A MÉLANGE OF DIAMETER HELLY-TYPE THEOREMS

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Abstract. A Helly-type theorem for diameter provides a bound on the diameter of the intersection of a finite family of convex sets in $\mathbb{R}^d$ given some information on the diameter of the intersection of all sufficiently small subfamilies. We prove fractional and colorful versions of a longstanding conjecture by Bárány, Katchalski, and Pach. We also show that a norm in $\mathbb{R}^d$ admits an exact Helly-type theorem for diameter if and only if its unit ball is a polytope and prove a colorful version for those that do. Finally, we prove Helly-type theorems for the property of "containing k colinear integer points."

1. Introduction

Helly's theorem is one of the most prominent results on the intersection properties of families of convex sets [23, 33]. It says that *if the intersection of every $d+1$ or fewer elements of a finite family of convex sets in $\mathbb{R}^d$ is nonempty, then the intersection of the entire family is nonempty.* This result has many extensions and generalizations, including topological, colorful, and fractional variants (see, for example, [3, 24] and the references therein).

Quantitative versions of Helly's theorem guarantee that the intersection of a family of convex sets is not just nonempty but "large" in some quantifiable sense. Bárány, Katchalski, and Pach initiated this direction of research [8, 9] when they proved that *if the intersection of every $2d$ or fewer elements of a finite family of convex sets in $\mathbb{R}^d$ has volume greater than or equal to 1, then the intersection of the entire family has volume at least d^{-2d^2}.*

Naszódi [32] improved the guarantee of the volume in the intersection to d^{-2d}, mostly settling this volumetric variant. His approach, based on sparsification of John decompositions of the identity, has been improved in several articles [12, 13, 16, 22, 25]. A constellation of related results that adjust the function measuring the size of the intersection, the cardinality of the subfamily intersection, or the guarantee in the conclusion have since been proven [18, 27, 34, 35, 37]. Bárány, Katchalski, and Pach conjectured a Helly-type theorem for diameter as well, which remains open.

Conjecture 1.1 (Bárány, Katchalski, Pach 1982 [8])**.** *Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If the intersection of every $2d$ or fewer members of $\mathcal{F}$ has diameter greater than or equal to 1, then the intersection of $\mathcal{F}$ has diameter greater than or equal to $cd^{-1/2}$, for some absolute constant $c > 0$.*

Bárány, Katchaslki, and Pach showed that the diameter of the intersection is at least d^{-2d}. Brazitikos [14] improved this to $d^{-11/2}$, providing the first polynomial bound on the diameter of the intersection, and Ivanov and Naszódi [26] recently obtained a bound of $(2d)^{-3}$. Brazitikos [13] also proved, under the hypothesis that the intersections of subfamilies of size αd have diameter at least 1 (for some large enough absolute constant α), that the intersection of the entire family has diameter

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at least $cd^{-3/2}$ and strengthened the bound to $cd^{-1/2}$ under the further assumption that each set is centrally symmetric. Asymptotically optimal bounds are known for much larger subfamily intersection sizes [37].

In this manuscript we prove several new Helly-type theorems for the diameter. Our proof techniques are based on new parametrizations of these problems and do not use properties of ellipsoids. The analytic tools we use are basic results about volume concentration of balls in $\mathbb{R}^d$.

Our first result proves that the colorful strengthening of Conjecture 1.1 holds for at least a large subfamily.

Theorem 1.2. *There exists a decreasing function $\gamma\colon (0,\sqrt{2}) \to (0,1]$ such that $\gamma(c) \to 1$ as $c \to 0$ and the following holds for every $c \in (0,\sqrt{2})$, $\alpha \in (0,1]$, and $d \geq 2$. Let $\beta = 1 - \big(1-\alpha\cdot\gamma(c)\big)^{1/2d}$. Assume that $\mathcal{F}_1,\dots,\mathcal{F}_{2d}$ are finite families of convex sets in $\mathbb{R}^d$ and set $N = \prod_{i=1}^{2d}|\mathcal{F}_i|$. If $\bigcap_{i=1}^{2d} F_i$ has diameter greater than or equal to 1 for at least αN different $2d$-tuples $(F_i)_{i=1}^{2d}$ with $F_i \in \mathcal{F}_i$ for each i, then there exists an index $k \in [2d]$ and a subfamily $\mathcal{G} \subseteq \mathcal{F}_k$ such that $|\mathcal{G}| \geq \beta|\mathcal{F}_k|$ and the diameter of $\bigcap \mathcal{G}$ is greater than or equal to $cd^{-1/2}$.*

We exclude the case $d = 1$ since the real line has an exact diameter Helly-type theorem, while all higher dimensions do not. The function γ is related to volume concentration properties of d-dimensional balls, and is described in Section 4.

The description "colorful," which dates back to Lovász's original colorful Helly theorem [6], is derived from thinking of each family $\mathcal{F}_i$ as having a particular color. Then, if the intersections of sufficiently many colorful collections (containing one set of each color) have large diameter, there is a large monochromatic family whose intersection has large diameter. Taking $\mathcal{F}_i = \mathcal{F}$ for each $i \in [2d]$ gives a traditional fractional version of Conjecture 1.1. In this respect, the conjecture seems to be an outlier among its peers: while the proof of many fractional Helly-type theorems relies on the corresponding "standard" Helly-type theorem, in this case the standard theorem appears more difficult.

Crucially, $\beta \to 1$ as $\alpha \to 1$ and $c \to 0$ in Theorem 1.2, so the size of the subfamily can be arbitrarily close to that of the original set. The value of β is a consequence of Bulavka, Goodarzi, and Tancer's recent work on the colorful fractional Helly theorem [15], which builds on research by Kalai [28, 29]; Bárány, Fodor, Montejano, Oliveros, and Pór [7]; and Kim [31].

We make no assumptions on the convex sets involved. The proof techniques based on John ellipsoids have, for diameter Helly theorems, a loss factor on the diameter at least as large as the Banach-Mazur distance between the intersection of the family $\mathcal{F}$ and its John ellipsoid. This can be linear in the dimension d. (The results of Fernández Vidal, Galicer, and Mrezbacher [22] show the factors obtained if we interpolate between several results regarding John decompositions of the identity.) We obtain estimates with a loss factor proportional to $d^{1/2}$ by first controlling the diameter in fixed directions and then using analytic aspects of convex sets.

Theorem 1.2 suggests that it might be possible to extend the diameter conjecture to a version that is both fractional and colorful.

Conjecture 1.3. *There is an absolute constant $c > 0$ such that, for each $d \geq 2$ and $\alpha \in (0,1]$, there is a constant $\beta > 0$ that satisfies the following. Assume that $\mathcal{F}_1,\dots,\mathcal{F}_{2d}$ are finite families of convex sets in $\mathbb{R}^d$ and set $N = \prod_{i=1}^{2d}|\mathcal{F}_i|$. If $\bigcap_{i=1}^{2d} F_i$ has diameter greater than or equal to 1 for at least αN different $2d$-tuples $(F_i)_{i=1}^{2d}$ with $F_i \in \mathcal{F}_i$ for each i, then there exists an index $k \in [2d]$ and a subfamily $\mathcal{G} \subseteq \mathcal{F}_k$ such that $|\mathcal{G}| \geq \beta|\mathcal{F}_k|$ and the diameter of $\bigcap \mathcal{G}$ is greater than or equal to $cd^{-1/2}$. Furthermore, $\beta \to 1$ as $\alpha \to 1$ for each fixed d.*

Even the case $\alpha = 1$ is open and interesting.

In another direction, any Helly-type theorem for diameter necessarily entails some loss—it is not possible to conclude that intersection of the entire family has diameter at least 1 even by checking arbitrarily large subfamilies [37]. Sarkar, Xue, and Soberón [35] suggested that this may be a consequence of the norm used to measure diameter, and that the ℓ_1 norm may give exact Helly-type diameter results. We show that this is indeed the case.

Theorem 1.4. *Let ρ be a norm in $\mathbb{R}^d$ whose unit ball is a polytope with k facets, and let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If the intersection of every kd or fewer members of $\mathcal{F}$ has ρ-diameter greater than or equal to 1, then $\bigcap \mathcal{F}$ has ρ-diameter greater than or equal to 1. Moreover, this statement is not true if kd is replaced by $kd - 1$.*

In particular, there is an exact diameter Helly-type theorem in the ℓ_1-norm, although the intersection condition on subfamilies is necessarily exponential. We present three proofs of Theorem 1.4, one of which implies a colorful version (see Theorem 3.4). In Theorem 3.7, we prove that no other norm admits an exact Helly-type theorem for diameter, thus characterizing the norms for which an exact theorem is possible.

The particular case of the ℓ_∞ norm implies a different relaxation of Conjecture 1.1.

Corollary 1.5. *Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If the intersection of every $2d^2$ or fewer elements of $\mathcal{F}$ has diameter greater than or equal to 1, then $\bigcap \mathcal{F}$ has diameter greater than or equal to $d^{-1/2}$.*

If we relax Conjecture 1.1 to checking subfamilies of quadratic cardinality in the dimension, then an application of Nászodi's method guarantees only that the diameter of $\bigcap \mathcal{F}$ is at least d^{-1} (see, e.g., [22, Theorem 1.4]). To obtain a bound of $d^{-1/2}$, the method would require that each set be centrally symmetric.

Finally, we investigate a discrete analogue of diameter Helly-type theorems. Doignon extended Helly's theorem to the integer lattice [21], showing that *if the intersection of every 2^d or fewer elements of a finite family of convex sets in $\mathbb{R}^d$ contains an integer point, then the entire intersection also contains an integer point.* This result was proved independently by Bell [11] and by Scarf [36]. In most cases, the aim of quantitative Helly-type theorems for the integer lattice is to bound the number of integer points in the intersection of a family of convex sets [2, 4, 19, 20].

Such work can be thought of as Helly-type theorems for "discrete volume." We think of a convex set as having large "discrete diameter" if it contains many colinear integer points. In contrast to most continuous diameters, there is an exact Helly-type theorem for discrete diameter.

Theorem 1.6. *Let k be a positive integer and $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If the intersection of every 4^d or fewer elements of $\mathcal{F}$ contains k colinear integer points, then $\bigcap \mathcal{F}$ contains k colinear integer points.*

Our proof also implies a colorful version of Theorem 1.6. Doignon's theorem shows that the size of the subfamilies in the hypothesis is necessarily exponential in the dimension, but this size can be significantly reduced if it suffices to maintain a bound on the diameter of a large subfamily of $\mathcal{F}$.

Corollary 1.7. *For every positive integer d and real number $\alpha \in (0, 1]$, there exists a real number $\beta = \beta(\alpha, d) > 0$ such that the following holds. Assume that $\mathcal{F}$ is a finite family of convex sets in $\mathbb{R}^d$ and let k be a positive integer. If $\bigcap \mathcal{H}$ contains at least k colinear integer points for at least $\alpha\binom{|\mathcal{F}|}{2d+1}$ subcollections $\mathcal{H} \subseteq \mathcal{F}$ of $2d + 1$*

sets, then there exists a subfamily $\mathcal{G} \subseteq \mathcal{F}$ such that $|\mathcal{G}| \geq \beta|\mathcal{F}|$ whose intersection contains k colinear integer points.

Since Doignon's theorem is optimal, the proportion $\beta(1,d)$ is necessarily strictly less than 1.

We present the proof of Theorem 1.6 in Section 2. Our results for general norms are collected in Section 3, and Section 4 contains the proof of Theorem 1.2. We conclude in Section 5 by discussing additional aspects one of our key building blocks (Theorem 3.1) and presenting some open questions.

2. Discrete diameter results

The proofs in this section employ similar methods to those of Sarkar, Xue, and Soberón in [35], in which a suitable parametrization reduces quantitative Helly-type theorems to standard Helly-type theorems in higher-dimensional spaces.

We denote the standard inner product in $\mathbb{R}^d$ by $\langle \cdot, \cdot \rangle$.

Proof of Theorem 1.6. Let $v \in \mathbb{R}^d$ be a vector whose components are algebraically independent. In particular, $\langle v, z\rangle \neq 0$ for every $z \in \mathbb{Z}^d \setminus \{0\}$. For every convex set $K \subseteq \mathbb{R}^d$, we define the set

$$S(K) = \big\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x \in K,\ x + (k-1)y \in K,\ \langle v, y\rangle > 0\big\},$$

which is convex and nonempty. Moreover, if $x \in K$, $x+(k-1)y \in K$, and $\langle y, v\rangle \neq 0$, then either $(x,y) \in S(K)$ or $(x+(k-1)y, -y) \in S(K)$. Now consider the family

$$\mathcal{G} = \{S(K) : K \in \mathcal{F}\}.$$

The conditions of the theorem imply that the intersection of every 2^{2d} or fewer sets in $\mathcal{G}$ contains a point of $\mathbb{Z}^{2d}$ in their intersection. By Doignon's theorem, $\bigcap \mathcal{G}$ contains an integer point. If (x,y) is such a point, then $y \neq 0$ and the k colinear integer points $x, x+y, \ldots, x+(k-1)y$ are contained in every member of $\mathcal{F}$. $\square$

The proof above is quite malleable. For example, replacing Doignon's theorem with its colorful version (proved by De Loera, La Haye, Oliveros, and Roldán-Pensado [17]) yields a colorful version of Theorem 1.6. To obtain Corollary 1.7, we replace Doignon's theorem by the following fractional version.

Theorem 2.1 (Bárány, Matoušek 2003 [10])**.** *For every positive integer d and real number $\alpha \in (0,1]$ there exists a real number $\beta = \beta(\alpha, d) > 0$ such that the following is true. If $\mathcal{F}$ is a finite family of convex sets such that $\bigcap \mathcal{H}$ contains an integer point for at least $\alpha\binom{|\mathcal{F}|}{d+1}$ subcollections $\mathcal{H} \subseteq \mathcal{F}$ of $d+1$ sets, then there is a subfamily $\mathcal{G} \subseteq \mathcal{F}$ with at least $\beta|\mathcal{F}|$ sets whose intersection contains an integer point.*

It is unclear whether the number 4^d in Theorem 1.6 is optimal. Since the case $k=1$ is Doignon's theorem, 4^d cannot be replaced by anything smaller than 2^d. The following construction, which generalizes a construction communicated by Gennadiy Averkov for $d=2$, improves the lower bound to $d2^d$.

Claim 2.2. *For each $d \geq 1$, there exists a finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ such that the intersection of any $d2^d - 1$ sets in $\mathcal{F}$ contains three colinear integer points but $\bigcap \mathcal{F}$ does not.*

Proof. Let $R \subseteq \{0,1,2,3\}^d$ be the collection of integer points where exactly one coordinate is in $\{0,3\}$ and let $Q = \{1,2\}^d$. That is, $Q \cup R$ is the set of integer points in the hypercube $[0,3]^d$ that do not lie on its $(d-2)$-skeleton. We define

$$\mathcal{F} = \{\operatorname{conv}(Q \cup R \setminus \{x\}) : x \in R\},$$

which is a collection of $d2^d$ sets. The intersection of any $d2^d - 1$ sets in $\mathcal{F}$ contains every point in Q and one point in R, so it contains 3 colinear integer points. But the integer points in $\bigcap \mathcal{F}$ are exactly those in Q, which does not contain 3 colinear integer points. $\square$

3. Diameter results for general norms

In this section, ρ represents a norm on $\mathbb{R}^d$. The *ρ-diameter* of a compact set $K \subseteq \mathbb{R}^d$ is

$$\operatorname{diam}_\rho(K) = \max\{\rho(x-y) : x, y \in K\}.$$

Given a vector $v \in \mathbb{R}^d \setminus \{0\}$, the *$v$-width* of K is $\max_{x,y\in K}\langle x-y, v\rangle$. Suppose $K \subseteq \mathbb{R}^d$ is a centrally symmetric polytope with k facets, and for each facet L_i, let v_i be the vector such that $\{x \in \mathbb{R}^d : \langle x, v_i\rangle = 1\}$ is the hyperplane containing L_i. We assume that L_i and $L_{(k/2)+i}$ are opposing facets, so $v_{(k/2)+i} = -v_i$. If K is the unit ball of ρ, then

$$\rho(x) = \max_{1\le i\le k/2} |\langle x, v_i\rangle| \tag{3.1}$$

for every $x \in \mathbb{R}^d$.

The proof in Section 2 can be adapted to simplify the proof of a Helly-type theorem for v-width by the second author [37].

Theorem 3.1. *Let v be a nonzero vector in $\mathbb{R}^d$ and $\mathcal{F}$ be a finite family of compact convex sets in $\mathbb{R}^d$. If the intersection of every $2d$ sets in $\mathcal{F}$ has v-width greater than or equal to 1, then $\bigcap \mathcal{F}$ has v-width greater than or equal to 1.*

Proof. For every convex set $K \subseteq \mathbb{R}^d$, let

$$S(K) = \big\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x \in K,\ x+y \in K,\ \langle y, v\rangle = 1\big\}.$$

The set $S(K)$ is convex and nonempty. Consider the set $\mathcal{G} = \{S(K) : K \in \mathcal{F}\}$; this family is contained in an affine subspace of dimension $2d-1$ by the condition $\langle y, v\rangle = 1$. The hypothesis of the theorem implies that every $2d$ or fewer elements of $\mathcal{G}$ intersect, so $\bigcap \mathcal{G}$ is nonempty by Helly's theorem. If $(x,y) \in \bigcap \mathcal{G}$, then x, $x+y \in F$ for every set $F \in \mathcal{F}$, which shows that the v-width of $\bigcap \mathcal{F}$ is at least 1. $\square$

The earlier proof of Theorem 3.1 does not use the parametrization above, but instead relies on simple arguments of "sweeping hyperplanes". While this technique can be used to prove colorful or fractional versions, the fractional versions fall short of optimal bounds [37].

In contrast, we can substitute the use of Helly's theorem with almost any of its generalizations in the proof above to obtain corresponding versions of Theorem 3.1. For example, we can use the optimal colorful and fractional Helly theorem.

Theorem 3.2 (Bulavka, Goodrazi, Tancer 2020 [15])**.** *For every positive integer d and real number $\alpha \in (0,1]$, the following holds with $\beta = 1-(1-\alpha)^{1/(d+1)}$. Assume that $\mathcal{F}_1, \dots, \mathcal{F}_{d+1}$ are finite families of convex set in $\mathbb{R}^d$ and set $N = \prod_{i=1}^{d+1}|\mathcal{F}_i|$. If $\bigcap_{i=1}^{d+1} F_i$ is not empty for at least αN different $(d+1)$-tuples $(F_i)_{i=1}^{d+1}$ with $F_i \in \mathcal{F}_i$ for each i, then there exists an index $k \in [d+1]$ and a subfamily $\mathcal{G} \subseteq \mathcal{F}_k$ such that $|\mathcal{G}| \ge \beta|\mathcal{F}_k|$ whose intersection is not empty.*

We immediately obtain the following theorem.

Theorem 3.3. *For every positive integer d and real number $\alpha \in (0,1]$, the following holds with $\beta = 1-(1-\alpha)^{1/2d}$ for every nonzero vector $v \in \mathbb{R}^d$. Assume that $\mathcal{F}_1, \dots, \mathcal{F}_{2d}$ are finite families of convex sets in $\mathbb{R}^d$ and set $N = \prod_{i=1}^{2d}|\mathcal{F}_i|$. If $\bigcap_{i=1}^{2d} F_i$ has v-width greater than or equal to 1 for at least αN different $2d$-tuples $(F_i)_{i=1}^{2d}$ with $F_i \in \mathcal{F}_i$ for each i, then there exists an index $k \in [2d]$ and a subfamily $\mathcal{G} \subseteq \mathcal{F}_k$ such that $|\mathcal{G}| \ge \beta|\mathcal{F}_k|$ whose intersection has v-width greater than or equal to 1.*

Two instances of theorem above were already known: one with $\beta = \alpha/2d$ and $\mathcal{F}_1 = \cdots = \mathcal{F}_{2d}$ (fractional with a weaker constant), and one with $\beta = \alpha = 1$ (colorful but not fractional) [37]. The explicit bound $\beta = 1 - (1-\alpha)^{1/2d}$ seems out of reach of the earlier proof even for $\mathcal{F}_1 = \cdots = \mathcal{F}_{2d}$. We discuss in Section 5 some additional consequences that cannot be obtained with the original proof of Theorem 3.1. We now use the Helly-type theorem for v-width to prove a colorful version of Theorem 1.4.

Theorem 3.4. *Let ρ be a norm in $\mathbb{R}^d$ whose unit ball is a polytope with k facets, and let $\mathcal{F}_1, \ldots, \mathcal{F}_{kd}$ be finite families of convex sets in $\mathbb{R}^d$. If $\bigcap_{i=1}^{kd} F_i$ has ρ-diameter at least 1 for every kd-tuple $(F_i)_{i=1}^{kd}$ such that $F_i \in \mathcal{F}_i$ for each i, then there exists an index $l \in [kd]$ such that $\bigcap \mathcal{F}_l$ has ρ-diameter at least 1. Moreover, the same statement is not true if kd is replaced by $kd-1$.*

Proof. We prove the contrapositive. Assume that $\mathcal{F}_1, \ldots, \mathcal{F}_{kd}$ are finite families of convex sets in $\mathbb{R}^d$ such that $\bigcap \mathcal{F}_i$ has ρ-diameter at most 1 for each $i \in [kd]$. We want to find a colorful kd-tuple whose intersection has ρ-diameter at most 1.

Let P be the unit ball of ρ. For each facet L_j of P, let v_j be the vector in $\mathbb{R}^d$ such that $\langle x, v_j \rangle = 1$ for every $x \in L_j$. We choose a labelling of the facets so that $L_{(k/2)+j} = -L_j$ for each $j \in [k/2]$. From the assumption on ρ-width of $\mathcal{F}_i$ and equation (3.1), the v_j-width of $\bigcap \mathcal{F}_i$ is at most 1 for each $i \in [kd]$ and $j \in [k/2]$. Applying the contrapositive of the colorful version of Theorem 3.1 (Theorem 3.3 with $\alpha = 1$) to the collection of $2d$ families $\mathcal{F}_{2(j-1)d+1}, \ldots, \mathcal{F}_{2jd}$ implies that there is a set $F_{i+2(j-1)d} \in \mathcal{F}_{i+2(j-1)d}$ for each $i \in [2d]$ such that $\bigcap_{i=1}^{2d} F_{2(j-1)d+i}$ has v_j-width less than or equal to 1.

Let $\mathcal{G}$ denote the family $\{F_1, \ldots, F_{kd}\}$. By construction, $\mathcal{G}$ has exactly one element from each $\mathcal{F}_i$. Its intersection $\bigcap \mathcal{G}$ has v_j-width at most 1 for every $j \in [k/2]$, so the ρ-diameter of $\bigcap \mathcal{G}$ is at most 1 by (3.1).

Now we prove optimality. Consider a set $\{x_1, \ldots, x_k\}$ of points in $\mathbb{R}^d$ such that x_i is in the relative interior of L_i and $x_i = -x_{(k/2)+i}$. For each $i \in [k/2]$, choose d closed half-spaces such that

- each half-space contains every point in $\{x_j\}_{j \neq i}$,
- the intersection of the d half-spaces with L_i is the singleton $\{x_i\}$, and
- the intersection of any $d-1$ of them contains a point y such that $\langle y, v_i \rangle > 1$.

Let $\mathcal{F}$ be the collection all kd half-spaces. The intersection $\bigcap \mathcal{F}$ is contained in P, so its ρ-diameter is at most 2. For any subset $\mathcal{F}' \subseteq \mathcal{F}$ of size $kd-1$, there is a facet L_i of P with at most $d-1$ corresponding half-spaces in $\mathcal{F}'$. Therefore, there exists a point $\tilde{x} \in \bigcap \mathcal{F}'$ outside of P such that the segment $0\tilde{x}$ intersects L_i. We can choose $\tilde{x}$ close enough to x_i so that the segment between $-x_i$ and $\tilde{x}$ has ρ-length greater than 2. Therefore, the ρ-diameter $\bigcap \mathcal{F}'$ is greater than 2 for every subset $\mathcal{F}' \subseteq \mathcal{F}$ of size $kd-1$, while the ρ-diameter of $\bigcap \mathcal{F}$ is at most 2. Taking $\mathcal{F}_i = \mathcal{F}$ for each $i \in [kd-1]$ shows the optimality of the parameter kd. □

Setting $\mathcal{F}_i = \mathcal{F}$ for each $i \in [kd]$ in Theorem 3.4 proves the Helly-type statement in Theorem 1.4, and the proof the optimality of kd also carries over to the monochromatic version.

Theorem 1.4 implies the following more general version of Corollary 1.5.

Theorem 3.5. *Let $p \geq 1$ and $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If the intersection of every $2d^2$ or fewer sets in $\mathcal{F}$ has ℓ_p-diameter greater than or equal to 1, then $\bigcap \mathcal{F}$ has ℓ_p-diameter greater than or equal to $d^{-1/p}$.*

Proof. A set in $\mathbb{R}^d$ with ℓ_p-diameter at least 1 has ℓ_∞-diameter at least $d^{-1/p}$. Since the unit ball in the ℓ_∞ norm is a polytope with $2d$ facets, we can employ Theorem

1.4 to conclude that the ℓ_∞-diameter of $\bigcap \mathcal{F}$ is at least $d^{-1/p}$. The ℓ_p-diameter of $\mathcal{F}$ is at least $d^{-1/p}$ as well. □

The ℓ_1 norm is a useful lens with which to compare our results. Theorem 3.5 says that we can bound the ℓ_1-diameter of $\bigcap \mathcal{F}$ by d^{-1} if we know that the intersection of every $2d^2$ sets in $\mathcal{F}$ has ℓ_1-diameter greater than or equal to 1, whereas Theorem 1.4 says that we can bound the ℓ_1-diameter of $\bigcap \mathcal{F}$ by 1 if we know that the intersection of every $d2^d$ in $\mathcal{F}$ sets has ℓ_1-diameter greater than or equal to 1. Neither of these consequences implies the other.

We now present two additional proofs of the Helly-type statement in Theorem 1.4. The first proof uses the following lemma, in which the boundary of a set $K \subseteq \mathbb{R}^d$ is denoted by ∂K.

Lemma 3.6. *Let $P \subseteq \mathbb{R}^d$ be a centrally symmetric polytope with k facets and $\mathcal{G}$ be a finite family of sets in $\mathbb{R}^{2d}$ such that*

(1) *for every facet L of P and every $K \in \mathcal{G}$, the intersection $K \cap (\mathbb{R}^d \times L)$ is convex, and*
(2) *if $x, y \in \mathbb{R}^d$, $K \in \mathcal{G}$, and $(x, y) \in K$, then $(x + y, -y) \in K$.*

If the intersection of every kd or fewer sets in $\mathcal{G}$ contains a point in $\mathbb{R}^d \times \partial P$, then $\bigcap \mathcal{G}$ contains a point in $\mathbb{R}^d \times \partial P$.

Proof. The general approach is similar to that of Radon's proof of Helly's theorem [33]. We proceed by induction on $|\mathcal{G}|$. If $|\mathcal{G}| \le kd$, there is nothing to prove, so we assume the result holds for all collections of convex sets with $n \ge kd$ members. Let $G = \{K_1, \dots, K_{n+1}\}$ be a collection of $n + 1$ convex sets in $\mathbb{R}^{2d}$ that satisfies the hypothesis of the lemma. The induction hypothesis implies that for each $i \in [n+1]$ there is a point $(x_i, y_i) \in \bigcap(\mathcal{G} \setminus K_i)$ such that $y_i \in \partial P$. Grouping the facets of P by opposing pairs, there must be a pair of facets $L, -L$ whose union contains at least

$$\frac{n+1}{k/2} > 2d$$

points in $\{y_i\}_{i=1}^{n+1}$. By replacing (x_i, y_i) by $(x_i + y_i, -y_i)$ if necessary, the facet L contains at least $2d+1$ points in $\{y_i\}_{i=1}^{n+1}$. Therefore $\mathbb{R}^d \times L$ contains at least $2d+1$ points in $\{(x_i, y_i)\}_{i=1}^{n+1}$. Since $\mathbb{R}^d \times L$ is a $(2d-1)$-dimensional affine subspace, applying Radon's lemma to these $2d+1$ points yields a partition of them into two sets A and B whose convex hulls intersect. Any point in $\operatorname{conv}(A) \cap \operatorname{conv}(B)$ is in $\bigcap \mathcal{G}$ as well as $\mathbb{R}^d \times L \subseteq \mathbb{R}^d \times \partial P$. □

Second proof of Theorem 1.4. Let P be the unit ball of ρ. For each convex set $K \subseteq \mathbb{R}^d$, let

$$S(K) = \big\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x \in K,\ x + y \in K,\ \rho(y) = 1\big\}.$$

The conditions of Theorem 1.4 ensure that we can apply Lemma 3.6 to the family $\mathcal{G} = \{S(K) : K \in \mathcal{F}\}$. Given a point $(x, y) \in \bigcap \mathcal{G} \cap (\mathbb{R}^d \times \partial P)$, every set in $\mathcal{F}$ contains the segment between x and $x + y$; since $\rho(y) = 1$, the ρ-diameter of $\bigcap \mathcal{G}$ is at least 1. □

Our final proof of Theorem 1.4 relies on a limit argument.

Third proof of Theorem 1.4. We may assume without loss of generality that every set in $\mathcal{F}$ is compact. We denote by K^n the n-fold product of the set K. Let $f\colon (\mathbb{R}^d)^k \to \mathbb{R}$ be defined by

$$f(x_1, y_1, \dots, x_{k/2}, y_{k/2}) = \sum_{i=1}^{k/2} \langle y_i - x_i, v_i \rangle.$$

If a set $K \subseteq \mathbb{R}^d$ has ρ-diameter greater than or equal to 1, then it has v_i-width at least 1 for some $i \in [k/2]$. Therefore there exists a point $\bar{x} \in K^k$ such that $f(\bar{x}) \geq 1$. For each $K \in \mathcal{F}$, consider the set

$$S(K) = \{\bar{x} \in K^k : f(\bar{x}) = 1\},$$

which is convex and lies in an affine subspace of dimension $kd - 1$. An application of Helly's theorem implies that $(\bigcap \mathcal{F})^k$ contains a point $\bar{a}_1$ with $f(\bar{a}_1) = 1$.

Now consider the function $g\colon (\mathbb{R}^d)^k \to \mathbb{R}$ defined by

$$g(x_1, y_1, \ldots, x_{k/2}, y_{k/2}) = \max_{1 \leq i \leq k/2} \langle y_i - x_i, v_i \rangle.$$

If $g(\bar{a}_1) = 1$, we are done. Otherwise, equation (3.1) implies that that for each kd-tuple $\mathcal{F}' \subseteq \mathcal{F}$, there are two points $x, y \in \bigcap \mathcal{F}'$ and an index $i \in [k/2]$ such that $\langle y - x, v_i \rangle \geq 1$. Replacing the corresponding coordinates of $\bar{a}_1$ by x, y, we obtain a new point $\bar{x} \in (\bigcap \mathcal{F}')^k$ such that $f(\bar{x}) \geq 1 + (1 - g(\bar{a}_1))$.

Bootstrapping the previous arguments, we can find a point $\bar{a}_2 \in (\bigcap \mathcal{F})^k$ such that $f(\bar{a}_2) = 2 - g(\bar{a}_1)$. Iterating this argument creates a sequence $(\bar{a}_n)_{n=1}^{\infty}$ in $(\bigcap \mathcal{F})^k$ such that

$$f(\bar{a}_n) \geq n - \sum_{i=1}^{n-1} g(\bar{a}_i)$$

for each $n \in \mathbb{N}$. Let $\beta_n = \max\{g(\bar{a}_1), \ldots, g(\bar{a}_n)\}$. We have

$$\beta_n \geq g(\bar{a}_n) \geq \frac{f(\bar{a}_n)}{d} \geq \frac{n - \sum_{i=1}^{n-1} g(\bar{a}_i)}{d} \geq \frac{n - (n-1)\beta_n}{d}.$$

In other words, the ρ-diameter of $\bigcap \mathcal{F}$ is at least $\beta_n \geq n/(d+n-1)$ for every $n \geq 1$. Taking the limit as $n \to \infty$ finishes the proof. □

The next result shows that there is no exact Helly-type theorem for diameter for any norm whose unit ball is not a polytope.

Theorem 3.7. *Let ρ be a norm in $\mathbb{R}^d$ whose unit ball is not a polytope. Then, for every integer n there exists a finite family $\mathcal{G}$ of convex sets such that the intersection of every n or fewer sets in $\mathcal{G}$ has ρ-diameter greater than or equal to 1, but the ρ-diameter of $\bigcap \mathcal{G}$ is strictly less than 1.*

Proof. Let P be the unit ball of ρ and $\mathcal{F}$ be the infinite family of closed containment-minimal half-spaces that contain P. We parametrize $\mathcal{F}$ using the unit sphere S^{d-1} by associating each vector $x \in S^{d-1}$ with the half-space $H_x \in \mathcal{F}$ whose bounding hyperplane is perpendicular to it and that contains an infinite ray in the direction of x. For any finite family $\mathcal{F}' \subseteq \mathcal{F}$, the unit ball P is strictly contained in $\bigcap \mathcal{F}'$, so the ρ-diameter of $\bigcap \mathcal{F}'$ is strictly larger than 2.

We define a function $f\colon (S^{d-1})^n \to \mathbb{R}$ by

$$f(x_1, \ldots, x_n) = \min\left\{100,\ \operatorname{diam}_\rho\Big(\bigcap_{i=1}^{n} H_{x_i}\Big)\right\}.$$

The minimum ensures that f is well-defined when $\bigcap_{i=1}^{n} H_{x_i}$ is unbounded. The function f is continuous, and $f(x_1, \ldots, x_n) > 2$ for every n-tuple in $(S^{d-1})^n$. Since the domain of f is compact, f attains a minimum value $s_n > 2$.

Let $\varepsilon = (s_n - 2)/3$. Standard results on approximation of convex sets by polytopes show that there exists a polytope Q such that $P \subset Q \subset (1+\varepsilon)P$. In particular, $\operatorname{diam}_\rho(Q) \leq (1+\varepsilon)\operatorname{diam}_\rho(P) = 2(1+\varepsilon) < s_n$.

We define $\mathcal{G} \subseteq \mathcal{F}$ to be the family of half-spaces in $\mathcal{F}$ whose bounding hyperplanes are parallel to some facet of Q, scaled by a factor of $1/s_n$. The intersection of every

n or fewer sets in $\mathcal{G}$ has ρ-diameter greater than or equal to 1. But $\bigcap \mathcal{G} \subseteq (1/s_n)Q$, so its ρ-diameter is strictly less than 1. $\square$

4. Diameter results for $2d$-tuples

We combine Theorem 3.1 with volume concentration properties of balls to prove Theorem 1.2. The properties described below can be found in Keith Ball's expository notes [5].

Let B be a ball centered at the origin, $c > 0$, and u be a unit vector. The *c-cap* of B in the direction u is

$$C(B, c, u) = \{x \in B : \langle x, u\rangle \geq c\}.$$

For two unit vectors u and v, we have that $v \in C(B, c, u)$ if and only if $u \in C(B, c, v)$.

Let B_d be the unit ball in $\mathbb{R}^d$, and let r_d be the radius of a volume-one ball in $\mathbb{R}^d$. Asymptotically, $r_d \sim d^{1/2}/\sqrt{2\pi e}$. For a fixed unit vector u and real number x, the $(d-1)$-dimensional volume of the intersection of $r_d B_d$ with the hyperplane $\{y \in \mathbb{R}^d : \langle u, y\rangle = x\}$ converges to $\sqrt{e}\exp(-\pi e x^2)$ as $d \to \infty$. In other words, the volume of the region $\{y \in \mathbb{R}^d : |\langle y, u\rangle| < x\} \cap r_d B_d$ converges as $d \to \infty$, and converges to zero as $x \to 0$. For any fixed constant c, we define

$$\gamma(c) = \inf_{d\geq 2} \operatorname{vol}\left[C\left(r_d B_d, \frac{c r_d}{\sqrt{d}}, u\right) \cup C\left(r_d B_d, \frac{c r_d}{\sqrt{d}}, -u\right)\right] \tag{4.1}$$

$$= \frac{1}{\operatorname{vol}(B_d)} \inf_{d\geq 2} \operatorname{vol}\left[C\left(B_d, \frac{c}{\sqrt{d}}, u\right) \cup C\left(B_d, \frac{c}{\sqrt{d}}, -u\right)\right] \tag{4.2}$$

This is the remaining volume of $r_d B_d$ after removing a slab centered at the origin with width approximately $2c/\sqrt{2\pi e}$ (see Figure 1). From the discussion above, $\gamma(c) \to 1$ as $c \to 0$. Equation (4.2) shows that $\gamma(c)$ is the fraction of the volume of two opposite $(cd^{-1/2})$-caps in the unit sphere. If $c < \sqrt{2}$, the volume of $C\big(r_d B_d, c r_d d^{-1/2}, u\big)$ is strictly positive for each $d \geq 2$ and tends to a positive limit as $d \to \infty$. So $\gamma(c) > 0$ for every $c \in (0, \sqrt{2})$.

Proof of Theorem 1.2. Recall that $N = \prod_{i=1}^{2d} |\mathcal{F}_i|$. For each colorful collection $\mathcal{F}' = (F_i)_{i=1}^{2d}$ whose intersection has diameter greater than or equal to 1, assign a unit vector $u_{\mathcal{F}'}$ such that $\bigcap \mathcal{F}'$ contains a unit segment with direction $u_{\mathcal{F}'}$. Let $\mathcal{G}$ be the collection of sets

$$C\Big(B_d, cd^{-1/2}, u_{\mathcal{F}'}\Big) \cup C\Big(B_d, cd^{-1/2}, -u_{\mathcal{F}'}\Big)$$

where $\mathcal{F}'$ is a $2d$-tuple of $\mathcal{F}$ whose intersection has diameter greater than or equal to 1. Each such set covers at least a $\gamma(c)$ fraction of the volume of the unit ball. Therefore, the total volume covered amongst all sets in $\mathcal{G}$ is at least

$$\gamma(c) \cdot \alpha N.$$

Since $\bigcup \mathcal{G}$ does not contain the origin, there is a nonzero point x in the unit ball covered at least $\gamma(c) \cdot \alpha N$ times by $\mathcal{G}$. Setting $v = x/||x||$, the set

$$C\Big(B_d, cd^{-1/2}, v\Big) \cup C\Big(B_d, cd^{-1/2}, -v\Big)$$

contains at least $\gamma(c) \cdot \alpha N$ different vectors $u_{\mathcal{F}'}$. Thus, the intersection of at least $\gamma(c) \cdot \alpha N$ different colorful $2d$-tuples of $\mathcal{F}$ have v-width greater than or equal to $cd^{-1/2}$. An application of Theorem 3.3 finishes the proof. $\square$

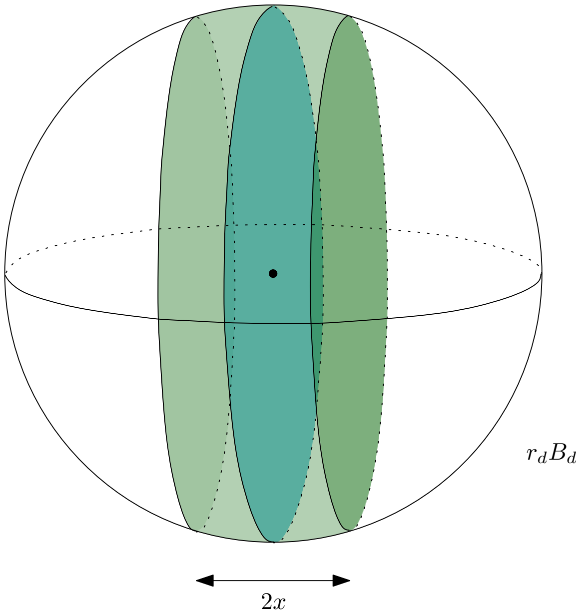

FIGURE 1. The volume of the region in $r_d B_d$ between two parallel hyperplanes at distance x from the origin converges as $d \to \infty$.

5. ADDITIONAL REMARKS

Kalai and Meshulam proved a topological extension of the colorful Helly theorem [30] in which the conditions on the intersecting subfamilies of convex sets are given by an arbitrary matroid. Their theorem is called "topological" because it is actually a statement about simplicial complexes that satisfy certain topological properties; in particular, the nerve complexes of convex sets satisfy these conditions. Given a finite set V, a *matroid* or *matroidal complex* M on V is a family of subsets of V with three properties:

- $\emptyset \in M$,
- if $A \subseteq B$ and $B \in M$, then $A \in M$, and
- if $A, B \in M$ and $|B| > |A|$, then there exists an element $a \in B \setminus A$ such that $A \cup \{a\} \in M$.

We call the sets in M *independent.* For a subset $V' \subseteq V$ we denote by $\rho(V')$ the *rank* of V', which is the cardinality of the largest independent set contained in V'. A direct application of Kalai and Meshulam's main result in our proof of Theorem 3.1 yields the following theorem.

Theorem 5.1. *Let M be a matroid on a finite set V with rank function ρ, and let v be a non-zero vector in $\mathbb{R}^d$. For each $x \in V$, let F_x be a convex set in $\mathbb{R}^d$. If the v-width of $\bigcap_{x\in V'} F_x$ is at least 1 for each set $V' \in M$, then there exists a set $\tau \subseteq V$ such that $\rho(V \setminus \tau) \leq 2d-1$ for which $\bigcap_{x\in\tau} F_x$ has v-width at least 1.*

As an example, consider M to be a partition matroid with rank $2d$. In this case, the vertices are split into $2d$ pairwise disjoint subsets: $V = V_1 \cup \cdots \cup V_{2d}$. A subset $V' \subseteq V$ is independent if an only if it contains at most one element from each V_i.

Using this matroid in Theorem 5.1 implies the colorful version of Theorem 3.1 (i.e., the case $\alpha = 1$ for Theorem 3.3). It is unclear what a "fractional and topological" Helly theorem would mean, but the theorem above indicates that there should be a topological Helly for the diameter.

Conjecture 5.2. *There exists an absolute constant $c > 0$ such that the following statement holds. Let M be a matroid on a finite set V with rank function ρ and for each $x \in V$, let F_x be a convex set in $\mathbb{R}^d$. If the diameter of $\bigcap_{x\in V'} F_x$ is at least 1 for each set $V' \in M$, then there exists a set $\tau \subseteq V$ such that $\rho(V \setminus \tau) \le 2d - 1$ for which $\bigcap_{x\in\tau} F_x$ has diameter at least $cd^{-1/2}$.*

If true, the conjecture above would imply the Bárány, Katchalski, Pach conjecture. We prove a weaker version of this conjecture.

Theorem 5.3. *Let M be a matroid on a finite set V with rank function ρ. For each $x \in V$, let F_x be a convex set in $\mathbb{R}^d$. If the diameter of $\bigcap_{x\in V'} F_x$ is at least 1 for each set $V' \in M$, then there exists a set $\tau \subseteq V$ such that $\rho(V \setminus \tau) \le \frac{d(d+3)}{2} - 1$ for which $\bigcap_{x\in\tau} F_x$ has diameter at least d^{-1}.*

Proof. Let $\mathcal{P}_d$ be the set of $d \times d$ positive semidefinite matrices whose trace is equal to 1. The dimension of this space is $\frac{d(d+1)}{2} - 1$. For each convex set $K \subseteq \mathbb{R}^d$, let

$$S(K) = \{(a, A) \in \mathbb{R}^d \times \mathcal{P}_d : a + (1/2)AB_d \subseteq K\} \subseteq \mathbb{R}^{[d(d+3)/2]-1}.$$

Notice that $S(K)$ is convex. Also, the diameter of $a + (1/2)AB_d$ is equal to the largest eigenvalue of A. We now apply the topological Helly theorem to the family of sets $\{S(F_x) : x \in V\}$. This implies that there exists a set $\tau \subseteq V$ such that $\rho(V \setminus \tau) \le \frac{d(d+3)}{2} - 1$ and $\bigcap_{x\in\tau} F_x$ contains an ellipsoid of the form $a + (1/2)AB_d$ for some $a \in \mathbb{R}^d$ and $A \in \mathcal{P}_d$. The trace of A is equal to 1, so its largest eigenvalue is at least $1/d$. In other words, the diameter of the ellipsoid is at least $1/d$. □

Other results that follow from similar parametrizations of ellipsoids can be found in [35]. Even though this result falls short of the conjecture mentioned, it implies the bounds which are a direct consequence of John's theorem (see, e.g., the discussion after Theorem 1.4 of [22]). For example, suppose $\mathcal{F}$ is a finite family of convex sets in which the intersection of any $\frac{d(d+3)}{2}$ of its members has diameter at least 1. We define a matroid M that has $\frac{d(d+3)}{2}$ vertices corresponding to each set in $\mathcal{F}$ and declare a set of vertices independent if its cardinality is at most $\frac{d(d+3)}{2}$. Theorem 5.3 implies that the diameter of the intersection of the whole family is at least d^{-1}.

It would be interesting to see extensions of v-width and diameter results in further directions. For example, Adiprasito, Bárány, Mustafa, and Terpai [1] introduced versions of Carathéodory's, Helly's, and Tverberg's theorems whose conclusions are independent of the ambient dimension. What might versions of Helly's theorem for v-width and diameter look like within this framework?

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