the finite difference

$$\begin{aligned}\mathcal{E}[D^\tau] - \mathcal{E}[D] &= \underbrace{\int_{\mathbb{R}^d} 2(1_{D^\tau} - 1_D)\Big(1_D * K - \frac{\Omega}{2}g(x)\Big)\,dx}_{=:I_1} \\ &\quad + \underbrace{\int_{\mathbb{R}^d} (1_{D^\tau} - 1_D)\big((1_{D^\tau} - 1_D) * K\big)\,dx}_{=:I_2},\end{aligned}$$

where in the equality we used that $\int 1_D(1_{D^\tau} * K)\,dx = \int 1_{D^\tau}(1_D * K)\,dx$ for any radial kernel K.

Let us control the term I_1 first. First note that (4.4) implies that the integrand is supported in $B_\tau[D]$. Next we claim that **(HK)** implies $1_D * K - \frac{\Omega}{2}g \in C^{0,\delta'}(\mathbb{R}^d)$ for $\delta' := \min\{\delta, 1\}$, where $C^{0,1}$ stands for *Lipschitz continuity*. The proof is a simple potential theory estimate, which we provide below for completeness. For any $x, z \in \mathbb{R}^d$,

$$\begin{aligned}&\big|(1_D * K)(x+z) - (1_D * K)(x)\big| \\ &\quad = \Big|\int_{\mathbb{R}^d} 1_D(x-y)\big(K(y+z) - K(y)\big)\,dy\Big| \\ &\quad \le \int_{|y|<2|z|} \big|K(y+z) - K(y)\big|\,dy + \int_{|y|>2|z|} 1_D(x-y)\big|K(y+z) - K(y)\big|\,dy \\ &\quad =: J_1 + J_2.\end{aligned}$$

Since $(1_D * K)(x) \in L^\infty(\mathbb{R}^d)$, one can moreover assume that $|z| < \frac{1}{3}$; then, a crude estimate gives

$$J_1 \le \int_{|y|<2|z|} \big|K(y+z)\big| + \big|K(y)\big|\,dy \le 2\int_{|y|<3|z|} \big|K(y)\big|\,dy \le C(d)|z|^\delta,$$

where in the last step we used that **(HK)** and $K(1) = 0$ imply $|K(y)| \le C|y|^{-d+\delta}$ for $|y| \le 1$. For J_2, note that **(HK)** and the mean value theorem gives

$$\big|K(y+z)-K(y)\big| \le C|y|^{-d-1+\delta}|z| \quad \text{for all } |y|>2|z|,$$

and plugging it into the integral gives $J_2 \le C(d,|D|)|z|^{\delta}$. Putting the estimates for J_1 and J_2 together gives that $1_D * K \in C^{0,\delta'}(\mathbb{R}^d)$ for $\delta' = \min\{\delta, 1\}$, and combining this with the assumption $g \in C^1(\mathbb{R}^d)$ gives $1_D * K - \frac{\Omega}{2} g \in C^{0,\delta'}(\mathbb{R}^d)$.

In addition, by (4.3), we have $1_D * K - \frac{\Omega}{2} g(x) \equiv C_0$ on ∂D for some constant C_0. Thus we have

$$\Big|1_D * K - \frac{\Omega}{2} g(x) - C_0\Big| \le C\big(\delta', d, |D|\big)\tau^{\delta'} \quad \text{in } B^{\tau}[D]$$

for some constant $C>0$, where we used the Hölder continuity of $1_D * K - \frac{\Omega}{2} g$ and the definition of $B^{\tau}[D]$. This leads to

$$|I_1| \le 2\big|B^{\tau}[D]\big| \sup_{x\in B^{\tau}[D]} \Big|1_D * K - \frac{\Omega}{2} g(x) - C_0\Big| \le C\big(\delta', d, |D|\big) M \tau^{1+\delta'},$$

where in the first inequality we used that $\int_{B_\tau} (1_D - 1_{D^\tau}) C_0 \, dx = 0$, which follows from Lemma 4.1(a), and in the second inequality we used **(HD)**.

Next, using (4.4) we control I_2 by the crude bound

$$\begin{aligned} |I_2| &\le \int_{\mathbb{R}^d} 1_{B^{\tau}[D]} \big|(1_{B^{\tau}[D]} * K)\big| \, dx \\ &\le \big|B^{\tau}[D]\big| \|1_{B^{\tau}[D]} * K\|_{\infty} \\ &\le M\tau \big\|(1_{B^{\tau}[D]})^* * K\big\|_{\infty}, \end{aligned}$$

where the last step follows from the Hardy–Littlewood inequality, where $(1_{B^{\tau}[D]})^*$ is the radial decreasing rearrangement of $1_{B^{\tau}[D]}$. By **(HD)**, $(1_{B^{\tau}[D]})^*$ is a characteristic function of a ball whose radius is bounded by $C(d)(M\tau)^{1/d}$, thus

$$\begin{aligned} \big\|(1_{B^{\tau}[D]})^* * K\big\|_{\infty} &\le \int_0^{C(d)(M\tau)^{1/d}} \big|K(r)\big| \omega_d r^{d-1} \, dr \\ &\le \int_0^{C(d)(M\tau)^{1/d}} \omega_d r^{-1+\delta} \, dx \le C(d)(M\tau)^{\frac{\delta}{d}}, \end{aligned}$$

and plugging it into the I_2 estimate gives

$$|I_2| \le C(d) M^{\frac{d+\delta}{d}} \tau^{1+\frac{\delta}{d}}.$$

Putting the estimates of I_1 and I_2 together directly yields

$$\frac{|\mathcal{E}[D^{\tau}] - \mathcal{E}[D]|}{\tau} \le C\big(\delta', d, M, |D|\big) \tau^{\min\{\frac{\delta}{d}, \delta'\}},$$

and since $\delta > 0$, we have $\frac{d^+}{d\tau} \mathcal{E}[D^{\tau}]|_{\tau=0} = 0$.

Now, we use another way to calculate $\frac{d^+}{d\tau}\mathcal{E}[D^\tau]|_{\tau=0}$. Let us deal with the $\Omega < 0$ case first. Since K is radial and increasing in r, it has been shown in [8, Corollary 2] and [63, Theorem 3.7] that the interaction energy $\mathcal{I}[D^\tau] = \int_{D^\tau}\int_{D^\tau} K(x-y)\,dx\,dy$ is nonincreasing along the continuous Steiner symmetrization, leading to

$$\frac{d^+}{d\tau}\mathcal{I}[D^\tau] \le 0 \quad \text{for all } \tau \ge 0.$$

For the other term $\mathcal{V}[D^\tau] = (-\Omega)\int_{D^\tau} g(x)\,dx$, by the assumptions that $g'(r) > 0$ for all $r > 0$ and D is not Steiner symmetric about $\{x_1 = 0\}$, we can use [13, Lemma 2.22] to show that, for $\Omega < 0$,

$$\frac{d^+}{d\tau}\mathcal{V}[D^\tau]|_{\tau=0} = (-\Omega)\frac{d^+}{d\tau}\int_{D^\tau} g(x)\,dx|_{\tau=0} < 0.$$

Adding them together gives

$$\frac{d^+}{d\tau}\mathcal{E}[D^\tau]|_{\tau=0} < 0,$$

leading to a contradiction with (4.5).

In the $\Omega = 0$ case, recall that we assume that the center of mass of D is at the origin. Thus if D is not Steiner symmetric about $\{x_1 = 0\}$, then the same proof as [13, Proposition 2.15] gives that $\mathcal{I}[D]$ must be decreasing to the first order for a short time, leading to

$$\frac{d^+}{d\tau}\mathcal{E}[D^\tau]|_{\tau=0} = \frac{d^+}{d\tau}\int_{D^\tau}\int_{D^\tau} K\big(|x-y|\big)\,dx\,dy|_{\tau=0} < 0,$$

again contradicting (4.5). We point out that although the proposition was stated for continuous densities, the same proof works for the patch setting. In addition, although [13] only dealt with the kernels no more singular than Newtonian potential, the proof indeed holds for all kernels K satisfying **(HK)**: see [14, Theorem 6] for an extension to all Riesz potentials $K_{\alpha,d}$ with $\alpha \in (0,2)$. □

The above theorem immediately leads to the following result concerning a simply connected stationary/rotating patch solution with $\Omega \le 0$.

THEOREM 4.4
Let $D \subset \mathbb{R}^2$ be a bounded, simply connected domain with rectifiable boundary. If 1_D is a V-state for (1.2) for some $\alpha \in [0,2)$ with angular velocity $\Omega \le 0$, then D must be a disk. In addition, the disk must be centered at the origin if $\Omega < 0$.

Proof
We have $1_D * K - \frac{\Omega}{2}|x|^2 = C$ for some constant C on ∂D. For the Euler equation,

$K = \frac{1}{2\pi} \ln |x|$. For the g-SQG equation, $K = -C_\alpha |x|^{-\alpha}$. In both cases, the proof follows from Theorem 4.2. □

Remark 4.5
As we discussed in the beginning of this subsection, in the case of gSQG with $\alpha \in (0, 2)$, Theorem 4.4 is not true if the simply connected assumption is dropped, due to the nonradial patches in [28] and [41] bifurcating from annuli.

4.3. *Smooth solutions with simply connected level sets with* $\Omega \le 0$

The rest of this section is devoted to the smooth setting. We will show that any nonnegative smooth rotating solution of the Euler or gSQG equation with angular velocity $\Omega \le 0$ must be radial, under the additional assumption that all the super-level sets U^h

$$U^h := \left\{ x \in \mathbb{R}^d : \omega_0(x) > h \right\} \tag{4.6}$$

are simply connected for any $h > 0$. We believe that the simply connected assumption is necessary, since it is likely that the bifurcation arguments from annuli in [28] and [41] can be extended to the smooth setting as well, using a similar argument as in [18] or [19].

THEOREM 4.6
Let $\omega(x) \in C^1(\mathbb{R}^2)$ be nonnegative and compactly supported. In addition, assume that the super-level set U^h as in (4.6) is simply connected for all $h \in (0, \sup \omega)$. Assume that K satisfies **(HK)**. *If for some $\Omega \le 0$, we have*

$$\omega * K - \frac{\Omega}{2}|x|^2 = C_0(h) \quad \text{on } \partial U^h \text{ for all } h \in (0, \sup \omega), \tag{4.7}$$

then ω is radially decreasing up to a translation. Moreover, it is centered at the origin if $\Omega < 0$.

Proof
We prove it by contradiction. For the $\Omega < 0$ case, we assume without loss of generality that ω is not symmetric decreasing about the line $x_1 = 0$. For the $\Omega = 0$ case, similar to Remark 4.3, we assume without loss of generality that the center of mass is at the origin, and then we assume that ω is not symmetric decreasing about the line $x_1 = 0$.

For any $\tau \ge 0$, we define the continuous Steiner symmetrization $\omega^\tau(x)$ in the same way as [13, Definition 2.12]:

$$\omega^\tau(x) := \int_0^{h_0} 1_{S^\tau[U^h]}(x)\, dh,$$

where $h_0 := \sup \omega$, and $S^\tau[U^h]$ is the continuous Steiner symmetrization of the super-level set U^h at time $\tau \geq 0$.

Consider the energy functional

$$\mathcal{E}[\omega] := \underbrace{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \omega(x)\omega(y)K(x-y)\,dx\,dy}_{=:\mathcal{I}[\omega]} + \underbrace{(-\Omega)\int_{\mathbb{R}^2}\omega(x)|x|^2\,dx}_{=:\mathcal{V}[\omega]}.$$

We proceed similarly as in Theorem 4.2 to compute $\frac{d^+}{d\tau}\mathcal{E}[\omega^\tau]$ in two different ways. We first rewrite the finite difference $\mathcal{E}[\omega^\tau] - \mathcal{E}[\omega]$ as

$$\begin{aligned}
&\mathcal{E}[\omega^\tau] - \mathcal{E}[\omega] \\
&\quad = \int_{\mathbb{R}^2} 2\big(\omega^\tau(x) - \omega(x)\big)\Big(\omega * K - \frac{\Omega}{2}|x|^2\Big)\,dx \\
&\qquad + \iint_{\mathbb{R}^2\times\mathbb{R}^2} \big(\omega^\tau(x) - \omega(x)\big)\big(\omega^\tau(y) - \omega(y)\big)K(x-y)\,dx\,dy \\
&\quad =: I_1 + I_2.
\end{aligned} \tag{4.8}$$

Since $\omega \in C_c^1(\mathbb{R}^2)$ and K satisfies **(HK)** (hence is locally integrable), one can easily check that $\omega * K - \frac{\Omega}{2}|x|^2$ is Lipschitz in $\tilde{D} := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \operatorname{supp}\omega) \leq 1\}$. Note that we have $\operatorname{supp}\omega^\tau \in \tilde{D}$ for all $\tau \in [0,1]$. Combining this fact with the assumption (4.7), there exists $C_1 > 0$ independent of h such that

$$\Big|(\omega * K)(x) - \frac{\Omega}{2}|x|^2 - C_0(h)\Big| \leq C_1\tau \quad \text{on } S^\tau[U^h]\triangle U^h \text{ for all } h \in (0,h_0). \tag{4.9}$$

Let us first rewrite I_1 as

$$I_1 = 2\int_0^{h_0}\int_{\mathbb{R}^2}\big(1_{S^\tau[U^h]}(x) - 1_{U^h}(x)\big)\Big((\omega * K)(x) - \frac{\Omega}{2}|x|^2\Big)\,dxdh.$$

By Lemma 4.1(a), we have $\int_{\mathbb{R}^2}(1_{S^\tau[U^h]}(x) - 1_{U^h}(x))\,dx = 0$ for all $h \in (0,h_0)$. Thus we can control I_1 as

$$\begin{aligned}
|I_1| &= \Big|2\int_0^{h_0}\int_{\mathbb{R}^2}\big(1_{S^\tau[U^h]}(x) - 1_{U^h}(x)\big)\Big((\omega * K)(x) - \frac{\Omega}{2}|x|^2 - C_0(h)\Big)\,dx\,dh\Big| \\
&\leq 2C_1\tau\int_0^{h_0}\big|\big(S^\tau[U^h]\big)\triangle U^h\big|\,dh \\
&\leq 2C_1\tau\int_0^{h_0} 2|\partial U^h|\tau\,dh \\
&= 4C_1\tau^2\int_{\operatorname{supp}\omega}|\nabla\omega|\,dx \leq C(\omega)\tau^2.
\end{aligned} \tag{4.10}$$

Here in the second line we used (4.9); in the third line we used Lemma 4.1(b) and the property (4.2) in two dimensions; and in the fourth line we used the coarea formula and the fact that $\omega \in C_c^1$.

We next move on to I_2. Since $|\nabla\omega|$ is bounded, Lemma 4.1(b) leads to

$$\begin{aligned}\big|\omega^\tau(x) - \omega(x)\big| &= \Big|\int_0^\infty 1_{S^\tau[U^h]}(x) - 1_{U^h}(x)\,dh\Big| \\ &\le C\,\|\nabla\omega\|_{L^\infty}\tau \quad \text{for all } x \in \mathbb{R}^2,\end{aligned}$$

and $\operatorname{supp}\omega^\tau \in \tilde{D}$ for all $\tau \in [0,1]$. Thus

$$\begin{aligned}|I_2| &\le \|\omega^\tau - \omega\|_{L^1}\big\|(\omega^\tau - \omega) * K\big\|_{L^\infty} \\ &\le \|\omega^\tau - \omega\|_{L^1}\|\omega^\tau - \omega\|_{L^\infty}\int_{\tilde{D}}\big|K(x)\big|\,dx \\ &\le C(\omega)\tau^2.\end{aligned}$$

Combining the estimates for I_1 and I_2 gives $\mathcal{E}[\omega^\tau] - \mathcal{E}[\omega] \le C(\omega)\tau^2$ for all $\tau \in [0,1]$, thus

$$\frac{d^+}{d\tau}\mathcal{E}[\omega^\tau]|_{\tau=0} = \frac{d^+}{d\tau}(I_1 + I_2)|_{\tau=0} = 0. \tag{4.11}$$

On the other hand, we compute $\frac{d^+}{d\tau}\mathcal{E}[\omega^\tau]|_{\tau=0}$ in a different way as

$$\frac{d^+}{d\tau}\mathcal{E}[\omega^\tau]|_{\tau=0} = \frac{d^+}{d\tau}\big(\mathcal{I}[\omega^\tau] + \mathcal{V}[\omega^\tau]\big)|_{\tau=0}.$$

In the $\Omega < 0$ case, similarly as in Theorem 4.2, we have that $\mathcal{I}[\omega^\tau]$ is nonincreasing along the continuous Steiner symmetrization by [8, Corollary 2] and [63, Theorem 3.7], thus

$$\frac{d^+}{d\tau}\mathcal{I}[\omega^\tau] \le 0 \quad \text{for all } \tau > 0.$$

For $\mathcal{V}[\omega^\tau]$, by the assumption that ω is not symmetric decreasing about $\{x_1 = 0\}$, we again use [13, Lemma 2.22] to show that, for $\Omega < 0$,

$$\frac{d^+}{d\tau}\mathcal{V}[\omega^\tau] = (-\Omega)\frac{d^+}{d\tau}\int_{\mathbb{R}^2}\omega^\tau(x)|x|^2\,dx|_{\tau=0} < 0.$$

Adding them together gives $\frac{d^+}{d\tau}\mathcal{E}[D^\tau]|_{\tau=0} < 0$, contradicting (4.11).

In the $\Omega = 0$ case, we assume that the center of mass of ω is at the origin. Thus if ω is not symmetric decreasing about $\{x_1 = 0\}$, then the same proof as [13, Proposition 2.15] gives that $\mathcal{I}[D]$ must be decreasing to the first order for a short time (again, the proof holds for all kernels K satisfying **(HK)**; see [14, Theorem 6] for extensions to Riesz kernels $K_{\alpha,d}$ with $\alpha \in (0,2)$). This gives $\frac{d^+}{d\tau}\mathcal{E}[D^\tau]|_{\tau=0} < 0$, again contradicting (4.11). □

The above theorem immediately gives the following corollary concerning the V-states for the Euler and gSQG equations.

COROLLARY 4.7
Assume that $\omega(x) \in C^1(\mathbb{R}^2)$ is a nonnegative, compactly supported V-state satisfying the Euler equation or the gSQG equation for some $\alpha \in (0,2)$ with $\Omega \le 0$. In addition, assume that the super-level set U^h as in (4.6) is simply connected for all $h \in (0, \sup \omega)$. Then ω must be radially decreasing if $\Omega < 0$, and radially decreasing up to a translation if $\Omega = 0$.

Proof
For the Euler equation, $K = \frac{1}{2\pi}\ln|x|$. For the gSQG equation, $K = -C_\alpha|x|^{-\alpha}$. In both cases, the proof follows from Theorem 4.6. □

5. Radial symmetry of rotating gSQG solutions with $\Omega > \Omega_\alpha$

In this section, we focus on rotating gSQG patches with area π and $\alpha \neq 0$. As we discussed in the Introduction, for $\alpha \in [0,2)$, there exist rotating patches bifurcating from the unit disk at angular velocities $\Omega_m^\alpha = 2^{\alpha-1}\frac{\Gamma(1-\alpha)}{\Gamma(1-\frac{\alpha}{2})^2}\left(\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})} - \frac{\Gamma(m+\frac{\alpha}{2})}{\Gamma(m+1-\frac{\alpha}{2})}\right)$, where Ω_m^α is increasing in m for any fixed $\alpha \in [0,2)$. Let us denote $\Omega_\alpha := \lim_{m\to\infty}\Omega_m^\alpha$. If $\alpha \in (0,1)$, then we have that

$$\Omega_\alpha = 2^{\alpha-1}\frac{\Gamma(1-\alpha)}{\Gamma(1-\frac{\alpha}{2})^2}\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})}. \tag{5.1}$$

Note that Ω_α is a continuous function of α for $\alpha \in (0,1)$, with $\Omega_0 = \frac{1}{2}$, and $\Omega_\alpha = +\infty$ for all $\alpha \in [1,2)$.

A natural question is whether there can be rotating patches with area π with $\Omega \ge \Omega_\alpha$ for $\alpha \in (0,1)$. Note that the area constraint is necessary for all $\alpha > 0$: if D is a rotating gSQG patch for $\alpha \in (0,2)$ with angular velocity Ω, then one can easily check that its scaling $\lambda D = \{\lambda x : x \in D\}$ is a rotating patch with angular velocity $\lambda^{-\alpha}\Omega$.

In Theorem 2.14, we showed that for the 2D Euler case ($\alpha = 0$), every rotating patch with $\Omega \ge \Omega_0 = \frac{1}{2}$ must be a disk. In this section, our goal is to show that every *simply connected* rotating patch with area π with $\Omega \ge \Omega_\alpha$ for $\alpha \in (0,1)$ must be a disk. Whether there exist nonsimply connected or disconnected rotating patches with $\Omega \ge \Omega_\alpha$ for $\alpha \in (0,1)$ is still an open question.

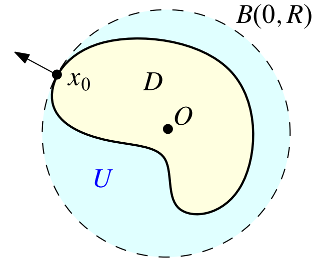

Figure 7. (Color online) Illustration of the set U and the point x_0.

Below is the main theorem of this section. Recall that for $\alpha \in (0,2)$, $K_\alpha = -C_\alpha |x|^{-\alpha}$ is the fundamental solution for $-(-\Delta)^{-1+\frac{\alpha}{2}}$, where $C_\alpha = \frac{1}{2\pi}\frac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})}$.

THEOREM 5.1
Let $D \subset \mathbb{R}^2$ be a bounded, simply connected patch with C^1 boundary. Let us denote $R := \max_{x\in D}|x|$. Assume that D is a uniformly rotating patch with angular velocity Ω of the gSQG equation with $\alpha \in (0,1)$, that is,

$$1_D * K_\alpha - \frac{\Omega}{2}|x|^2 = C \quad \text{on } \partial D. \tag{5.2}$$

Let $\Omega_c(R) := R^{-\alpha}\Omega_\alpha$. If $\Omega \geq \Omega_c(R)$, then D must coincide with $B(0,R)$.

Remark 5.2
(a) Note that all sets $D \subset \mathbb{R}^2$ with area π must have $R \geq 1$. In this case we have $\Omega_c(R) \leq \Omega_\alpha$, thus Theorem 5.1 immediately implies that all simply connected rotating patches with area π and $\Omega \geq \Omega_\alpha$ must be a disk.

(b) Note that the constant Ω_α is sharp, since there exist patches bifurcating from a disk of radius 1 at velocities Ω_m^α, which can get arbitrarily close to Ω_α as $m \to \infty$ (see [47, Theorem 1.4]).

Proof
Towards a contradiction, assume that $D \neq B(0,R)$. Let $x_0 \in \partial D$ be the farthest point from 0. Then we have that $D \subset B(0,R)$, and let $U := B(0,R) \setminus D$. See Figure 7 for an illustration of U and x_0. Then (5.2) can be rewritten as

$$1_U * K_\alpha = 1_{B(0,R)} * K_\alpha - \frac{\Omega}{2}|x|^2 - C \quad \text{on } \partial D. \tag{5.3}$$

The key idea of this proof is to use two different ways to compute $\nabla(1_U * K_\alpha)(x_0)\cdot x_0$, and obtain a contradiction if $\Omega \geq \Omega_c(R)$. On the one hand,

$$\nabla(1_U * K_\alpha)(x_0)\cdot x_0 = \alpha C_\alpha \int_U \frac{(x_0-y)\cdot x_0}{|x_0-y|^{\alpha+2}}\,dy > 0, \tag{5.4}$$

where we used the fact that $(x_0-y)\cdot x_0 > 0$ for all $y\in U\subset B(0,R)$ since the two vectors point to the same half-plane.

On the other hand, we claim that the following properties hold for $1_U * K_\alpha$:

(1) $\Delta(1_U * K_\alpha) < 0$ in D,

(2) along ∂D, the minimum of $1_U * K_\alpha$ is achieved at x_0.

To show property (1), using the fact that $K_\alpha = -C_\alpha|x|^{-\alpha}$ is the fundamental solution for $-(-\Delta)^{-1+\frac{\alpha}{2}}$, we have $1_U * K_\alpha = -(-\Delta)^{-1+\frac{\alpha}{2}}1_U$, thus $\Delta(1_U * K_\alpha) = (-\Delta)^{\alpha/2}1_U$. Thus for any $x\in D$, using the singular integral definition of the fractional Laplacian (see [62, Theorem 1.1, Definition (e)]) and the fact that $1_U\equiv 0$ in D, we have

$$\begin{aligned}(-\Delta)^{\alpha/2}1_U(x) &= C_1(\alpha)\int_{\mathbb{R}^2}\frac{1_U(x)-1_U(y)}{|x-y|^{2+\alpha}}\,dy\\ &= C_1(\alpha)\int_{\mathbb{R}^2}\frac{0-1_U(y)}{|x-y|^{2+\alpha}}\,dy < 0 \quad \text{for } x\in D\end{aligned}$$

for some constant $C_1(\alpha)>0$. Note that despite the denominator being singular, the integral indeed converges for all $x\in D$, due to the fact that D is open and the integrand is identically zero in D which yields

$$\Delta(1_U * K_\alpha)(x) = (-\Delta)^{\alpha/2}1_U(x) < 0 \quad \text{in } D. \tag{5.5}$$

We now move on to property (2). Due to (5.3) and the fact that x_0 is the outermost point on ∂D, it suffices to show that the radial function $1_{B(0,R)} * K_\alpha - \frac{\Omega}{2}|x|^2$ is nonincreasing in $|x|$ for all $\Omega\ge\Omega_c(R)$. We prove this in Proposition 5.3 right after this theorem.

The above claims allow us to apply the maximum principle to $1_U * K_\alpha$ (recall that $1_U * K_\alpha$ is superharmonic in D by property (1)), which yields that the minimum of $1_U * K_\alpha$ in $\overline{D}$ is also achieved at x_0, thus

$$\nabla(1_U * K_\alpha)(x_0)\cdot\vec{n}(x_0)\le 0,$$

where $\vec{n}(x_0)$ is the outer normal of D at x_0. Since $\vec{n}(x_0) = x_0/|x_0|$, the above inequality contradicts with (5.4). As a result, D must coincide with $B(0,R)$. $\square$

Now we prove the proposition that was used in the proof of the above theorem.

PROPOSITION 5.3
For a fixed $\alpha\in(0,1)$ and $R>0$, let $\Omega_c(R)$ be the smallest number such that

$$g_R(x) := 1_{B(0,R)} * K_\alpha - \frac{\Omega_c}{2}|x|^2$$

is nonincreasing in $|x|$. Then we have $\Omega_c(R) = R^{-\alpha}\Omega_\alpha$, with Ω_α given in (5.1).

Proof

Recall that $K_\alpha = -C_\alpha|x|^{-\alpha}$ with $C_\alpha = \frac{1}{2\pi}\frac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})}$. Since $|x|^2$ and $1_{B(0,R)} * K_\alpha$ are both radially symmetric and increasing in $|x|$, we have

$$\Omega_c(R) = 2C_\alpha \sup_{|x_1|\neq|x_2|} \frac{\int_{B(0,R)}|x_2-y|^{-\alpha}\,dy - \int_{B(0,R)}|x_1-y|^{-\alpha}\,dy}{|x_1|^2-|x_2|^2}.$$

Let us denote the fraction above by $F(x_1,x_2)$. We claim that the $\sup_{|x_1|\neq|x_2|} F(x_1,x_2)$ is attained when $|x_1| = R$, and $|x_2| \to R$.

To prove the claim, we first compute $I(x) := \int_{B(0,R)}|x-y|^{-\alpha}\,dy$. Taking the Fourier transform:

$$I(x) = CR^{2-\alpha}\int_0^\infty r^{a-2}J_1(r)J_0\Big(\frac{|x|r}{R}\Big)\,dr,$$

where C is some positive constant. By Sonine–Schafheitlin's formula (see [89, p. 401]) and by continuity, we obtain

$$I(x) = \begin{cases} CR^{2-\alpha}2^{\alpha-2}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})}{}_2F_1(\frac{\alpha}{2}-1,\frac{\alpha}{2},1,\frac{|x|^2}{R^2}) & \text{if } |x|\le R,\\ CR^{2-\alpha}2^{\alpha-2}|x|^{-\alpha}R^{\alpha}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}{}_2F_1(\frac{\alpha}{2},\frac{\alpha}{2},2,\frac{R^2}{|x|^2}) & \text{if } |x|>R. \end{cases}$$

By the mean value theorem, it is enough to check that $\min J(z) = J(R^2)$, where

$$\begin{aligned} J(z) &= \begin{cases} \frac{d}{dz}({}_2F_1(\frac{\alpha}{2}-1,\frac{\alpha}{2},1,\frac{z}{R^2})) & \text{if } z\le R^2,\\ \frac{d}{dz}((1-\frac{\alpha}{2})z^{-\frac{\alpha}{2}}R^{\alpha}_2F_1(\frac{\alpha}{2},\frac{\alpha}{2},2,\frac{R^2}{z})) & \text{if } z>R^2, \end{cases}\\ &= \begin{cases} \frac{\alpha(\alpha-2)}{4}\frac{1}{R^2}{}_2F_1(\frac{\alpha}{2},1+\frac{\alpha}{2},2,\frac{z}{R^2}) & \text{if } z\le R^2,\\ \frac{\alpha(\alpha-2)}{4}z^{-1-\frac{\alpha}{2}}R^{\alpha}_2F_1(\frac{\alpha}{2},1+\frac{\alpha}{2},2,\frac{R^2}{z}) & \text{if } z>R^2. \end{cases} \end{aligned}$$

Writing the series expansion (resp., at $z=0$ and $z=\infty$) of the hypergeometric series:

$$\begin{aligned} &\frac{\alpha(\alpha-2)}{4}\frac{1}{R^2}{}_2F_1\Big(\frac{\alpha}{2},1+\frac{\alpha}{2},2,z\Big)\\ &\quad= \frac{1}{R^2}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{\alpha}{2}+n)\Gamma(n+1+\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2}-1)\Gamma(\frac{\alpha}{2})\Gamma(1+n)\Gamma(2+n)}\Big(\frac{z}{R^2}\Big)^n,\\ &\frac{\alpha(\alpha-2)}{4}z^{-1-\frac{\alpha}{2}}R^{\alpha}_2F_1\Big(\frac{\alpha}{2},1+\frac{\alpha}{2},2,\frac{1}{z}\Big) \end{aligned}$$

$$= \Big(\frac{1}{z}\Big)^{\frac{\alpha}{2}} R^{\alpha-2} \sum_{n=1}^{\infty} \frac{\Gamma(\frac{\alpha}{2}+n)\Gamma(n-1+\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2}-1)\Gamma(\frac{\alpha}{2})\Gamma(n)\Gamma(n+1)} \Big(\frac{R^2}{z}\Big)^n,$$

which are both minimized at $z = R^2$ since every coefficient is negative. This proves the claim.

The claim immediately implies that

$$\Omega_c(R) = -\frac{C_\alpha}{R} \frac{d}{d|x|} \int_{B(0,R)} |x-y|^{-\alpha}\, dy|_{|x|=R}, \tag{5.6}$$

where $\frac{d}{d|x|}$ denotes the derivative in the radial variable (recall that $\int_{B(0,R)} |x-y|^{-\alpha}\, dy$ is radially symmetric). To compute the derivative at $|x| = R$, we can simply compute the partial derivative in the x_1 direction at the point $(R, 0)$:

$$\begin{aligned}
&\frac{\partial}{\partial x_1} \int_{B(0,R)} |x-y|^{-\alpha}\, dy|_{x=(R,0)} \\
&\quad = -\alpha \int_{B(0,R)} \big((R-y_1)^2 + y_2^2\big)^{-\frac{\alpha}{2}-1} (R-y_1)\, dy_1\, dy_2 \\
&\quad = -2 \int_0^R \big((R-y_1)^2 + y_2{}^2\big)^{-\frac{\alpha}{2}} \Big|_{y_1=-\sqrt{R^2-y_2{}^2}}^{y_1=\sqrt{R^2-y_2{}^2}} dy_2 \\
&\quad = -2^{1-\frac{\alpha}{2}} R^{1-\alpha} \Big(\int_0^1 (1-\sqrt{1-u^2})^{-\frac{\alpha}{2}}\, du - \int_0^1 (1+\sqrt{1-u^2})^{-\frac{\alpha}{2}}\, du \Big) \\
&\quad = -2^{1-\frac{\alpha}{2}} R^{1-\alpha} \Big(\int_0^{\frac{\pi}{2}} (1-\cos\theta)^{-\frac{\alpha}{2}} \cos\theta\, d\theta - \int_0^{\frac{\pi}{2}} (1+\cos\theta)^{-\frac{\alpha}{2}} \cos\theta\, d\theta \Big) \\
&\quad =: -2^{1-\frac{\alpha}{2}} R^{1-\alpha} (I_1 - I_2),
\end{aligned} \tag{5.7}$$

where in the third line we used the identity $(R \pm \sqrt{R^2 - y_2^2})^2 + y_2^2 = 2R^2(1 \pm \sqrt{1-(R^{-1}y_2)^2})$, as well as the substitution $u = R^{-1} y_2$.

Using a substitution $\theta = 2\beta$, we rewrite I_1 as

$$I_1 = 2\int_0^{\frac{\pi}{4}} \big(1-\cos(2\beta)\big)^{-\frac{\alpha}{2}} \cos(2\beta)\, d\beta = 2^{1-\frac{\alpha}{2}} \int_0^{\frac{\pi}{4}} (\sin\beta)^{-\alpha} (1-2\sin^2\beta)\, d\beta.$$

Likewise, the substitution $\theta = \pi - 2\beta$ allows us to rewrite $-I_2$ as

$$-I_2 = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \big(1-\cos(2\beta)\big)^{-\frac{\alpha}{2}} \cos(2\beta)\, d\beta = 2^{1-\frac{\alpha}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin\beta)^{-\alpha} (1-2\sin^2\beta)\, d\beta.$$

Adding the above two identities for I_1 and $-I_2$ together gives

$$
\begin{aligned}
I_1 - I_2 &= 2^{1-\frac{\alpha}{2}} \int_0^{\frac{\pi}{2}} (\sin\beta)^{-\alpha}(1 - 2\sin^2\beta)\, d\beta \\
&= 2^{1-\frac{\alpha}{2}} \Big(\frac{1}{2} B\Big(\frac{1-\alpha}{2}, \frac{1}{2}\Big) - B\Big(\frac{3-\alpha}{2}, \frac{1}{2}\Big)\Big) \\
&= 2^{-\frac{\alpha}{2}} \frac{\Gamma(\frac{1-\alpha}{2})\Gamma(\frac{1}{2})}{\Gamma(1-\frac{\alpha}{2})} - 2^{1-\frac{\alpha}{2}} \frac{\Gamma(\frac{3-\alpha}{2})\Gamma(\frac{1}{2})}{\Gamma(2-\frac{\alpha}{2})},
\end{aligned}
$$

where B stands for the beta function. Here the second identity follows from the property that $B(x,y) = 2\int_0^{\pi/2} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}\, d\theta$, and the third line follows from the property that $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. According to the properties of the gamma function $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$, we have

$$
I_1 - I_2 = 2^{-1+\frac{\alpha}{2}} \frac{\alpha}{2-\alpha} \cdot \frac{2\pi\Gamma(1-\alpha)}{\Gamma(1-\frac{\alpha}{2})^2}. \tag{5.8}
$$

Finally, plugging this into (5.7) and (5.6) gives

$$
\begin{aligned}
\Omega_c(R) &= R^{-\alpha} C_\alpha 2^{1-\frac{\alpha}{2}} (I_1 - I_2) \\
&= R^{-\alpha} \frac{1}{2\pi} \frac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})} \frac{\alpha}{2-\alpha} \Big(\frac{2\pi\Gamma(1-\alpha)}{\Gamma(1-\frac{\alpha}{2})^2} \Big) \\
&= R^{-\alpha} \frac{2^{\alpha-1}\Gamma(1-\alpha)\Gamma(\frac{\alpha}{2}+1)}{\Gamma(1-\frac{\alpha}{2})^2\Gamma(2-\frac{\alpha}{2})} = R^{-\alpha}\Omega_\alpha,
\end{aligned}
$$

finishing the proof. □

At the end of this section, we point out that Theorem 5.1 directly gives the following quantitative estimate: if a simply connected patch D rotates with angular velocity $\Omega \in (0, \Omega_\alpha)$ that is very close to Ω_α, then D must be very close to a disk in terms of symmetric difference.

COROLLARY 5.4
Assume that $0 < \alpha < 1$. *Let* D *be a rotating patch with area* π *and angular velocity* $\Omega \in (0, \Omega_\alpha)$*, and let* B *be the unit disk. Then we have*

$$
|D \triangle B| \le 2\pi \Big(\Big(\frac{\Omega_\alpha}{\Omega}\Big)^{2/\alpha} - 1 \Big).
$$

Note that for a fixed $\alpha \in (0,1)$*, the right-hand side goes to 0 as* $\Omega \nearrow \Omega_\alpha$.

Proof
Denote $R := \max_{x\in D} |x|$. If D is a rotating patch with angular velocity Ω and is not a

disk, then Theorem 5.1 gives that $\Omega \le R^{-\alpha}\Omega_\alpha$, which gives that $R \le (\frac{\Omega_\alpha}{\Omega})^{1/\alpha}$. Thus $D \subset B(0, (\frac{\Omega_\alpha}{\Omega})^{1/\alpha})$, which implies that the symmetric difference $D \triangle B$ satisfies

$$|D \triangle B| = 2|D \setminus B| \le 2\left|B\left(0, \left(\frac{\Omega_\alpha}{\Omega}\right)^{1/\alpha}\right) \setminus B\right| = 2\pi\left(\left(\frac{\Omega_\alpha}{\Omega}\right)^{2/\alpha} - 1\right). \qquad \square$$

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