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Quantitative stability and error estimates for optimal transport plans

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Optimal transport maps and plans between two absolutely continuous measures μ and ν can be approximated by solving semidiscrete or fully discrete optimal transport problems. These two problems ensue from approximating μ or both μ and ν by Dirac measures. Extending an idea from Gigli (2011, On Hölder continuity-in-time of the optimal transport map towards measures along a curve. *Proc. Edinb. Math. Soc. (2)*, 54, 401–409), we characterize how transport plans change under the perturbation of both μ and ν. We apply this insight to prove error estimates for semidiscrete and fully discrete algorithms in terms of errors solely arising from approximating measures. We obtain weighted L^2 error estimates for both types of algorithms with a convergence rate $O(h^{1/2})$. This coincides with the rate in Theorem 5.4 in Berman (2018, Convergence rates for discretized Monge–Ampère equations and quantitative stability of optimal transport. Preprint available at arXiv:1803.00785) for semidiscrete methods, but the error notion is different.

Keywords: optimal transport; transport plans; quantitative stability; error estimates; Monge–Ampère; Oliker–Prussner method; linear programming method.

Dedicated to the memory of John W. Barrett

1. Introduction

The optimal transport (OT) problem, first proposed by Monge (1781) and later generalized by Kantorovich (1942), has been extensively studied from the theoretical point of view (Brenier, 1987, 1989; Caffarelli, 1992a,b, 1996; Villani, 2009; Gigli, 2011; Santambrogio, 2015). It has a wide variety of applications in economics (Chiappori *et al.*, 2010; Beiglböck *et al.*, 2013), fluid mechanics (Brenier, 1989), meteorology (Cullen & Douglas, 1999; Cullen & Maroofi, 2003), image processing (Rubner *et al.*, 2000; Rabin *et al.*, 2010), transport (Xia, 2003; Carlier & Santambrogio, 2005) and optic design (Gutiérrez & Huang, 2009; Prins *et al.*, 2014). In this section, we briefly introduce the basic theory and numerical methods of OT and point out our contribution to the numerical analysis of OT in this paper.

1.1 *Introduction to optimal transport*

If X, Y are subdomains of $\mathbb{R}^d$ and $\mu \in P(X), \nu \in P(Y)$ are given probability measures, the Monge formulation of OT is to find an optimal map $T : X \to Y$ that minimizes the transport cost, i.e.

$$\inf_{T:T_\#\mu=\nu} \int_X c\,(x, T(x))\ \mathrm{d}\mu(x), \tag{1.1}$$

where the cost function $c(x, y) : X \times Y \to [0, \infty)$ is given. Hereafter, $T_\#\mu$ denotes the pushforward of measure μ through T, namely $T_\#\mu = \nu$ means that for any measurable set $A \subset Y$, we have $\nu(A) = \mu(T^{-1}(A))$, or equivalently

$$\int_X \phi(T(x))\,\mathrm{d}\mu(x) = \int_Y \phi(y)\,\mathrm{d}\nu(y) \tag{1.2}$$

for all continuous functions $\phi : Y \to \mathbb{R}$. Since this problem is difficult to study, and sometimes the optimal map does not exist, Kantorovich (1942) generalized the notion of transport map and considered the following problem:

$$\inf_{\gamma\in\Pi(\mu,\nu)} \int_{X\times Y} c(x, y)\ \mathrm{d}\gamma(x, y), \tag{1.3}$$

where $\Pi(\mu, \nu)$ is the set of transport plans between μ and ν, namely

$$\Pi(\mu, \nu) := \{\gamma \in P(X \times Y)\ :\ (\pi_1)_\#\gamma = \mu,\ (\pi_2)_\#\gamma = \nu\}.$$

Here, π_1 and π_2 are the projections defined as $\pi_1(x, y) = x, \pi_2(x, y) = y$. This definition implies that for $\gamma \in \Pi(\mu, \nu)$ and any measurable sets $A \subset X, B \subset Y$, we have $\gamma(A \times Y) = \mu(A), \gamma(X \times B) = \nu(B)$. For $X = Y = \mathbb{R}^d$ and $1 \le p < \infty$, let $P_p(\mathbb{R}^d) \subset P(\mathbb{R}^d)$ be the set of probability measures with bounded p moment, i.e.

$$P_p(\mathbb{R}^d) := \left\{\mu \in P(\mathbb{R}^d)\colon \int_{\mathbb{R}^d} |x|^p\ \mathrm{d}\mu(x) < \infty\right\}, \tag{1.4}$$

which clearly contains those probabilities measures with bounded support. If $\mu, \nu \in P_p(\mathbb{R}^d)$ then $\gamma \in P_p(\mathbb{R}^{2d})$ for all $\gamma \in \Pi(\mu, \nu)$ because

$$\int_{\mathbb{R}^{2d}} \left(|x|^p + |y|^p\right) \mathrm{d}\gamma(x, y) = \int_{\mathbb{R}^d} |x|^p\ \mathrm{d}\mu(x) + \int_{\mathbb{R}^d} |y|^p\ \mathrm{d}\nu(y) < \infty. \tag{1.5}$$

Moreover, if $c(x, y) = |x - y|^p$, the Kantorovich problem (1.3) always has a finite minimum value. In fact, this defines the well-known Wasserstein metric on $P_p(\mathbb{R}^d)$:

$$W_p(\mu, \nu) := \left(\min_{\gamma\in\Pi(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x - y|^p\ \mathrm{d}\gamma(x, y)\right)^{1/p} \quad \forall\, \mu, \nu \in P_p(\mathbb{R}^d).$$

In addition, for quadratic cost, i.e. $p = 2$, Brenier (1987) shows that there exists a unique optimal map $T = \nabla\varphi$ for a convex function φ (T is uniquely determined μ-a.e.) provided that μ gives no mass to $(d-1)$-surfaces of class C^2. If μ, ν are absolutely continuous measures with respect to Lebesgue measure with densities $f, g \geq 0$, then

$$\mu(S) = \int_S f(x)\,\mathrm{d}x = \nu(\nabla\varphi(S)) = \int_{\nabla\varphi(S)} g(y)\,\mathrm{d}y = \int_S g(\nabla\varphi(x))\det D^2\varphi(x)\,\mathrm{d}x,$$

whence φ satisfies the generalized Monge–Ampère equation

$$g(\nabla\varphi(x))\det D^2\varphi(x) = f(x) \quad \forall x \in X, \tag{1.6}$$

with second-type boundary condition $\nabla\varphi(X) = Y$ (Figalli, 2017, Section 4.6) provided that the domains $X, Y \subset \mathbb{R}^d$ are chosen so that

$$X = \{x \in \mathbb{R}^d : f(x) > 0\}, \quad Y = \{y \in \mathbb{R}^d : g(y) > 0\}. \tag{1.7}$$

The optimal transport map T induces an optimal transport plan $\gamma \in P(X, Y)$ given by $\gamma = (id, T)_\# \mu$, where *id* denotes the identity map, namely

$$\gamma(A) = \mu\big\{x \in X : \quad (x, T(x)) \in A\big\}$$

for all measurable sets $A \subset X \times Y$. Similar results also hold for any $p \in (1, \infty)$, and for general costs $c(x, y)$ satisfying a twist condition (Villani, 2009, Chapter 10). However, the quantitative stability and error estimates studied in this paper are restricted to the case $p = 2$. Their generalization to $p \neq 2$ and more general costs is open.

1.2 *Numerical methods for OT and our contribution*

In view of the numerous and diverse applications of OT, developing fast and sound numerical methods for OT is of paramount importance. Several algorithms do exist, ranging from those inspired by partial differential equations (PDEs) and variational techniques for absolutely continuous measures μ and ν (Benamou & Brenier, 2000; Angenent *et al.*, 2003; Froese, 2012; Benamou *et al.*, 2014; Papadakis *et al.*, 2014; Lindsey & Rubinstein, 2017; Benamou & Duval, 2019 Hamfeldt, 2019) to those approximating one or both measures by Dirac masses and then solving the approximate OT (Mérigot, 2011; Burkard *et al.*, 2012; Cuturi, 2013; Schmitzer & Schnörr, 2013; Benamou *et al.*, 2015; Lévy, 2015; Oberman & Ruan, 2015; Schmitzer, 2016; Kitagawa *et al.*, 2019). However, intrinsic difficulties make their numerical analysis far from complete.

In this paper, we develop stability and error analyses for quadratic costs that account for the effect of approximating measures μ and ν by Dirac masses. To this end, we extend the stability estimates of Gigli (2011), originally derived for optimal maps, to optimal plans γ. This is critical because optimal maps might not exist for Dirac measures. We do not develop new techniques to approximate γ.

If at least one of the two measures is discrete then it is possible to solve for the exact transport map numerically without further approximations, which includes semidiscrete algorithms (Aurenhammer *et al.*, 1998; Mérigot, 2011; Lévy, 2015; Kitagawa *et al.*, 2019) and fully discrete methods (Burkard *et al.*, 2012; Schmitzer & Schnörr, 2013; Schmitzer, 2016). For these methods, since their errors solely come from approximating absolutely continuous measures with discrete measures, our results directly lead to error estimates. We also point out that an error estimate for a semidiscrete method was recently obtained by Berman (2018), but there are no such results for fully discrete schemes.

We now describe our results. Let X, Y be compact sets of $\mathbb{R}^d$ and μ, ν be absolutely continuous measures with respect to the Lebesgue measure with densities $f \in L^1(X), g \in L^1(Y)$, respectively. Let

$$\mu_h = \sum_{i=1}^{N} f_i \delta_{x_i}, \quad \nu_h = \sum_{j=1}^{M} g_j \delta_{y_j}$$

be approximations of μ, ν governed by a parameter $h > 0$, which means the Wasserstein distances satisfy

$$W_2(\mu, \mu_h) \leq h, \quad W_2(\nu, \nu_h) \leq h.$$

To make this more concrete, we briefly introduce one way to obtain an approximation μ_h of μ. Choose $(x_i)_{i=1}^N$ such that $X \subset \cup_{i=1}^N B_h(x_i)$, where $B_r(x)$ denotes the open ball with radius r centered at x. Then, consider Voronoi tessellations: let

$$V_i := \left\{ x \in X : |x - x_i| = \min_{1 \leq j \leq N} |x - x_j| \right\} \tag{1.8}$$

be the Voronoi cell for x_i and

$$f_i := \mu(V_i), \qquad \mu_h := \sum_{i=1}^{N} f_i \delta_{x_i}.$$

Notice that we may assume $f_i > 0$ since otherwise for $f_i = 0$ we could just drop the Dirac measure at x_i. Define the map $U_h : X \to (x_i)_{i=1}^N$ such that $U_h(x) = x_i$ for all $x \in V_i$. It can be easily checked that this map U_h is well defined and satisfies $|U_h(x) - x| \leq h$ a.e. in X because the intersection between V_i and V_j for $i \neq j$ is of zero Lebesgue measure and $X \subset \cup_{i=1}^N B_h(x_i)$; in particular, $V_i \subset B_h(x_i)$ for all $1 \leq i \leq N$. Therefore, U_h is a transport map from μ to μ_h, and thus

$$W_2(\mu, \mu_h) \leq \left(\int_X |x - U_h(x)|^2 \, \mathrm{d}\mu(x) \right)^{1/2} \leq \left(\int_X h^2 \, \mathrm{d}\mu(x) \right)^{1/2} = h. \tag{1.9}$$

Similarly, we could approximate the absolutely continuous measure ν with density g and bounded support Y with $\nu_h = \sum_{j=1}^M g_j \delta_{y_j}$ satisfying $W_2(\nu, \nu_h) \leq h$.

The semidiscrete algorithms solve the OT between μ_h and ν upon finding a nodal function $\varphi_h : (x_i)_{i=1}^N \to \mathbb{R}$ such that

$$f_i = \nu(F_i) = \int_{F_i} g(y) \, \mathrm{d}y, \qquad F_i := \partial \varphi_h(x_i) \cap Y, \tag{1.10}$$

where the discrete subdifferential is given by

$$\partial \varphi_h(x_i) := \left\{ y \in \mathbb{R}^d : \quad \varphi_h(x_j) \geq \varphi_h(x_i) + y \cdot (x_j - x_i) \quad \forall j = 1, \cdots, N \right\}.$$

This type of discretization was introduced by Oliker & Prussner (1989) for Dirichlet boundary conditions, whereas error estimates have been derived in Neilan & Zhang (2018); Nochetto & Zhang (2019). The set F_i coincides with the subdifferential of the convex envelope $\Gamma(\varphi_h)$ associated with φ_h (Nochetto & Zhang, 2019, Lemma 2.1). The function $\Gamma(\varphi_h)$ is piecewise linear and induces a mesh with nodes $(x_i)_{i=1}^N$ whose elements may be quite elongated; see (Nochetto & Zhang, 2019, Section 2.2). We also refer to Berman (2018), who has derived error estimates for the second-type boundary condition involving $\Gamma(\varphi_h)$.

Denote the barycenter of $\partial\varphi_h(x_i)$ with respect to the measure ν by m_i, namely $m_i := f_i^{-1}\int_{F_i} yg(y)\,\mathrm{d}y$, and define the map T_h such that $T_h(x_i) = m_i$ for all $1 \le i \le N$. Under proper assumptions on measures (μ,ν), to be stated in Section 2, we prove a weighted L^2 error estimate in Theorem 4.2 for the exact optimal transport map T

$$\left(\sum_{i=1}^N f_i\,|T(x_i) - T_h(x_i)|^2\right)^{1/2} \le C(\mu,\nu)h^{1/2}. \tag{1.11}$$

This rate of convergence coincides with that in (Berman, 2018, Theorem 5.4) for $\nabla\Gamma(\varphi_h)$, but the error notion is different; we refer to Section 4 for details. Our approach is tailored to discrete transport plans and thus extends to fully discrete methods.

Fully discrete methods aim to find the discrete optimal transport plan

$$\gamma_h = \sum_{i=1}^N\sum_{j=1}^M \gamma_{h,ij}\,\delta_{x_i}\,\delta_{y_j}$$

between μ_h and ν_h through the constrained minimization problem

$$\min_{\gamma_h}\sum_{i,j=1}^{N,M}\gamma_{h,ij}c_{ij}: \quad \gamma_{h,ij}\ge 0,\;\; \sum_{i=1}^N\gamma_{h,ij} = g_j,\;\; \sum_{j=1}^M\gamma_{h,ij} = f_i. \tag{1.12}$$

If we construct the map $T_h(x_i) := \frac{1}{f_i}\sum_{j=1}^M\gamma_{h,ij}y_j$ from γ_h for $1\le i\le N$, then we also obtain a weighted L^2 error estimate in Theorem 5.1 for the optimal map T

$$\left(\sum_{i=1}^N f_i\,|T(x_i) - T_h(x_i)|^2\right)^{1/2} \le C(\mu,\nu)\,h^{1/2}, \tag{1.13}$$

under suitable assumptions on measures (μ,ν) described in Section 2. This is a new error estimate for fully discrete schemes, and the convergence rate in (1.13) is the same as (1.11) for semidiscrete methods.

1.3 *Outline*

In Section 2, we introduce the notion of λ-regularity and show that it leads to W^1_∞-regularity of the transport map T. We prove in Section 3 that λ-regularity implies a stability bound characterizing how the optimal transport plan γ changes under perturbations of both μ and ν; this hinges on an idea from

Gigli (2011). We measure the change of transport plans using either a weighted L^2 norm in Theorem 3.5 or the Wasserstein metric in Corollary 3.9. We apply the stability bound to derive error estimates in Sections 4 and 5. For semidiscrete schemes, we obtain weighted L^2 error estimates in Theorem 4.1 and Corollary 4.2 of Section 4 with a convergence rate $O(h^{1/2})$. We also compare our geometric estimate of Corollary 4.2 with a similar one due to Berman (2018); however, the two error notions are different. Moreover, we obtain in Theorem 5.1 of Section 5 an entirely new error estimate of order $O(h^{1/2})$ for fully discrete schemes with errors measured in both the weighted L^2 norm and the Wasserstein distance.

2. Regularity and nondegeneracy

We now discuss the assumptions we make on measures $\mu, \nu \in P_2(\mathbb{R}^d)$ in order to prove quantitative stability bounds for optimal transport plans and error estimates for numerical methods.

ASSUMPTION 2.1 (λ-regularity). We say that (μ, ν, φ) is λ-*regular* for $\lambda > 0$ if $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a convex function such that $\nabla\varphi$ is the optimal transport map from μ to ν and φ is λ-smooth in the sense that

$$\varphi((1-t)x_0 + tx_1) + \frac{t(1-t)\lambda}{2}|x_0 - x_1|^2 \geq (1-t)\varphi(x_0) + t\varphi(x_1), \tag{2.1}$$

for any $x_0, x_1 \in \mathbb{R}^d$ and $t \in [0,1]$, i.e. $\frac{\lambda}{2}|x|^2 - \varphi(x)$ is convex. We also say (μ, ν) is λ-regular if there exists a φ such that (μ, ν, φ) is λ-regular.

We will see below that (2.1) implies $\varphi \in W^2_\infty(\mathbb{R}^d)$; note that if $\varphi \in C^2(\mathbb{R}^d)$ then $D^2\varphi(x) \leq \lambda I$ for all $x \in \mathbb{R}^d$. Although we require φ to be defined in the whole $\mathbb{R}^d$ in Assumption 2.1, this is not restrictive because any convex function $\varphi : X \to \mathbb{R}$ defined in a bounded convex set X can be extended as in (Figalli, 2017, Theorem 4.23)

$$E\varphi(z) = \sup_{x\in X, p\in\partial\varphi(x)} \varphi(x) + \langle p, z - x\rangle.$$

Then, one can show that $E\varphi = \varphi$ in X, and $E\varphi$ satisfies (2.1) for any $x_0, x_1 \in \mathbb{R}^d$ if (2.1) holds for φ and any $x_0, x_1 \in X$.

Now, we introduce the *Legendre transform* φ^* of φ, which is defined by

$$\varphi^*(y) = \sup_{z\in\mathbb{R}^d} \langle y, z\rangle - \varphi(z). \tag{2.2}$$

Since φ is convex, given $y \in \partial\varphi(x)$ we readily get for all $z \in \mathbb{R}^d$

$$\varphi(x) + \langle y, z - x\rangle \leq \varphi(z) \quad \Rightarrow \quad \langle y, z\rangle - \varphi(z) \leq \langle y, x\rangle - \varphi(x),$$

whence in view of (2.2) the following two key properties of φ^* are valid:

$$\varphi^*(y) = \langle y, x\rangle - \varphi(x) < +\infty \quad \forall\, y \in \partial\varphi(x),$$

and $y \in \partial\varphi(x)$ if and only if $x \in \partial\varphi^*(y)$. To see that $y \in \partial\varphi(x)$ implies $x \in \partial\varphi^*(y)$, let $z \in \mathbb{R}^d$ be arbitrary and notice that (2.2) for $\varphi^*(z)$ yields

$$\varphi^*(y) + \langle x, z - y\rangle = \langle x, z\rangle - \varphi(x) \le \varphi^*(z) \quad \Rightarrow \quad x \in \partial\varphi^*(y).$$

Consequently, if φ and φ^* are of class C^1 then $\nabla\varphi^* = (\nabla\varphi)^{-1}$ is the inverse of the transport map $\nabla\varphi$. Moreover,

$$-\infty < \int_Y \varphi^*(y)\,\mathrm{d}\nu(y) < +\infty, \tag{2.3}$$

provided (μ, ν, φ) is λ-regular. In fact, since $(\nabla\varphi)_{\#}\mu = \nu$, in view of (1.2) we have

$$\int_Y \varphi^*(y)\,\mathrm{d}\nu(y) = \int_X \varphi^*(\nabla\varphi(x))\,\mathrm{d}\mu(x) = \int_X \Big(\langle\nabla\varphi(x), x\rangle - \varphi(x)\Big)\,\mathrm{d}\mu(x).$$

We exploit the convexity of φ and $\frac{\lambda}{2}|\cdot|^2 - \varphi$ to obtain

$$\varphi(0) - \frac{\lambda}{2}|x|^2 \le \varphi(x) - \langle\nabla\varphi(x), x\rangle \le \varphi(0).$$

Since $\mu \in P_2(\mathbb{R}^d)$, (1.4) imply that the following integrals are finite and yield (2.3)

$$-\int_X \varphi(0)\,\mathrm{d}\mu(x) \le \int_X \Big(\langle\nabla\varphi(x), x\rangle - \varphi(x)\Big)\,\mathrm{d}\mu(x) \le \int_X \Big(-\varphi(0) + \frac{\lambda}{2}|x|^2\Big)\,\mathrm{d}\mu(x).$$

The following lemma states that λ-regularity of a convex function φ is equivalent to uniform convexity of its Legendre transform φ^* (Azé & Penot, 1995, Proposition 2.6).

Lemma 2.2 (λ-regularity vs uniform convexity). If $\varphi : \mathbb{R}^d \to \mathbb{R}$ is convex then the following statements are valid:

(a) if φ is λ-smooth then its Legendre transform φ^* must satisfy

$$\varphi^*((1-t)y_0 + ty_1) + \frac{t(1-t)}{2\lambda}|y_0 - y_1|^2 \le (1-t)\varphi^*(y_0) + t\varphi^*(y_1) \tag{2.4}$$

for all $y_0, y_1 \in \mathbb{R}^d$ and $t \in [0, 1]$, i.e. $\varphi^*(y) - \frac{1}{2\lambda}|y|^2$ is convex;

(b) if $\varphi(y) - \frac{1}{2\lambda}|y|^2$ is convex then its Legendre transform φ^* is λ-smooth.

Proof. For completeness, we repeat the proof of (Azé & Penot, 1995, Proposition 2.6) to show (a); the proof of (b) is similar. Given $y_0, y_1 \in \mathbb{R}^d$ and $t \in [0, 1]$, for any $x_0, v \in \mathbb{R}^d$ we set $x_t := x_0 + tv$ and

$y_t := y_0 + t(y_1 - y_0)$ and use (2.2) to write

$$\begin{aligned}(1-t)\varphi^*(y_0) + t\varphi^*(y_1) &\geq (1-t)\langle y_0, x_0\rangle + t\langle y_1, x_1\rangle - (1-t)\varphi(x_0) - t\varphi(x_1)\\ &\geq (1-t)\langle y_0, x_0\rangle + t\langle y_1, x_1\rangle - \varphi(x_t) - \frac{t(1-t)\lambda}{2}|v|^2\\ &= \langle y_t, x_0 + tv\rangle - \varphi(x_0 + tv) + t(1-t)\langle y_1 - y_0, v\rangle - \frac{t(1-t)\lambda}{2}|v|^2.\end{aligned}$$

We exploit that x_0, v are arbitrary. Taking supremum with respect to x_0, we get

$$(1-t)\varphi^*(y_0) + t\varphi^*(y_1) \geq \varphi^*(y_t) + t(1-t)\langle y_1 - y_0, v\rangle - \frac{t(1-t)\lambda}{2}|v|^2.$$

Maximizing the last two terms with respect to v, we obtain $v = \frac{1}{\lambda}(y_1 - y_0)$ and

$$(1-t)\varphi^*(y_0) + t\varphi^*(y_1) \geq \varphi^*(y_t) + \frac{t(1-t)}{2\lambda}|y_0 - y_1|^2,$$

which is the asserted inequality (2.4). That $\varphi^*(y) - \frac{1}{2\lambda}|y|^2$ is convex is a straightforward calculation that completes the proof. □

LEMMA 2.3 (W^2_∞-regularity). A λ-smooth convex map φ is of class $W^2_\infty(\mathbb{R}^d)$ and the Lipschitz constant of $T = \nabla\varphi$ is λ, namely

$$|T(x_1) - T(x_2)| \leq \lambda|x_1 - x_2| \quad \forall x_1, x_2 \in \mathbb{R}^d. \tag{2.5}$$

Proof. According to Lemma 2.2(a), the function $\psi(y) := \varphi^*(y) - \frac{1}{2\lambda}|y|^2$ is convex. If $x_1 \in \partial\varphi^*(y_1)$ and $x_2 \in \partial\varphi^*(y_2)$, then

$$\begin{aligned} x_1 - \frac{1}{\lambda}y_1 \in \partial\psi(y_1) \quad &\Rightarrow \quad \psi(y_2) \geq \psi(y_1) + \left\langle x_1 - \frac{1}{\lambda}y_1, y_2 - y_1\right\rangle,\\ x_2 - \frac{1}{\lambda}y_2 \in \partial\psi(y_2) \quad &\Rightarrow \quad \psi(y_1) \geq \psi(y_2) + \left\langle x_2 - \frac{1}{\lambda}y_2, y_1 - y_2\right\rangle.\end{aligned}$$

Adding the two inequalities and rearranging terms yield

$$\frac{1}{\lambda}|y_1 - y_2|^2 \leq \langle y_1 - y_2, x_1 - x_2\rangle \leq |y_1 - y_2|\,|x_1 - x_2|.$$

This implies the assertion (2.5). □

It is now apparent from Lemma 2.3 that the constant λ dictates the stability of the transport map $T = \nabla\varphi$; λ is also the stability constant that appears in our error estimates of Sections 4 and 5. If $\varphi \in C^2(\mathbb{R}^d)$ then (2.5) is equivalent to $D^2\varphi(x) \leq \lambda I$ for all x. As we have already mentioned at the end of Section 1.1, the optimal map from μ to ν for quadratic cost is given by $\nabla\varphi$ under suitable assumptions on μ. Caffarelli's regularity results (Caffarelli, 1992a,b, 1996) provide sufficient conditions for λ-regularity: if both $\overline{X} = \mathrm{supp}(\mu)$ and $\overline{Y} = \mathrm{supp}(\nu)$ are uniformly convex domains of $\mathbb{R}^d$ with C^2 boundary and the densities f, g of μ, ν are bounded away from 0 and $f \in C^{0,\alpha}(\overline{X})$, $g \in C^{0,\alpha}(\overline{Y})$, then the solution φ of the corresponding second boundary value Monge–Ampère problem (1.6) is of class $C^{2,\alpha}(\overline{X})$. This implies that (μ, ν, φ) is λ-regular for some $\lambda > 0$ if we extend φ to $\mathbb{R}^d$. Moreover, the same assumptions imply that (ν, μ, φ^*) is ξ-*regular* for the Legendre transform $\varphi^* \in C^{2,\alpha}(\overline{Y})$ of φ and some $\xi > 0$.

3. Quantitative stability of optimal transport plans

In this section, we generalize the quantitative stability bounds of (Gigli, 2011, Corollary 3.4) for optimal transport maps and show some consequences of our theorem under suitable assumptions on measures μ and ν. They will be useful in Sections 4 and 5 to derive error estimates. The stability estimate in (Gigli, 2011, Corollary 3.4) is based on bounding the L^2 difference between an optimal transport map and another feasible transport map by the difference between their transport costs. Proposition 3.1 below is just a generalization of this important property in that it replaces transport maps by transport plans. This is essential in deriving error estimates because optimal maps might not exist for semidiscrete and fully discrete optimal transport problems; see Section 1.2 as well as (Santambrogio, 2015, Section 1.4) for some examples.

Proposition 3.1 (stability of transport plans). Let (μ, ν, φ) be λ-regular for $\lambda > 0$, and $T = \nabla\varphi$ be the optimal transport map from μ to ν. Then, for any transport plan $\gamma \in \Pi(\mu, \nu)$, there holds

$$\int_{X\times Y} |y - T(x)|^2 \,\mathrm{d}\gamma(x, y) \leq \lambda \left(\int_{X\times Y} |y - x|^2 \,\mathrm{d}\gamma(x, y) - \int_X |T(x) - x|^2 \,\mathrm{d}\mu(x) \right). \tag{3.1}$$

Proof. We proceed in the same way as in Gigli (2011). If φ^* is the Legendre transform of φ then $\int_Y \varphi^*(y)\,\mathrm{d}\nu$ is finite from (2.3). Since $T_\# \mu = \nu$, using (1.2), we have

$$0 = \int_Y \varphi^*(y)\,\mathrm{d}\nu(y) - \int_X \varphi^*(T(x))\,\mathrm{d}\mu(x) = \int_{X\times Y} \big(\varphi^*(y) - \varphi^*(T(x))\big)\,\mathrm{d}\gamma(x, y).$$

In view of Lemma 2.3 (W^2_∞-regularity), the map $T = \nabla\varphi$ is of class W^1_∞. It thus follows that $x = \nabla\varphi^*(T(x))$ for all $x \in X$ because $\nabla\varphi^* = (\nabla\varphi)^{-1}$. Using

$$2\langle z, z - y\rangle = |z|^2 - |y^2| + |z - y|^2 \quad \forall y, z \in \mathbb{R}^d,$$

together with Lemma 2.2 (λ-regularity vs. uniform convexity), namely $x \mapsto \varphi^*(x) - \frac{1}{2\lambda}|x|^2$ is convex, we further obtain that

$$\begin{aligned} 0 &= \int_{X\times Y} \big(\varphi^*(y) - \varphi^*(T(x))\big)\,\mathrm{d}\gamma(x, y) \\ &\geq \int_{X\times Y} \langle x,\ y - T(x)\rangle\,\mathrm{d}\gamma(x, y) + \frac{1}{2\lambda}\int_{X\times Y} |y - T(x)|^2\,\mathrm{d}\gamma(x, y). \end{aligned}$$

Noticing that

$$2\langle x,\ y-T(x)\rangle = |T(x)-x|^2 - |y-x|^2 - |T(x)|^2 + |y|^2,$$

we deduce

$$2\int_{X\times Y}\langle x,\ y-T(x)\rangle\,\mathrm{d}\gamma(x,y) = \int_X |T(x)-x|^2\,\mathrm{d}\mu(x)$$
$$-\int_{X\times Y}|y-x|^2\,\mathrm{d}\gamma(x,y) - \int_X |T(x)|^2\,\mathrm{d}\mu(x) + \int_Y |y|^2\,\mathrm{d}\nu(y).$$

In view of (1.2), the last two terms are equal and cancel out. Consequently,

$$2\int_{X\times Y}\langle x,\ y-T(x)\rangle\,\mathrm{d}\gamma(x,y) = \int_X |T(x)-x|^2\,\mathrm{d}\mu(x) - \int_{X\times Y}|y-x|^2\,\mathrm{d}\gamma(x,y),$$

whence

$$0 \geq \int_X |T(x)-x|^2\,\mathrm{d}\mu - \int_{X\times Y}|y-x|^2\,\mathrm{d}\gamma(x,y) + \frac{1}{\lambda}\int_{X\times Y}|y-T(x)|^2\,\mathrm{d}\gamma(x,y).$$

Rearranging the equation above gives (3.1). □

Remark 3.2 (interpretation of (3.1)). Notice that the right-hand side in (3.1) without factor λ is the difference of transport costs between the given transport plan γ and the optimal transport map T. The factor λ acts as a stability constant.

Remark 3.3 (stability of transport maps). We point out that the left-hand side of (3.1) can be seen as the square of a weighted L^2-error between the transport plan γ and the optimal map T. To understand this, notice that if the plan γ in Proposition 3.1 is induced by a map $S:\mathbb{R}^d\to\mathbb{R}^d$, i.e. $(id,S)_\#\mu=\gamma$, then (3.1) can be written as

$$\int_X |S(x)-T(x)|^2\,\mathrm{d}\mu(x) \leq \lambda\left(\int_X |S(x)-x|^2\,\mathrm{d}\mu(x) - \int_X |T(x)-x|^2\,\mathrm{d}\mu(x)\right),$$

which is the same as (Gigli, 2011, Proposition 3.3). This is an estimate of $\|S-T\|^2_{L^2_\mu(X)}$.

Let us recall the gluing lemma for measures (see (Villani, 2009, p.23)), which plays an important role in proving the triangle inequality of Wasserstein distance in the theory of optimal transport. We refer to (Santambrogio, 2015, Lemma 5.5) for a proof.

Lemma 3.4 (gluing of measures). Let (X_i,μ_i) be probability spaces for $i=1,2,3$, with $X_i\subset\mathbb{R}^d$, and $\gamma_{1,2}\in\Pi(\mu_1,\mu_2)$, $\gamma_{2,3}\in\Pi(\mu_2,\mu_3)$. Then, there exists (at least) one measure $\gamma\in P(X_1\times X_2\times X_3)$ such that $(\pi_{1,2})_\#\gamma=\gamma_{1,2}$ and $(\pi_{2,3})_\#\gamma=\gamma_{2,3}$, where $\pi_{i,j}$ is the projection defined as $\pi_{i,j}(x_1,x_2,x_3)=(x_i,x_j)$.

We use this lemma to glue the measures μ,ν with their approximations μ_h,ν_h as follows. Given optimal transport plans $\gamma_h\in\Pi(\mu_h,\nu_h)$ between μ_h and ν_h, $\alpha_h\in\Pi(\mu,\mu_h)$ between μ and μ_h, and

$\beta_h \in \Pi(\nu_h, \nu)$ between ν_h and ν, we let $\Gamma_h \in P(X_1 \times X_2 \times X_3 \times X_4)$ be a probability measure so that

$$(\pi_{1,2})_\# \Gamma_h = \alpha_h, \quad (\pi_{2,3})_\# \Gamma_h = \gamma_h, \quad (\pi_{3,4})_\# \Gamma_h = \beta_h, \tag{3.2}$$

where $X_1 = X_2 = X$ and $X_3 = X_4 = Y$; Lemma 3.4 guarantees the existence of Γ_h. This in particular implies that projections π_i on the coordinate x_i satisfy

$$(\pi_1)_\# \Gamma_h = \mu, \quad (\pi_2)_\# \Gamma_h = \mu_h, \quad (\pi_3)_\# \Gamma_h = \nu_h, \quad (\pi_4)_\# \Gamma_h = \nu, \tag{3.3}$$

along with $\sigma := (\pi_{1,4})_\# \Gamma_h \in \Pi(\mu, \nu)$. Moreover, the following formula holds

$$\int_{X_1 \times X_2 \times X_3 \times X_4} F(x_1, x_2)\, \mathrm{d}\Gamma_h(x_1, x_2, x_3, x_4) = \int_{X_1 \times X_2} F(x_1, x_2)\, \mathrm{d}\alpha_h(x_1, x_2), \tag{3.4}$$

for $(x_1, x_2; \alpha_h)$ and similar expressions are valid for $(x_2, x_3; \gamma_h)$, $(x_3, x_4; \beta_h)$ and $(x_1, x_4; \sigma)$. If $X = Y = \mathbb{R}^d$, and $\mu, \mu_h, \nu_h, \nu \in P_2(\mathbb{R}^d)$ then $\alpha_h, \beta_h, \gamma_h, \sigma \in P_2(\mathbb{R}^{2d})$ according to (1.5).

Now, we state and prove our main perturbation estimates for transport plans.

THEOREM 3.5 (perturbation of optimal transport plans). Let $T = \nabla\varphi$ be the optimal transport map from μ to ν and (μ, ν, φ) be λ-regular for $\lambda > 0$. Let $\mu_h, \nu_h \in P_2(\mathbb{R}^d)$ be approximations of μ, ν, and let $\alpha_h \in \Pi(\mu, \mu_h)$, $\beta_h \in \Pi(\nu_h, \nu)$ be two transport plans between μ, μ_h and ν_h, ν with L^2-errors

$$e_{\alpha_h} := \left(\int_{X^2} |x - x'|^2\, \mathrm{d}\alpha_h(x, x')\right)^{\frac{1}{2}}, \quad e_{\beta_h} := \left(\int_{Y^2} |y' - y|^2\, \mathrm{d}\beta_h(y', y)\right)^{\frac{1}{2}}, \tag{3.5}$$

and let $e_h := e_{\alpha_h} + e_{\beta_h}$. If γ_h is an optimal transport plan between μ_h and ν_h, then

$$\left(\int_{X \times Y} |T(x') - y'|^2\, \mathrm{d}\gamma_h(x', y')\right)^{\frac{1}{2}} \le 2\lambda^{\frac{1}{2}}\, e_h^{\frac{1}{2}} \Big(W_2(\mu, \nu) + e_h\Big)^{\frac{1}{2}} + \lambda e_{\alpha_h} + e_{\beta_h}. \tag{3.6}$$

REMARK 3.6 (interpretation of (3.6)). We emphasize that, in view of Remark 3.3 (stability of transport maps), the left-hand side of (3.6) can be viewed as the square of the L^2-error between the optimal map T and the transport plan γ_h. The quantity e_h is a measure of the approximation errors between μ, μ_h and ν, ν_h. A typical choice of α_h, β_h is to take the optimal transport plans between μ, μ_h and ν_h, ν respectively, which implies $e_h = W_2(\mu, \mu_h) + W_2(\nu_h, \nu)$.

REMARK 3.7 (non-uniqueness). In general, an optimal transport plan γ_h between μ_h and ν_h may be neither unique nor induced by a map, no matter how small e_h is; for instance, if $X = \{(\pm 1, 0)\}_{i=1}^2$, $Y = \{(0, \pm 1)\}_{j=1}^2$ then any plan from X to Y has the same cost. However, (3.6) in Theorem 3.5 shows that all optimal transport plans are somewhat close to the map T in a certain L^2-sense dictated by $e_h^{1/2}$.

Proof of Theorem 3.5. The proof proceeds as in (Gigli, 2011, Corollary 3.4). Let $\Gamma_h \in P_2(X^2 \times Y^2)$ be a measure gluing μ, μ_h, ν_h, ν and satisfying (3.2) and (3.3). Since $\sigma = (\pi_{1,4})_\# \Gamma_h \in \Pi(\mu, \nu)$ is a transport plan between μ and ν, Proposition 3.1 (stability of transport plans) gives

$$\int_{X\times Y} |T(x)-y|^2 \,\mathrm{d}\sigma(x,y) \le \lambda \left(\int_{X\times Y} |x-y|^2 \,\mathrm{d}\sigma(x,y) - W_2^2(\mu,\nu) \right), \tag{3.7}$$

according to the definition of Wasserstein distance $W_2^2(\mu,\nu) = \int_X |T(x)-x|^2 \,\mathrm{d}\mu(x)$ and the fact that T is the optimal map from μ to ν. Since $\gamma_h = (\pi_{2,3})_\# \Gamma_h \in \Pi(\mu_h,\nu_h)$ is an optimal transport plan between μ_h and ν_h, i.e.

$$W_2^2(\mu_h,\nu_h) = \int_{X\times Y} |x'-y'|^2 \,\mathrm{d}\gamma_h(x',y'),$$

applying the triangle inequality, we deduce

$$W_2(\mu_h,\nu_h) \le W_2(\mu_h,\mu) + W_2(\mu,\nu) + W_2(\nu,\nu_h) \le W_2(\mu,\nu) + e_h. \tag{3.8}$$

Combining the relation $\sigma = (\pi_{1,4})_\# \Gamma_h$ between σ and Γ_h, namely

$$\left(\int_{X\times Y} |x-y|^2 \,\mathrm{d}\sigma(x,y) \right)^{\frac{1}{2}} = \left(\int_{X^2\times Y^2} |x-y|^2 \,\mathrm{d}\Gamma_h(x,x',y',y) \right)^{\frac{1}{2}},$$

with the triangle inequality yields

$$\begin{aligned} \left(\int_{X\times Y} |x-y|^2 \,\mathrm{d}\sigma(x,y) \right)^{\frac{1}{2}} \le & \left(\int_{X^2\times Y^2} |x-x'|^2 \,\mathrm{d}\Gamma_h(x,x',y',y) \right)^{\frac{1}{2}} \\ & + \left(\int_{X^2\times Y^2} |x'-y'|^2 \,\mathrm{d}\Gamma_h(x,x',y',y) \right)^{\frac{1}{2}} \\ & + \left(\int_{X^2\times Y^2} |y'-y|^2 \,\mathrm{d}\Gamma_h(x,x',y',y) \right)^{\frac{1}{2}}. \end{aligned}$$

In view of (3.2) and (3.8), this expression can be equivalently rewritten as

$$\begin{aligned} \left(\int_{X\times Y} |x-y|^2 \,\mathrm{d}\sigma(x,y) \right)^{\frac{1}{2}} \le & \left(\int_{X\times Y} |x-x'|^2 \,\mathrm{d}\alpha_h(x,x') \right)^{\frac{1}{2}} \\ & + \left(\int_{X\times Y} |x'-y'|^2 \,\mathrm{d}\gamma_h(x',y') \right)^{\frac{1}{2}} \\ & + \left(\int_{X\times Y} |y'-y|^2 \,\mathrm{d}\beta_h(y',y) \right)^{\frac{1}{2}} \le W_2(\mu,\nu) + 2e_h. \end{aligned}$$

This implies the inequality

$$\int_{X\times Y} |x-y|^2 \,\mathrm{d}\sigma(x,y) - W_2^2(\mu,\nu) \le 4W_2(\mu,\nu)e_h + 4e_h^2,$$

whence substitution into (3.7) immediately gives

$$\left(\int_{X\times Y} |T(x)-y|^2 \,\mathrm{d}\sigma(x,y)\right)^{\frac{1}{2}} \le 2\lambda^{\frac{1}{2}}\, e_h^{\frac{1}{2}}\Big(W_2(\mu,\nu)+e_h\Big)^{\frac{1}{2}}. \tag{3.9}$$

To prove (3.6), we first recall (3.2) and (3.3) to write

$$\int_{X\times Y} |T(x')-y'|^2 \,\mathrm{d}\gamma_h(x',y') = \int_{X^2\times Y^2} |T(x')-y'|^2 \,\mathrm{d}\Gamma_h(x,x',y',y),$$
$$\int_{X\times Y} |T(x)-y|^2 \,\mathrm{d}\sigma(x,y) = \int_{X^2\times Y^2} |T(x)-y|^2 \,\mathrm{d}\Gamma_h(x,x',y',y),$$

and next utilize the triangle inequality together with the Lipschitz property (2.5) of the optimal transport map T to obtain

$$\begin{aligned}
&\left(\int_{X\times Y} |T(x')-y'|^2 \,\mathrm{d}\gamma_h(x',y')\right)^{\frac{1}{2}} - \left(\int_{X\times Y} |T(x)-y|^2 \,\mathrm{d}\sigma(x,y)\right)^{\frac{1}{2}}\\
&\le \left(\int_{X^2\times Y^2} |T(x')-T(x)|^2 \,\mathrm{d}\Gamma_h(x,x',y',y)\right)^{\frac{1}{2}} + \left(\int_{X^2\times Y^2} |y-y'|^2 \,\mathrm{d}\Gamma_h(x,x',y',y)\right)^{\frac{1}{2}}\\
&\le \lambda\left(\int_{X\times Y} |x'-x|^2 \,\mathrm{d}\alpha_h(x,x')\right)^{\frac{1}{2}} + \left(\int_{X\times Y} |y-y'|^2 \,\mathrm{d}\beta_h(y',y)\right)^{\frac{1}{2}} = \lambda e_{\alpha_h} + e_{\beta_h}.
\end{aligned}$$

Rearranging the above inequality and using (3.9) prove (3.6). □

One could also consider a plan $\gamma_h \in \Pi(\mu_h,\nu_h)$ that is not exactly optimal, but close to the optimal transport plan. For this case, we prove the following corollary.

COROLLARY 3.8 (perturbation of transport plans in L^2). Let μ,ν,μ_h,ν_h and φ be as in Theorem 3.5 (perturbation of optimal transport plans). Let $\gamma_h \in \Pi(\mu_h,\nu_h)$ be a transport plan between μ_h and ν_h, and let $\widetilde{e}_h$ be given by

$$\widetilde{e}_h := e_h + \frac{\varepsilon_h}{2}, \quad \varepsilon_h := \left(\int_{X\times Y} |x'-y'|^2 \,\mathrm{d}\gamma_h(x',y')\right)^{\frac{1}{2}} - W_2(\mu_h,\nu_h), \tag{3.10}$$

and $e_h = e_{\alpha_h} + e_{\beta_h}$ be defined in (3.5). We then have

$$\left(\int_{X\times Y} |T(x') - y'|^2 \, \mathrm{d}\gamma_h(x', y')\right)^{\frac{1}{2}} \le 2\lambda^{\frac{1}{2}} \, \widetilde{e}_h^{\frac{1}{2}} \Big(W_2(\mu, \nu) + \widetilde{e}_h\Big)^{\frac{1}{2}} + \lambda e_{\alpha_h} + e_{\beta_h}. \tag{3.11}$$

Proof. We argue as in Theorem 3.5, except that instead of (3.8) we now have

$$\left(\int_{X\times Y} |x' - y'|^2 \, \mathrm{d}\gamma_h(x', y')\right)^{\frac{1}{2}} \le W_2(\mu, \nu) + e_h + \varepsilon_h,$$

whence (3.9) becomes

$$\left(\int_{X\times Y} |T(x) - y|^2 \, \mathrm{d}\sigma(x, y)\right)^{\frac{1}{2}} \le 2\,\lambda^{\frac{1}{2}} \, \widetilde{e}_h^{\frac{1}{2}} \Big(W_2(\mu, \nu) + \widetilde{e}_h\Big)^{\frac{1}{2}}. \tag{3.12}$$

The rest of the proof continues as in Theorem 3.5. □

Instead of measuring the L^2-error in (3.11) between the optimal transport map T from μ to ν and the optimal transport plan γ_h from μ_h to ν_h, we could alternatively characterize the discrepancy between the corresponding plans γ and γ_h in terms of the Wasserstein distance. This is possible in light of (1.5) because $\gamma, \gamma_h \in P_2(\mathbb{R}^{2d})$ provided $\mu, \nu \in P_2(\mathbb{R}^d)$ and $\mu_h, \nu_h \in P_2(\mathbb{R}^d)$, respectively.

COROLLARY 3.9 (perturbation of transport plans in Wasserstein metric). Let $T = \nabla\varphi$ be the optimal transport map from μ to ν and (μ, ν, φ) be λ-regular for $\lambda > 0$. Let $\gamma = (id, T)_{\#}\mu$ be the optimal transport plan induced by T. Let $\mu_h, \nu_h \in P_2(\mathbb{R}^d)$ be approximations to μ, ν and $\gamma_h \in \Pi(\mu_h, \nu_h)$ be a transport plan between μ_h and ν_h. Then, we have

$$W_2(\gamma, \gamma_h) \le 2\,\lambda^{\frac{1}{2}} \, \widetilde{e}_h^{\frac{1}{2}} \Big(W_2(\mu, \nu) + \widetilde{e}_h\Big)^{\frac{1}{2}} + e_h,$$

where $e_h = W_2(\mu, \mu_h) + W_2(\nu, \nu_h)$ and $\widetilde{e}_h$ is defined in (3.10).

Proof. Let $\alpha_h \in \Pi(\mu, \mu_h), \beta_h = \Pi(\nu_h, \nu)$ be the optimal transport plans for the corresponding OT, and $\Gamma_h \in P_2(\mathbb{R}^4)$ be a gluing measure satisfying (3.2) and (3.3). We need to construct a suitable transport plan $\widetilde{\Gamma}_h \in \Pi(\gamma, \gamma_h)$ from γ to γ_h and use the fact that the corresponding transport cost dominates $W_2(\gamma, \gamma_h)$. To this end, we introduce the map $S : \mathbb{R}^{4d} \to \mathbb{R}^{2d}$ defined as $S(x, x', y', y) := (x, T(x))$, and consider the pushforward of measure Γ_h through S

$$S_{\#}\Gamma_h = (id, T)_{\#}\big((\pi_1)_{\#}\Gamma_h\big) = (id, T)_{\#}\mu = \gamma.$$

Since $(\pi_{2,3})_{\#}\Gamma_h = \gamma_h$ according to (3.2), we infer that the map $\widetilde{S} := (S, \pi_{2,3}) : \mathbb{R}^{4d} \to \mathbb{R}^{4d}$ defined by

$$\widetilde{S}(x, x', y', y) = (z_1, z_2), \quad z_1 = (x, T(x)), \quad z_2 = (x', y')$$

induces a pushforward measure $\widetilde{\Gamma}_h := \widetilde{S}_\# \Gamma_h \in \Pi(\gamma, \gamma_h)$ of Γ_h through $\widetilde{S}$ that happens to be a transport plan between γ and γ_h. This implies

$$
\begin{aligned}
W_2(\gamma, \gamma_h) &\leq \left(\int_{\mathbb{R}^{4d}} |z_1 - z_2|^2 \, \mathrm{d}\widetilde{\Gamma}_h(z_1, z_2) \right)^{\frac{1}{2}} \\
&= \left(\int_{\mathbb{R}^{4d}} \left| S(x, x', y', y) - \pi_{2,3}(x, x', y', y) \right|^2 \mathrm{d}\Gamma_h(x, x', y', y) \right)^{\frac{1}{2}} \\
&= \left(\int_{\mathbb{R}^{4d}} \left(|x - x'|^2 + |T(x) - y'|^2 \right) \mathrm{d}\Gamma_h(x, x', y', y) \right)^{\frac{1}{2}}.
\end{aligned}
$$

Using the triangle inequality along with (3.4) yields

$$
\begin{aligned}
W_2(\gamma, \gamma_h) &\leq \left(\int_{\mathbb{R}^{4d}} |x - x'|^2 \, \mathrm{d}\Gamma_h(x, x', y', y) \right)^{\frac{1}{2}} \\
&\quad + \left(\int_{\mathbb{R}^{4d}} |y' - y|^2 \, \mathrm{d}\Gamma_h(x, x', y', y) \right)^{\frac{1}{2}} \\
&\quad + \left(\int_{\mathbb{R}^{4d}} |T(x) - y|^2 \, \mathrm{d}\Gamma_h(x, x', y', y) \right)^{\frac{1}{2}} \\
&= W_2(\mu, \mu_h) + W_2(\nu, \nu_h) + \left(\int_{\mathbb{R}^{4d}} |T(x) - y|^2 \, \mathrm{d}\Gamma_h(x, x', y', y) \right)^{\frac{1}{2}}
\end{aligned}
$$

because α_h and β_h are optimal transport plans. For the last term, we recall that $\sigma = (\pi_{1,4})_\# \Gamma_h$ and employ (3.4) again, together with (3.12), to arrive at

$$
\begin{aligned}
\left(\int_{\mathbb{R}^{4d}} |T(x) - y|^2 \, \mathrm{d}\Gamma_h(x, x', y', y) \right)^{\frac{1}{2}} &= \left(\int_{\mathbb{R}^{2d}} |T(x) - y|^2 \, \mathrm{d}\sigma(x, y) \right)^{\frac{1}{2}} \\
&\leq 2\lambda^{\frac{1}{2}} \, \widetilde{e}_h^{\frac{1}{2}} \left(W_2(\mu, \nu) + \widetilde{e}_h \right)^{\frac{1}{2}}.
\end{aligned}
$$

This finishes the proof of the corollary. □

In the next two sections, we explain how to use the preceding stability estimates to obtain some useful error estimates for numerical solutions of OT between the probability measures μ, ν with densities f, g, supported in $\overline{X}, \overline{Y} \subset \mathbb{R}^d$, respectively, and quadratic cost $c(x, y) = |x - y|^2$.

4. Error estimates for semi-discrete schemes

In this section, we present and prove error estimates for semidiscrete schemes such as those in Aurenhammer *et al.* (1998); Mérigot (2011); Lévy (2015); Kitagawa *et al.* (2019). We conclude our discussion with comparisons between our results and those of Berman (2018).

Let $\mu_h = \sum_{i=1}^N f_i \delta_{x_i}$ ($f_i > 0$) be an approximation of μ satisfying $W_2(\mu, \mu_h) \le h$. One way to obtain such an approximation is explained in Section 1.2. The semidiscrete schemes are designed to compute the optimal transport plan between the *discrete* measure μ_h and the *continuous* measure ν. The semidiscrete problem is equivalent to finding a nodal function $\varphi_h := \left(\varphi_h(x_i)\right)_{i=1}^N \in \mathbb{R}^N$ so that

$$\nu\left(F_i\right) = f_i, \quad F_i := \partial\varphi_h(x_i) \cap Y \qquad i = 1, \cdots, N, \tag{4.1}$$

where $Y = \{y \in \mathbb{R}^d : g(y) > 0\}$ and the discrete subdifferential set is given by

$$\partial\varphi_h(x_i) := \{y \in \mathbb{R}^d : \quad \varphi_h(x_i) + (x_j - x_i)\cdot y \le \varphi_h(x_j) \quad \forall j = 1, \cdots, N\}.$$

The convex sets F_i are called *Laguerre cells (or generalized Voronoi cells)*. We do not discuss methods to determine φ_h for which we refer to Aurenhammer *et al.* (1998); Mérigot (2011); Lévy (2015); Kitagawa *et al.* (2019). As pointed out in (Berman, 2018, Lemma 5.3), vector $\varphi_h \in \mathbb{R}^N$ is unique up to a constant if Y is a Lipschitz domain, which is a direct consequence of results in Kitagawa *et al.* (2019). Once we have found φ_h, the optimal transport map from ν to μ_h is given by the map S_h

$$S_h(y) := x_i \quad \text{a.e. } y \in F_i,$$

and the induced optimal transport plan $\gamma_h \in P(X \times Y)$ reads $\gamma_h = (S_h, id)_\# \nu$ and

$$\gamma_h(E) = \nu\big\{y \in Y : \left(S_h(y), y\right) \in E\big\} \quad \Rightarrow \quad \gamma_h\left(\{x_i\} \times F_i\right) = \nu(F_i) \tag{4.2}$$

for all Borel sets $E \subset X \times Y$. We also point out that there is no optimal map from μ_h to ν since every node x_i is associated with the Laguerre cell F_i, which has infinite number of points (multivalued function). We are now ready to estimate the difference of γ_h and the continuous optimal map T.

THEOREM 4.1 (convergence rate for semidiscrete schemes). Let the triple (μ, ν, φ) be λ-regular for $\lambda > 0$ where the measures μ to ν have non-negative densities f and g. Let $T = \nabla\varphi$ be the optimal transport map from μ to ν. Let $\mu_h = \sum_{i=1}^N f_i \delta_{x_i}$ be an approximation of μ with $f_i > 0$. If φ_h is a nodal function that solves (4.1) and γ_h is the corresponding semidiscrete optimal plan (4.2), then we have

$$\int_{X\times Y} |T(x) - y|^2 \, \mathrm{d}\gamma_h(x, y) = \sum_{i=1}^N \int_{F_i} |T(x_i) - y|^2 g(y) \, \mathrm{d}y \le E_h^2, \tag{4.3}$$

where

$$E_h := 2\lambda^{\frac{1}{2}} \; W_2(\mu, \mu_h)^{\frac{1}{2}} \Big(W_2(\mu, \nu) + W_2(\mu, \mu_h)\Big)^{\frac{1}{2}} + \lambda W_2(\mu, \mu_h). \tag{4.4}$$

Moreover, if μ_h satisfies $W_2(\mu, \mu_h) \le h$ then there exists a constant C depending on (μ, ν) such that for $h \le \lambda^{-1}$ we have

$$\int_{X\times Y} |T(x) - y|^2 \, \mathrm{d}\gamma_h(x, y) = \sum_{i=1}^N \int_{F_i} |T(x_i) - y|^2 g(y) \, \mathrm{d}y \le C\lambda h. \tag{4.5}$$

Proof. The first equality in (4.3) follows from the relation between $\gamma_h = (S_h, id)_\# \nu$ and φ_h expressed in (4.1) and (4.2)

$$\int_{X\times Y} |T(x) - y|^2 \,\mathrm{d}\gamma_h(x,y) = \int_Y |T\left(S_h(y)\right) - y|^2 \,\mathrm{d}\nu(y)$$
$$= \sum_{i=1}^N \int_{F_i} |T\left(S_h(y)\right) - y|^2 \,\mathrm{d}\nu(y) = \sum_{i=1}^N \int_{F_i} |T(x_i) - y|^2 g(y)\,\mathrm{d}y.$$

Now, we resort to Theorem 3.5 (perturbation of optimal transport plans). Since there is no discretization of ν, we let $\beta_h = (id, id)_\# \nu$ and α_h be the optimal transport plan between μ and μ_h. Then, (3.6) in Theorem 3.5 implies that

$$\int_{X\times Y} |T(x') - y'|^2 \,\mathrm{d}\gamma_h(x', y') \le \left(2\,\lambda^{\frac{1}{2}}\, e_h^{\frac{1}{2}} \left(W_2(\mu,\nu) + e_h\right)^{\frac{1}{2}} + \lambda e_{\alpha_h} + e_{\beta_h}\right)^2,$$

with $e_h = e_{\alpha_h} = W_2(\mu, \mu_h)$ and $e_{\beta_h} = 0$. This proves (4.3), whereas (4.5) follows directly from $W_2(\mu, \mu_h) \le h$ and $h \le \lambda^{-1}$. □

We recall that the optimal transport plan $\alpha_h = (id, U_h)_{\#\mu}$ between μ and μ_h defined before (1.9) satisfies $W_2(\mu, \mu_h) \le h$ according to (1.9). In addition, our estimates (4.3) and (4.5) contain geometric information about the Laguerre cells F_i provided g is strictly positive. To see this, for each $1 \le i \le N$ we introduce the *center of mass (or barycenter)* m_i of F_i with respect to the measure ν

$$m_i := \frac{1}{f_i} \int_{F_i} y\, g(y)\,\mathrm{d}y. \tag{4.6}$$

COROLLARY 4.2 (convergence rate for center of mass and Laguerre cells). We make the same hypotheses as in Theorem 4.1 and further assume that $g(y) \in [c_1, c_2]$ for all $y \in Y$ for some $c_2 \ge c_1 > 0$, and that Y is a bounded convex set. Then, there exists a constant C depending on d and c_1, c_2 such that

$$\sum_{i=1}^N f_i\left(|T(x_i) - m_i|^2 + C\,\mathrm{diam}(F_i)^2\right) \le E_h^2, \tag{4.7}$$

where F_i, m_i and E_h are defined in (4.1), (4.6) and (4.4), respectively. Moreover, if μ_h satisfies $W_2(\mu, \mu_h) \le h$, then there exists a constant C depending on (d, μ, ν, c_1, c_2) such that for $h \le \lambda^{-1}$ we have

$$\sum_{i=1}^N f_i\, |T(x_i) - m_i|^2 \le C\lambda h, \quad \sum_{i=1}^N f_i\,\mathrm{diam}\left(F_i\right)^2 \le C\lambda h. \tag{4.8}$$

Proof. Using the facts that $f_i = \nu(F_i) = \int_{F_i} g(y)\,\mathrm{d}y$ and $\int_{F_i} (m_i - y) g(y)\,\mathrm{d}y = 0$, which are valid due to (4.1) and (4.6), we infer that

$$\int_{F_i} \langle T(x_i) - m_i, m_i - y\rangle g(y)\,\mathrm{d}y = 0,$$

whence

$$\int_{F_i} |T(x_i) - y|^2 g(y)\,\mathrm{d}y = \int_{F_i} \Big(|T(x_i) - m_i|^2 + |m_i - y|^2\Big) g(y)\,\mathrm{d}y$$
$$= f_i |T(x_i) - m_i|^2 + \int_{F_i} |m_i - y|^2 g(y)\,\mathrm{d}y.$$

Therefore, to prove (4.7) it remains to show that there exists a constant C such that

$$C f_i \operatorname{diam}\big(F_i\big)^2 \le \int_{F_i} |m_i - y|^2 g(y)\,\mathrm{d}y.$$

Notice that $F_i = \partial\varphi_h(x_i) \cap Y$ is convex because both $\partial\varphi_h(x_i)$ and Y are convex. Let $\widetilde{m}_i := |F_i|^{-1} \int_{F_i} y\,\mathrm{d}y$ be the center of mass of F_i with respect to the Lebesgue measure, and apply (Gutiérrez, 2016, Theorem 1.8.2) to deduce the existence of an ellipsoid E centered at $\widetilde{m}_i$ such that $d^{-3/2}E \subset F_i \subset E$, where αE denotes the α-dilation of E with respect to $\widetilde{m}_i$. Then, we have

$$\int_{F_i} |m_i - y|^2 g(y)\,\mathrm{d}y \ge c_1 \int_{F_i} |m_i - y|^2\,\mathrm{d}y \ge c_1 \int_{F_i} |\widetilde{m}_i - y|^2\,\mathrm{d}y$$
$$\ge c_1 \int_{\mathrm{d}^{-3/2}E} |\widetilde{m}_i - y|^2\,\mathrm{d}y \ge c_d c_1 |E| \operatorname{diam}(E)^2,$$

where c_d is a constant depending on d. In fact, we will prove below that for any ellipsoid E_0 centered at z, we have

$$\int_{E_0} |y - z|^2\,\mathrm{d}y \ge c_d' |E_0| \operatorname{diam}(E_0)^2, \tag{4.9}$$

where c_d' is a constant depending on d. We finally utilize the properties

$$f_i = \int_{F_i} g(y)\,\mathrm{d}y \le c_2 |E|,$$

and $\operatorname{diam}(E) \ge \operatorname{diam}(F_i)$ because $F_i \subset E$, to obtain

$$c_d c_1 |E| \operatorname{diam}(E)^2 \ge \frac{c_d c_1}{c_2} f_i \operatorname{diam}(F_i)^2.$$

This finishes our proof of (4.7) by choosing $C = c_d c_1 c_2^{-1}$. Finally, (4.8) is a direct consequence of (4.7) because $E_h^2 \le C\lambda h$ whenever $W_2(\mu, \mu_h) \le h$ and $h \le \lambda^{-1}$.

It remains to show the geometric property (4.9). Let E_0 be given by

$$E_0 = \left\{ y \in \mathbb{R}^d : \quad \sum_{i=1}^{d} \frac{x_i^2}{a_i^2} \le 1 \right\}$$

and $a_1 = \max_{1\le j\le d} a_j$; hence, $\mathrm{diam}(E_0) = 2a_1$ and the barycenter z of E_0 coincides with the origin. We consider now the subset of E_0

$$\widetilde{E}_0 = \left\{ y \in E_0 : \quad \frac{1}{3}a_1 \le y_1 \le \frac{2}{3}a_1 \right\},$$

which satisfies

$$\widetilde{E}_0 \supset \left\{ y_1 \in \mathbb{R} : \quad \frac{1}{3}a_1 \le y_1 \le \frac{2}{3}a_1 \right\} \times \left\{ (y_j)_{j=2}^d : \quad \sum_{j=2}^d \frac{y_j^2}{a_j^2} \le \frac{5}{9} \right\} = \widetilde{E}_1.$$

We see that $E_0 \subset E_1 = [-a_1, a_1] \times \left\{ (y_j)_{j=2}^d : \sum_{j=2}^d \frac{y_j^2}{a_j^2} \le 1 \right\}$, whence $|E_0| < |E_1| \le c_d |\widetilde{E}_0|$, the last inequality being a consequence of a scaling argument between E_1 and $\widetilde{E}_1$. Since $|y| \ge \frac{a_1}{3} = \frac{1}{6}\mathrm{diam}(E_0)$ in $\widetilde{E}_0$, we infer that

$$\int_{E_0} |y|^2 \,\mathrm{d}y \ge \int_{\widetilde{E}_0} |y|^2 \,\mathrm{d}y \ge \frac{1}{36}\, \mathrm{diam}(E_0)^2 |\widetilde{E}_0| \ge c_d' \,\mathrm{diam}(E_0)^2 |E_0|.$$

This shows (4.9) and concludes the proof. □

Remark 4.3 (discrete transport map). To compensate for not having an optimal transport map from μ_h to ν, because $(x_i \mapsto F_i)_{i=1}^N$ is a multivalued function, we could define the discrete transport map T_h to be

$$T_h(x_i) = m_i \quad \forall\, i = 1, \cdots, N.$$

Expression (4.8) immediately yields the weighted L^2 error estimate

$$\sum_{i=1}^N f_i \left| T(x_i) - T_h(x_i) \right|^2 \le C\lambda h.$$

Remark 4.4 (mean errors). We point out that since $\sum_{i=1}^N f_i = 1$, (4.8) shows that on average the pointwise error $|T(x_i) - m_i|^2$ and $\mathrm{diam}(F_i)^2$ are of order $O(h)$.

Remark 4.5 (approximate transport plans). In practice, the semidiscrete schemes may not solve (4.1) exactly, but rather approximately: let $\widetilde{\varphi}_h$ solve

$$\nu(\widetilde{F}_i) = \widetilde{f}_i, \quad \widetilde{F}_i := \partial\widetilde{\varphi}_h(x_i) \cap Y \qquad i = 1, \cdots, N,$$

with $\widetilde{f}_i$ close to f_i, and induce an optimal plan $\widetilde{\gamma}_h$ between $\widetilde{\mu}_h = \sum_{i=1}^N \widetilde{f}_i \delta_{x_i}$ and ν. We note that the error estimate (4.5) in Theorem 4.1 (convergence rate for semidiscete schemes) only requires $W_2(\mu, \widetilde{\mu}_h) \le Ch$, and thus deduce that as long as the approximate measure $\widetilde{\mu}_h$ satisfies $W_2(\mu_h, \widetilde{\mu}_h) \le Ch$, we are still able to obtain

$$\sum_{i=1}^{N} \int_{\widetilde{F}_i} |T(x_i) - y|^2 g(y)\, \mathrm{d}y \le Ch$$

because $W_2(\mu, \widetilde{\mu}_h) \le W_2(\mu, \mu_h) + W_2(\mu_h, \widetilde{\mu}_h) \le Ch$. This shows that enforcing (4.1) exactly may not be computationally profitable.

We conclude this section comparing our results of Theorem 4.1 and Corollary 4.2 with that of (Berman, 2018, Theorem 5.4). The nodal function $\varphi_h = (\varphi_h(x_i))_{i=1}^N$ that satisfies (4.1) also induces the convex function $\Gamma(\varphi_h)$, in fact the convex envelope of $(\varphi_h(x_i))_{i=1}^N$. For convenience, we still denote $\Gamma(\varphi_h)$ by φ_h and observe that, being piecewise linear, it dictates a partition $\mathcal{T}_h$ of X into simplices with vertices $(x_i)_{i=1}^N$. However, this partition $\mathcal{T}_h$ of X might not be shape-regular in general even for a quasi-uniform distribution of nodes $(x_i)_{i=1}^N$; we refer to (Nochetto & Zhang, 2019, Section 2.2). Therefore, there is no direct relation between elements K_j of $\mathcal{T}_h$ containing x_i and the Voronoi cells V_i in (1.8). By assuming that $\lambda^{-1} I \le D^2\varphi \le \lambda I$ for some $\lambda > 0$ and the density g satisfies $g(y) \ge c_1 > 0$ for all $y \in Y$, Berman obtains the following error estimate for the piecewise constant function $\nabla\varphi_h$ over $\mathcal{T}_h$ (Berman, 2018, Theorem 5.4)

$$\|\nabla\varphi - \nabla\varphi_h\|_{L^2(X)} \le Ch^{\frac{1}{2}}, \tag{4.10}$$

where the constant C depends on λ, c_1 and other information. We see that this rate of convergence is similar to those in Theorem 4.1 and Corollary 4.2, but the error notion is different. To explore this fact, we rewrite (4.10) as follows

$$\sum_{K_j \in \mathcal{T}_h} \int_{K_j} |\nabla\varphi(x) - \nabla\varphi_h(x)|^2\, \mathrm{d}x \le Ch.$$

Since the Hessian $D^2\varphi$ is uniformly bounded both from above and below, we infer that $\nabla\varphi$ and $(\nabla\varphi)^{-1}$ are both Lipschitz with Lipschitz constants proportional to λ and λ^{-1}. Therefore, the previous inequality is equivalent to

$$\sum_{K_j \in \mathcal{T}_h} \int_{K_j} |x - (\nabla\varphi)^{-1}\nabla\varphi_h(x)|^2\, \mathrm{d}x \le C\lambda^2 h.$$

Let $z_j = |K_j|^{-1} \int_{K_j} x\, \mathrm{d}x$ be the barycenter of K_j with respect to the Lebesgue measure. An argument similar to the proof of Corollary 4.2 yields

$$|K_j|\, |z_j - (\nabla\varphi)^{-1} y_j|^2 + |K_j|\, \mathrm{diam}(K_j)^2 \lesssim \int_{K_j} |x - (\nabla\varphi)^{-1}\nabla\varphi_h(x)|^2\, \mathrm{d}x,$$

where $y_j = \nabla\varphi_h(x)$ is the constant slope for $x \in K_j$. Exploiting again that $T = \nabla\varphi$ is Lipschitz, we see that (4.10) leads to

$$\sum_{K_j \in \mathcal{T}_h} |K_j|\, \left|T(z_j) - y_j\right|^2 \le C\lambda^4 h, \quad \sum_{K_j \in \mathcal{T}_h} |K_j|\, \mathrm{diam}(K_j)^2 \le C\lambda^2 h. \tag{4.11}$$

This looks similar to (4.8), but involving simplices K_j instead of Laguerre cells F_i; (4.8) and (4.11) are, however, intrinsically different. Several comments are in order.

- The Laguerre cell F_i is the convex hull of all vectors $y_j = \nabla\varphi|_{K_j}$ where $K_j \in \mathcal{T}_h$ are the simplices containing x_i, the so-called star (or patch) ω_i (Nochetto & Zhang, 2018, Section 5.3). Therefore, m_i in (4.8) can be replaced by any y_j for $K_j \subset \omega_i$ because of the occurrence of $\operatorname{diam}(F_i)$ in (4.8). However, there is no immediate relation between $f_i = \nu(F_i)$ and $|\omega_i|$ or $\mu(\omega)$.
- Both estimates (4.8) and (4.11) require a lower positive bound for g. However, (4.5) just entails λ-regularity of (μ, ν, φ), but not a lower bound on either g or f; note that the assumption $f_i > 0$ in Theorem 4.1 is for convenience because if $f_i = 0$ we could simply drop the Dirac mass at x_i.
- The estimate (4.5) applies to discrete transport plans and extends to fully discrete schemes; see Section 5. It is not clear what a fully discrete version of (4.11) could be if the measure ν is further discretized into a sum of Dirac measures.
- The derivation of Theorem 4.1 and Corollary 4.2 is purely analytical and hinges on the quantitative stability estimates of Gigli (2011). In contrast, the proof of (4.11) in Berman (2018) uses a complexification argument to deduce estimates from well-known inequalities in Kähler geometry and pluripotential theory.
- Both (4.8) and (4.11) are meaningful ways to measure the error between the continuous optimal map T and the semidiscrete one T_h. Our error notion is natural in dealing with optimal transport maps and plans, while the error notion in Berman (2018) is perhaps more natural in the setting of Monge–Ampère equations.

5. Error estimates for fully discrete schemes

If we approximate both measures μ, ν by discrete measures $\mu_h = \sum_{i=1}^N f_i \delta_{x_i}$ and $\nu_h = \sum_{j=1}^M g_j \delta_{y_j}$, respectively, then we can consider the following *linear programming problem* to approximate the optimal transport map T (Cuturi, 2013; Schmitzer & Schnörr, 2013; Benamou *et al.*, 2015; Oberman & Ruan, 2015; Schmitzer, 2016): find the discrete measure $\gamma_h = \sum_{i=1}^N \sum_{j=1}^M \gamma_{h,ij} \delta_{x_i} \delta_{y_j}$ such that

$$\min_{\gamma_{h,ij}} \sum_{i,j=1}^{N,M} \gamma_{h,ij}\, c_{ij} : \quad \gamma_{h,ij} \ge 0, \ \sum_{i=1}^N \gamma_{h,ij} = g_j, \ \sum_{j=1}^M \gamma_{h,ij} = f_i, \tag{5.1}$$

where $c_{ij} = |x_i - y_j|^2$. One thing worth pointing out here is that the solution of the above problem may not be unique even though the continuous problem has a unique optimal map T (and plan γ); see Remark 3.7 (non-uniqueness). The minimum transport cost of (5.1) is equal to $W_2^2(\mu_h, \nu_h)$ according to the definition, but in practice we may not get an exact optimal plan γ_h, whence the cost $\sum_{i,j} \gamma_{h,ij} c_{ij}$ might be larger than $W_2^2(\mu_h, \nu_h)$.

Since the discrete plan γ_h may not be induced by a map, i.e. for some i there might exist more than one j such that $\gamma_{h,ij} > 0$, one may need to define a map T_h from γ_h to approximate the optimal transport map T between μ and ν. One way is to use a barycentric projection (Oberman & Ruan, 2015, Definition 5) and define T_h to be

$$T_h(x_i) := \frac{1}{f_i} \sum_{j=1}^M \gamma_{h,ij}\, y_j. \tag{5.2}$$

This is a discrete counterpart of (4.6). In general, the quantity $T_h(x_i)$ may not belong to the set $\{y_j : j = 1, \cdots, M\}$. The following theorem quantifies the errors committed in approximating the optimal plan γ by γ_h and the optimal map T by the map T_h generated from γ_h.

THEOREM 5.1 (convergence rate for fully discrete schemes). Let the triple (μ, ν, φ) be λ-regular for $\lambda > 0$ where the measures μ to ν have non-negative densities f and g. Let $T = \nabla\varphi$ be the optimal transport map from μ to ν. Let the approximations $\mu_h = \sum_{i=1}^{N} f_i\delta_{x_i}$, $\nu_h = \sum_{j=1}^{M} g_j\delta_{y_j}$ of μ, ν satisfy $W_2(\mu, \mu_h), W_2(\nu, \nu_h) \le C_1 h$. If $\gamma_h = \sum_{i,j=1}^{N,M} \gamma_{h,ij}\delta_{x_i}\delta_{y_j} \in \Pi(\mu_h, \nu_h)$ is a discrete transport plan so that

$$\left(\sum_{i,j=1}^{N,M} \gamma_{h,ij}\, c_{ij}\right)^{\frac{1}{2}} - W_2(\mu_h, \nu_h) \le C_2 h,$$

then for $h \le \min\{\lambda, \lambda^{-1}\}$ we have

$$\left(\int_{X\times Y} |T(x) - y|^2 \,\mathrm{d}\gamma_h(x, y)\right)^{\frac{1}{2}} = \left(\sum_{i,j=1}^{N,M} \gamma_{h,ij}\, |T(x_i) - y_j|^2\right)^{\frac{1}{2}} \le C\lambda^{\frac{1}{2}} h^{\frac{1}{2}}, \tag{5.3}$$

where C is a constant depending on (μ, ν, C_1, C_2). In addition, the Wasserstein distance between γ and γ_h satisfies

$$W_2(\gamma, \gamma_h) \le C\lambda^{\frac{1}{2}} h^{\frac{1}{2}}, \tag{5.4}$$

and the map T_h generated by γ_h according to (5.2) verifies

$$\left(\sum_{i=1}^{N} f_i |T(x_i) - T_h(x_i)|^2\right)^{\frac{1}{2}} \le C\lambda^{\frac{1}{2}} h^{\frac{1}{2}}. \tag{5.5}$$

Proof. The equality in (5.3) follows from the definition of $\gamma_h = \sum_{i,j=1}^{N,M} \gamma_{h,ij}\delta_{x_i}\delta_{y_j}$. By letting $\alpha_h \in \Pi(\mu, \mu_h), \beta_h \in \Pi(\nu_h, \nu)$ be the corresponding optimal transport plans between μ, μ_h and ν, ν_h, the inequality in (5.3) is a direct consequence of (3.11) in Corollary 3.8 (perturbation of transport plans in L^2) because

$$e_h = W_2(\mu, \mu_h) + W_2(\nu_h, \nu) \le 2C_1 h, \quad \varepsilon_h = \left(\sum_{i,j} \gamma_{h,ij}\, c_{ij}\right)^{\frac{1}{2}} - W_2(\mu_h, \nu_h) \le C_2 h,$$

and $h \le \min\{\lambda, \lambda^{-1}\}$. Similarly, (5.4) follows from Corollary 3.9 (perturbation of transport plans in Wasserstein metric). The proof of (5.5) is a discrete version of Corollary 4.2 (convergence rate for center of mass and Laguerre cells). We first notice that (5.1), together with the definition (5.2) of T_h, yields

$$\sum_{j=1}^{M} \gamma_{h,ij}\big(T_h(x_i) - y_j\big) = f_i T_h(x_i) - \sum_{j=1}^{M} \gamma_{h,ij}\, y_j = 0 \quad \forall i = 1, \cdots, N.$$

This implies the orthogonality relation

$$\sum_{j=1}^{M} \gamma_{h,ij} \langle T(x_i) - T_h(x_i), T_h(x_i) - y_j \rangle = 0,$$

whence

$$\sum_{i=1}^{N} \sum_{j=1}^{M} \gamma_{h,ij} \left|T(x_i) - y_j\right|^2 = \sum_{i=1}^{N} \sum_{j=1}^{M} \gamma_{h,ij} \left(\left|T(x_i) - T_h(x_i)\right|^2 + \left|T_h(x_i) - y_j\right|^2 \right).$$

Finally, we combine this equality and (5.3) to arrive at

$$\begin{aligned} C\lambda h &\geq \sum_{i=1}^{N} \sum_{j=1}^{M} \gamma_{h,ij} \left|T(x_i) - y_j\right|^2 \\ &\geq \sum_{i=1}^{N} \sum_{j=1}^{M} \gamma_{h,ij} \left|T(x_i) - T_h(x_i)\right|^2 \\ &= \sum_{i=1}^{N} f_i \left|T(x_i) - T_h(x_i)\right|^2. \end{aligned}$$

This shows (5.5) and thus finishes the proof of this theorem. □

Theorem 5.1 (convergence rates for fully discrete schemes) is the first result known to us that quantitatively measures the errors between γ and γ_h and between T and T_h. The convergence rate $O(h^{1/2})$ coincides with that for semidiscrete schemes developed in Section 4 and first proved by Berman (2018) for a different error notion. Moreover, Theorem 5.1 reveals that it is enough to solve the linear programming problem (5.1) approximately provided that the cost for γ_h is within $O(h)$ from the optimal cost. Therefore, approximate techniques from Cuturi (2013); Benamou *et al.* (2015); Oberman & Ruan (2015) are relevant in practice.

It is also worth pointing out that, according to Theorem 5.1, although γ_h is not generally sparse it must be close to the sparse plan $\widetilde{\gamma}_h := (S_h)_{\#\mu_h}$, the pushforward of μ_h by the map $S_h := (id, T_h) : \{x_i\}_{i=1}^N \subset X \to X \times Y$ and given by

$$\widetilde{\gamma}_h(A) = \mu_h\big(S_h^{-1}(A)\big) = \sum_{(x_i, T_h(x_i)) \in A} f_i$$

for all measurable sets $A \subset X \times Y$. We hope this observation might provide insight on acceleration processes for solving problem (5.1) that take advantage of sparsity of γ_h, as shown for instance in Oberman & Ruan (2015).

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