BALANCED CONVEX PARTITIONS OF LINES IN THE PLANE

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Abstract. We prove an extension of a ham sandwich theorem for families of lines in the plane by Dujmović and Langerman. Given two sets A, B of n lines each in the plane, we prove that it is possible to partition the plane into r closed convex regions such that the following holds. For each region C of the partition there is a subset of $c_r n^{1/r}$ lines of A whose pairwise intersections are in C, and the same holds for B. In this statement c_r only depends on r. We also prove that the dependence on n is optimal. For a single set A of n lines, we prove that there exists a partition of the plane into r parts using $r-1$ vertical lines such that each region contains the pairwise intersections of a set of $n^{1/r}$ lines of A. The value $n^{1/r}$ is optimal.

1. Introduction

A general measure partition problem deals with the way we can split points or measures in Euclidean spaces. Given a set of rules to split the ambient space, we are interested to know if we can divide a given set of points in a prescribed way. The quintessential result of this kind is the classic ham sandwich theorem.

Theorem. *We are given d sets of points in $\mathbb{R}^d$ in general position. If each set has an even cardinality, there exists a hyperplane that simultaneously splits each set exactly by half.*

The proof of a mass partition result usually boils down to understanding topological properties of the space of partitions [Živ17]. The methods developed to tackle measure partitions problems have broad applications in combinatorial topology.

In this manuscript we are interested in extensions of the ham sandwich theorem for convex partitions of the plane. A convex partition of $\mathbb{R}^2$ into r parts is a family of closed sets $C_1, \ldots, C_r \subset \mathbb{R}^2$ such that

- the sets cover $\mathbb{R}^2$, so $\cup_{i=1}^r C_i = \mathbb{R}^2$,
- the interiors of the sets are pairwise disjoint, and
- each C_i is a closed convex set.

The ham sandwich theorem has been generalized to convex partitions of the plane. The following theorem was proven independently by Ito, Uehara, and Yokoyama [IUY00], by Bespamyatnikh, Kirkpatrick, and Snoeyink [BKS00], and by Sakai [Sak02].

Theorem 1.1. *Let A_1, A_2 be two sets of points in $\mathbb{R}^2$, in general position. If the cardinality of each set is a multiple of r, there exists a closed convex partition of the plane into r sets $C_1, \ldots, C_r$ such that for each $i \in \{1,2\}, j \in \{1,2,\ldots,r\}$ we have*

$$|A_i \cap C_j| = \frac{1}{r}|A_i|.$$

In other words, each set is partitioned evenly.

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The continuous version of the theorem above has been generalized to convex partitions of $\mathbb{R}^d$ with d measures [Sob12, KHA14, BZ14]. The high-dimensional versions of Karasev, Hubard, and Aronov [KHA14], and of Blagojević and Ziegler [BZ14] hold in a much more general setting. These were motivated by a problem of Nandakumar and Rao, which has been recently solved [AAK18]. A discrete version in high dimensions was recently established [BRSZ19]. Theorem 1.1 can be bootstrapped to obtain partitions of measures where each part has positive size in an arbitrary number of measures [BPSZ19]. The planar version has applications to drawings of political district maps [Hum11, Sob17].

In this manuscript we are interested in splitting families of lines in $\mathbb{R}^2$ instead of families of points. Our main result is an extension of Theorem 1.1 to families of lines. We use the following definition for what it means to split a family of lines. Given a set L of lines in the plane such that no two lines of L are parallel, let $I(L)$ be the set of pairwise intersection points of L. We call $I(L)$ the **incidence set** of L.

Definition 1. *Given a closed set $K \subset \mathbb{R}^d$, and a set of lines L in the plane such that no two sets of lines of L are parallel, we say that K **encloses** L if*

$$I(L) \subset K.$$

If K is convex, the condition above is equivalent to $\operatorname{conv}(I(L)) \subset K$. A ham sandwich theorem with this definition was proved by Dujmović and Langerman.

Theorem 1.2 (Dujmović, Langerman 2013 [DL13])**.** *Given two finite sets A, B of lines each in the plane, if no two lines of $A \cup B$ are parallel, there exists a line ℓ such that each of the two closed half-spaces it defines encloses a subset of at least $\sqrt{|A|}$ lines of A and a subset of at least $\sqrt{|B|}$ lines of B.*

Given two sets of lines, we obtain convex partitions of the plane where each part encloses a large subset of lines. Partitions related to families of lines have been studied before in other settings. For instance, the celebrated polynomial partitioning method of Guth and Katz shows the existence of an equipartition of $I(L)$ using a low-degree polynomial, where no line intersects too many regions [GK15, Gut16]. A recent work of Schnider proves extensions of the ham sandwich theorem for families of lines in $\mathbb{R}^3$ [Sch19], under a different interpretation of separation of lines. In order to state our main result, we need the following definition.

Definition 2. *We say that a set of lines L in the plane is in general position if*

- *No two lines in L are parallel*
- *No three lines in L are concurrent*
- *If three points in $I(L)$ are colinear, they belong to the same line in L.*

With this, our main result is the following.

Theorem 1.3. *Let A, B be two finite sets of lines in $\mathbb{R}^2$ such that $A \cup B$ is in general position, and let r be a fixed positive integer. Then, there is a closed convex partition $(C_1, \ldots, C_r)$ of $\mathbb{R}^2$ into r parts such that for all $j \in \{1, \ldots, r\}$ there exist sets $A_j \subset A$, $B_j \subset B$ such that $I(A_j) \subset C_j, I(B_j) \subset C_j$ and*

$$|A_j| \geq r^{\ln(2/3)}|A|^{1/r} - 2r, \qquad |B_j| \geq r^{\ln(2/3)}|B|^{1/r} - 2r.$$

Notice that, if r is a power of two, a repeated application of Theorem 1.2 implies the existence of a convex partition into r parts where each part encloses $|A|^{1/r}$ lines of A and $|B|^{1/r}$ lines of B. We do not know if the leading factor $r^{\ln(2/3)} \sim r^{-0.405}$ is necessary in general. The dependence of Theorem 1.3 on $|A|$ and $|B|$ is optimal up to that factor, as the following theorem shows. If we are interested in splitting a single set, Theorem 2.2 is an optimal result in which the partition can only be made by vertical strips.

Theorem 1.4. *Let n, r be positive integers. There exist sets A, B of n lines in the plane in general position such that for every closed convex partition $C_1, \dots, C_r$ of the plane into r sets, there exists some $j \in \{1, \dots, r\}$ such that either C_j encloses at most $\lceil n^{1/r} \rceil$ lines of A or C_j encloses at most $\lceil n^{1/r} \rceil$ lines of B.*

The proof of Theorem 1.3 is similar to the first proof of Theorem 1.1 [BKS00]. The main idea is to use only partitions of $\mathbb{R}^2$ into two or three convex pieces, and subdivide each piece until we obtain the desired partition. The only topological tool needed is the Knaster–Kuratowski–Mazurkiewicz theorem [KKM29] for a triangle.

In Section 2 we prove an Erdős–Szekeres type lemma which is crucial in the main proof. In Section 3 we discuss properties of partitions of $\mathbb{R}^2$ into two or three parts. In Section 4 we show how, given two families of lines, we can find a point from which three rays make a closed convex partition of the plane into three parts which are useful to prove our main results. In Section 5 we prove Theorem 1.3, and we prove our upper bounds in Section 6. Finally, in Section 7 we include remarks in the proof. We discuss the extensions of the Erdős–Szekeres theorem which would allow us to extend other proofs of Theorem 1.1 to the setting with lines.

2. Partitions into two and three parts

Given a set A of lines in the plane, and $K \subset \mathbb{R}^d$, we consider

$$\mu_A(K) = \max\{|A'| : A' \subset A, I(A') \subset K\}.$$

We now prove a couple of properties of μ_A for convex partitions of $\mathbb{R}^2$ into two or three parts. We use the Erdős–Szekeres theorem [ES35], which says that *every sequence of $ab+1$ real numbers contains either a monotone non-decreasing sequence of length $a+1$ or a monotone non-increasing subsequence of length $b+1$.*

Lemma 2.1. *Let A be a set of n lines in $\mathbb{R}^2$ in general position. Let ℓ be another line in the plane, which defines a convex partition of $\mathbb{R}^2$ into two closed half-planes C_1, C_2. Then we have*

$$\mu_A(C_1)\mu_A(C_2) \geq n.$$

Proof. First, assume that ℓ does not contain any point of $I(A)$ and is not parallel to any line in ℓ. We may assume that ℓ is a vertical line. We order the points of the form $\ell \cap a$ with lines in $a \in A$ by their vertical coordinates, from bottom to top. Suppose that $\ell \cap a$ is the i-th point. We define x_i to be the slope of a. Consider the sequence $(x_1, \dots, x_n)$. An increasing subsequence defines a subset of A enclosed by the left side of ℓ, while a decreasing subsequence defines a subset of A enclosed by the right side of ℓ. Therefore, the Erdős–Szekeres theorem finishes this case.

If $\ell \in A$, then we apply the argument above to $A \setminus \ell$ and add ℓ to each enclosed set. If $\ell \notin A$ and it either contains a point of $I(A)$, is parallel to a line in A, or both, a standard approximation argument finishes the proof. This is done by constructing a sequence of lines $\ell_1, \ell_2, \dots$, such that $\ell_i \to \ell$ as $i \to \infty$, and none of ℓ_i is parallel to a line of A nor contains a point of $I(A)$. If ℓ_i splits the plane into C_1^i, C_2^i, then we have $\mu_A(C_1^i) \to \mu_A(C_i)$ and $\mu_A(C_2^i) \to \mu_A(C_2)$. Since $\mu_A(C_1^i)\mu_A(C_2^i) \geq n$ for all i, we also have this inequality in the limit. □

A repeated application of this lemma gives the following result. Its optimality is proved in Theorem 6.1.

Theorem 2.2. *Let A be a set of n lines in $\mathbb{R}^2$ in general position. There exists a set of $r-1$ vertical lines that split $\mathbb{R}^2$ into r closed regions $C_1, \dots, C_r$ such that $\mu_A(C_i) \geq n^{1/r}$ for each $i = 1, \dots, r$.*

Proof. We proceed by induction on r. For $r = 1$, we take $C_1 = \mathbb{R}^2$ and we are done. Assume $r \geq 2$ and the result is known for $r-1$. We choose the left-most vertical line ℓ, which splits the plane into two regions C_1 and C_2. Assume C_1 is the closed left side of ℓ and C_2 is the closed right side of ℓ. We start with ℓ far enough to the left so that $\mu_A(C_1) = 0$ and $\mu_A(C_2) = n$. We start moving ℓ continuously to the right until the first moment when $\mu_A(C_1) \geq n^{1/r}$. If ℓ is placed any further on the left, we would have $\mu_A(C_1) < n^{1/r}$ and so by Lemma 2.1 we would have $\mu_A(C_2) > n^{(r-1)/r}$. Therefore, at the first position in which $\mu_A(C_1) \geq n^{1/r}$ we have $\mu_A(C_2) \geq n^{(r-1)/r}$. Let A' be a set of cardinality $\mu_A(C_2)$ whose pairwise intersections are contained in C_2. If we apply the induction hypothesis to A', we can find $r-2$ vertical lines that split C_2 into $r-1$ regions $C_1', \ldots, C_{r-1}'$ so that

$$\mu_A(C_i') \geq \mu_{A'}(C_i') \geq |A'|^{1/(r-1)} \geq \left(n^{(r-1)/r}\right)^{1/(r-1)} = n^{1/r}.$$

Therefore, $C_1, C_1', \ldots, C_{r-1}'$ is the partition we wanted. □

For the next lemma we use Dilworth's theorem [Dil50], which says that *in any finite partially ordered set X, the smallest chain decomposition has exactly as many parts as the largest anti-chain in X*. A chain decomposition is a partition of X into totally ordered sets. An anti-chain is a set of element in which no two are comparable. In particular, Dilworth's theorem implies that the product of the cardinality of the largest chain and the cardinality of the largest anti-chain is greater than or equal to $|X|$.

Lemma 2.3. *Let A be a set of n lines in $\mathbb{R}^2$ in general position. Let r_1, r_2 be two rays that star from a common point p. The broken line $r_1 \cup r_2$ splits $\mathbb{R}^2$ into two closed sets C_1, C_2. Then,*

$$\mu_A(C_1)\mu_A(C_2) \geq \frac{2n}{3}.$$

Proof. We assume that r_1 and r_2 are not contained in lines of A, that they do not contain points of $I(A)$ and that they are not parallel to lines of A. A standard approximation argument shows that we do not lose generality by making these assumptions. We may also assume without loss of generality that r_2 is the positive y-axis, r_2 form an acute angle with the positive x-axis, and C_1 is convex. See Figure 1. We assign one of four different "types" to each line in A as follows:

- Type 1: lines intersecting r_1 but not r_2
- Type 2: lines intersecting r_2 but not r_1
- Type 3: lines intersecting both r_1 and r_2
- Type 4: lines intersecting neither r_1 nor r_2.

Then, we consider the sets

$$A_1 = \{a \in A : a \text{ has type 2,3, or 4}\}$$
$$A_2 = \{a \in A : a \text{ has type 1,3, or 4}\}$$
$$A_3 = \{a \in A : a \text{ has type 1 or 2}\}$$

Notice that $|A_1| + |A_2| + |A_3| = 2n$, so at least one of the sets has cardinality greater than or equal to $\frac{2n}{3}$. We are going to define a partial order on each of the three sets in such a way that the elements of any chain pairwise intersect in C_1 and the element of any anti-chain pairwise intersect in C_2. An application of Dilworth's theorem to the largest set among A_1, A_2, A_3 will give us the desired conclusion.

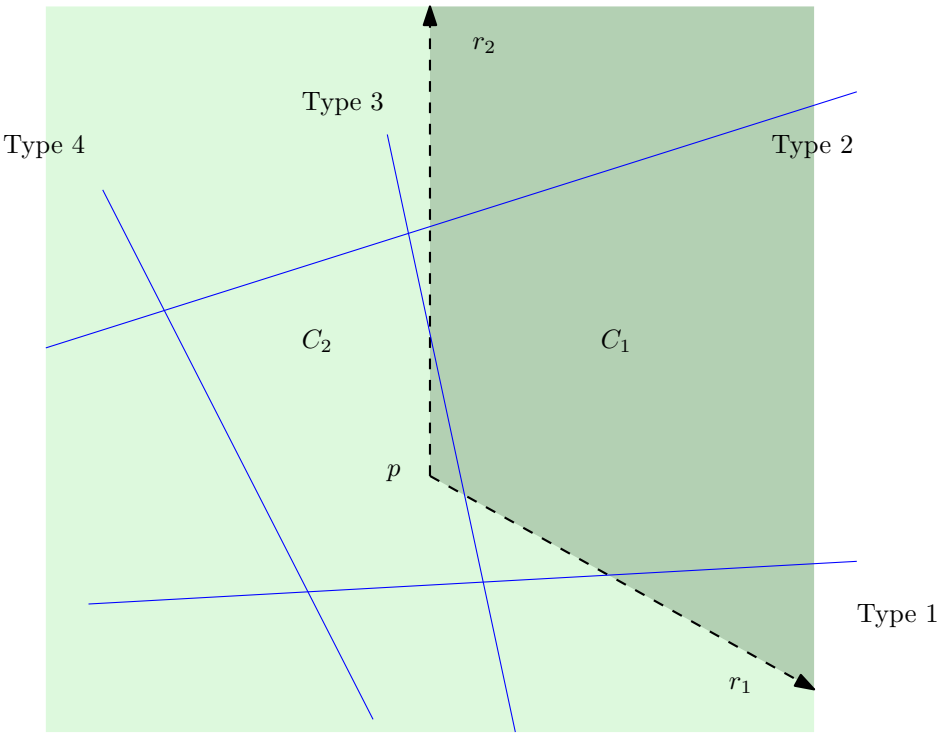

FIGURE 1. Type assignment for the proof of Lemma 2.3

Let us start by defining a partial order on A_1. For a line ℓ that intersects r_2, we define

$$d(\ell) = \text{dist}(p, r_2 \cap \ell).$$
$$s(\ell) = \text{ the slope of } \ell.$$

We say that ℓ_1 is comparable to ℓ_2 if either $\ell_1 = \ell_2$ or $\ell_1 \cap \ell_2 \in C_1$. Furthermore, if ℓ_1 and ℓ_2 are comparable, we say that $\ell_1 \leq \ell_2$ if either

- ℓ_1 is of type 2 and ℓ_2 is of type 3 or
- ℓ_1 and ℓ_2 are of the same type and $d(\ell_1) \leq d(\ell_2)$.

In order to show that this is a partial order, we prove it is transitive. Suppose that $\ell_1 \leq \ell_2$ and $\ell_2 \leq \ell_3$.

Case 1. The lines ℓ_1, ℓ_2, ℓ_3 are all of type 2. In this case, we know $d(\ell_1) \leq d(\ell_2)$ and $d(\ell_2) \leq d(\ell_3)$, so $d(\ell_1) \leq d(\ell_3)$. We just need to know that $\ell_1 \cap \ell_3 \in C_1$. Since $d(\ell_1) \leq d(\ell_2)$ and $\ell_1 \cap \ell_2$ has a positive x-coordinate, then $s(\ell_1) \geq s(\ell_2)$. Similarly, $s(\ell_2) \geq s(\ell_3)$, so $s(\ell_1) \geq s(\ell_3)$. Therefore, $\ell_1 \cap \ell_3$ has a positive x-coordinate. Since neither ℓ_1 nor ℓ_3 intersect r_1, their intersection is is C_1.

Case 2. The lines ℓ_1, ℓ_2 are of type 2 and ℓ_3 is of type 3. It suffices to show that $\ell_1 \cap \ell_3 \in C_1$. This is equivalent to showing that $d(\ell_1) \leq d(\ell_3)$. However, $d(\ell_1) \leq d(\ell_2)$ and $d(\ell_2) \leq d(\ell_3)$, so we obtain the desired inequality.

Case 3. The line ℓ_1 is of type 2 and the lines ℓ_2, ℓ_3 are of type 3. We just need to show that $\ell_1 \cap \ell_3 \in C_1$. This reduces to showing $d(\ell_1) \leq d(\ell_3)$, which can be done as in Case 2.

Case 4. The lines ℓ_1, ℓ_2, ℓ_3 are all of type 3. We obtain $d(\ell_1) \leq d(\ell_3)$ as case 1, so we just need to show that $\ell_1 \cap \ell_3 \in C_1$. This reduces to showing that the

intersections $\ell_1 \cap r_1, \ell_3 \cap r_1$ appear in reverse order as $\ell_1 \cap r_2, \ell_3 \cap r_2$. However, since this happens for the pairs of lines (ℓ_1, ℓ_2) and (ℓ_2, ℓ_3), we are done.

The order for A_2 can be defined in an analogous way, replacing the roles of r_1 and r_2.

Let us define the order of A_3. If ℓ is a line that intersects r_1, we define

$$d'(\ell) = \operatorname{dist}(p, \ell \cap r_1)$$

We say that two lines ℓ_1, ℓ_2 in A_3 are comparable if either $\ell_1 = \ell_2$ or $\ell_1 \cap \ell_2 \in C_1$. Moreover, if they are comparable we say that $\ell_1 \leq \ell_2$ if either

- The lines ℓ_1, ℓ_2 are both of type 2 and $d(\ell_1) \leq d(\ell_2)$,
- The lines ℓ_1, ℓ_2 are both of type 1 and $d'(\ell_1) \geq d'(\ell_2)$, or
- The line ℓ_1 is of type 1 and the line ℓ_2 is of type 2.

Let us prove transitivity. If $\ell_1 \leq \ell_2$ and $\ell_2 \leq \ell_3$, the work above shows that $\ell_1 \leq \ell_3$ if the three lines are of the same type. Therefore, we only need to check two cases, in which the line ℓ_1 is of type 1, the line ℓ_3 is of type 2, and the line ℓ_2 is either of type 2 or type 1.

Case 1. The line ℓ_2 is of type 2. Because $\ell_1 \leq \ell_2$, we have that $s(\ell_1) \geq s(\ell_2)$. Since $\ell_2 \cap \ell_3 \in C_1$, we have that $s(\ell_2) \geq s(\ell_3)$. Therefore, $s(\ell_1) \geq s(\ell_3)$. This means that $\ell_1 \cap \ell_3 \in C_1$.

Case 2. The line ℓ_2 is of type 1. This case is analogous to Case 1 if we swap the roles of r_1 and r_2 and reverse the order.

□

Lemma 2.4. *Let A be a set of n lines in $\mathbb{R}^2$ in general position. Let (C_1, C_2, C_3) be a convex partition of $\mathbb{R}^2$ into three closed convex parts. Then,*

$$\mu_A(C_1)\mu_A(C_2)\mu_A(C_3) \geq \frac{2n}{3}$$

Proof. If (C_1, C_2, C_3) is formed by two parallel lines, we can apply Lemma 2.1 twice to finish. If not, then (C_1, C_2, C_3) is formed by three rays coming out of the same point. Consider the partition $(C_1, C_2 \cup C_3)$. By Lemma 2.3, we know that

$$\mu_A(C_1)\mu_A(C_2 \cup C_3) \geq \frac{2n}{3}.$$

Let $A' \subset A$ be the set of lines that realizes $\mu_A(C_2 \cup C_3)$. Let ℓ be the line that contains the ray spitting C_2 and C_3. We can apply Lemma 2.1 to A' and ℓ to obtain

$$\mu_A(C_2)\mu_A(C_3) \geq \mu_{A'}(C_2)\mu_{A'}(C_3) \geq |A'| = \mu_A(C_2 \cup C_3),$$

which concludes the proof. □

3. Equitable cuttings.

The gist of the proof for Theorem 1.3 is to show that for each value of r and two finite sets of lines A, B, each in general position, we always have at least one of the two following situations.

- There exists two positive integers r_1, r_2 such that $r_1 + r_2 = r$ and there is a convex partition of the plane into two parts (C_1, C_2) such that

$$\mu_A(C_i) \geq \left(\frac{2|A|}{3}\right)^{r_i/r} - 2, \quad \mu_B(C_i) \geq \left(\frac{2|B|}{3}\right)^{r_i/r} - 2 \qquad \text{for } i = 1, 2$$

- There exists three positive integers r_1, r_2, r_3 such that $r_1 + r_2 + r_3 = r$ and there is a convex partition of the plane into three parts (C_1, C_2, C_3) such that

$$\mu_A(C_i) \geq \left(\frac{2|A|}{3}\right)^{r_i/r} - 2, \quad \mu_B(C_i) \geq \left(\frac{2|B|}{3}\right)^{r_i/r} - 2 \qquad \text{for } i = 1, 2, 3$$

The constant 2/3 factors are the reason why we have the $r^{\ln(2/3)}$ factor in the main theorem. We will call the first type of partition an **equitable** (r_1, r_2) **cut**, and the second type an **equitable** (r_1, r_2, r_3) **cut**.

The rest of the paper will focus on proving the following lemma.

Lemma 3.1. *Let A, B be two finite sets of points in the plane, each in general position, and $r \geq 2$ be a positive integer. Then, there either exists a pair (r_1, r_2) of positive integers with sum r for which there is an equitable (r_1, r_2) cut, or there exists a triple (r_1, r_2, r_3) of positive integers with sum r for which there is an equitable (r_1, r_2, r_3) cut.*

Let us first show that Lemma 3.1 implies Theorem 1.3.

Proof that Lemma 3.1 implies Theorem 1.3. First, notice that $r^{\ln(2/3)} = \left(\frac{2}{3}\right)^{\ln r}$. We prove Theorem 1.3 by strong induction on r. When $r = 1$, the result is clear. If $r \geq 2$, suppose that there is a pair (r_1, r_2) with sum r for which there is an equitable (r_1, r_2) cut. The case for a triple will be analogous. Denote by (C_1, C_2) the two sets of the partition.

We know that $\mu_A(C_i) \geq \left(\frac{2|A|}{3}\right)^{r_i/r} - 2$. Let $A_i \subset A$ of cardinality $\mu_A(C_i)$ be enclosed by C_i. We apply Theorem 1.3 to A_i to find a convex partition $(C_1^i, \ldots, C_{r_i}^i)$ of the plane into r_i parts. We know that

$$\begin{aligned}
\mu_A(C_j^i \cap C_i) \geq \mu_{A_i}(C_j^i \cap C_i) &\geq \left(\frac{2}{3}\right)^{\ln(r_i)} |A_i|^{1/r_i} - 2r_i \geq \left(\frac{2}{3}\right)^{\ln(r_i)} \left(\left[\frac{2|A|}{3}\right]^{r_i/r} - 2\right)^{1/r_i} - 2r_i \\
&\geq \left(\frac{2}{3}\right)^{\ln(r_i)} \left(\left[\frac{2|A|}{3}\right]^{r_i/r}\right)^{1/r_i} - 2(r_i + 1) \geq \left(\frac{2}{3}\right)^{\ln(r-1)+1/r} |A|^{1/r} - 2r \\
&\geq \left(\frac{2}{3}\right)^{\ln r} |A|^{1/r} - 2r,
\end{aligned}$$

where the last inequality follows from the mean value theorem. Equivalent arguments work for B. Therefore, the partition formed by the sets $C_j^i \cap C_i$ for $i = 1, 2$, $j = 1, \ldots, r_i$ is the one we are looking for. □

Definition 3. *Let r_1 be an integer with $1 \leq r_1 \leq r - 1$. We say that a closed half-plane H is r_1-critical for A if*

- *H encloses a subset of A of cardinality at least $\left(\frac{2|A|}{3}\right)^{r_1/r} - 2$ and*
- *The interior of H does not enclose a subset of A of cardinality at least $\left(\frac{2|A|}{3}\right)^{r_1/r} - 2$.*

Notice that the boundary line of an r_1-critical halfspace is a support line of a set of the form $\mathrm{conv}(I(A'))$, where $A' \subset A$ and $|A'| = \left\lceil \left(\frac{2|A|}{3}\right)^{r_1/r} \right\rceil - 2$, and its interior contains no other such set.

Lemma 3.2. *Let A be a finite set of lines in the plane in general position and r_1, r be two positive integers with $1 \leq r_1 \leq r - 1$. Suppose that a line ℓ induces a convex*

partition into two closed half-spaces (C_1, C_2). *If* C_1 *is* r_1*-critical, then*

$$\mu_A(C_2) > \left(\frac{3}{2}\right)^{(r_1/r)} |A|^{1-(r_1/r)}$$

Proof. Let (C_1', C_2') be a convex partition of the plane such that $C_1' \subset C_1$ and C_1' contains the same points of $I(A)$ as the interior of C_1. Therefore, $\mu_A(C_1') < \left(\frac{2|A|}{3}\right)^{r_1/r}$ and $\mu_A(C_2) = \mu_A(C_2')$. However, by Lemma 2.1, $\mu_A(C_1')\mu_A(C_2') \geq |A|$, so we get the desired conclusion. □

Lemma 3.3. *Let* A, B *be two finite sets of lines in the plane, each in general position, and* $r \geq 2$ *be a positive integer. Let* $1 \leq r_1 \leq r-1$ *be an integer. If there are two* r_1*-critical half-spaces* H *and* H' *for* A*, such that* $\mu_B(H) \geq \left(\frac{2|B|}{3}\right)^{r_1/r} - 2$ *and* $\mu_B(H') \leq \left(\frac{2|B|}{3}\right)^{r_1/r} - 2$*, then there exists a closed half-space* H'' *which is* r_1*-critical for* A *and such that* $\mu_B(H'') = \left\lceil \left(\frac{2|B|}{3}\right)^{r_1/r} \right\rceil - 2$.

Proof. Notice that for every direction there is a unique oriented line that defines a r_1-critical half-plane for A on the left side of the line. Moreover, as the direction changes, this line changes continuously, since it is defined as a minimum of several support functions of convex sets. Therefore, we can go from H to H' by a continuous change of the boundary line, while always maintaining an r_1-critical half-plane for A. The value of $\mu_B(\cdot)$ on this half-plane can only change by increments or decrements of one, as B is in general position. Therefore, at some point $\mu_B(\cdot)$ has the required value. □

If the conditions of the Lemma 3.3 are satisfied, define $r_2 = r - r_1$. Then, we have a partition of $\mathbb{R}^2$ into two closed convex parts (C_1, C_2) such that

$$\mu_A(C_1) \geq \left(\frac{2|A|}{3}\right)^{r_1/r} - 2, \qquad \mu_A(C_2) \geq \left(\frac{3}{2}\right)^{(r_1/r)} |A|^{1-(r_1/r)} \geq \left(\frac{2|A|}{3}\right)^{r_2/r} - 2$$

$$\mu_B(C_1) \geq \left(\frac{2|B|}{3}\right)^{r_1/r} - 2, \qquad \mu_B(C_2) \geq \left(\frac{3}{2}\right)^{(r_1/r)} |B|^{1-(r_1/r)} \geq \left(\frac{2|B|}{3}\right)^{r_2/r} - 2$$

This means we have an equitable (r_1, r_2) cut. Therefore, if A, B are two finite sets of lines in the plane, each in general position, such that there is no equitable (r_1, r_2) cut, we either have that

- Every r_1-critical half-plane H for A satisfies $\mu_B(H) > \left(\frac{2|B|}{3}\right)^{r_1/r} - 2$, or
- Every r_1-critical half-plane H for A satisfies $\mu_B(H) < \left(\frac{2|B|}{3}\right)^{r_1/r} - 2$.

We can assign a sign to r_1 depending on which scenario above holds true. We will say that r_1 is positive for A if the first one happens, and negative for A otherwise. Notice that r_1 is positive for A if and only if it is negative for B.

Now we can use the following theorem.

Theorem 3.4 (Theorem 9 in [BKS00])**.** *If every element of* $1, 2, \ldots, r-1$ *is given a sign, there exists either a pair* (r_1, r_2) *with* $1 \leq r_1 \leq r_2 \leq r-1$ *such that* r_1, r_2 *have the same sign and* $r_1 + r_2 = r$*, or a triple* (r_1, r_2, r_3) *with* $1 \leq r_1 \leq r_2 \leq r_3 \leq r-1$ *such that* r_1, r_2, r_3 *have the same sign and* $r_1 + r_2 + r_3 = r$*. Moreover, we can further assume that* $r_i \leq 2r/3$ *for all* i *in both cases.*

Lemma 3.5. *Let* A, B *be two finite sets of lines in the plane, each in general position. Let* r_1, r_2, r *be positive integers such that* $r_1 + r_2 = r$*. If there is no equitable* (r_1, r_2) *cut, then the signs given to* r_1 *and* r_2 *are different.*

Proof. Suppose that there are such r_1, r_2 with the same sign and we look for a contradiction. By swapping the role of A and B, we may assume that both r_1, r_2 have positive sign. Let x_0 be the value such that the half-plane $H_0 = \{(x, y) : x \leq x_0\}$ is r_1-critical for A. Let x_1 be the value for which the half-plane $H_1 = \{(x, y) : x \geq x_1\}$ is r_2-critical for A. By Lemma 2.1, we know $x_0 \leq x_1$ (otherwise, the vertical line whose equation is $x = (x_0 + x_1)/2$ splits the plane into two half-planes whose combined μ_A-sizes are too small). Therefore, any vertical line between these two half-planes induces an (r_1, r_2) equitable cut, which is the contradiction we wanted. □

If we do not have an equitable (r_1, r_r) cut for any pair such that $r_1 + r_2 = r$, then Lemma 3.5 and Theorem 3.4 imply that there is a triple (r_1, r_2, r_3) of positive integers with sum r and each r_i has the same sign. We will use this triple to show that there is an equitable (r_1, r_2, r_3)-cut. The next section contains a technical construction used in the proof of Lemma 3.1. Given a point p, we construct three rays emanating from p that create a partition of the plane into three convex sets. Each of these sets will have a large size with respect to $\mu_A(\cdot)$, for a fixed set of lines A. In Section 5 we show the existence of a particular point p such that each of these sets also has a large value with respect to $\mu_B(\cdot)$, for a second set of lines B.

4. Region of convex canonical cuts

For this section, consider A to be a finite set of lines in general position and r a positive integer. By applying an appropriate rotation we may assume that no two points of $I(A)$ have the same x-coordinate. Let (r_1, r_2, r_3) be a triple of positive integers such that $r_1 + r_2 + r_3 = r$. Let x_0, x_1 be the numbers such that the half-planes

$$H_0 = \{(x, y) : x \leq x_0\}$$
$$H_1 = \{(x, y) : x \geq x_1\}$$

are r_1-critical and r_2-critical for A, respectively. We know that $x_0 \leq x_1$. For each point $p = (x, y)$ such that $x_0 \leq x \leq x_1$ we are going to define a **canonical p-cutting**. This is going to be a partition of the plane into three parts (C_1, C_2, C_3).

For convenience, let $M_i = \left\lceil \left(\frac{2|A|}{3}\right)^{r_i/r} \right\rceil - 2$ for $i = 1, 2, 3$. In order to find our partition, the main idea is to construct three rays starting from p. The first ray r_0 is pointing downwards. Given an angle α_1, we define C_1 to be the region made by a clockwise angle of α_1 starting at r_0. We choose α_1 to be the minimum number such that $\mu_A(C_1) = M_1$. Notice that due to the location of p, we know that $\alpha_1 \leq \pi$ (i.e., C_1 is convex). We define α_2, C_2 equivalently on the other side with α_2 now begin a counter-clockwise angle such that $\mu_A(C_2) = M_2$ and α_2 is minimal with that property. The region C_3 is the top region, which may or may not be convex. See Figure 2.

The main issue with this construction is that α_1, α_2 are not continuous as functions of p, which is undesirable. The discontinuities may occur when p shares the x-coordinate of a point of $I(A)$. We will refine the definitions of α_1 and α_2 to avoid this problem. We will redefine the values of α_1, α_2 when the x-coordinate of p is equal to x_0, when it is equal to x_1, and when it it very close to the x-coordinate of a point in $I(A)$, but not very close to x_0 or x_1. The modifications make sure that $\mu_A(C_1)$ and $\mu_A(C_2)$ are large enough for our purposes. We prove in Claim 4.1 that $\mu_A(C_3)$ is also large.

Let $p_0 = (x', y')$ be a point of $I(A)$ such that $x_0 < x' < x_1$. Let ε be a small positive real number such that the no closed vertical strip of width 2ε contains more

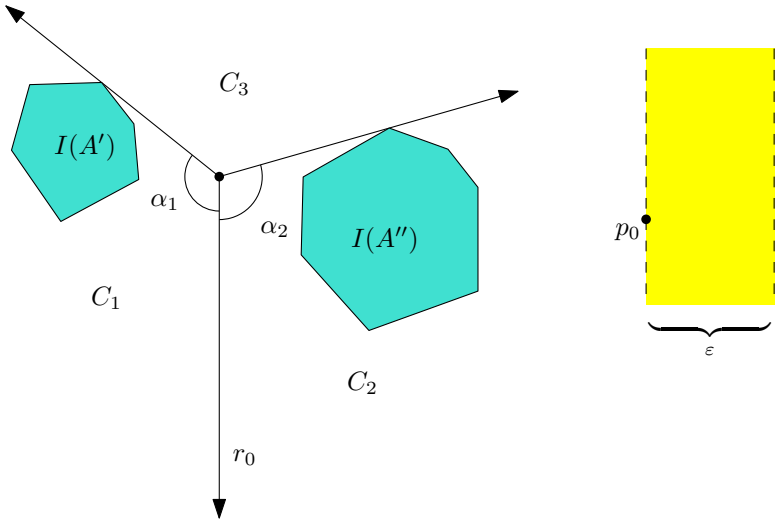

FIGURE 2. Construction of the p-canonical cutting. For points in the vertical strip of width ε on the right side of a point p_0 of $I(A)$, we need to redefine the value of α_1.

than one point of $I(A)$. If $p = (x, y)$ and $p = (x' + t\varepsilon, y)$ for some $t \in [0, 1]$, we redefine α_1. Let

- α_1' be the minimum angle such that C_1 encloses a subset $A' \subset A$ of size M_1.
- α_1'' be the minimum angle such that C_1 encloses a subset $A' \subset A$ of size M_1 but such that $p_0 \notin I(A')$.

In order to define C_1, we use an angle of $\alpha_1 = t\alpha_1' + (1-t)\alpha_1''$. Since $\alpha_1'' \geq \alpha_1'$, we know that $\alpha_1 \geq \alpha_1'$, so $\mu_A(C_1) \geq M_1$. However, if $\mu_A(C_1) \geq M_1 + 2$, by removing at most one line from the set realizing $\mu_A(C_1)$, we can assume that p_0 is not part of its incidence set. This would imply that $\alpha_1 > \alpha_1''$, which is a contradiction. Therefore, $M_1 + 1 \geq \mu_A(C_1)$.

The value of α_2 is not changed in this region. If $p = (x' - t\varepsilon, y)$, then we do an analogous modification by swapping the roles of C_1, C_2. The reader may notice that now the angles α_1, α_2 are continuous functions of the point p.

Along the line $x = x_0$ and the region $x_0 \leq x \leq x_0 + \varepsilon$ we will redefine α_1. For this, assume that we have another set B of lines in general position, and that $\mu_B(H_0) \geq \left\lceil \left(\frac{2|B|}{3}\right)^{r_1/r} \right\rceil - 2$. Also assume that $I(B)$ has no points on the line $x = x_0$.

Since H_0 is r_1-critical for A, it means that there is a set $A_0 \subset A$ such that H_0 encloses A_0, the cardinality of A_0 is exactly M_1 and there is a unique point $p_0 = (x_0, y_0)$ in $I(A_0)$. Notice that α_1 is not continuous in the line $x = x_0$. At any point $p = (x_0, y)$ with $y < y_0$, $\alpha_1 = \pi$. However, at p_0, α_1 defines a ray r_1 whose slope is equal to the slope of the top tangent of $\text{conv}(I(A_0))$ at p_0 (if there are multiple sets A_0 that satisfy the properties above, then the slope is the minimum of the top tangents to those sets). For a point $p = (x_0, y)$ we define

- α_1' to be the minimum angle such that $\mu_A(C_1) \geq M_1$ and
- β_1 to be the minimum angle such that $\mu_B(C_1) \geq \left\lceil \left(\frac{2|B|}{3}\right)^{r_1/r} \right\rceil - 2$.

Now we define the angle $\tilde{a}_1 = \tilde{a}_1(x_0, y)$ of C as

$$\tilde{a}_1 = \begin{cases} \pi & \text{if } y < y_0 \\ t\max\{\alpha_1', \beta_1\} + (1-t)\pi & \text{if } y = y_0 + t\varepsilon, \qquad t \in [0,1] \\ \max\{\alpha_1', \beta_1\} & \text{if } y > y_0 + \varepsilon \end{cases}$$

The region C_1 defined by angle $\tilde{\alpha}_1$ satisfies $\mu_A(C_1) = M_1$ (since H_0 was r_1-critical) and $\mu_B(C_1) \geq \left\lceil \left(\frac{2|B|}{3}\right)^{r_1/r} \right\rceil - 2$.

Then, for $p = (x_0 + s\varepsilon, y)$ for some $s \in [0,1]$, we define

- α_1'' the minimum angle such that the region C_1 satisfies $\mu_A(C_1) \geq M_1$.
- $\alpha_1 = s\alpha_1'' + (1-s)\tilde{\alpha}_1(x_0, y)$.

The angle α_1 is now a continuous function, $M_1 + 1 \geq \mu_A(C_1) \geq M_1$ and the behavior of C_1 on the line $x = x_0$ is the one we described above. We do an analogous definition for α_2. Now we are ready to define the region of the points p we are interested in.

Let

$$R = \{p = (x, y) \in \mathbb{R}^2 : x_0 \leq x \leq x_1 \text{ and the region } C_3 \text{ of the canonical } p\text{-cutting is convex}\}.$$

Notice that the top boundary of R is defined by the equation $\alpha_1 + \alpha_2 = \pi$. The region R is bounded above and unbounded below. For every point on the top boundary the region C_3 is a half-plane. Moreover, the continuity of α_1 and α_2 implies that there are no vertical segments in the boundary of R except for those contained on the lines $x = x_0$ and $x = x_1$.

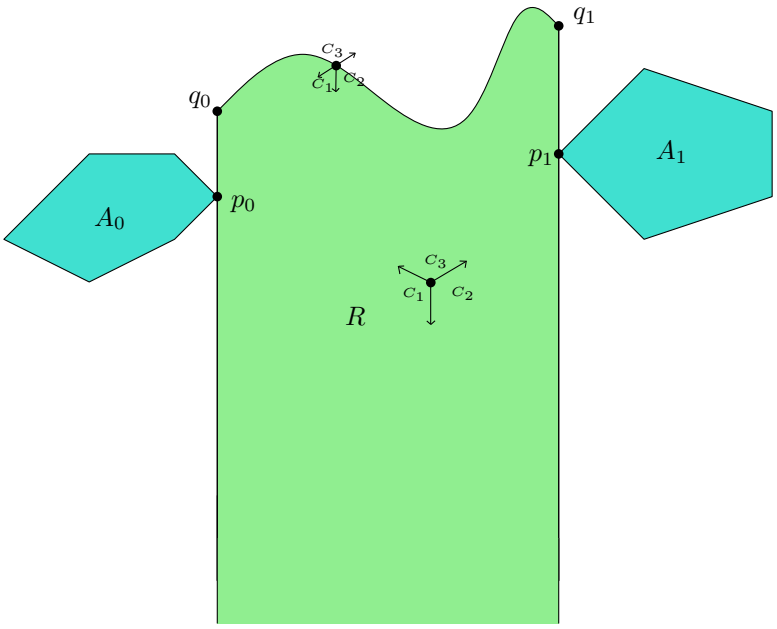

FIGURE 3. An example of the region R, with a couple of canonical p-cuttings marked.

Claim 4.1. *For every point p of R, we have $\mu_A(C_3) \geq M_3$.*

Proof. If we apply Lemma 2.4, we know that $\mu_A(C_1)\mu_A(C_2)\mu_A(C_3) \geq \frac{2|A|}{3}$. We also know that $(M_1+1)(M_2+1)(M_3+1) < \frac{2|A|}{3}$. Therefore, since $\mu_A(C_1) \leq M_1 + 1$ and $\mu_A(C_2) \leq M_2 + 1$, we have $\mu_A(C_3) \geq M_3$. □

Claim 4.1 actually proves that $\mu_A(C_3) > M_3$. However, the excess is not enough to improve directly the main results in this manuscript.

5. Proof of Lemma 3.1

With the construction of the region R in the previous section, we are ready to prove Lemma 3.1.

Proof of Lemma 3.1. Let A, B be two sets of lines in $\mathbb{R}^2$, each in general position. We may assume that no two points of $I(A) \cup I(B)$ share an x-coordinate. If there is no (r_1, r_2)-equitable cut for any pair (r_1, r_2) of positive integers with sum r, we may assign to every integer in $\{1, \dots, r-1\}$ a sign as in Section 3. By Theorem 3.4 and Lemma 3.5, there exists a triple (r_1, r_2, r_3) of positive integers whose sum is r and each r_i has the same sign. By swapping the roles of A and B, we can assume that the sign of each r_i is positive.

For each $i \in \{1, 2, 3\}$, let $N_i = \left(\frac{2|B|}{3}\right)^{r_i/r} - 2$.

We are going to color the points of R, where each color is represented by an element of $\{1, 2, 3\}$. The point $p \in R$ is going to be colored by color i if and only if in the canonical cutting of p, we have $\mu_B(C_i) \geq N_i$. Points are allowed to have multiple colors. By Claim 4.1, it is sufficient to prove that a point p has all three colors to finish the proof. Notice the following lemma.

Claim 5.1. *Every point of R has at least one color.*

Proof. This follows from Lemma 2.4 and the fact that $N_1 N_2 N_3 < \frac{2|B|}{3}$. □

Claim 5.2. *Every point of the top boundary of R has color 3, every point of R on the line $x = x_0$ has color 1 and every point of R on the line $x = x_1$ has color 2.*

Proof. Since r_1, r_2, r_3 all have positive sign, we know that $\mu_B(H_0) \geq N_1$ and $\mu_B(H_1) \geq N_2$. By the definition of the canonical cuttings on the lines $x = x_0$ and $x = x_1$, we have the claim on the left and right boundaries. For every point p on the top boundary, we know that C_3 is a half-plane with $\mu_A(C_3) \geq M_3$. This means that C_3 contains a r_3-critical halfplane C_3'. Therefore, $\mu_B(C_3) \geq \mu_B(C_3') \geq N_3$, due to the sign of r_3. □

Claim 5.3. *The color classes are closed sets.*

Proof. Take any converging sequence of points of color i. There must be a subsequence on which C_i encloses the same subsets of B. By the continuity of the angles α_1, α_2, on the point of convergence we also have that C_i encloses those subsets. □

Claim 5.4. *There is a sufficiently small value $\tilde{y}$ such that every point on the intersection of the line $y = \tilde{y}$ and the region R has colors 1 and 2.*

Proof. For every value of x, there has to be a y_x such that if $y < y_x$, then C_1 encloses A_0 for the canonical cutting of $p = (x, y)$. Moreover, the value of y_x is a continuous function of x. Therefore, it attains a minimum value $\tilde{y}_1$ on $[x_0, x_1]$. Points in R with y-coordinate $\tilde{y}_1$ or less will have color 1. Similarly, we can find a $\tilde{y}_2$ such that points in R with y-coordinate $\tilde{y}_2$ or less will have color 2. □

We define $R' = \{p = (x, y) : p \in R,\ y \geq \tilde{y}\}$. Now, we are able to apply the classic Knaster–Kuratowski–Mazurkiewicz (KKM) theorem in dimension two.

Theorem (Knaster, Kuratowski, Mazurkiewicz [KKM29])**.** *Let Δ be a triangle with vertices $1, 2, 3$. Suppose that Δ is colored with colors $\{1, 2, 3\}$ such that every vertex i has color i, and every point on a side ij has at least one of the colors i or j. If every color class is a closed set, then there is a point with all three colors.*

We can choose a point on the left side of the boundary of R' to be vertex 1, a point on the right side of the boundary of R' to be vertex 2, and a point on the top boundary to be vertex 3. The application of the KKM theorem finishes the proof. □

6. Upper bounds

In order to obtain upper bounds for our results, let's start with a single set of lines that is hard to split using vertical strips. Consider $[a] = \{1, \dots, a\}$.

Theorem 6.1. *Let a and r be positive integers. There is a set A of a^r lines in the plane in general position such that for any partition of the plane into r closed vertical strips $A_1, \dots, A_r$, there exists an $i \in [r]$ such that*

$$\mu_A(A_i) \leq a.$$

Proof. Let us first construct a set X of a^r points in the plane. We consider $v_1, \dots, v_r$ nonzero vectors in $\mathbb{R}^2$, none of which are vertical or horizontal, with the following properties

- The slope of v_i is l_i, and $l_1 < l_2 < \ldots < l_r$
- The norms of the vectors satisfy $||v_1|| < \ldots < ||v_r||$. Moreover, $||v_{i+1}||$ is significantly larger than $||v_i||$, in a way that will be made precise below.

Then, let

$$X = \left\{ \sum_{i=1}^{r} x_i v_i : (x_1, \dots, x_r) \in [a]^r \right\}.$$

Let $\varepsilon > 0$ be a sufficiently small number such that the intervals $I_i = [l_i - \varepsilon, l_i + \varepsilon]$ are pairwise disjoint. For two points $p = \sum_{i=1}^{r} x_i v_i$ and $q = \sum_{i=1}^{r} y_i v_i$ in X, we consider

$$m(p, q) = \max\{i \in [a] : x_i \neq y_i\}.$$

We want the sequence $||v_1||, \dots, ||v_r||$ to grow fast enough so that, if $j = m(p, q)$, then the slope of $p - q$ lies in the interval $[l_j - \varepsilon, l_j + \varepsilon] = I_j$.

Now we construct the set A of lines (given by their equations) as

$$A = \{y = mx + c : (m, c) \in X\}$$

If we are given two lines with equations $y = m_1 x + c_1$ and $y = m_2 x + c_2$, the x-coordinate of their intersection is given by $x = \frac{c_2 - c_1}{m_1 - m_2}$, which is the negative of the slope between (m_1, c_1) and (m_2, c_2). Now, consider a partition of $\mathbb{R}^2$ into r vertical strips. The boundary of these strips is given by the $r - 1$ lines $x = t_1, x = t_2, \dots, x = t_{r-1}$, for some real $t_1 \leq \ldots \leq t_{r-1}$. A simple inductive argument shows that at least one of the intervals $[-\infty, t_1], [t_1, t_2], \dots, [t_{r-1}, \infty]$ intersects at most one of the intervals $-I_1, \dots, -I_r$. Suppose that $[t_i, t_{i+1}]$ intersects only $-I_j$.

This means that if a subset $A' \subset A$ is enclosed by the strip between the lines $x = t_i, x = t_{i+1}$, these lines came from a subset $X' \subset X$ whose pairwise slopes are contained in I_j. This, in turn, implies that for $p, q \in X'$ we have that $m(p, q) = j$, so all the points of X' differ in the j-th coordinate (as vectors in $[a]^r$). Thus, $|A'| = |X'| \leq a$, as we wanted. If we want our set of lines to be in general position, a small perturbation of A gives us the desired set. □

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. We may assume without loss of generality that $|A|$ is the r-th power of an integer. We set A to be the example from Theorem 6.1. We set B to be such that $I(B)$ is contained in a disk of small radius. If we start to translate the set B upwards, and $C_1, C_2, \dots, C_r$ is a convex partition of the plane where each C_i intersects both $\operatorname{conv} I(B)$ and $\operatorname{conv} I(A)$, then the boundary between all

pairs C_i, C_j is either above $\text{conv}(B)$ or very close to a vertical line. Therefore, for a sufficiently high $I(B)$, the properties that the set A satisfies complete the proof. □

7. Remarks

The proof of our main results follows the ideas from Bespamyatnikh, Kirkpatrick, and Snoeyink [BKS00]. It would be interesting to see if it is possible to obtain a proof that uses stronger topological tools, as we have for point sets [Sob12, KHA14, BZ14]. The problem boils down to the following question, which would be a nice extension of the Erdős–Szekeres theorem.

Open question 7.1. *Determine if the following statement is true. Given a finite set A of lines in the plane and a convex partition $(C_1, \dots, C_r)$ of the plane that comes from a power diagram, the following equation holds*

$$\mu_A(C_1)\mu_A(C_2)\dots\mu_A(C_r) \geq |A|.$$

Actually, it would be sufficient to have

$$\max_{1\leq i\leq r} \mu_A(C_i) \geq |A|^{1/r}.$$

One may also wonder what happens in higher dimensions. We say that a set of hyperplanes A in $\mathbb{R}^d$ is in general position if every d of its normal vectors are linearly independent and no $d+1$ hyperplanes of A share a point. We denote by $I(A)$ the set of all points that come from the intersection of d hyperplanes in A, and say that K encloses A if $I(A) \subset K$. It is natural to ask the following question.

Open question 7.2. *Given n, d, r positive integers, find the smallest value of $M = f_{d,r}(n)$ such that the following holds. Given d sets $A_1, \dots, A_d$ of M hyperplanes each in $\mathbb{R}^d$, there exists a convex partition of $\mathbb{R}^d$ into r parts $(C_1, \dots, C_r)$ such that each C_j encloses a subset of at least n hyperplanes of each A_i.*

Dujmović and Langerman proved the existence of such a function when $r = 2$, and the rate of growth of $f_{d,2}$ has been bounded by Conlon, Fox, Pach, Sudakov, and Suk [CFP+14] to

$$f_{d,2}(n) \leq \text{twr}_{d-1}(c_d n^2 \log n),$$

where the tower function $\text{twr}_{d-1}(\cdot)$ is the composition of the function 2^x with itself $d-1$ times, and c_d is a constant that depends only on d.

If one is interested to see if the constant leading constant $r^{\ln(2/3)}$ can be replaced by 1 with the current proof, we would need to remove the need for $2/3$ in Lemma 2.3 or in Lemma 2.4. However, this replacement cannot be done for Lemma 2.3. Figure 4 shows an example of five lines and two rays such that none of the two sides of the broken line encloses more than two lines.

The loss of $2r$ lines in Theorem 1.3 comes from requiring that the angles of the canonical 3-cuttings are continuous. Without this assumption, we are not able to guarantee that the coloring of the region R' satisfies the properties of the KKM theorem. Moreover, it would also allow the top boundary of R to have vertical segments, which can also make the analysis more difficult (the arguments presented would only show that the top point of each vertical segment has color 3, instead of the whole segment).

8. Acknowledgments

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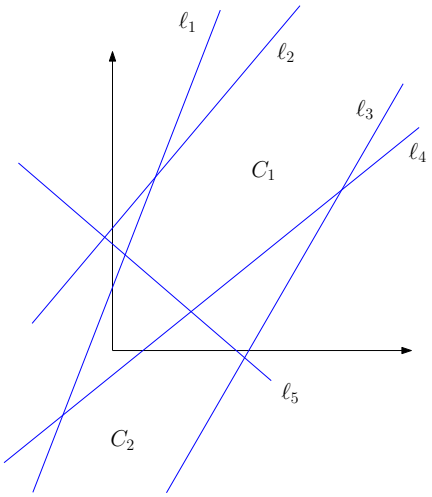

FIGURE 4. Every triangle made by three blue lines intersects the broken black line. The points $\ell_1 \cap \ell_3$ and $\ell_2 \cap \ell_4$ are in C_2.

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