

# The Loewner framework for nonlinear identification and reduction of Hammerstein cascaded dynamical systems

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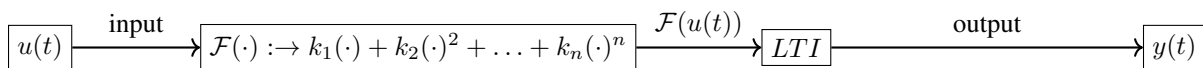
We present an algorithm for data-driven identification and reduction of nonlinear cascaded systems with Hammerstein structure. The proposed algorithm relies on the Loewner framework (LF) which constitutes a non-intrusive algorithm for identification and reduction of dynamical systems based on interpolation. We address the following problem: the actuator (control input) enters a static nonlinear block. Then, this processed signal is used as an input for a linear time-invariant system (LTI). Additionally, it is considered that the orders of the linear transfer function and of the static nonlinearity are not a priori known.

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## 1 Introduction

In some engineering applications that deal with the study of dynamical control systems, the control input enters the differential equations in a nonlinear fashion [5]. It is of interest to identify the hidden nonlinearity while at the same time reduction is needed for robust simulations and control design [1]. The LF [2–4] constitutes a non-intrusive method that uses only input-output data. The matrix pencil composed of two Loewner matrices reveals the minimality (in terms of McMillan degree) of the LTI system. By means of a singular value decomposition (SVD), one can find left and right projection matrices that are used to construct a low order model.

The Hammerstein system is characterized by two blocks connected in series, where the static nonlinear (memoryless) block is followed by a linear time-invariant system (LTI) as in Fig. 1. The scalar control input- $u(t)$  is used as an argument to the static nonlinearity- $\mathcal{F}$  and then the signal  $\mathcal{F}(u(t))$  passes through a linear time-invariant (LTI) system. The static polynomial map approximates other non-polynomial maps (Taylor series expansion) s.a.  $\tanh(\cdot)$ ,  $\exp(\cdot)$ , etc. The aim is to identify the cascaded system by estimating the coefficients of the polynomial map  $k_i$ ,  $i = 1, 2, \dots, n$  and the hidden LTI system by using only input-output data  $(u(t), y(t))$ ,  $t \geq 0$ .



**Fig. 1:** The input-output scheme of a cascaded system with a static nonlinear (polynomial) map of  $n^{\text{th}}$  order followed by an LTI. The connection describes a Hammerstein nonlinear model.

The steady state output solution can be computed explicitly with the convolution integral<sup>1</sup>, the impulse response  $h(t)$ ,  $t \geq 0$  and the linear transfer function  $H(j\omega)$ ,  $j\omega \in \mathbb{C}$  of the LTI as:

$$\begin{aligned} y(t) &= ((k_1 u(t) + k_2 u^2(t) + \dots + k_n u^n(t)) \star h)(t) = k_1 (u \star h)(t) + k_2 (u^2 \star h)(t) + \dots + k_n (u^n \star h)(t) \\ &= k_1 \int_{-\infty}^{\infty} h(\tau) u(t - \tau) d\tau + \dots + k_n \int_{-\infty}^{\infty} h(\tau) u^n(t - \tau) d\tau = \sum_{i=1}^n k_i \int_{-\infty}^{\infty} h(\tau) u^i(t - \tau) d\tau. \end{aligned} \quad (1)$$

Let the singleton real input be defined as  $u(t) = A \cos(\omega t) = \alpha e^{j\omega t} + \alpha e^{-j\omega t}$  with the amplitude  $\alpha = A/2$ , the imaginary unit  $j$ , the driving frequency  $\omega > 0$  and time  $t \geq 0$ . By substituting the above input in Eq. (1) and by making use of the binomial theorem, we conclude that:

$$y(t) = \sum_{i=1}^n k_i \int_{-\infty}^{\infty} h(\tau) \left( \alpha e^{j\omega(t-\tau)} + \alpha e^{-j\omega(t-\tau)} \right)^i d\tau = \sum_{i=1}^n \sum_{m=0}^i k_i \alpha^i \frac{i!}{(i-m)!m!} H(j\omega(2m-i)) e^{j\omega(2m-i)t} \quad (2)$$

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<sup>1</sup>  $(f \star g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau$



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At frequency  $\omega$  the  $\ell^{\text{th}}$  harmonic is computed by applying the single-sided Fourier transform in Eq. (2) as:

$$Y_{\omega,\ell}(j\ell\omega) = H(j\ell\omega)\delta(j\ell\omega) \sum_{\ell \leq i \neq 0}^n k_i \phi_{i,\ell}, \ell = 0, \dots, n, \quad (3)$$

$$\phi_{i,\ell} = \begin{cases} 2\alpha^i \cdot {}^i C_{(i+\ell)/2}, & (i \geq \ell) \text{ and } (i+\ell)(\text{even}) \\ 0, & (\text{else}) \end{cases}, \text{ where } {}^n C_m = \frac{n!}{(n-m)!m!}$$

## 2 The Loewner-Hammerstein identification method

As we have computed the total output of the Hammerstein cascaded system, we proceed with the method of determining the unknowns from input-output data. The symmetry in Eq. (3) allows the cancellation of the unknown contribution of the transfer function. Thus, we first determine the unknown coefficients  $k_i$ , and afterwards, we fit the LTI system by means of the LF. For this purpose, it is important to define the following invariant frequency quantities  $\lambda_{p,q}$ .

**Definition 2.1** (Frequency invariant quantities)

The  $Y_{p,q}$  denotes the  $q^{\text{th}}$  harmonic at  $p$  frequency.

$$\lambda_{p,q} = \frac{Y_{p,q}}{Y_{q,p}} = \frac{\sum_{i=p}^n k_i \phi_{i,p}}{\sum_{i=q}^n k_i \phi_{i,q}}, p \neq q. \quad (4)$$

The entries  $\lambda_{p,q}$  are independent of  $\omega$ .

| input \ harmonic                          | 1 <sup>st</sup> | 2 <sup>nd</sup> | 3 <sup>rd</sup> | 4 <sup>th</sup> | ...      | $n^{\text{th}}$ |
|-------------------------------------------|-----------------|-----------------|-----------------|-----------------|----------|-----------------|
| $1\omega \rightarrow \boxed{\mathcal{N}}$ | $Y_{1\omega,1}$ | $Y_{1\omega,2}$ | $Y_{1\omega,3}$ | $Y_{1\omega,4}$ | $\cdots$ | $Y_{1\omega,n}$ |
| $2\omega \rightarrow \boxed{\mathcal{N}}$ | $Y_{2\omega,1}$ | $Y_{2\omega,2}$ | $Y_{2\omega,3}$ | $Y_{2\omega,4}$ | $\cdots$ | $Y_{2\omega,n}$ |
| $\vdots$                                  | $\vdots$        | $\vdots$        | $\vdots$        | $\vdots$        | $\ddots$ | $\vdots$        |
| $n\omega \rightarrow \boxed{\mathcal{N}}$ | $Y_{n\omega,1}$ | $Y_{n\omega,2}$ | $Y_{n\omega,3}$ | $Y_{n\omega,4}$ | $\cdots$ | $Y_{n\omega,n}$ |

The above harmonic map allows the construction of the following linear system. Due to the mixing linearities (i.e.  $k_1 u(t)$  and  $(u \star h)(t)$ ), we can fix  $k_1$  to an arbitrary value. For  $p = 1$  and  $q = 2, \dots, n$  results:

$$\begin{bmatrix} \phi_{21} - \lambda_{12}\phi_{22} & \phi_{31} - \lambda_{12}\phi_{32} & \cdots & \phi_{n1} - \lambda_{12}\phi_{n2} \\ \phi_{21} & \phi_{31} - \lambda_{13}\phi_{32} & \cdots & \phi_{n1} - \lambda_{13}\phi_{n3} \\ \phi_{21} & \phi_{31} & \cdots & \phi_{n1} - \lambda_{14}\phi_{n4} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{21} & \phi_{31} & \cdots & \phi_{n1} - \lambda_{1n}\phi_{nn} \end{bmatrix} \begin{bmatrix} k_2 \\ k_3 \\ k_4 \\ \vdots \\ k_n \end{bmatrix} = -k_1 \phi_{11} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \forall k_1 \in \mathbb{R} \setminus \{0\}. \quad (5)$$

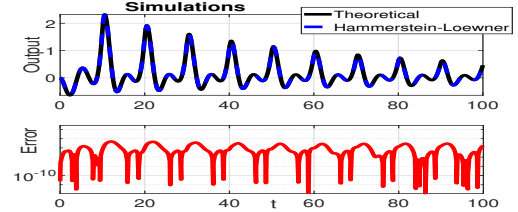
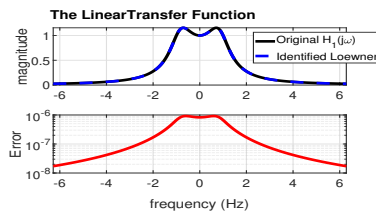
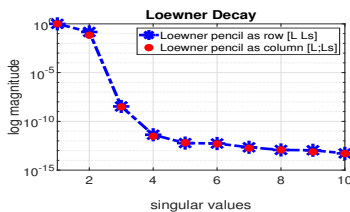
Finally, as we have identify the scaled ( $k_1$  arbitrary) coefficient vector  $\mathbf{k} = (k_1, k_2, \dots, k_n)$ , we can transform the above harmonic map into a measurement map for the linear transfer function as  $H(j\ell\omega) = Y_{\omega,\ell} / \sum_{\ell \leq i \neq 0}^n k_i \phi_{i,\ell}$ . The identification and reduction of the LTI system is done by applying the LF [2–4].

### Algorithm 1: Hammerstein identification with the Loewner framework

**Input:** Apply signals  $u(t) = \alpha \cos(\omega_i t)$  with driving frequencies  $\omega_i$ ,  $i = 1, \dots, n$  where  $n$  is the maximum nonzero harmonic index.

**Output:** An identified Hammerstein system.

- 1 Apply FT and measure  $U(j\omega_i)$ ,  $Y_{1\text{st}}(j\omega_i)$ ,  $Y_{2\text{nd}}(2j\omega_i)$ ,  $\dots$ ,  $Y_{n\text{th}}(nj\omega_i)$  from the power spectrum.
- 2 Fix  $k_1$  to an arbitrary value and determine the scaled coefficient vector  $\mathbf{k} = (k_1, k_2, \dots, k_n)$  by solving the system in Eq. (5).
- 3 Estimate the measurements of the linear transfer function from  $H(j\ell\omega) = Y_{\omega,\ell} / \sum_{\ell \leq i \neq 0}^n k_i \phi_{i,\ell}$ .
- 4 Apply the linear Loewner framework for identification and reduction of the LTI.



**Fig. 2:** The singular value decay of the Loewner matrices.  $\sigma_3/\sigma_1 \sim 1e-10$ .

**Fig. 3:** The identified linear transfer function with  $\|H - H_r\|_\infty \sim 1e-7$ .

**Fig. 4:** The simulated identified Hammerstein system in comparison with the original one.  $\|y - y_r\|_\infty \sim 1e-7$ .

## 3 Numerical example

To illustrate the proposed method, we choose the following static nonlinearity as  $\mathcal{F}(\cdot) = e^{(\cdot)} - 1$  along with the transfer function  $H(j\omega) = 1/((j\omega)^2 + j\omega + 1)$ . By exciting the system with  $u(t) = 2 \cos(\omega_i t)$ ,  $\omega_i = 2\pi[1, 2, \dots, 10]$  and with collecting the steady state snapshots, we perform Fourier transform for each signal and we solve the linear system in Eq. (5) for  $n = 10$  ( $Y_{10\text{th}} \sim 1e-10$ ). The solution is  $\hat{\mathbf{k}} = (1, 0.5, 0.167, 0.0417, 0.0083, 0.0014, 1.9676e-4, 2.4001e-5, 3.1005e-6, 3.7401e-7)$  which constitutes a very good approximation of the Taylor expansion coefficients of the  $\mathcal{F}$ . Next, we estimate the linear transfer function with the LF [2–4]. The singular value decay in Fig. 2 allows the assignment of the order  $r = 2$  ( $dt = 1e-3$ ). In Fig. 3 for the biased  $k_1 = 1$  solution, we show that the identification of the transfer function is accurate. The time

domain simulation in Fig. 4 is independent of the choice of  $k_1$ . The large input as  $u(t) = 2\text{sawtooth}(0.1 \cdot 2\pi t)e^{-0.01t} \cos(0.1 \cdot 2\pi t)$  certifies that the method is able to perform well under large inputs for nonlinear Hammerstein systems.

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