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Model reduction of linear hybrid systems

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Abstract— **The paper proposes a model reduction algorithm for linear hybrid systems, i.e., hybrid systems with externally induced discrete events, with linear continuous subsystems, and linear reset maps. The model reduction algorithm is based on balanced truncation. Moreover, the paper also proves an analytical error bound for the difference between the input-output behaviors of the original and the reduced-order model. This error bound is formulated in terms of singular values of the Gramians used for model reduction.**

I. Introduction

In this paper, we propose a model reduction method for linear hybrid systems with external switching. A linear hybrid system is a hybrid system with continuous states that are governed by linear differential equations, the reset maps are linear, and the discrete-events are external inputs. Linear hybrid systems can be viewed as a generalization of linear switched systems [1], [2]. In contrast to linear switched systems, state jumps are allowed, and additionally, the change of discrete states is supposed to follow the transition structure of a Moore automaton. Linear hybrid systems occur in several applications; a well known class of piecewise-affine systems is directly related to linear hybrid systems, as the former can be viewed as a feedback interconnection of the latter with a discrete-event generator. The model reduction method we propose is based on balanced truncation, performed for each linear subsystem. The corresponding Gramians have to satisfy certain linear matrix inequalities (LMIs). In addition to the novel algorithm, we propose an analytic error bound for the difference between the input-output behaviors of the original and of the reduced-order models. This error bound is a direct counterpart of the well-known error bound for balanced truncation of linear systems [3], and it involves the singular values of the Gramians.

To the best of our knowledge, the contribution of the paper is new. Indeed, the existing methods for model reduction of hybrid systems can be grouped into the following categories.

LMI-based methods These methods compute the matrices of the reduced-order model by solving a set of LMIs. The disadvantage is that the proposed conditions are only sufficient, and the trade-off between the dimension of the reduced model and the error bound is not clear. Moreover, the computational complexity of solving those LMIs might be too high. Without claiming completeness, we mention the following papers [4], [5], [6], [7]. First of all, the cited papers do not deal with linear reset maps. Moreover, in contrast to the cited papers, the current paper proposes a method, whose applicability depends on the existence of solution for a few simple LMIs which are necessary to find the observability/controllability Gramians. Once the existence of these Gramians is assured, the model reduction method can be applied. Moreover, there is an analytic error bound and the trade-off between the approximation error and the dimension of the reduced system is formalized in terms of the singular values of those Gramians.

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Methods based on local Gramians

The algorithms that belong to this class are based on finding observability/controllability Gramians for each linear subsystem. They are solutions of LMIs derived by relaxing the classical Lyapunov-like equations for observability/controllability Gramians. The disadvantage of these methods is that often there are no error bounds or the reduced-order model need not be well-posed. Examples of such papers include [8], [9], [10], [11], [12], [13]. Note that, to the best of our knowledge, the only algorithm which always yields a well-posed linear switched system of the same type as the original one and for which there exists an analytic error bound is the one proposed in [13]. Nevertheless, this algorithm provides an error bound only for sufficiently slow switching signals, i.e., switching sequences with a suitable minimal dwell time. The method proposed in this paper is an extension of [13]. The main differences between the current paper and those in [13] are stated below:

- In contrast to [13], the error bound of this paper no longer uses the assumption of minimum dwell time. However, this comes at price, as the LMIs involved are more conservative.
- The discrete states are no longer assumed to be inputs, but they are states of the system and they are assumed to evolve according to a Moore-automaton. However, the Moore-automaton is driven by discrete events which are external inputs. That is, the system class considered in this paper is more general than that in [13].

More recently, a balancing truncation method for linear switched systems that are characterized by constrained switching scenarios was proposed in [14]. The technique is based on defining generalized Gramians for each discrete mode, specifically tailored to particular switching scenarios.

Methods based on common Gramians These methods rely on finding the same observability/controllability Gramian for each linear subsystem. In most contributions, the Gramians are derived as solutions of a suitable LMI. Such

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algorithms were described in [15], [16] and an analytic error bound was derived in [17]. The results of this paper can also be viewed as a direct extension of [17]. In particular, when applied to a linear switched system of the type studied in [17], the results of the present paper boil down to those of [17]. With respect to [17], the main novelty of the present paper is that it considers a system class which is much larger than the one of [17]. Nevertheless, some methods that do not rely on solving LMIs are also available. For example, in [18] a balancing procedure based on recasting the original linear switched system as an envelope linear time-invariant system with no switching was proposed. Additionally, a balancing procedure based on reformulating the original system as a bilinear system with no switching was presented in [19].

Moment matching The idea behind these algorithms is to find a reduced-order switched system such that certain coefficients of the series expansions of the input-output maps of the original and the reduced-order system coincide. The series expansion can be the Taylor series with respect to switching times, in which case the so-called Markov parameters are matched. Alternatively, the series expansion can be a Laurent-series expansion of a multivariate Laplace transform of the input-output map around a certain frequency. The former approach was pursued in [20], [21], [22] , the latter in [23]. While those methods do not allow for analytical error bounds, under suitable assumption it can be guaranteed that the reduced model will have the same input-output behavior for certain switching signals [20], [21], [22]. A somewhat different approach is that of [24], which considers switched systems with autonomous switching and it proposed a model reduction procedure which guarantees that the reduced model has the same steady-state output response to certain inputs as the original model.

The results of the present paper are based on balanced truncation. As a result, in contrast to the cited papers, we are able to propose an analytic error bound. Moreover, the class of systems considered in this paper is much larger than that of the cited papers. In particular, we allow reset maps and the evolution of the discrete states is governed by a Moore-automaton.

The paper is structured as follows. In Section II-B we introduce the notation and present the formal definition of linear hybrid systems and of some related concepts. In Section III we present a balanced truncation algorithm for model reduction and an analytical error bound for this algorithm. In Section IV we present a numerical example to illustrate the proposed algorithm.

II. Preliminaries

A. Notation

Let $\mathbb{N}$ denote the set of natural numbers including 0, and $\mathbb{R}_+ = [0,+\infty)$ denote the positive *real time-axis*. We denote by $PC(A,B)$ the set of all *piecewise-continuous maps* $A \to B$, and by $L_2(A,B)$ the set of all *Lebesgue measurable maps* $A \to B$. The L_2-norm and Euclidean 2-norm are denoted by $\|\cdot\|_{L_2}$ and $\|\cdot\|_2$ respectively.

B. Linear hybrid systems: definition and basic concepts

Definition 1 (LHS): A *linear hybrid system* H (abbreviated as LHS) is a tuple

$$H = (Q,\Gamma,O,\delta,\lambda,\{n_q,A_q,B_q,C_q\}_{q\in Q}, \{M_{q_1,\gamma,q_2}\}_{q_2\in Q,\gamma\in\Gamma,q_1=\delta(q_2,\gamma)},h_0), \quad (1)$$

where

1) Q is a finite set, called *the set of discrete states*,
2) Γ is a finite set, called *the set of discrete events*,
3) O is a finite set, called *the set of discrete outputs*,
4) $\delta : Q\times\Gamma \to Q$ is a function called the *discrete state-transition map*,
5) $\lambda : Q \to O$ is a function called the *discrete readout map*,
6) $\Sigma_q = (A_q,B_q,C_q)$, $q \in Q$ is *the linear system in the discrete state* q and $A_q \in \mathbb{R}^{n_q\times n_q}, B_q \in \mathbb{R}^{n_q\times m}, C_q \in \mathbb{R}^{p\times n_q}$ are the matrices of this linear system,
7) $M_{q_1,\gamma,q_2} \in \mathbb{R}^{n_{q_1}\times n_{q_2}}$ are matrices for all $q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2,\gamma)$, which are called *reset maps*,
8) $h_0 = (q_0,x_0)$ is the *initial state*, where $q_0 \in Q$ and $x_0 \in \mathbb{R}^{n_{q_0}}$.

The space $\mathbb{R}^{n_q}$, $q \in Q$, $0 < n_q \in \mathbb{N}$, is called the *continuous state space associated with the discrete state q*, $\mathbb{R}^m$ is called the *continuous input space*, $\mathbb{R}^p$ is called the *continuous output space*. The *state space* $\mathscr{H}_H$ *of H* is the set $\mathscr{H}_H = \bigcup_{q\in Q}\{q\}\times\mathbb{R}^{n_q}$.

Notation 1: An element $x \in \mathscr{H}_H$ comprises of a pair $x = (q,x_q)$ with $q \in Q$ and $x_q \in \mathbb{R}^{n_q}$. In some places throughout this article, we will suppress the notation and write instead of x_q, simply x (whenever it is clear from the content which discrete mode is associated with x).

Note also that the linear control systems associated with different discrete states may have different state-spaces, but they have the same input and output space. The intuition behind the definition of a linear hybrid system is provided in what follows. We associate a linear system

$$\Sigma_q : \begin{cases} \dot{x} = A_q x + B_q u \\ y = C_q x \end{cases}, \quad (2)$$

with each discrete state $q \in Q$. As long as we are in the discrete state q, the state x and the continuous output y develops according to (2). The discrete state can change only if a discrete event $\gamma \in \Gamma$ takes place. If a discrete event γ occurs at time t, then the new discrete state q^+ is determined by applying the discrete state-transition map δ to q, i.e., $q^+ = \delta(q,\gamma)$. The new continuous-state $x^+(t) \in \mathbb{R}^{n_{q^+}}$ is computed from the current continuous state $x(t^-) = \lim_{s\uparrow t} x(s)$ by applying the *reset map* $M_{q^+,\gamma,q}$ to $x(t^-)$, i.e., $x^+(t) = M_{q_+,\gamma,q}x(t^-)$. After the transition, the continuous state x and the continuous output y evolve according to the linear system associated with the new discrete state q^+, started from the initial state $x^+(t)$. Finally, when in a discrete state $q \in Q$, the system produces a discrete output $o = \lambda(q)$.

Note that *the discrete events are external inputs. All the continuous subsystems are defined with the same inputs and outputs, but on possibly different state-spaces.* Below we will formalize the intuition described above, by defining input-to-state and input-output maps for LHS . To this end, we need the following.

Definition 2 (Timed sequences): A *timed sequence of discrete events* is an infinite sequence over the set $(\Gamma \times \mathbb{R}_+)$, i.e., a timed sequence w is a sequence of the form

$$w = (\gamma_1, t_1)(\gamma_2, t_2) \cdots (\gamma_k, t_k) \cdots, \tag{3}$$

where $\gamma_i \in \Gamma$, $k > 0$ are discrete events, and $t_i \in \mathbb{R}_+$ are time instances, and $\lim_{k\to\infty} \sum_{i=1}^{k} t_i = \infty$. We denote the set of timed sequences of discrete events by $\Gamma^{\infty}_{\text{timed}}$.
The interpretation of a timed sequence $w \in \Gamma^{\infty}_{\text{timed}}$ as above will be stated in what follows. If w is of the form (3), then w represents the scenario, when the event γ_i took place *after* the event γ_{i-1} and t_i *is the time which has passed between the arrival of* γ_{i-1} *and the arrival of* γ_i, i.e., t_i is the difference of the arrival times of γ_i and γ_{i-1}. Hence, $t_i \geq 0$ but we allow $t_i = 0$, i.e., γ_i can arrive instantly after γ_{i-1}. If $i = 1$, then t_1 is simply the time when the first event γ_1 arrived.

Notation 2 (Inputs $\mathbf{U}$*):* Denote by $\mathbf{U} = L_2(\mathbb{R}_+, \mathbb{R}^m) \times \Gamma^{\infty}_{\text{timed}}$ the set of inputs of a LHS .

If $(u, w) \in \mathbf{U}$, then u represents the continuous-valued input to be fed to the system, where w is the timed-event sequence. Below we define the notion of input-to-state and input-output maps for LHSs . These functions map elements from $\mathbf{U}$ to states and outputs, respectively.

In the rest of this section, H *denotes a LHS of the form introduced in* (1).

Definition 3 (Input-to-state map): The *input-to-state map of* H *induced by a state* $h = (q_I, x_I) \in \mathscr{H}_H$ *of* H, where $q_I \in Q$ and $x_I \in \mathbb{R}^{n_{q_I}}$, is the function $\xi_{H,h} : \mathbf{U} \to PC(\mathbb{R}_+, \mathscr{H}_H) \times PC(\mathbb{R}_+, Q)$such that the following holds. For any $(u, w) \in \mathbf{U}$, where w is of the form (3), define $T_0 = 0, T_i = \sum_{j=1}^{i} t_j$, $i \in \mathbb{N}$. Then $\xi_{H,h}(u, w) = (x, q)$ such that

1) $q(t) = q_i$, $t \in [T_i, T_{i+1})$, where $q_0 = q_I$ and $q_{i+1} = \delta(q_i, \gamma_{i+1})$ for all $i \in \mathbb{N}$.
2) The restriction of x to $[0, T_1)$ is the unique solution (in the sense of Carathéodory) of the differential equation $\dot{z}(t) = A_{q_I} z(t) + B_{q_I} u(t)$, $z(0) = x_I$ on $[0, T_1)$, and the restriction of x to $[T_i, T_{i+1})$ for $i > 0$ is the unique solution (in the sense of Carathéodory) of the differential equation $\dot{z}(s) = A_{q_i} z(s) + B_{q_i} u(s)$, $z(T_i) = M_{q_{i+1}, \gamma_{i+1}, q_i} \lim_{t \uparrow T_i} x(t)$.

Definition 4 (Input-output map): The *input-output map of the system* H *induced by a state* $h \in \mathscr{H}_H$ *of* H is the function $\upsilon_{H,h} : \mathbf{U} \to PC(\mathbb{R}_+, O) \times PC(\mathbb{R}_+, \mathbb{R}^p)$defined as follows: for all $(u, w) \in \mathbf{U}$, $\upsilon_{H,h}(u, w) = (\mathbf{o}, y)$, such that if $(q, x) = \xi_{H,h}(u, w)$, then

$$\mathbf{o}(t) = \lambda(q(t)), \; y(t) = C_{q(t)} x(t).$$

The input-output map $\upsilon_{H,h}$ induced by the initial state h_0 is called the *input-output map* of H and it is denoted by υ_H.

III. Balanced truncation

Consider an LHS H of the form (1) with initial condition $h_0 = (q_0, x_0)$ such that $x_0 = 0$.

Definition 5: A collection $\{\mathscr{Q}_q\}_{q \in Q}$ of positive definite matrices is called a collection of generalized observability Gramians of H, if for all $q \in Q$,

$$\begin{aligned} &A_q^T \mathscr{Q}_q + \mathscr{Q}_q A_q + C_q^T C_q < 0, \\ &\forall \gamma \in \Gamma, \; q^+ = \delta(q, \gamma) : M^T_{q^+, \gamma, q} \mathscr{Q}_{q^+} M_{q^+, \gamma, q} - \mathscr{Q}_q \leqslant 0. \end{aligned} \tag{4}$$

Definition 6: A collection $\{\mathscr{P}_q\}_{q \in Q}$ of positive definite matrices is called a collection of generalized reachability Gramians of H, if for all $q \in Q$,

$$\begin{aligned} &A_q \mathscr{P}_q + \mathscr{P}_q A_q^T + B_q B_q^T < 0, \\ &\forall \gamma \in \Gamma, \; q^+ = \delta(q, \gamma) : M_{q^+, \gamma, q} \mathscr{P}_q M^T_{q^+, \gamma, q} - \mathscr{P}_{q^+} \leqslant 0. \end{aligned} \tag{5}$$

Remark 1: The LMIs in (4) can be rewritten as follows

$$\begin{aligned} &\forall x \in \mathbb{R}^{n_q} : 2(A_q x)^T \mathscr{Q}_q x \leq -\|C_q x\|_2^2, \\ &x^T M^T_{q^+, \gamma, q} \mathscr{Q}_{q^+} M_{q^+, \gamma, q} x \leq x^T \mathscr{Q}_q x. \end{aligned} \tag{6}$$

The LMIs in (5) can be rewritten as follows

$$\begin{aligned} &\forall x \in \mathbb{R}^{n_q}, u \in \mathbb{R}^m : \; 2(A_q x + B_q u)^T \mathscr{P}_q^{-1} x \leq \|u\|_2^2, \\ &x^T M^T_{q^+, \gamma, q} \mathscr{P}^{-1}_{q^+} M_{q^+, \gamma, q} x \leq x^T \mathscr{P}_q^{-1} x. \end{aligned} \tag{7}$$

Definition 7: We say that the LHS H is quadratically stable, if there exists a collection collection $\{P_q\}_{q \in Q}$ of positive definite matrices, such that

$$\begin{aligned} &A_q^T P_q + P_q A_q < 0, \\ &\forall \gamma \in \Gamma, \; q^+ = \delta(q, \gamma) : M^T_{q^+, \gamma, q} P_{q^+} M_{q^+, \gamma, q} - P_q \leq 0. \end{aligned} \tag{8}$$

For completness we recall the following lemmas (see [25] for proofs and further discussion).

Lemma 1 (Stability and Gramians): H is quadratically stable iff there exist generalized observability Gramians iff there exist generalized controllability Gramians.

Lemma 2: [Observability Gramian and output energy] If $\{\mathscr{Q}_q\}_{q \in Q}$ are observability Gramians, $h_0 = (q_0, x_0)$, $(q, x) = \xi_{H,h_0}(0, w)$, $(\mathbf{o}, y) = \upsilon_{H,h_0}(0, w)$, i.e., x, y are the continuous state and output trajectories of H if started from the initial state h_0 and fed with the timed sequence w and zero continuous input $u = 0$, then

$$\int_0^{\infty} \|y(s)\|_2^2 ds \leq x_0^T \mathscr{Q}_{q_0} x_0.$$

Lemma 3: [Controllability Gramian and input energy] If $\{\mathscr{P}_q\}_{q \in Q}$ are reachability Gramians, $h_0 = (q_0, 0)$, $(q, x) = \xi_{H,h_0}(u, w)$, i.e., x, q are the continuous and discrete state trajectories of H if started from the initial state h_0 and fed with the timed sequence w and continuous input u, then

$$x(t) \mathscr{P}^{-1}_{q(t)} x(t) \leq \int_0^t \|u(s)\|_2^2.$$

Next, formulate a balanced model reduction procedure.

Procedure 1:
1) Compute reachabilility and observability Gramians $\{\mathscr{P}_q > 0\}_{q \in Q}$ and $\{\mathscr{Q}_q > 0\}_{q \in Q}$ which satisfy (5), and, respectively (4).
2) Find square factor matrices $\mathbf{U}_q$ so that $\mathscr{P}_q = \mathbf{U}_q \mathbf{U}_q^T$. Additionally, compute the eigenvalue decomposition of the symmetric matrix $\mathbf{U}_q^T \mathscr{Q}_q \mathbf{U}_q$, as

$$\mathbf{U}_q^T \mathscr{Q}_q \mathbf{U}_q = \mathbf{V}_q \Lambda_q^2 \mathbf{V}_q^T,$$

where

$$\Lambda_q = \text{diag}(\sigma_{q,1}, \ldots, \sigma_{q,n_q}),$$

is a diagonal matrix with the real entries sorted in decreasing order, i.e., $\sigma_{q,1} \geq \sigma_{q,2} \geq \cdots \geq \sigma_{q,n_q}$.

3) Construct the transformation matrices $\mathbf{S}_q \in \mathbb{R}^{n_q \times n_q}$ as follows

$$\mathbf{S}_q = \Lambda_q^{1/2} \mathbf{V}_q^T \mathbf{U}_q^{-1}. \tag{9}$$

Define the matrices (with $q_1 = \delta(q_2, \gamma),\ q_2 \in Q$)

$$\begin{aligned} \bar{A}_q = \mathbf{S}_q A_q \mathbf{S}_q^{-1}, \quad \bar{B}_q = \mathbf{S}_q B_q, \quad \bar{C}_q = C_q \mathbf{S}_q^{-1}, \\ \bar{M}_{q_2,\gamma,q_1} = \mathbf{S}_{q_2} M_{q_2,\gamma,q_1} \mathbf{S}_{q_1}^{-1}. \end{aligned} \tag{10}$$

4) Choose the truncation orders $0 < r_q \leq n_q$ and consider the partitioning

$$\begin{aligned} &\bar{A}_q = \begin{bmatrix} \bar{A}_q^{11} & \bar{A}_q^{12} \\ \bar{A}_q^{21} & \bar{A}_q^{22} \end{bmatrix}, \bar{B}_q = \begin{bmatrix} \bar{B}_q^{1} \\ \bar{B}_q^{2} \end{bmatrix}, \bar{C}_q = \begin{bmatrix} \bar{C}_q^{1} & \bar{C}_q^{2} \end{bmatrix},\ r_q < n_q, \\ &\bar{M}_{q_1,\gamma,q_2} = \begin{bmatrix} \bar{M}_{q_1,\gamma,q_2}^{11} & \bar{M}_{q_1,\gamma,q_2}^{12} \\ \bar{M}_{q_1,\gamma,q_2}^{21} & \bar{M}_{q_1,\gamma,q_2}^{22} \end{bmatrix} \ \text{ if } r_{q_1} < n_{q_1}, r_{q_2} < n_{q_2}, \\ &\bar{M}_{q_1,\gamma,q_2} = \begin{bmatrix} \bar{M}_{q_1,\gamma,q_2}^{11} & \bar{M}_{q_1,\gamma,q_2}^{12} \end{bmatrix} \ \text{ if } r_{q_1} = n_{q_1}, r_{q_2} < n_{q_2}, \\ &\bar{M}_{q_1,\gamma,q_2} = \begin{bmatrix} \bar{M}_{q_1,\gamma,q_2}^{11} \\ \bar{M}_{q_1,\gamma,q_2}^{21} \end{bmatrix} \ \text{ if } r_{q_1} < n_{q_1}, r_{q_2} = n_{q_2}, \end{aligned} \tag{11}$$

where $\bar{A}_q^{11} \in \mathbb{R}^{r_q \times r_q}$, $\bar{M}_{q_1,\gamma,q_2}^{11} \in \mathbb{R}^{r_{q_1} \times r_{q_2}}$, $\bar{B}_q^1 \in \mathbb{R}^{r_q \times m}$, and $\bar{C}_q^1 \in \mathbb{R}^{p \times r_q}$.

5) Define the reduced model

$$\begin{aligned} \hat{H} = (Q, \Gamma, O, \delta, \lambda, \{r_q, \hat{A}_q, \hat{B}_q, \hat{C}_q\}_{q \in Q}, \\ \{\hat{M}_{q_1,\gamma,q_2}\}_{q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2,\gamma)}, (q_0, 0)), \end{aligned}$$

where

$$\begin{aligned} &\hat{A}_q = \bar{A}_q^{11}, \quad \hat{B}_q = \bar{B}_q^1, \quad \hat{C}_q = \bar{C}_q^1, \quad \text{if } r_q \leq n_q, \\ &\hat{M}_{q_1,\gamma,q_2} = \bar{M}_{q_1,\gamma,q_2}^{11}, \text{ if } r_{q_1} < n_{q_1} \text{ or } \ r_{q_2} < n_{q_2}, \\ &\hat{A}_q = \bar{A}_q, \quad \hat{B}_q = \bar{B}_q, \quad \hat{C}_q = \bar{C}_q, \quad \text{if } \ r_q = n_q, \\ &\hat{M}_{q_1,\gamma,q_2} = \bar{M}_{q_1,\gamma,q_2}, \text{ if } r_{q_1} = n_{q_1} \text{ and } r_{q_2} = n_{q_2}. \end{aligned} \tag{12}$$

The proofs of the next two lemmas can be found in [25].

Lemma 4 (Balanced realization): Consider the LHS

$$\begin{aligned} \bar{H} = (Q, \Gamma, O, \delta, \lambda, \{r_q, \bar{A}_q, \bar{B}_q, \bar{C}_q\}_{q \in Q}, \\ \{\bar{M}_{q_1,\gamma,q_2}\}_{q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2,\gamma)}, (q_0, 0)). \end{aligned}$$

Then $\{\Lambda_q\}_{q \in Q}$ are both generalized reachability and observability Gramians of $\bar{H}$.

In what follows, we will say that an LHS is *balanced*, if it has generalized reachability Gramians $\{\mathscr{P}_q\}_{q \in Q}$, generalized observability Gramians $\{\mathscr{Q}_q\}_{q \in Q}$, and for all $q \in Q$, the matrices $\mathscr{Q}_q$ and $\mathscr{P}_q$ are equal and are diagonal. Lemma 4 says that $\bar{H}$ is balanced. In fact, more is true.

Lemma 5 (Preservation of balancing and stability): The reduced-order model $\hat{H}$ is balanced, and its generalized observability and reachability Gramians are $\{\hat{\Lambda}_q\}_{q \in Q}$, $\hat{\Lambda}_q = \text{diag}(\sigma_{q,1}, \ldots, \sigma_{q,r_q})$. In particular, $\hat{H}$ is quadratically stable.

Theorem 1 (Error bound): For any $(u, w) \in \mathbf{U}$, consider the outputs $(\mathbf{o}, y) = \upsilon_H(u, w)$ and $(\hat{\mathbf{o}}, \hat{y}) = \upsilon_{\hat{H}}(u, w)$ generated by H and $\hat{H}$, respectively under input u and timed sequence w from the corresponding initial state. Then $\hat{\mathbf{o}} = \mathbf{o}$, and

$$\|y - \hat{y}\|_{L_2} \leq 2 \Big(\sum_{q \in Q} \sum_{i=1}^{n_q - r_q} \sigma_{q, r_q + i} \Big) \|u\|_{L_2}.$$

Remark 2 (Relationship with the linear case): Note that the proof of Theorem 1 does not boil down to applying the classical error bound to each linear subsystem. In fact, classical error bounds assume that the linear system in question is started with zero initial state. However, when a discrete state transition occurs, the initial state of the active linear system is inherited from the final state of the previously active linear system, and hence this initial state is not zero in general. The proof of Theorem 1 is designed to take care of this fact; it relies on showing that a certain piecewise-quadratic form is a storage function with a certain supply rate for the linear hybrid system which describes the difference between the original and the reduced-order model. The corresponding inequalities take into account the non-zero initial state of each linear subsystem, in the same way as it is done when defining Lyapunov functions for hybrid systems [2].

First we prove Theorem 1 for the case when $n_q - r_q \leq 1$ for all $q \in Q$. More precisely, for each $q \in Q$, consider the decomposition

$$\Lambda_q = \begin{bmatrix} \hat{\Lambda}_q & 0 \\ 0 & \beta_q \end{bmatrix}, \quad \beta_q \in \mathbb{R}. \tag{13}$$

Define $\beta = \min_{q \in Q} \beta_q$ and for each $q \in Q$, define

$$r_q = \begin{cases} n_q - 1 & \text{if } \beta_q = \beta, \\ n_q & \text{otherwise} \end{cases}.$$

Consider the reduced-order model $\hat{H}$ from Procedure 1 for this choice of r_q.

Theorem 2 (One step error bound): For any $(u, w) \in \mathbf{U}$, consider the outputs $(\mathbf{o}, y) = \upsilon_H(u, w)$ and $(\hat{\mathbf{o}}, \hat{y}) = \upsilon_{\hat{H}}(u, w)$ generated by H and $\hat{H}$ respectively under the input u and timed event sequence w from the corresponding initial state. Then $\hat{\mathbf{o}} = \mathbf{o}$, and

$$\|y - \hat{y}\|_{L_2} \leq 2\beta \|u\|_{L_2}.$$

Theorem 1 follows by repeated application of Theorem 2. The proof of Theorem 2 is done via a sequence of lemmas. In order to state these lemmas, we introduce the following notation. Consider the balanced LHS $\bar{H}$ from Lemma 4. Note that the LHSs $\bar{H}$ and H are isomorphic, and hence they have the same input-output map. Consider now the state trajectory $(q, \bar{x}) = \xi_{\bar{H}, h_0}(u, w)$ of $\bar{H}$ and the state trajectory $(\hat{q}, \hat{x}) = \xi_{\hat{H}, \hat{h}_0}(u, w)$, where $\hat{h}_0 = (q_0, 0)$ is the initial state of $\hat{H}$. It is easy to see that $q = \hat{q}$.

For any $t \in \mathbb{R}_+$ such that $r_{q(t)} = n_{q(t)} - 1$, consider the partitioning

$$\bar{x}(t) = \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix},$$

with $\bar{x}_1(t) \in \mathbb{R}^{r_{q_i}}$, $\bar{x}_2(t) \in \mathbb{R}$. Define the functions

$$\begin{aligned} x_o(t) &= \begin{cases} \begin{bmatrix} \bar{x}_1(t) - \hat{x}(t) \\ \bar{x}_2(t) \end{bmatrix}, & r_{q(t)} = n_{q(t)} - 1 \\ \bar{x}(t) - \hat{x}(t) & \text{otherwise} \end{cases}, \\ x_c(t) &= \begin{cases} \begin{bmatrix} \bar{x}_1(t) + \hat{x}(t) \\ \bar{x}_2(t) \end{bmatrix}, & r_{q(t)} = n_{q(t)} - 1 \\ \bar{x}(t) + \hat{x}(t) & \text{otherwise} \end{cases}. \end{aligned} \tag{14}$$

Note that the following holds:

$$y(t)-\hat{y}(t)=C_{q(t)}x_o(t).$$

Define the function

$$V(x_o(t),x_c(t))=x_o(t)^T\Lambda_{q(t)}x_o(t)+\beta^2x_c(t)^T\Lambda_{q(t)}^{-1}x_c(t). \quad (15)$$

It can then be shown that (see [25] for proofs)

Lemma 6: The temporal derivative of the function V, as defined in (15), satisfies

$$\frac{\partial V(x_o(t),x_c(t))}{\partial t}\leqslant 4\beta^2\|u(t)\|_2^2-\|y(t)-\hat{y}(t)\|_2^2, \quad (16)$$

for all $t\in[T_{i-1},T_i)$.

Lemma 7: For all $i\in\mathbb{N}$,

$$V(x(T_{i+1}),\hat{x}(T_{i+1}))\leq V(x(T_{i+1}^-),\hat{x}(T_{i+1}^-)), \quad (17)$$

where $x(T_{i+1}^-)=\lim_{t\uparrow T_{i+1}}x(t)$, and $\hat{x}(T_{i+1}^-)=\lim_{t\uparrow T_{i+1}}\hat{x}(t)$.

Proof: [Proof of Lemma 7] Note that $q_i=q(t)$ for all $t\in[T_i,T_{i+1})$ and that $\delta(q_i,\gamma_{i+1})=q_{i+1}$. Moreover, by virtue of $\{\Lambda_q\}_{q\in Q}$ being generalized observability and reachability Gramians for $\bar{H}$, and Remark 1, the following holds

$$\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^T\Lambda_{q_{i+1}}^{-1}\bar{M}_{q_{i+1},\gamma_{i+1},q_i}<\Lambda_{q_i}^{-1}, \quad (18)$$
$$\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^T\Lambda_{q_{i+1}}\bar{M}_{q_{i+1},\gamma_{i+1},q_i}<\Lambda_{q_i}. \quad (19)$$

In order to prove (17), the following cases have to be distinguished.

Assume that $r_{q_{i+1}}=n_{q_i+1}$, i.e., no truncation takes place in mode q_{i+1}. In this case, $x(T_{i+1})=\bar{M}_{q_{i+1},\gamma_{i+1},q_i}x(T_{i+1}^-)$, and

$$\hat{x}(T_{i+1})=\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{11}\hat{x}(T_{i+1}^-)=\bar{M}_{q_{i+1},\gamma_{i+1},q_i}\begin{bmatrix}\hat{x}(T_{i+1}^-)\\0\end{bmatrix}, \quad (20)$$

if $r_{q_i}=n_{q_i}-1$, and

$$\hat{x}(T_{i+1})=\bar{M}_{q_{i+1},\gamma_{i+1},q_i}\hat{x}(T_{i+1}^-), \quad (21)$$

if $r_{q_i}=n_{q_i}$. It then follows that

$$\begin{aligned}x_c(T_{i+1})&=\bar{M}_{q_{i+1},\gamma_{i+1},q_i}x_c(T_{i+1}^-),\\ x_o(T_{i+1})&=\bar{M}_{q_{i+1},\gamma_{i+1},q_i}x_o(T_{i+1}^-).\end{aligned} \quad (22)$$

From (22) it then follows that

$$\begin{aligned}V(x(T_{i+1}),\hat{x}(T_{i+1}))=&x_o^T(T_{i+1}^-)\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^T\Lambda_{q_{i+1}}\bar{M}_{q_{i+1},\gamma_{i+1},q_i}x_o(T_{i+1}^-)\\&+\beta^2x_c^T(T_{i+1}^-)\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^T\Lambda_{q_{i+1}}^{-1}\bar{M}_{q_{i+1},\gamma_{i+1},q_i}x_c(T_{i+1}^-).\end{aligned} \quad (23)$$

From (19)-(18) it follows that

$$x_o^T(T_{i+1}^-)\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^T\Lambda_{q_{i+1}}\bar{M}_{q_{i+1},\gamma_{i+1},q_i}x_o(T_{i+1}^-)\leq x_o^T(T_{i+1}^-)\Lambda_{q_i}x_o(T_{i+1}^-),$$
$$x_c^T(T_{i+1}^-)\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^T\Lambda_{q_{i+1}}^{-1}\bar{M}_{q_{i+1},\gamma_{i+1},q_i}x_c(T_{i+1}^-)\leq x_c^T(T_{i+1}^-)\Lambda_{q_i}^{-1}x_c(T_{i+1}^-).$$

Hence, from (23), it follows that (17) holds.

Consider now the case when $r_{q_{i+1}}=n_{q_{i+1}}-1$, i.e., in mode q_{i+1} truncation takes place. In this case, $x(T_{i+1})=\bar{M}_{q_{i+1},\gamma_{i+1},q_i}x(T_{i+1}^-)$, and

$$\begin{aligned}\hat{x}(T_{i+1})&=\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{11}\hat{x}(T_{i+1}^-)\\&=\bar{M}_{q_{i+1},\gamma_{i+1},q_i}\begin{bmatrix}\hat{x}(T_{i+1}^-)\\0\end{bmatrix}-\begin{bmatrix}0\\\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{21}\end{bmatrix}\hat{x}(T_{i+1}^-),\end{aligned} \quad (24)$$

if $r_{q_i}=n_{q_i}-1$, and

$$\begin{aligned}\hat{x}(T_{i+1})&=\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{11}\hat{x}(T_{i+1}^-)\\&=\bar{M}_{q_{i+1},\gamma_{i+1},q_i}\hat{x}(T_{i+1}^-)-\begin{bmatrix}0\\\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{21}\end{bmatrix}\hat{x}(T_{i+1}^-),\end{aligned} \quad (25)$$

if $r_q=n_q$. It then follows that

$$\begin{aligned}x_c(T_{i+1})&=\bar{M}_{q_{i+1},\gamma_{i+1},q_i}x_c(T_{i+1}^-)-\begin{bmatrix}0\\\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{21}\end{bmatrix}\hat{x}(T_{i+1}^-),\\ x_o(T_{i+1})&=\bar{M}_{q_{i+1},\gamma_{i+1},q_i}x_o(T_{i+1}^-)+\begin{bmatrix}0\\\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{21}\end{bmatrix}\hat{x}(T_{i+1}^-).\end{aligned} \quad (26)$$

From (26) it then follows that

$$\begin{aligned}&x_o^T(T_{i+1})\Lambda_{q_{i+1}}x_o(T_{i+1})=\\&x_o^T(T_{i+1}^-)\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^T\Lambda_{q_{i+1}}\bar{M}_{q_{i+1},\gamma_{i+1},q_i}x_o(T_{i+1}^-)+\\&2x_o^T(T_{i+1}^{-1})\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^T\Lambda_{q_{i+1}}\begin{bmatrix}0\\\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{21}\end{bmatrix}\hat{x}(T_{i+1}^-)\\&+\left(\begin{bmatrix}0\\\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{21}\end{bmatrix}\hat{x}(T_{i+1}^-)\right)^T\Lambda_{q_{i+1}}\begin{bmatrix}0\\\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{21}\end{bmatrix}\hat{x}(T_{i+1}^-).\end{aligned} \quad (27)$$

Since $\Lambda_{q_{i+1}}=\begin{bmatrix}\hat{\Lambda}_{q_{i+1}}&0\\0&\beta_{q_{i+1}}\end{bmatrix}$, it follows that

$$\begin{aligned}&\left(\begin{bmatrix}0\\\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{21}\end{bmatrix}\hat{x}(T_{i+1}^-)\right)^T\Lambda_{q_{i+1}}\begin{bmatrix}0\\\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{21}\end{bmatrix}\hat{x}(T_{i+1}^-)\\&\qquad=\beta_{q_{i+1}}\|\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{21}\hat{x}(T_{i+1}^-)\|_2^2.\end{aligned}$$

Moreover,

$$\begin{aligned}&2x_o^T(T_{i+1}^-)\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^T\Lambda_{q_{i+1}}\begin{bmatrix}0\\\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{21}\end{bmatrix}\hat{x}(T_{i+1}^-)=\\&\qquad\gamma_o-2\beta_{q_{i+1}}\|\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{21}\hat{x}(T_{i+1}^-)\|_2^2,\end{aligned}$$

where

$$\gamma_o=\begin{cases}2\beta_{q_{i+1}}\left(\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{21}x_1(T_{i+1}^-)+\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{22}x_2(T_{i+1}^-)\right)^T\\ \times\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{21}\hat{x}(T_{i+1}^-)\text{ if }r_{q_i}=n_{q_i}-1\\ 2\beta_{q_{i+1}}\left(\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{21}x(T_{i+1}^-)\right)^T\\ \times\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{21}\hat{x}(T_{i+1}^-)\text{ if }r_{q_i}=n_{q_i}\end{cases}.$$

Hence, it follows that

$$\begin{aligned}&x_o^T(T_{i+1})\Lambda_{q_{i+1}}x_o(T_{i+1})\\&=x_o^T(T_{i+1}^-)\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^T\Lambda_{q_{i+1}}\bar{M}_{q_{i+1},\gamma_{i+1},q_i}x_o(T_{i+1}^-)\\&\quad+\gamma_o-\beta_{q_{i+1}}\|\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{21}\hat{x}(T_{i+1}^-)\|_2^2.\end{aligned} \quad (28)$$

With a similar reasoning,

$$\begin{aligned}&x_c^T(T_{i+1})\Lambda_{q_{i+1}}^{-1}x_c(T_{i+1})\\&=x_c^T(T_{i+1}^-)\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^T\Lambda_{q_{i+1}}^{-1}\bar{M}_{q_{i+1},\gamma_{i+1},q_i}x_c(T_{i+1}^-)\\&\quad-2x_c^T(T_{i+1}^{-1})\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^T\Lambda_{q_{i+1}}^{-1}\begin{bmatrix}0\\\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{21}\end{bmatrix}\hat{x}(T_{i+1}^-)\\&\quad+(\begin{bmatrix}0\\\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{21}\end{bmatrix}\hat{x}(T_{i+1}^-))^T\Lambda_{q_{i+1}}^{-1}\begin{bmatrix}0\\\bar{M}_{q_{i+1},\gamma_{i+1},q_i}^{21}\end{bmatrix}\hat{x}(T_{i+1}^-).\end{aligned} \quad (29)$$

Since $\Lambda_{q_{i+1}}^{-1} = \begin{bmatrix} \hat{\Lambda}_{q_{i+1}}^{-1} & 0 \\ 0 & \beta_{q_{i+1}}^{-1} \end{bmatrix}$, we can again write that

$$\left(\begin{bmatrix} 0 \\ \bar{M}^{21}_{q_{i+1},\gamma_{i+1},q_i} \end{bmatrix} \hat{x}(T_{i+1}^-)\right)^T \Lambda_{q_{i+1}}^{-1} \begin{bmatrix} 0 \\ \bar{M}^{21}_{q_{i+1},\gamma_{i+1},q_i} \end{bmatrix} \hat{x}(T_{i+1}^-) = \beta_{q_{i+1}}^{-1} \|\bar{M}^{21}_{q_{i+1},\gamma_{i+1},q_i} \hat{x}(T_{i+1}^-)\|_2^2,$$

and

$$2x_c^T(T_{i+1}^-)\bar{M}^T_{q_{i+1},\gamma_{i+1},q_i}\Lambda_{q_{i+1}}^{-1} \begin{bmatrix} 0 \\ \bar{M}^{21}_{q_{i+1},\gamma_{i+1},q_i} \end{bmatrix} \hat{x}(T_{i+1}^-) = \gamma_c + 2\beta_{q_{i+1}}^{-1} \|\bar{M}^{21}_{q_{i+1},\gamma_{i+1},q_i} \hat{x}(T_{i+1}^-)\|_2^2,$$

where

$$\gamma_c = \begin{cases} 2\beta_{q_{i+1}}^{-1} \left(\bar{M}^{21}_{q_{i+1},\gamma_{i+1},q_i} x_1(T_{i+1}^-) + \bar{M}^{22}_{q_{i+1},\gamma_{i+1},q_i} x_2(T_{i+1}^-)\right)^T \\ \times \bar{M}^{21}_{q_{i+1},\gamma_{i+1},q_i} \hat{x}(T_{i+1}^-) \text{ if } r_{q_i} = n_{q_i} - 1 \\ 2\beta_{q_{i+1}}^{-1} \left(\bar{M}^{21}_{q_{i+1},\gamma_{i+1},q_i} x(T_{i+1}^-)\right)^T \\ \times \bar{M}^{21}_{q_{i+1},\gamma_{i+1},q_i} \hat{x}(T_{i+1}^-) \text{ if } r_{q_i} = n_{q_i} \end{cases},$$

and hence

$$\begin{aligned} & x_c^T(T_{i+1})\Lambda_{q_{i+1}}^{-1} x_c(T_{i+1}) \\ & = x_c^T(T_{i+1}^-)\bar{M}^T_{q_{i+1},\gamma_{i+1},q_i}\Lambda_{q_{i+1}}^{-1}\bar{M}_{q_{i+1},\gamma_{i+1},q_i} x_c(T_{i+1}^-) \\ & \quad - \gamma_c - \beta_{q_{i+1}}^{-1} \|\bar{M}^{21}_{q_{i+1},\gamma_{i+1},q_i} \hat{x}(T_{i+1}^-)\|_2^2. \end{aligned} \tag{30}$$

Note that $\beta = \beta_{q_{i+1}}$ since it was assumed that $r_{q_{i+1}} = n_{q_{i+1}} - 1$. Moreover, notice that $\beta_{q_{i+1}}^2 \gamma_c = \gamma_o$, hence by using (28), (30), (19) and (18) it follows

$$\begin{aligned} & V(x(T_{i+1}), \hat{x}(T_{i+1})) \\ & = x_o^T(T_{i+1}^-)\bar{M}^T_{q_{i+1},\gamma_{i+1},q_i}\Lambda_{q_{i+1}}\bar{M}_{q_{i+1},\gamma_{i+1},q_i} x_o(T_{i+1}^-) \\ & \quad + \beta^2 x_c^T(T_{i+1}^-)\bar{M}^T_{q_{i+1},\gamma_{i+1},q_i}\Lambda_{q_{i+1}}^{-1}\bar{M}_{q_{i+1},\gamma_{i+1},q_i} x_c(T_{i+1}^-) \\ & \quad - 2\beta \|\bar{M}^{21}_{q_{i+1},\gamma_{i+1},q_i} \hat{x}(T_{i+1}^-)\|_2^2 \\ & \leq x_o^T(T_{i+1}^-)\Lambda_{q_i} x_o(T_{i+1}^-) + \beta^2 x_c^T(T_{i+1}^-)\Lambda_{q_i}^{-1} x_c(T_{i+1}^-) \\ & = V(x(T_{i+1}^-), \hat{x}(T_{i+1}^-)). \end{aligned}$$

From (19) and (18), it then follows that (17) holds. ■

Proof: [Proof of Theorem 2] From Lemma 6 it follows that

$$\begin{aligned} V(x(s), \hat{x}(s)) - V(x(T_i), \hat{x}(T_i)) &= \int_{T_i}^{s} \frac{\partial V(x_o(t), x_c(t))}{\partial t} dt \\ & \leqslant 4\beta^2 \int_{T_i}^{s} \|u(t)\|_2^2 dt - \int_{T_i}^{s} \|y(t) - \hat{y}(t)\|_2^2 dt, \end{aligned}$$

and hence

$$\begin{aligned} & V(x(T_{i+1}^-), \hat{x}(T_{i+1}^-)) - V(x(T_i), \hat{x}(T_i)) \\ & \quad \leqslant 4\beta^2 \int_{T_i}^{T_{i+1}} \|u(t)\|_2^2 dt - \int_{T_i}^{T_{i+1}} \|y(t) - \hat{y}(t)\|_2^2 dt. \end{aligned}$$

By Lemma 7, $V(x(T_{i+1}), \hat{x}(T_{i+1})) \leq V(x(T_{i+1}^-), \hat{x}(T_{i+1}^-))$ and hence

$$\begin{aligned} & V(x(T_{i+1}), \hat{x}(T_{i+1})) - V(x(T_i), \hat{x}(T_i)) \\ & \quad \leqslant 4\beta^2 \int_{T_i}^{T_{i+1}} \|u(t)\|_2^2 dt - \int_{T_i}^{T_{i+1}} \|y(t) - \hat{y}(t)\|_2^2 dt. \end{aligned}$$

By summing up the inequalities above,

$$\begin{aligned} & V(x(T_k), \hat{x}(T_k)) - V(x(T_0), \hat{x}(T_0)) \\ & = \sum_{i=0}^{k-1} V(x(T_{i+1}), \hat{x}(T_{i+1})) - V(x(T_i), \hat{x}(T_i)) \\ & \leq \sum_{i=0}^{k-1} 4\beta^2 \int_{T_i}^{T_{i+1}} \|u(t)\|_2^2 dt \\ & \quad - \int_{T_i}^{T_{i+1}} \|y(t) - \hat{y}(t)\|_2^2 dt \\ & = 4\beta^2 \int_{T_0}^{T_k} \|u(t)\|_2^2 dt - \int_{T_0}^{T_k} \|y(t) - \hat{y}(t)\|_2^2 dt. \end{aligned}$$

Using that $T_0 = 0$, $x(0) = 0$, $\hat{x}(0) = 0$, and $V(0,0) = 0$ and $V(x(T_k), \hat{x}(T_k)) \geq 0$, it follows that

$$\begin{aligned} & 0 \leq 4\beta^2 \int_0^{T_k} \|u(t)\|_2^2 dt - \int_{T_0}^{T_k} \|y(t)dt - \hat{y}(t)\|_2^2 dt \Leftrightarrow \\ & \int_{T_0}^{T_k} \|y(t)dt - \hat{y}(t)\|_2^2 dt \leq 4\beta^2 \int_0^{T_k} \|u(t)\|_2^2 dt. \end{aligned}$$

Since $\lim_{k \to \infty} T_k = \infty$, the statement of the theorem follows ■

IV. Numerical Examples

In this section, we analyze the practical applicability of the proposed MOR procedure. We consider a low-order artificial example represented by a linear hybrid systems with four subsystems.

First, we characterize the discrete dynamics. The discrete state-transition map $\delta : Q \times \Gamma \to Q$ can be described in two ways, explicitly, i.e.,

$$\begin{cases} \textbf{Mode } \mathbf{q_1}: & \delta(q_1, 0) = q_4, \quad \delta(q_1, 1) = q_2, \\ \textbf{Mode } \mathbf{q_2}: & \delta(q_2, 0) = q_3, \quad \delta(q_2, 1) = q_4, \\ \textbf{Mode } \mathbf{q_3}: & \delta(q_3, 0) = q_4, \quad \delta(q_3, 1) = q_1, \\ \textbf{Mode } \mathbf{q_4}: & \delta(q_4, 0) = q_2, \quad \delta(q_4, 1) = q_3. \end{cases}$$

or using a directed graph, i.e., as in Fig. 1.

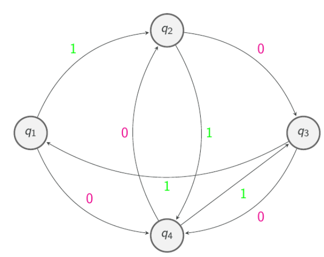

Fig. 1. Directed graph representation of the state transition map.

Next, we explicitly introduce the chosen discrete event signal $\gamma : \mathbb{R}_+ \to \Gamma$ and also the discrete state trajectory $q : \mathbb{R}_+ \to Q$

$$\gamma(t) = \begin{cases} 1, \; t \in [0, T_1), \\ 0, \; t \in [T_1, T_2), \\ 1, \; t \in [T_2, T_3), \\ \dots \\ 1, \; t \in [T_{10}, T_{11}). \end{cases} \quad q(t) = \begin{cases} q_2, \; t \in [0, T_1), \\ q_3, \; t \in [T_1, T_2), \\ q_1, \; t \in [T_2, T_3), \\ \dots \\ q_4, \; t \in [T_{10}, T_{11}), \end{cases} \tag{31}$$

with given $T_1, \dots, T_{11}$ (see Fig. 2). Additionally, in Fig. 2, we depict the two signals introduced in (31), i.e., $\gamma(t)$ and $q(t)$

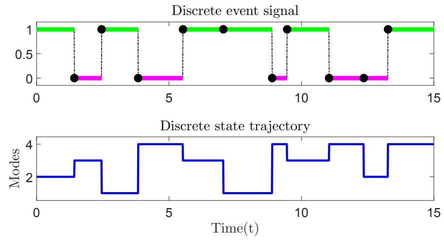

Fig. 2. The discrete event signal $\gamma(t)$ (up) and the discrete state trajectory $q(t)$ (down).

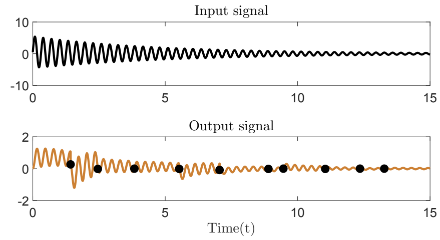

Fig. 3. The control input $u(t)$ (up) and the observed output $y(t)$ (down).

as a function of time (the time interval for this application was chosen to be $[0,15]$ seconds).

Finally, we proceed to the description of the continuous dynamics. Hence, the system matrices $(A_q, B_q, C_q), 1 \le q \le 4$ corresponding to the linear hybrid system under consideration are written as follows:

$$A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix},\ A_2 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix},\ A_4 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ B_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix},$$

$$B_4 = \begin{bmatrix} 2 \\ -1 \end{bmatrix},\ C_1 = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix},\ C_2 = \begin{bmatrix} 1 & \frac{3}{2} \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix},\ C_4 = \begin{bmatrix} 2 & 1 \end{bmatrix}.$$

Additionally, the reset maps are given by the following matrices

$$M_{4,0,1} = \frac{1}{\tau}\begin{bmatrix} 0 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix},\ M_{2,1,1} = \frac{1}{\tau}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$M_{3,0,2} = \frac{1}{\tau}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix},\ M_{4,1,2} = \frac{1}{\tau}\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix},$$

$$M_{4,0,3} = \frac{1}{\tau}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ M_{1,1,3} = \frac{1}{\tau}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},$$

$$M_{2,0,4} = \frac{1}{\tau}\begin{bmatrix} -1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix},\ M_{3,1,4} = \frac{1}{\tau}\begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Note that the parameter $\tau > 0$ was used in scaling the reset maps shown above. More precisely, in what follows, the value $\tau = 3$ was chosen for numerical computations.

We perform a time-domain simulation by using as continuous control input, the function $u(t) = 5\sin(20t)e^{-t/5} + 0.5e^{-t/2}$. In Fig. 3, we depict both the control input $u(t)$ and the observed output $y(t)$ (as introduced in (2)) The next step is to find appropriate Gramians to be used in the balanced truncation procedure. We start by first computing the observability Gramians.

We are looking for positive definite matrices that satisfy the conditions in (4). Hence, for each mode, we explicitly state the corresponding LMIs:

- Mode 1: $\begin{cases} A_1^T \mathscr{Q}_1 + \mathscr{Q}_1 A_1 + C_1^T C_1 < 0, \\ M_{4,0,1}^T \mathscr{Q}_4 M_{4,0,1} - \mathscr{Q}_1 \leqslant 0, \\ M_{2,1,1}^T \mathscr{Q}_2 M_{2,1,1} - \mathscr{Q}_1 \leqslant 0. \end{cases}$
- Mode 2: $\begin{cases} A_2^T \mathscr{Q}_2 + \mathscr{Q}_2 A_2 + C_2^T C_2 < 0, \\ M_{3,0,2}^T \mathscr{Q}_3 M_{3,0,2} - \mathscr{Q}_2 \leqslant 0, \\ M_{4,1,2}^T \mathscr{Q}_4 M_{4,1,2} - \mathscr{Q}_2 \leqslant 0. \end{cases}$
- Mode 3: $\begin{cases} A_3^T \mathscr{Q}_3 + \mathscr{Q}_3 A_3 + C_3^T C_3 < 0, \\ M_{4,0,3}^T \mathscr{Q}_4 M_{4,0,3} - \mathscr{Q}_3 \leqslant 0, \\ M_{1,1,3}^T \mathscr{Q}_1 M_{1,1,3} - \mathscr{Q}_3 \leqslant 0. \end{cases}$
- Mode 4: $\begin{cases} A_4^T \mathscr{Q}_4 + \mathscr{Q}_4 A_4 + C_4^T C_4 < 0, \\ M_{2,0,4}^T \mathscr{Q}_2 M_{2,0,4} - \mathscr{Q}_4 \leqslant 0, \\ M_{3,1,4}^T \mathscr{Q}_3 M_{3,1,4} - \mathscr{Q}_4 \leqslant 0. \end{cases}$

Note that, for the choice of parameter $\tau = 1$, the above systems of LMIs could not be solved (by means of the optimization software provided in [26] and [27]).

Nevertheless, when choosing $\tau = 3$, we were able to find a valid solution, i.e., a collection of positive definite matrices $\{\mathscr{Q}_1, \mathscr{Q}_2, \mathscr{Q}_3, \mathscr{Q}_4\}$. More precisely, we could find:

$$\mathscr{Q}_1 = \begin{bmatrix} 3.2662 & -0.1118 & 0.0733 \\ -0.1118 & 1.7564 & -0.0693 \\ 0.0733 & -0.0693 & 1.4755 \end{bmatrix},$$

$$\mathscr{Q}_3 = \begin{bmatrix} 1.7873 & -0.0041 & 0.0752 \\ -0.0041 & 3.4766 & 0.1468 \\ 0.0752 & 0.1468 & 2.4182 \end{bmatrix},$$

$$\mathscr{Q}_2 = \begin{bmatrix} 2.4546 & -0.0023 \\ -0.0023 & 4.0827 \end{bmatrix},\ \mathscr{Q}_4 = \begin{bmatrix} 3.9745 & 0.6789 \\ 0.6789 & 4.6925 \end{bmatrix}.$$

Next, we need to find positive definite matrices $\mathscr{P}_i$ that satisfy the conditions in (5). For each mode, we will state the corresponding LMIs:

- Mode 1: $\begin{cases} A_1 \mathscr{P}_1 + \mathscr{P}_1 A_1^T + B_1 B_1^T < 0, \\ M_{1,1,3} \mathscr{P}_3 M_{1,1,3}^T - \mathscr{P}_1 \leqslant 0, \end{cases}$
- Mode 2: $\begin{cases} A_2 \mathscr{P}_2 + \mathscr{P}_2 A_2^T + B_2 B_2^T < 0, \\ M_{2,0,4} \mathscr{P}_4 M_{2,0,4}^T - \mathscr{P}_2 \leqslant 0, \\ M_{2,1,1} \mathscr{P}_1 M_{2,1,1}^T - \mathscr{P}_2 \leqslant 0. \end{cases}$
- Mode 3: $\begin{cases} A_3 \mathscr{P}_3 + \mathscr{P}_3 A_3^T + B_3 B_3^T < 0, \\ M_{3,0,2} \mathscr{P}_2 M_{3,0,2}^T - \mathscr{P}_3 \leqslant 0, \\ M_{3,1,4} \mathscr{P}_4 M_{3,1,4}^T - \mathscr{P}_3 \leqslant 0. \end{cases}$
- Mode 4: $\begin{cases} A_4 \mathscr{P}_4 + \mathscr{P}_4 A_4^T + B_4 B_4^T < 0, \\ M_{4,0,1} \mathscr{P}_1 M_{4,0,1}^T - \mathscr{P}_4 \leqslant 0, \\ M_{4,0,3} \mathscr{P}_3 M_{4,0,3}^T - \mathscr{P}_4 \leqslant 0, \\ M_{4,1,2} \mathscr{P}_2 M_{4,1,2}^T - \mathscr{P}_4 \leqslant 0. \end{cases}$

Again, for $\tau = 3$, we could find the following matrices

$$\mathscr{P}_1 = \begin{bmatrix} 5.3173 & -0.1332 & 0.3859 \\ -0.1332 & 2.3055 & -0.0914 \\ 0.3859 & -0.0914 & 1.9288 \end{bmatrix},$$

$$\mathscr{P}_3 = \begin{bmatrix} 3.1234 & -0.0344 & 0.3250 \\ -0.0344 & 5.2759 & 0.5661 \\ 0.3250 & 0.5661 & 4.5523 \end{bmatrix},$$

$$\mathscr{P}_2 = \begin{bmatrix} 3.8471 & 0.1453 \\ 0.1453 & 5.3503 \end{bmatrix},\ \mathscr{P}_4 = \begin{bmatrix} 6.2062 & -0.3344 \\ -0.3344 & 7.4608 \end{bmatrix}.$$

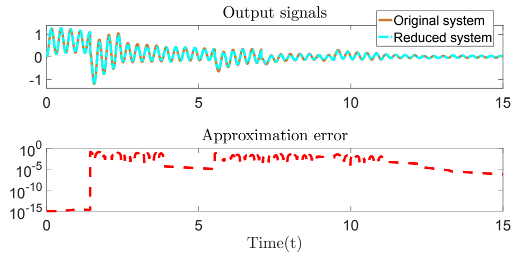

Fig. 4. The observed outputs for the original and reduced systems and the deviation between them (for the first choice of r_k's).

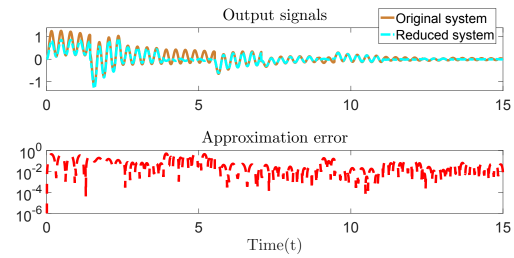

Fig. 5. The observed outputs for the original and reduced systems and the deviation between them (for the second choice of r_k's).

Next, we present the Gramians in balanced representation, i.e., the diagonal matrices Λ_q from step 2 of Procedure 1.

$$\Lambda_1 = \text{diag}(4.1894, 2.0184, 1.6542), \ \Lambda_2 = \text{diag}(4.6754, 3.0703),$$
$$\Lambda_3 = \text{diag}(4.3741, 3.2543, 2.3291), \ \Lambda_4 = \text{diag}(5.9718, 4.8538).$$

By choosing the reduction orders to be $r_1 = 2, r_2 = 2, r_3 = 2$ and $r_4 = 2$ (a dimension reduction is performed only for the first and third mode), we put together a reduced-order linear hybrid system. The time-domain simulation results are depicted in Fig. 4.

Next, we reduce the dimension of the systems corresponding to the second and forth modes as well. Hence, choose reduction orders $r_1 = 2, r_2 = 1, r_3 = 2$ and $r_4 = 1$. The time-domain simulations results are depicted in Fig. 5.

V. Conclusion

In this paper, we have proposed a balanced truncation procedure for reducing linear hybrid systems. For each linear subsystem, specific Gramian matrices were computed by solving particular LMIs. An analytical error bound in terms of singular values of the Gramians was also provided.

We demonstrated the effectiveness of the procedure through a numerical example. Extensions that could be further developed include adapting the proposed method to the case of hybrid systems with mild nonlinearities, e.g., systems with bilinear or stochastic behavior.

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